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The boundary behavior of $\mathcal{Q}_{p,q}$-homeomorphisms

This article studies systematically the boundary correspondence problem for $\mathcal{Q}_{p,q}$-homeomorphisms. The presented example demonstrates a deformation of the Euclidean boundary with the weight function degenerating on the boundary.

Bibliography: 72 titles.

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Introduction

In this section, we briefly survey the articles dealing with the boundary behavior of mappings in quasiconformal analysis. Consider two domains $D, D' \subset \mathbb{R}^2$ bounded by Jordan curves and a conformal mapping $f\colon D \to D'$. The classical result, established independently by Carathéodory [1] and Osgood and Taylor [2], asserts that f extends to the boundary, giving a homeomorphism $\overline{f}\colon \overline{D} \to \overline{D'}$. The Jordan condition for the boundary is necessary, which is easy to see in the example of a slit disk. Nevertheless, a homeomorphic extension is possible for some generalized boundary accounting for the geometry of the domain. This construction, introduced by Carathéodory [1] and called the *prime end boundary*, initiated intensive applications of the geometric approach to study the boundary behavior of mappings.

Carathéodory's prime end theory received developments on the plane $\mathbb{R}^2$ [3], [4] and in the space $\mathbb{R}^n$ for $n > 2$ [5], [6], in studying Dirichlet problems for elliptic equations [7], and in the theory of dynamical systems [8], [9]. For more detailed surveys of the available results and literature, see [10]–[13].

A natural development of these questions is to study the boundary behavior of quasiconformal mappings in space. It requires a more refined analysis of the geometric properties of domains. Indeed, in the higher-dimensional case there exist a Jordan domain and a quasiconformal mapping admitting no homeomorphic extension to the boundary of this domain [14]. In some questions it turned out helpful to describe the geometric properties of domains using the concept of modulus of

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a curve family [15]. With that, a simple classification of boundary points was introduced: for instance, the properties of the boundary to be quasiconformally flat or quasiconformally accessible in [16], [17], or properties $P1$ and $P2$ of [18]. This approach became widely used in the last decade to study the geometric properties of mappings. Let us mention only some articles concerning the boundary correspondence of quasiconformal mappings [19], [20], Q-homeomorphisms, see the book [21] and the articles [13], [22] (a more detailed discussion appears in Section 4), as well as the mappings satisfying generalized modular inequalities [23].

An alternative functional-geometric approach to study the boundary behavior of quasiconformal mappings is based on the relation between the Euclidean geometry of the domain and the functional space L^1_n via the concept of the variational capacity of a condenser. This approach was founded in [24]–[26] and applied also to studying mappings which are not quasiconformal [27]. As [17] shows, the functional-geometric approach can be interpreted in the language of moduli of curve families.

The three main approaches to the boundary behavior of mappings, using prime ends, geometric description, and functional-geometric definition, form an hierarchy, as each of them adequately describes the boundary behavior of certain classes of mappings. This article studies the problem of boundary correspondence for $\mathcal{Q}_{p,q}$-homeomorphisms, whose fundamental properties were established in [29]–[34]. To this end, we complete the domains in special capacity metrics on the image and the preimage, associated with the geometry of a suitable Sobolev class. The elements adjoined to the domain in the completion of the corresponding metric space constitute an improper boundary, which we call the *capacity boundary* H_ρ.

In § 2 the study of the boundary behavior of the homeomorphism $f \in \mathcal{Q}_{p,q}$ defined in § 1 consists in:

(1) continuing f to the capacity boundary H_ρ, with the main result stated as Theorem 2.22;

(2) establishing a connection between the elements of the capacity boundary and the points of the Euclidean boundary of the domain, see Theorem 2.37 and Corollaries 2.38 and 2.39.

In § 3 we compare the approaches stated in the languages of moduli and capacity. In § 4 we contrast the conclusions of this article with the main results of other approaches to the problem of boundary behavior of mappings. Some applications of our results are given in § 5.

This article naturally enters the line of publications [28]–[36], preceded by the results of [37]–[39] and the articles cited in the bibliographies in [28]–[34] and arising on the crossroads of the theory of Sobolev function spaces [40], [41] and geometric theory of functions [18], [42]–[48]. Some results of this series of articles have found applications in nonlinear elasticity, see [49].

§ 1. Classes of $\mathcal{Q}_{p,q}$-homeomorphisms

In what follows D and D' stand for domains (open connected sets) in $\mathbb{R}^n$. The norm $|x|_p$ of a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is defined as $|x|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ for $p \in [1, \infty)$ and $|x|_\infty = \max_{k=1,\dots,n} |x_k|$. A ball in the norm $|x|_2$ is a Euclidean ball, while in the norm $|x|_\infty$ it is a Euclidean cube.

1.1. Definitions of Sobolev spaces and the capacity of condensers. For the general theory of Sobolev spaces, the reader is referred to [40], [41]. We recall that a function $u\colon D\to\mathbb{R}$ is of *Sobolev class* $L^1_p(D)$ if $u\in L_{1,\mathrm{loc}}(D)$, meaning that $u\in L_1(U)$ for every domain U compactly embedded into D, written $U\Subset D$, and it has the generalized derivatives $\partial u/dx_j\in L_{1,\mathrm{loc}}(D)$ for every $j=1,\dots,n$ and finite seminorm

$$\|u\mid L^1_p(D)\|=\left(\int_D|\nabla u(y)|^p\,dy\right)^{1/p},\qquad 1\leqslant p\leqslant\infty,$$

where $\nabla u(y)=(\partial u/dx_1,\partial u/dx_2,\dots,\partial u/dx_n)$ is the generalized gradient of u. A mapping $\varphi=(\varphi_1,\dots,\varphi_n)\colon D\to\mathbb{R}^n$ belongs to the *Sobolev class* $W^1_{p,\mathrm{loc}}(D;\mathbb{R}^n)$ whenever $\varphi_j(x)\in L_{p,\mathrm{loc}}(D)$ and $\partial\varphi_j/dx_i\in L_{p,\mathrm{loc}}(D)$ for all j, $i=1,\dots,n$.

We say that a mapping $\varphi\colon D\to\mathbb{R}^n$ of Sobolev class $W^1_{1,\mathrm{loc}}(D;\mathbb{R}^n)$ is a mapping with *finite distortion* whenever

$$D\varphi(x)=0 \text{ almost everywhere (a. e.) on the set } Z=\{x\in D\colon \det D\varphi(x)=0\}. \tag{1.1}$$

(Meaning $\det D\varphi(x)=0$ at all points of Z except for a set of Lebesgue measure zero.)

Here and henceforth $D\varphi(x)=(\partial\varphi_j(x)/\partial x_i)^n_{i,j=1}$ stands for the Jacobi matrix of the mapping φ at $x\in D$, while $|D\varphi(x)|$, for its Euclidean operator norm, and $\det D\varphi(x)$, for its determinant, the Jacobian.

A locally integrable function $\omega\colon D'\to\mathbb{R}$ is called a *weight* whenever $0<\omega(y)<\infty$ for a. e. $y\in D'$. A function $u\colon D'\to\mathbb{R}$ belongs to the *weighted Sobolev class* $L^1_p(D';\omega)$, with $p\in[1,\infty)$, if $u\in L_{1,\mathrm{loc}}(D')$ and $\partial u/\partial y_j\in L_p(D';\omega)$ for every $j=1,\dots,n$. The seminorm of a function $u\in L^1_p(D';\omega)$ is then defined as

$$\|u\mid L^1_p(D';\omega)\|=\left(\int_{D'}|\nabla u(y)|^p\omega(y)\,dy\right)^{1/p}. \tag{1.2}$$

In the case $\omega\equiv 1$ instead of $L^1_p(D';1)$ we write simply $L^1_p(D')$.

Henceforth the symbol $\mathrm{Lip}_{\mathrm{loc}}(D')$ stands for the space of locally Lipschitz functions on D'. It is obvious that

$$\mathrm{Lip}_{\mathrm{loc}}(D')=W^1_{\infty,\mathrm{loc}}(D')\cap C(D'),$$

where $W^1_{\infty,\mathrm{loc}}(D')$ is the space of locally bounded measurable functions on D' with locally bounded generalized derivative.

We say that a homeomorphism $\varphi\colon D\to D'$ *induces the bounded composition operator*

$$\varphi^*\colon L^1_p(D';\omega)\cap\mathrm{Lip}_{\mathrm{loc}}(D')\to L^1_q(D),\qquad 1\leqslant q\leqslant p<\infty,$$

acting as $D\ni x\mapsto(\varphi^*u)(x)=u(\varphi(x))$, whenever for some constant $K_{q,p}<\infty$ the inequality

$$\|\varphi^*u\mid L^1_q(D)\|\leqslant K_{q,p}\|u\mid L^1_p(D';\omega)\|$$

holds for every function $u\in L^1_p(D')\cap\mathrm{Lip}_{\mathrm{loc}}(D')$.

1.2. Condensers and their capacity in Sobolev spaces. A *condenser* in a domain $D \subset \mathbb{R}^n$ is a pair $\mathcal{E} = (F_1, F_0)$ of connected compact sets (continua) F_1, $F_0 \subset D$. For a continuum $F \subset U$, where $U \Subset D$ is an open connected compactly embedded set, we denote the condenser $\mathcal{E} = (F, \partial U)$ by $\mathcal{E} = (F, U)$.

A condenser $\mathcal{E} = (F, U)$ is called *annular* whenever the complement in $\mathbb{R}^n$ to the open set $U \setminus F$ consists of two closed sets each of which is connected: the bounded connected component is the continuum F, and the unbounded component is $\mathbb{R}^n \setminus U$.

A condenser $\mathcal{E} = (F, U)$ in $\mathbb{R}^n$ is called *spherical* whenever $U = B(x, R) = \{y \in \mathbb{R}^n \colon |y - x|_2 < R\}$ and $F = \overline{B(x, r)} = \{y \in \mathbb{R}^n \colon |y - x|_2 \leqslant r\}$, where $r < R$, and *cubical* whenever $U = Q(x, R) = \{y \in \mathbb{R}^n \colon |y - x|_\infty < R\}$ and $F = \overline{Q(x, r)} = \{y \in \mathbb{R}^n \colon |y - x|_\infty \leqslant r\}$, respectively.

Definition 1.1. A function $u \colon D \to \mathbb{R}$ of class $W^1_{1,\mathrm{loc}}(D)$ is called *admissible* for a condenser $\mathcal{E} = (F_1, F_0) \subset D$ whenever

(1) u is *continuous*,

(2) $u \equiv 1$ on F_1, and

(3) $u \equiv 0$ on F_0.

We denote the collection of admissible functions for a condenser $\mathcal{E} = (F_1, F_0)$ by $\mathcal{A}(\mathcal{E})$.

The *capacity* of a condenser $\mathcal{E} = (F_1, F_0)$ in the space $L^1_q(D)$ with $q \in [1, \infty)$ is defined as

$$\operatorname{cap}\bigl(\mathcal{E}; L^1_q(D)\bigr) = \inf_u \|u \mid L^1_q(D)\|^q, \tag{1.3}$$

where the infimum is taken over all admissible functions $u \in \mathcal{A}(\mathcal{E}) \cap L^1_q(D)$ for the condenser $\mathcal{E} = (F_1, F_0) \subset D$.

Let us now define the *weighted capacity* of a condenser $\mathcal{E} = (F_1, F_0) \subset D'$ in the space $L^1_p(D'; \omega)$ by analogy with (1.3):

$$\operatorname{cap}\bigl(\mathcal{E}; L^1_p(D'; \omega)\bigr) = \inf_u \|u \mid L^1_p(D'; \omega)\|^p,$$

where the infimum is over all admissible functions $u \in \mathcal{A}(\mathcal{E}) \cap \mathrm{Lip}_{\mathrm{loc}}(D') \cap L^1_p(D'; \omega)$ for the condenser $\mathcal{E} = (F_1, F_0)$.

See the books [41], [44], which present the properties of capacity in Sobolev spaces. For more details on the properties of weighted capacity (for a special class of admissible weights), see [50, Ch. 2].

The definition of capacity yields the following property.

Property 1.2 (Subordination principle). *Consider two condensers $\mathcal{E}' = (F_1', F_0')$ and $\mathcal{E} = (F_1, F_0)$ in a domain D' with the plates of the first condenser included in those of the second one, $F_1' \subset F_1$ and $F_0' \subset F_0$. Then*

$$\operatorname{cap}\bigl(\mathcal{E}'; L^1_p(D'; \omega)\bigr) \leqslant \operatorname{cap}\bigl(\mathcal{E}; L^1_p(D'; \omega)\bigr).$$

1.3. A quasi-additive set function and its properties. Denote by $\mathcal{O}(D)$ a system of open sets in D with the following properties:

(1) $D \in \mathcal{O}(D)$ and if the closure of an open ball B (cube Q) lies in D, then $B \in \mathcal{O}(D)$ $(Q \in \mathcal{O}(D))$;

(2) if $U_1, \dots, U_k \in \mathcal{O}(D)$ is a disjoint system of open sets, then $\bigcup_{i=1}^{k} U_i \in \mathcal{O}(D)$, where $k \in \mathbb{N}$ is an arbitrary number.

The choice of a ball or cube in this definition depends on the choice of a system of elementary sets with respect to which the set function is differentiated, see (1.6).

DEFINITION 1.3. A mapping $\Phi\colon \mathcal{O}(D) \to [0,\infty]$ is called a *quasi-additive* set function if

(1) for every point $x \in D$ there exists a number $\delta(x) \in (0,\infty)$ such that $\overline{B(x,\delta(x))} \subset D$ and $0 < \Phi(B(x,\delta)) < \infty$ for all $\delta \in (0,\delta(x))$, and the ball in this condition can be replaced with a cube;

(2) every finite tuple $\{U_i \in \mathcal{O}(D)\}$, for $i = 1, \dots, l$, of disjoint open sets with

$$\bigcup_{i=1}^{l} U_i \subset U, \quad \text{where } U \in \mathcal{O}(D), \text{ satisfies } \sum_{i=1}^{l} \Phi(U_i) \leqslant \Phi(U). \tag{1.4}$$

If every finite tuple $\{U_i \in \mathcal{O}(D)\}$ of pairwise disjoint open sets satisfies

$$\sum_{i=1}^{n} \Phi(U_i) = \Phi\biggl(\bigcup_{i=1}^{n} U_i\biggr), \tag{1.5}$$

then this set function is called *finitely additive*, while if (1.5) holds for every countable tuple $\{U_i \in \mathcal{O}(D)\}$ of disjoint open sets, then this set function is called *countably additive*. The function Φ is *monotone* whenever $\Phi(U_1) \leqslant \Phi(U_2)$ as soon as $U_1 \subset U_2 \subset D$ with $U_1, U_2 \in \mathcal{O}(D)$. Every quasi-additive set function is obviously monotone. A quasi-additive set function $\Phi\colon \mathcal{O}(D) \to [0,\infty]$ is called a *bounded quasi-additive set function* whenever $D \in \mathcal{O}(D)$ and $\Phi(D) < \infty$.

It is known, see [51]–[53] for instance, that every quasi-additive set function Φ defined on some system $\mathcal{O}(D')$ of open subsets of a domain D' is differentiable in the following sense: *for a. e. point $y \in D'$ there exists the finite derivative*[1]:

$$\lim_{\delta\to 0,\, y\in B_\delta} \frac{\Phi(B_\delta)}{\mathcal{H}^n(B_\delta)} = \Phi'(y); \tag{1.6}$$

and for every open set $U \in \mathcal{O}(D')$ we have

$$\int_U \Phi'(y)\,dy \leqslant \Phi(U). \tag{1.7}$$

1.4. Definition of the class of $\mathcal{Q}_{p,q}(D',\omega;D)$-homeomorphisms and their properties. Denote by $\mathcal{O}_{\mathrm{c}}(D')$ the minimal system of open sets in D', which contains:

(1) D';

(2) every open cube Q whenever $\overline{Q} \subset D'$;

(3) the complement $Q_2 \setminus \overline{Q}_1$ whenever $Q_1 \subset Q_2$ are two cubes with a common center and $\overline{Q}_2 \subset D'$.

In the following Definition 1.4 and Theorem 1.6, we consider the mapping $\Phi\colon \mathcal{O}_{\mathrm{c}}(D') \to [0,\infty)$ as the bounded quasi-additive set function.

[1] Here and henceforth B_δ is an arbitrary ball $B(z,\delta) \subset D'$ containing the point y. The ball in this proposition can be replaced with a cube.

Definition 1.4 [31]. Given two domains D, $D' \subset \mathbb{R}^n$, for $n \geqslant 2$, we say that a homeomorphism $f\colon D' \to D$ is of class[2] $\mathcal{CRQ}_{p,q}(D',\omega;D)$, where $1 < q \leqslant p < \infty$ for $n \geqslant 3$ and $1 \leqslant q \leqslant p < \infty$ for $n = 2$, while $\omega \in L_{1,\mathrm{loc}}(D')$ is a weight function, if there exist

(1) a constant $K_p > 0$ for $q = p$ or

(2) a bounded quasi-additive function $\Psi_{p,q}$ defined on the system $\mathcal{O}_{\mathrm{c}}(D')$ of open sets in D' for $q < p$

such that for every cubical condenser $\mathcal{E} = (\overline{Q(x,r)}, Q(x,R)) \subset D'$ with $0 < r < R$ with the image $f(\mathcal{E}) = (f(\overline{Q(x,r)}), f(Q(x,R)) \subset D$ we have

$$\begin{cases} \operatorname{cap}^{1/p}\big(f(\mathcal{E}); L^1_p(D)\big) \leqslant K_p \operatorname{cap}^{1/p}\big(\mathcal{E}; L^1_p(D';\omega)\big) & \text{if } q = p,\\ \operatorname{cap}^{1/q}\big(f(\mathcal{E}); L^1_q(D)\big) \leqslant \Psi_{p,q}(Q(x,R) \setminus \overline{Q(x,r)})^{1/\sigma} \operatorname{cap}^{1/p}\big(\mathcal{E}; L^1_p(D';\omega)\big) & \text{if } q < p, \end{cases} \tag{1.8}$$

where $1/\sigma = 1/q - 1/p$.

Definition 1.5 [31], [32]. Let D and D' be open sets in $\mathbb{R}^n$ with $n \geqslant 2$, $1 < q \leqslant p < \infty$ for $n \geqslant 3$ and $1 \leqslant q \leqslant p < \infty$ for $n = 2$, and $\omega \in L_{1,\mathrm{loc}}(D')$ be a weight function. We say that a homeomorphism $\varphi\colon D \to D'$ belongs to the class $\mathcal{Q}_{p,q}(D',\omega;D)$, whenever each condenser $\mathcal{E} = (F_1, F_0)$ in D' with the preimage $\varphi^{-1}(\mathcal{E}) = (\varphi^{-1}(F_1), \varphi^{-1}(F_0))$ in D satisfies

$$\operatorname{cap}^{1/q}\big(\varphi^{-1}(\mathcal{E}); L^1_q(D)\big) \leqslant \begin{cases} \widetilde{K}_p \operatorname{cap}^{1/p}\big(\mathcal{E}; L^1_p(D';\omega)\big), & 1 < q = p < \infty,\\ \widetilde{\Psi}(D' \setminus (F_0 \cup F_1))^{1/\sigma} \operatorname{cap}^{1/p}\big(\mathcal{E}; L^1_p(D';\omega)\big), & 1 < q < p < \infty, \end{cases} \tag{1.9}$$

where $1/\sigma = 1/q - 1/p$, while $\widetilde{\Psi}$ is some bounded quasi-additive set function defined on open subsets of D'.

It is easy to see that if $\varphi \in \mathcal{Q}_{p,q}(D',\omega;D)$, then $f = \varphi^{-1} \in \mathcal{CRQ}_{p,q}(D',\omega;D)$.

The following Theorem 1.6 gives an analytic description of the mappings with inverses of class $\mathcal{CRQ}_{p,q}(D',\omega;D)$.

Theorem 1.6 [33, Theorem 1]. *A homeomorphism $f\colon D' \to D$ belongs to the class $\mathcal{CRQ}_{p,q}(D',\omega;D)$ with $1 < q \leqslant p < \infty$ for $n \geqslant 3$ and $1 \leqslant q \leqslant p < \infty$ for $n = 2$ if and only if the inverse homeomorphism $\varphi = f^{-1}\colon D \to D'$ enjoys one of the following properties*:

(1) *the composition operator $\varphi^*\colon L^1_p(D';\omega) \cap \mathrm{Lip}_{\mathrm{loc}}(D') \to L^1_q(D)$, with $1 < q \leqslant p < \infty$, is bounded*;

(2) *the homeomorphism $\varphi\colon D \to D'$ is of class $\mathcal{Q}_{p,q}(D',\omega;D)$ in the sense of Definition 1.5, with some bounded quasi-additive set function $\widetilde{\Psi}$ defined on open subsets of D'*;

(3) *a homeomorphism $\varphi\colon D \to D'$*

(a) *is of Sobolev class $W^1_{q,\mathrm{loc}}(D)$*,

(b) *has finite distortion in the sense of* (1.1), *and*

[2]In the acronym $\mathcal{CRQ}$ the letters stand for the words “cube”, “ring”, and “quasiconformal”. Therefore, $\mathcal{CRQ}$ is quasiconformality determined by cubical condensers.

(c) *the operator distortion function*

$$D \ni x \mapsto K^{1,\omega}_{q,p}(x,\varphi) = \begin{cases} \dfrac{|D\varphi(x)|}{|\det D\varphi(x)|^{1/p}\omega^{1/p}(\varphi(x))} & \text{if } \det D\varphi(x) \neq 0, \\ 0 & \text{if } \det D\varphi(x) = 0 \end{cases} \tag{1.10}$$

belongs to $L_\sigma(D)$, *where* $1/\sigma = 1/q - 1/p$ *if* $1 < q < p < \infty$ *and* $\sigma = \infty$ *if* $q = p$;

(4) *if* $n = 2$, *then claims* (1)–(3) *also hold in the case* $1 = q \leqslant p < \infty$.

Note that Theorem 1.6 is a consequence of [29, Theorem 1], [30], and [31], [32, Theorem 1], see details in [33, Theorem 1]. The smallest quantities K_p and $\widetilde{K}_p$ (quasiadditive functions Ψ and $\widetilde{\Psi}$) in (1.8), (1.9) satisfy

$$\text{for } q = p \quad \|\varphi^*\| = \|K^{1,\omega}_{p,p}(\,\cdot\,) \mid L_\infty(D)\| = K_p = \widetilde{K}_p \tag{1.11}$$

$$\big(\text{for } q < p \quad \|\varphi^*_W\|^\sigma = \|K^{1,\omega}_{q,p}(\,\cdot\,) \mid L_\sigma(\varphi^{-1}(W))\|^\sigma = \Psi(W) = \widetilde{\Psi}(W)\big) \tag{1.12}$$

for an open set $W \subset D'$, where $\|\varphi^*_W\|$ is the norm of the restriction

$$\varphi_W \colon L^1_p(W;\omega) \cap \overset{\circ}{\mathrm{Lip}}_{\mathrm{loc}}(W) \to L^1_q(D);$$

here $\overset{\circ}{\mathrm{Lip}}_{\mathrm{loc}}(W)$ stands for the space of locally Lipschitz functions vanishing on the boundary of W, see [34, Theorem 4].

Let us formulate the following corollary of Theorem 1.6.

Corollary 1.7. *A homeomorphism* $f \colon D \to D'$ *is of class* $\mathcal{CRQ}_{p,q}(D',\omega;D)$ *with* $1 < q \leqslant p < \infty$ *for* $n \geqslant 3$ *and* $1 \leqslant q \leqslant p < \infty$ *for* $n = 2$ *if and only if* $\varphi = f^{-1}$ *is also of class* $\mathcal{Q}_{p,q}(D',\omega;D)$.

Therefore, from now on we use only $\mathcal{Q}_{p,q}(D',\omega;D)$ to refer to both classes $\mathcal{CRQ}_{p,q}(D',\omega;D)$ and $\mathcal{Q}_{p,q}(D',\omega;D)$.

The differential properties of mappings of the classes $\mathcal{Q}_{p,q}(D',\omega;D)$ are established in [30] and [31, Theorem 2].

Remark 1.8. The homeomorphisms $\varphi \colon D \to D'$ with $f = \varphi^{-1} \in \mathcal{Q}_{p,q}(D',\omega;D)$ in the cases

(1) $q = p = n$ and $\omega \equiv 1$ coincide with quasiconformal mappings [18], [42]–[45];

(2) $1 < q = p < \infty$ and $\omega \equiv 1$ were studied in [28];

(3) $1 < q < p < \infty$ and $\omega \equiv 1$ were studied in [28], [37]–[39].

Let us extract from Theorem 1.6 and Corollary 1.7 the following two examples of $Q_{p,q}$-homeomorphisms.

Example 1.9 [29], [32]. If a homeomorphism $\varphi \colon D \to D'$ induces a bounded composition operator $\varphi^* \colon L^1_p(D';\omega) \cap \mathrm{Lip}_{\mathrm{loc}}(D') \to L^1_q(D)$, with $1 < q \leqslant p < \infty$ for $n \geqslant 3$ and $1 \leqslant q \leqslant p < \infty$ for $n = 2$, then the inverse homeomorphism $f = \varphi^{-1} \colon D' \to D$ is of class $\mathcal{Q}_{p,q}(D',\omega;D)$.

Example 1.10 [29], [32]. Consider a homeomorphism $\varphi \colon D \to D'$ of Sobolev class $W^1_{q,\mathrm{loc}}(D)$ with finite distortion (1.1) and the operator function distortion (1.10) of class $L_\sigma(D)$, where $1/\sigma = 1/q - 1/p$ for $1 \leqslant q < p < \infty$ and $\sigma = \infty$ for $q = p$.

If $1 < q \leqslant p < \infty$ for $n \geqslant 3$ and $1 \leqslant q \leqslant p < \infty$ for $n = 2$, then the inverse homeomorphism $f = \varphi^{-1}\colon D' \to D$ is of class $\mathcal{Q}_{p,q}(D', \omega; D)$.

In addition to Examples 1.9 and 1.10, other classes of mappings in the family $\mathcal{Q}_{p,q}(D', \omega; D)$ were considered in [31]. Let us present some of them.

Example 1.11 [31, Example 3]. Consider a homeomorphism $\varphi\colon D \to D'$ of Sobolev class $W^1_{p,\mathrm{loc}}(D)$, where $1 < p < \infty$ for $n \geqslant 3$ and $1 \leqslant p < \infty$ for $n = 2$, with finite distortion. The inverse homeomorphism $f = \varphi^{-1}\colon D' \to D$ is of class $\mathcal{Q}_{p,p}(D', \omega; D)$ with the constant $K_p = 1$ and the weight function

$$D' \ni y \mapsto \omega(y) = \begin{cases} \dfrac{|D\varphi(\varphi^{-1}(y))|^p}{|\det D\varphi(\varphi^{-1}(y))|} & \text{if } y \in D' \setminus (Z' \cup \Sigma'), \\ 1 & \text{otherwise.} \end{cases} \tag{1.13}$$

Remark 1.12. As [31, Theorem 5] shows, the weight function (1.13) is locally integrable.

Example 1.13 [31, Example 4]. For $n-1 < s < \infty$ consider a homeomorphism $f\colon D' \to D$ of open domains $D', D \subset \mathbb{R}^n$, where $n \geqslant 2$, such that

(1) $f \in W^1_{n-1,\mathrm{loc}}(D')$;
(2) the mapping f has finite distortion;
(3) the outer distortion function

$$D' \ni y \mapsto K^{1,1}_{n-1,s}(y, f) = \begin{cases} \dfrac{|Df(y)|}{|\det Df(y)|^{1/s}} & \text{if } \det Df(y) \neq 0, \\ 0 & \text{if } \det Df(y) = 0 \end{cases} \tag{1.14}$$

lies in $L_\sigma(D)$, where $\sigma = (n-1)p$ with $p = s/(s-(n-1))$.

Then the inverse homeomorphism $\varphi = f^{-1}\colon D \to D'$ has the properties

(4) $\varphi \in W^1_{p,\mathrm{loc}}(D)$, $p = s/(s-(n-1))$;
(5) φ has finite distortion;

while the homeomorphism $f\colon D' \to D$

(6) is of class $\mathcal{Q}_{p,p}(D', \omega; D)$ with the constant $K_p = 1$ and the weight function $\omega \in L_{1,\mathrm{loc}}(D')$ defined as

$$\omega(y) = \begin{cases} \dfrac{|\operatorname{adj} Df(y)|^p}{|\det Df(y)|^{p-1}} & \text{if } y \in D' \setminus Z', \\ 1 & \text{otherwise,} \end{cases} \tag{1.15}$$

where $Z' = \{y \in D' \colon Df(y) = 0\}$.

Say that a mapping $f \in W^1_{1,\mathrm{loc}}(D')$ has *finite codistortion* if the adjoint matrix $\operatorname{adj} Df(y)$ of the differential equals 0 a.e. on the zero set of the Jacobian

$$Z = \{y \in D' \mid \det Df(y) = 0\}.$$

Example 1.14 [31, Example 5]. For $n-1 < s < \infty$, consider a homeomorphism $f\colon D' \to D$ of domains $D', D \subset \mathbb{R}^n$, with $n \geqslant 2$, such that

(1) $f \in W^1_{n-1,\mathrm{loc}}(D')$;

(2) the mapping f has finite codistortion;
(3) the inner distortion function

$$D' \ni y \mapsto \mathcal{K}^{1,1}_{n-1,s}(y,f) = \begin{cases} \dfrac{|\operatorname{adj} Df(y)|}{|\det Df(y)|^{(n-1)/s}} & \text{if } \det Df(y) \neq 0, \\ 0 & \text{if } \det Df(y) = 0 \end{cases} \tag{1.16}$$

belongs to $L_p(D')$, where $p = s/(s-(n-1))$ and $n-1 < s < \infty$.

Then the inverse homeomorphism $\varphi = f^{-1}\colon D \to D'$ has the properties
(4) $\varphi \in W^1_{p,\mathrm{loc}}(D)$ and $p = s/(s-(n-1))$;
(5) φ has finite distortion;
and the homeomorphism $f\colon D' \to D$
(6) is of class $\mathcal{Q}_{p,p}(D',\omega;D)$ with the constant $K_p = 1$ and the weight function (1.15);
(7) has finite distortion for $n-1 < s < n+1/(n-2)$.

Example 1.15 [35, Definition 11, Theorem 34]. A homeomorphism $f\colon D' \to D$ is called a homeomorphism *with inner bounded θ-weighted (s,r)-distortion*, or of class $\mathcal{ID}(D';s,r;\theta,1)$, where $n-1 < s \leqslant r < \infty$, whenever:
(1) $f \in W^1_{n-1,\mathrm{loc}}(D')$;
(2) the mapping f has finite codistortion;
(3) the function of local θ-weight (s,r)-distortion

$$D' \ni x \mapsto \mathcal{K}^{\theta,1}_{s,r}(x,f) = \begin{cases} \dfrac{\theta^{(n-1)/s}(x)|\operatorname{adj} Df(x)|}{|\det Df(x)|^{(n-1)/r}} & \text{if } \det Df(x) \neq 0, \\ 0 & \text{otherwise} \end{cases} \tag{1.17}$$

belongs to $L_\varrho(\Omega)$, where ϱ can be found from the condition $1/\varrho = (n-1)/s - (n-1)/r$, and $\varrho = \infty$ for $s = r$.

Then under the condition $n-1 < s \leqslant r < \infty$ and the local summability of the function $\omega(x) = \theta^{-(n-1)/(s-(n-1))}(x)$, the homeomorphism $f\colon D' \to D$ belongs to $\mathcal{Q}_{p,q}(D',\omega;D)$, where $q = r/(r-(n-1))$ and $p = s/(s-(n-1))$, for $1 < q \leqslant p < \infty$. Furthermore, the factors on the right-hand side of (1.8) are equal to $K_p = \|\mathcal{K}^{\theta,1}_{r,r}(\,\cdot\,,f) \mid L_\infty(\Omega)\|$ for $q = p$ and

$$\Psi_{p,q}(Q(x,R)\setminus\overline{Q(x,r)})^{1/\sigma} = \big\|\mathcal{K}^{\theta,1}_{s,r}(\,\cdot\,,f) \bigm| L_\varrho(Q(x,R)\setminus\overline{Q(x,r)})\big\| \quad \text{for} \quad q < p,$$

where $1/\sigma = 1/q - 1/p = 1/\varrho$.

Example 1.16 [36, Definition 3, Theorem 19]. A homeomorphism $f\colon D' \to D$ is of class $\mathcal{OD}(D';s,r;\theta,1)$, with $n-1 < s \leqslant r < \infty$, and is called a mapping *with outer bounded θ-weighted (s,r)-distortion*, whenever:
(1) $f \in W^1_{n-1,\mathrm{loc}}(D')$;
(2) the mapping f has finite distortion;
(3) the function of local θ-weighted (s,r)-distortion

$$D' \ni x \mapsto K^{\theta,1}_{s,r}(x,f) = \begin{cases} \dfrac{\theta^{1/s}(x)|Df(x)|}{|\det Df(x)|^{1/r}} & \text{if } \det Df(x) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

belongs to $L_\rho(D')$, where ρ can be found from the conditions $1/\rho = 1/s - 1/r$ and $\rho = \infty$ for $s = r$.

Then under the condition $n-1 < s \leqslant r < \infty$ and the local summability of $\omega(x) = \theta^{-(n-1)/(s-(n-1))}(x)$, the homeomorphism $f\colon D' \to D$ belongs to $\mathcal{Q}_{p,q}(D',\omega;D)$, where $q = r/(r-(n-1))$ and $p = s/(s-(n-1))$ with $1 < q \leqslant p < \infty$. The factors on the right-hand side of (1.8) are equal to $K_p = \|K^{\theta,1}_{r,r}(\,\cdot\,,f) \mid L_\infty(D')\|^{n-1}$ for $q = p$ and

$$\Psi_{p,q}(Q(x,R)\setminus\overline{Q(x,r)})^{1/\sigma} = \bigl\|K^{\theta,1}_{s,r}(\,\cdot\,,f) \bigm| L_\rho(Q(x,R)\setminus\overline{Q(x,r)})\bigr\|^{n-1}$$

for $q < p$, where $1/\sigma = 1/q - 1/p = (n-1)/\varrho$.

It is shown in [36, Theorem 8] that the inclusion

$$\mathcal{OD}(D';s,r;\theta,1) \subset \mathcal{ID}(D';s,r;\theta,1)$$

holds under the condition $n-1 < s \leqslant r < \infty$. Moreover, for every homeomorphism $f\colon D' \to D$ of class $\mathcal{OD}(D';s,r;\theta,1)$, with $n-1 < s \leqslant r < \infty$, we have

$$\|\mathcal{K}^{\theta,1}_{s,r}(\,\cdot\,,f) \mid L_\sigma(D')\| \leqslant \|K^{\theta,1}_{s,r}(\,\cdot\,,f) \mid L_\rho(D')\|^{n-1},$$

where the numbers ρ and σ are defined in Examples 1.15 and 1.16.

More examples of $\mathcal{OD}(D';s,r;\theta,1)$-homeomorphisms in $\mathbb{R}^2$ can be found in [54].

§ 2. Behavior of mappings with respect to the capacity metric

Fix two domains $D, D' \subset \mathbb{R}^n$, a locally integrable weight function $\omega\colon D' \to \mathbb{R}$ on D', and a mapping $f \in \mathcal{Q}_{p,q}(D',\omega;D)$ with $n-1 < q \leqslant p < \infty$.

Recall that Corollary 1.7 guarantees that f satisfies (1.9) for every condenser $\mathcal{E} = (F_1, F_0)$ in D'.

Fix some continuum $F_0 \subset D'$ with nonempty interior such that the open set $D' \setminus F_0$ is connected.

2.1. Capacity metric functions in domains for the homeomorphisms of class $\mathcal{Q}_{p,q}(D',\omega;D)$ for $n-1 < q \leqslant p \leqslant n$. Observe that in the case $n-1 < q \leqslant n$ the left-hand side of (1.9) is nonzero as long as the continuum $f(F_1)$ is distinct from a point. Indeed, we have the following proposition.

LEMMA 2.1. *In a domain $D \subset \mathbb{R}^n$ fix two balls $B_0 \Subset D$ and $B_1 \Subset D$ satisfying $\overline{B_0} \cap \overline{B_1} = \varnothing$. Then for $n-1 < q \leqslant n$, a fixed continuum $T_0 \subset B_0$, and an arbitrary continuum $T_1 \subset \overline{B_1}$, the relation*

$$\operatorname{cap}^{1/q}\bigl((T_1,T_0); L^1_q(D)\bigr) \to 0 \tag{2.1}$$

holds[3] *if and only if* $\operatorname{diam} T_1 \to 0$.

[3]In other words, the left-hand side of (1.9) is small if and only if $\operatorname{diam} T_1$ is small (under the condition that the continuum T_1 lies in some ball $B_1 \Subset D$ with $\overline{B_0} \cap \overline{B_1} = \varnothing$).

PROOF. Let us present the scheme of the proof of Lemma 2.1.

Necessity. By [48, Lemma 3], there is a John domain [48, Definition 8] $\Omega \in J(\alpha,\beta)$ compactly embedded into D, with some positive parameters α and β depending on D and the balls B_0 and B_1, which includes the closures of both balls. On the domain Ω under the conditions $1 \leqslant q < n$ and $q \leqslant q^* \leqslant nq/(n-q)$ we have the following Poincaré inequality [55, Theorems 4 and 9]:

$$\|u - c_u \mid L_{q^*}(\Omega)\| \leqslant C_\Omega \left(\frac{\alpha}{\beta}\right)^n (\operatorname{diam}\Omega)^{1-n/q+n/q^*} \|\nabla u \mid L_q(\Omega)\|, \tag{2.2}$$

where c_u and C_Ω are constants, with $C_\Omega > 0$ independent of u, α, and β. By (2.1) there exists a sequence of continua $T_{1,k} \subset B_1$ and admissible functions $u_k \in C(\Omega) \cap L^1_q(\Omega)$ for the capacity $\operatorname{cap}((T_{1,k}, T_0); L^1_q(\Omega))$ such that

$$u_k|_{T_{1,k}} = 1, \quad u_k|_{T_0} = 0, \quad 0 \leqslant u_k \leqslant 1 \quad \text{and} \quad \|\nabla u_k \mid L_q(\Omega)\| \to 0 \quad \text{as} \quad k \to \infty. \tag{2.3}$$

And hence, the inequality (2.2) implies that $\|u_k - c_{u_k} \mid L_{q^*}(\Omega)\| \to 0$ as $k \to \infty$. Note that the sequence of numbers $\{c_{u_k}\}$ is bounded. Indeed, if $\{c_{u_k}\}$ is not bounded then, due to the relations $0 \leqslant u_k \leqslant 1$, the left-hand side of (2.2) is also not bounded, which contradicts the right convergence in (2.3). Therefore, we may assume that c_{u_k} converges to some number c_0, and up to subsequence $u_k - c_{u_k} \to 0$ for a. e. $x \in \Omega$ as $k \to \infty$. Hence, $u_k \to c_0$ for a. e. $x \in \Omega$ as $k \to \infty$, and due to $u_k|_{B_0} \equiv 0$ we deduce $c_0 = 0$. In addition, Ω is a bounded domain, and the Lebesgue dominated convergence theorem shows that

$$\|u_l \mid L_q(\Omega)\| \to 0 \quad \text{as} \quad l \to \infty. \tag{2.4}$$

From (2.3) and (2.4) we infer that $\|u_l \mid W^1_q(\Omega)\| \to 0$ as $l \to \infty$. We can extend the restrictions $u_l|_{B_1}$ to the functions $\widetilde{u}_l \in W^1_q(\mathbb{R}^n)$ so that the extension operator is bounded. Therefore,

$$\|\widetilde{u}_l \mid W^1_q(\mathbb{R}^n)\| \to 0 \quad \text{as} \quad l \to \infty.$$

We obtain then that the capacity of the continua $T_{1,l}$ in the space $W^1_q(\mathbb{R}^n)$ of Bessel potentials is positive and tends to 0 as $l \to \infty$. For $n-1 < q < n$ the latter is possible only if $\operatorname{diam} T_{1,l} \to 0$ as $l \to \infty$; see the details in [56], [41].

The case $q = n$ reduces to the previous one using Hölder's inequality.

Sufficiency. Since Property 1.2 yields

$$\operatorname{cap}\big((T_1, T_0); L^1_q(D)\big) \leqslant \operatorname{cap}\big((T_1, \overline{B_0}); L^1_q(D)\big),$$

it suffices to prove that $\operatorname{cap}((T_1, \overline{B_0}); L^1_q(D)) \to 0$ as $\operatorname{diam} T_1 \to 0$.

Put $R = \operatorname{dist}(B_0, B_1)$ and suppose that the continuum T_1 satisfies $r_{T_1} < R$. Then we may assume that every admissible function for the condenser $(\overline{B(x, r_{T_1})}, B(x,R))$ is also admissible for the condenser (T_1, T_0), and so

$$\operatorname{cap}\big((T_1, T_0); L^1_q(D)\big) \leqslant \operatorname{cap}\big((\overline{B(x, r_{T_1})}, B(x,R)); L^1_q(B(x,R))\big).$$

From Example 2.7 below for $\alpha = 0$, we conclude

$$\operatorname{cap}\big((\overline{B(0,r)}, B(0,R)); L^1_q(B(0,R))\big) = \begin{cases} \sigma_{n-1}\left(\dfrac{n-q}{n-1}\right)^{q-1}\big(r^{(q-n)/(q-1)} - R^{(q-n)/(q-1)}\big)^{1-q} & \text{for } q<n,\\ \sigma_{n-1}\left(\ln\dfrac{R}{r}\right)^{1-n} & \text{for } q=n,\end{cases}$$

where $r \in (0,R)$, while σ_{n-1} is the measure of the unit $(n-1)$-dimensional sphere in the space $\mathbb{R}^n$. Thus,

$$\operatorname{cap}\big((\overline{B(x,r_{T_1})}, B(x,R)); L^1_q(B(x,R)\big) \to 0 \quad \text{as} \quad r_{T_1} \to 0,$$

and the proof of Lemma 2.1 is complete.

Corollary 2.2. *For $n-1<q\leqslant n$, the existence of a mapping $f \in \mathcal{Q}_{p,q}(D',\omega;D)$ is ensured by the condition*

$$\operatorname{cap}^{1/p}\big(\mathcal{E}; L^1_p(D';\omega)\big) \neq 0 \tag{2.5}$$

for an arbitrary condenser $\mathcal{E} = (\gamma, F_0)$, where $\gamma\colon [a,b] \to D' \setminus F_0$ is an arbitrary closed curve with distinct endpoints $x = \gamma(a)$ and $y = \gamma(b)$.

Proof. Since the continuum $F_0 \subset D'$ has nonempty interior, there exists a closed ball $\overline{B_0'} \subset F_0$ and a closed ball $\overline{B_1'} \subset D'$ centered on γ such that $\overline{B_1'}\cap\overline{B_0'} = \varnothing$. Consider the condenser $\mathcal{E} = (\gamma \cap \overline{B_1'}, \overline{B_0'})$. By (1.8), it suffices to show that

$$\operatorname{cap}\big(f(\mathcal{E}); L^1_q(D)\big) \neq 0. \tag{2.6}$$

The latter follows from Lemma 2.1. Indeed, there are closed disjoint balls $\overline{B_0''} \subset f(\overline{B_0'})$ and $\overline{B_1''} \subset f(\overline{B_1'})$ whose intersection $\gamma \cap \overline{B_1''}$ is a nondegenerate continuum. Then, Lemma 2.1 and (1.8) yield

$$0 \neq \operatorname{cap}\big((\gamma \cap \overline{B_1'}, \overline{B_0'}); L^1_q(D)\big) \leqslant \operatorname{cap}\big(f(\mathcal{E}); L^1_q(D)\big).$$

This justifies Corollary 2.2.

With (2.5) we can define a metric function similar to the one introduced in [25], [26, Ch. 5] in the unweighted case.

Definition 2.3. The *capacity* (ω,p)-*metric function* between two distinct points $x, y \in D' \setminus F_0$ with respect to F_0 is defined as

$$\rho^{\omega}_{p,F_0}(x,y) = \inf_{\overline{xy}} \operatorname{cap}^{1/p}\big((\overline{xy}, F_0); L^1_p(D';\omega)\big), \tag{2.7}$$

where the infimum is over all curves $\overline{xy}$ in $D' \setminus F_0$ with endpoints $x, y \in D' \setminus F_0$.

By analogy, we define the *capacity q-metric function* $\rho_{q,f(F_0)}(a,b)$ between two points $a, b \in D \setminus f(F_0)$ with respect to the continuum $f(F_0)$ in the image D':

$$\rho_{q,f(F_0)}(a,b) = \inf_{\overline{ab}} \operatorname{cap}^{1/q}\big((\overline{ab}, f(F_0)); L^1_q(D)\big). \tag{2.8}$$

Proposition 2.4. *If a homeomorphism $f\colon D'\to D$ belongs to $\mathcal{Q}_{p,q}(D',\omega;D)$, where $n-1<q\leqslant p\leqslant n$ for $n\geqslant 3$ and $1\leqslant q\leqslant p\leqslant 2$ for $n=2$, then the capacity metric functions satisfy*

$$\begin{cases}\rho_{p,f(F_0)}\big(f(x),f(y)\big)\leqslant K_p\rho^{\omega}_{p,F_0}(x,y) & \text{if } q=p,\\ \rho_{q,f(F_0)}\big(f(x),f(y)\big)\leqslant \Psi_{p,q}(D'\setminus F_0)^{1/\sigma}\rho^{\omega}_{p,F_0}(x,y) & \text{if } q<p,\end{cases}\tag{2.9}$$

for all points $x,y\in D'\setminus F_0$, where $1/\sigma=1/q-1/p$.

Proof. Take $\mathcal{E}=(\overline{xy},F_0)$ in D', then from (1.9) it follows that

$$\begin{aligned}\rho_{q,f(F_0)}(f(x),f(y))&\leqslant \operatorname{cap}^{1/q}\big((f(\overline{xy}),f(F_0));L^1_q(D)\big)\\ &\leqslant \Psi_{p,q}(D'\setminus F_0)^{1/\sigma}\operatorname{cap}^{1/p}\big((\overline{xy},F_0);L^1_p(D';\omega)\big)\end{aligned}$$

provided that $q<p$. Passing to the infimum over all curves $\overline{xy}\subset D'\setminus F_0$ with endpoints x and y, we arrive at the second inequality in (2.9).

The case $q=p$ is similar.

Proposition is proved.

Proposition 2.5. *In the case $n-1<q\leqslant p\leqslant n$ for $n\geqslant 3$ and $1\leqslant q\leqslant p\leqslant 2$ for $n=2$, the capacity (ω,p)-metric function $\rho^{\omega}_{p,F_0}(x,y)$ enjoys the properties*

(1) *$\rho^{\omega}_{p,F_0}(x,y)=\rho^{\omega}_{p,F_0}(y,x)$ for all points $x,y\in D'\setminus F_0$;*

(2) *$\rho^{\omega}_{p,F_0}(x,z)\leqslant\rho^{\omega}_{p,F_0}(x,y)+\rho^{\omega}_{p,F_0}(y,z)$ for all points $x,y,z\in D'\setminus F_0$.*

Proof. Property (1) is obvious.

To verify the second property, consider the case $x\neq z$, $x\neq y$, $y\neq z$; otherwise property (2) obviously holds. Fix $\varepsilon>0$ and some curves $\overline{xy}$ and $\overline{yz}$ with endpoints x, y and y, z, respectively, such that

$$\operatorname{cap}^{1/p}\big((\overline{xy},F_0);L^1_p(D';\omega)\big)<\rho^{\omega}_{p,F_0}(x,y)+\frac{\varepsilon}{4},\tag{2.10}$$

$$\operatorname{cap}^{1/p}\big((\overline{yz},F_0);L^1_p(D';\omega)\big)<\rho^{\omega}_{p,F_0}(y,z)+\frac{\varepsilon}{4}.\tag{2.11}$$

Take two functions u_1 and u_2 admissible for the capacities $\operatorname{cap}((\overline{xy},F_0);L^1_p(D';\omega))$ and $\operatorname{cap}((\overline{yz},F_0);L^1_p(D';\omega))$ such that

$$\bigg(\int_{D'}|\nabla u_1|^p(y)\omega(y)\,dy\bigg)^{1/p}<\operatorname{cap}^{1/p}\big((\overline{xy},F_0);L^1_p(D';\omega)\big)+\frac{\varepsilon}{4},\tag{2.12}$$

$$\bigg(\int_{D'}|\nabla u_2|^p(y)\omega(y)\,dy\bigg)^{1/p}<\operatorname{cap}^{1/p}\big((\overline{yz},F_0);L^1_p(D';\omega)\big)+\frac{\varepsilon}{4}.\tag{2.13}$$

It is easy to see that u_1+u_2 is admissible for the capacity $\operatorname{cap}^{1/p}((\overline{xy}\cup\overline{yz},F_0);L^1_p(D';\omega))$. Hence, from (2.10)–(2.13), we obtain

$$\begin{aligned}\rho^{\omega}_{p,F_0}(x,z)&\leqslant\operatorname{cap}^{1/p}\big((\overline{xy}\cup\overline{yz},F_0);L^1_p(D';\omega)\big)\leqslant\bigg(\int_{D'}|\nabla(u_1+u_2)|^p(y)\omega(y)\,dy\bigg)^{1/p}\\ &\leqslant\bigg(\int_{D'}|\nabla u_1|^p(y)\omega(y)\,dy\bigg)^{1/p}+\bigg(\int_{D'}|\nabla u_2|^p(y)\omega(y)\,dy\bigg)^{1/p}\\ &<\rho^{\omega}_{p,F_0}(x,y)+\rho^{\omega}_{p,F_0}(y,z)+\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is chosen arbitrarily, the triangle inequality is verified. Proposition is proved.

Recall that the metric function ρ^{ω}_{p,F_0} is defined in (2.7) for distinct points $x \neq y$ of the open set $D' \setminus F_0$. If $x = y \in D' \setminus F_0$, put

$$\rho^{\omega}_{p,F_0}(x,x) = \operatorname{cap}^{1/p}\big((\{x\}, F_0); L^1_p(D';\omega)\big). \tag{2.14}$$

For the capacity metric function ρ^{ω}_{p,F_0} to be a metric, we must ensure that

$$\rho^{\omega}_{p,F_0}(x,x) = 0 \tag{2.15}$$

for every point $x \in D' \setminus F_0$.

PROPOSITION 2.6. *Given $x \in D' \setminus F_0$, condition (2.15) holds if and only if*

$$\lim_{r\to 0} \operatorname{cap}\big((\overline{B(x,r)}, F_0); L^1_p(D';\omega)\big) = 0. \tag{2.16}$$

PROOF. Since the condenser $(\{x\}, F_0)$ is a part of the condenser $(\overline{B(x,r)}, F_0)$, Property 1.2 yields

$$\rho^{\omega}_{p,F_0}(x,x) \leqslant \lim_{r\to 0} \operatorname{cap}^{1/p}\big((\overline{B(x,r)}, F_0); L^1_p(D';\omega)\big).$$

Granted (2.16), this implies (2.15).

Suppose now that (2.15) holds: $\rho^{\omega}_{p,F_0}(x,x) = \operatorname{cap}^{1/p}((\{x\}, F_0); L^1_p(D';\omega)) = 0$. By the definition of capacity, for every $\varepsilon \in (0, 1/2)$, there exists a function $u_\varepsilon \in \operatorname{Lip}_{\mathrm{loc}}(D')$ such that $u_\varepsilon(y) \in [0,1]$ for all $y \in D'$, while $u_\varepsilon|_{F_0} = 0$, $u_\varepsilon(x) = 1$, and

$$\int_{D'} |\nabla u_\varepsilon|^p(y)\omega(y)\,dy < \varepsilon. \tag{2.17}$$

Since x is an interior point of $\{y \in D' \colon u_\varepsilon(y) > 1 - \varepsilon\}$, we have $B(x, r_0) \subset \{y \in D' \colon u_\varepsilon(y) > 1 - \varepsilon\}$ for some ball $B(x, r_0)$. Consequently, the function

$$\frac{\min(u_\varepsilon(y), 1-\varepsilon)}{1-\varepsilon}$$

is admissible for the capacity of the condenser $(\overline{B(x,r)}, F_0)$ provided that $r \in (0, r_0)$. Therefore,

$$\begin{aligned}\operatorname{cap}\big((\overline{B(x,r)}, F_0); L^1_p(D';\omega)\big) &\leqslant \frac{1}{(1-\varepsilon)^p}\int_{D'} \big|\nabla\big(\min(u_\varepsilon(y), 1-\varepsilon)\big)\big|^p \omega(y)\,dy \\ &\leqslant \frac{1}{(1-\varepsilon)^p}\int_{D'} |\nabla u_\varepsilon|^p(y)\omega(y)\,dy \leqslant \frac{\varepsilon}{(1-\varepsilon)^p} < 2^p\varepsilon\end{aligned}$$

by (2.17). Since $\varepsilon \in (0, 1/2)$ is arbitrary, (2.16) is justified.

This completes the proof of Proposition 2.6.

Observe that (2.16) always holds in the case $q \leqslant p \leqslant n$ and $\omega \equiv 1$. In the case of a nontrivial weight function condition (2.15) need not hold, see Examples 2.7 and 2.8.

Example 2.7 [50, Example 2.22]. Consider the domain $D' = B(0,2)$ with the weight $\omega(x) = |x|^\alpha$, where $\alpha > -n$, and $p > 1$. The capacity of the condenser $\mathcal{E} = (\overline{B(0,r)}, B(0,1))$ with $0 < r < 1$ in the space $L_p(D';\omega)$, where the weight function ω belongs to the special class of weight functions called admissible in [50], is

$$\operatorname{cap}\bigl((\overline{B(0,r)}, B(0,1)); L^1_p(D';\omega)\bigr) = \begin{cases} c(n,p,\alpha)\bigl|1 - r^{(p-n-\alpha)/(p-1)}\bigr|^{1-p} & \text{for } p-n-\alpha \neq 0,\\ \sigma_{n-1}\Bigl(\ln \dfrac{1}{r}\Bigr)^{1-p} & \text{for } p-n-\alpha = 0,\end{cases}$$

where σ_{n-1} is the measure of the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$, while $c(n,p,\alpha)$ is a constant depending only on n, p, and α. Since

$$\operatorname{cap}\bigl((\overline{B(0,r)}, B(0,1)); L^1_p(D';\omega)\bigr) \to \operatorname{cap}\bigl((\{0\}, B(0,1)); L^1_p(D';\omega)\bigr) \quad \text{as} \quad r \to 0,$$

the definition of the capacity metric function yields $\rho^\omega_{p,S(0,1)}(0,0) \neq 0$ if $p-n-\alpha > 0$.

In the following example, we construct a weight function for which condition (2.15) is violated on a countable dense subset of D'.

Example 2.8. Consider an arbitrary bounded domain $D' \subset \mathbb{R}^n$, a continuum F_0, and a number α satisfying $p - n - \alpha > 0$. To each point x_i of some countable dense subset of D' associate the function

$$D' \ni x \mapsto \omega_i(x) = \begin{cases} \omega(x - x_i) & \text{if } x \in B(x_i,2) \cap D',\\ 2^\alpha & \text{if } x \in D' \setminus B(x_i,2),\end{cases}$$

where ω is the weight function of Example 2.7. As the weight function on the domain D' consider

$$D' \ni x \mapsto \sigma(x) = \sum_{i=1}^{\infty} \frac{1}{2^i}\omega_i(x).$$

It is not difficult to check that the function σ is integrable on D'. Fix an index $j \in \mathbb{N}$ and a function $u \in \operatorname{Lip}_{\mathrm{loc}}(D') \cap L^1_p(D';\sigma)$ admissible for the capacity $\operatorname{cap}\bigl((\{x_j\}, F_0); L^1_p(D';\omega)\bigr)$. In view of the inequality

$$\frac{1}{2^{ip}} \int_{D'} |\nabla u(x)|^p \omega_i(x)\,dx \leqslant \int_{D'} |\nabla u(x)|^p \sigma(x)\,dx,$$

which is valid for every admissible function u mentioned above, the left-hand side of the last inequality is separated from zero by some constant independent of u. Therefore,

$$\rho^\sigma_{p,F_0}(x_j,x_j) = \operatorname{cap}^{1/p}\bigl((\{x_j\}, F_0); L^1_p(D';\sigma)\bigr) \neq 0$$

for every index $j \in \mathbb{N}$.

Example 2.9. Consider a bounded domain $D' \subset \mathbb{R}^n$, a point $x \in D'$, a continuum $F_0 \subset D' \setminus B(x, e^{-1})$, and a weight $\omega\colon D' \to [1,\infty)$ with $\omega \in \operatorname{BMO}(D')$. For $0 < r < e^{-2}$ define the function

$$u_r(y) = \begin{cases} 0 & \text{if } y \in D' \setminus B(x,e^{-1}),\\ \dfrac{\log(\log(1/|y|))}{\log(\log(1/r))} & \text{if } y \in D' \cap (B(x,e^{-1}) \setminus B(x,r)),\\ 1 & \text{if } y \in D' \cap B(x,r).\end{cases}$$

It is not difficult to verify that u_r belongs to the class of admissible functions $\mathcal{A}(B(x,r)\cap D', F_0)$. Then the definition of capacity yields

$$\rho^{\omega}_{n,F_0}(x,x) = \operatorname{cap}\big((\{x\},F_0); L^1_n(D';\omega)\big) = \lim_{r\to 0}\operatorname{cap}\big(\big(B(x,r)\cap D',F_0\big); L^1_n(D';\omega)\big)$$
$$\leqslant \lim_{r\to 0}\int_{D'} |\nabla u_r(y)|^n \omega(y)\,dy = 0.$$

The last equality holds thanks to the following estimate for $\omega \in \mathrm{BMO}(B(x,1))$ [21, Lemma 5.2]:

$$\int_{D'} |\nabla u_r(y)|^n \omega(y)\,dy \leqslant \frac{1}{\log(\log(1/r))}\int_{B(x,e^{-1})\setminus B(x,r)} \frac{\omega(y)\,dy}{|y|^n(\log(1/|y|))^n}$$
$$\leqslant \frac{C}{\log(\log(1/r))},$$

where the constant C depends only on n and ω, but is independent of r.

Examples 2.7–2.9 show that condition (2.15) depends on the properties of the weight function ω.

Henceforth, denote by $d(x,y)$ the Euclidean distance between two points $x,y\in\mathbb{R}^n$.

PROPOSITION 2.10. *Consider a homeomorphism $f\colon D'\to D$ belonging to the class $\mathcal{Q}_{p,q}(D',\omega;D)$, where $n-1<q\leqslant p\leqslant n$ for $n\geqslant 3$ and $1\leqslant q\leqslant p\leqslant 2$ for $n=2$.*

(1) *If $y\in D'\setminus F_0$ and $\rho^{\omega}_{p,F_0}(z_m,y)\to 0$ as $m\to\infty$ in the domain $D'\setminus F_0$, then*
   (a) *one of conditions (2.15) and (2.16) is met at the point y;*
   (b) *we have the convergence $d(z_m,y)\to 0$ as $m\to\infty$.*

(2) *Provided with (2.15) at $y\in D'\setminus F_0$, the convergence $d(z_m,y)\to 0$ as $m\to\infty$ implies the convergence $\rho^{\omega}_{p,F_0}(z_m,y)\to 0$ with respect to the capacity (ω,p)-metric function ρ^{ω}_{p,F_0} in the domain $D'\setminus F_0$.*

PROOF. (1) By Definition 2.3, for each $m\in\mathbb{N}$, there exists a continuous curve $\gamma_m\colon[0,1]\to D'\setminus F_0$ with endpoints $z_m=\gamma_m(0)$, $y=\gamma_m(1)\in D'\setminus F_0$ such that

$$\operatorname{cap}^{1/p}\big((\overline{\gamma_m},F_0); L^1_p(D';\omega)\big) \leqslant 2\rho^{\omega}_{p,F_0}(z_m,y), \tag{2.18}$$

where $\overline{\gamma_m}=\gamma_m([0,1])$ stands for the image of the curve $\gamma_m\colon[0,1]\to D'\setminus F_0$. Using the inequality

$$\operatorname{cap}^{1/p}\big((\{y\},F_0); L^1_p(D';\omega)\big) \leqslant \operatorname{cap}^{1/p}\big((\overline{z_m y},F_0); L^1_p(D';\omega)\big),$$

valid for all $m\in\mathbb{N}$, from (2.18) and the condition $\rho^{\omega}_{p,F_0}(z_m,y)\to 0$ as $m\to\infty$ in the domain $D'\setminus F_0$, we infer that

$$\operatorname{cap}^{1/p}\big((\{y\},F_0); L^1_p(D';\omega)\big) = 0.$$

Furthermore, from (2.9) and the condition $\rho^{\omega}_{p,F_0}(z_m,y)\to 0$ as $m\to\infty$ we find that $\rho_{q,f(F_0)}(f(z_m),f(y))\to 0$ as $m\to\infty$. By Lemma 2.1, the latter is possible if and only if $f(z_m)\to f(y)$ as $m\to\infty$. Hence, $z_m\to y$ as $m\to\infty$.

(2) Assume that condition (2.15) holds at $y \in D' \setminus F_0$ and $d(z_m, y) \to 0$ as $m \to \infty$ for some sequence $z_m \in D' \setminus F_0$. On assuming condition (2.15), Proposition 2.6 implies that

$$\lim_{r\to 0} \operatorname{cap}^{1/p}\bigl((\overline{B(y,r)}, F_0); L^1_p(D';\omega)\bigr) = 0. \tag{2.19}$$

For $z_m \in B(y,r)$, from the properties of capacity, we infer that

$$\rho^{\omega}_{p,F_0}(z_m, y) \leqslant \operatorname{cap}^{1/p}\bigl((\overline{B(y,r)}, F_0); L^1_p(D';\omega)\bigr),$$

and hence $\rho^{\omega}_{p,F_0}(z_m, y) \to 0$ as $m \to \infty$.

Proposition is proved.

Given a set $B \subset \mathbb{R}^n$, define the distance $\operatorname{dist}(y, B)$ from a point $y \in \mathbb{R}^n$ to B as $\inf_{z\in B} d(y,z)$, where $d(\,\cdot\,,\cdot\,)$ is the Euclidean distance. The following proposition generalizes Proposition 2.10.

Proposition 2.11. *Consider a homeomorphism $f\colon D' \to D$ belonging to the class $\mathcal{Q}_{p,q}(D',\omega; D)$, where $n-1 < q \leqslant p \leqslant n$ for $n \geqslant 3$ and $1 \leqslant q \leqslant p \leqslant 2$ for $n = 2$. If $\{y_l \in D' \setminus F_0\}$, for $l \in \mathbb{N}$, is a fundamental sequence with respect to the metric function ρ^{ω}_{p,F_0}, while y is one of its partial limits in the topology of the extended space $\overline{\mathbb{R}^n}$, then the following claims hold*:

(1) *if $y \in D' \setminus F_0$, then $d(y_l, y) \to 0$ as $l \to \infty$*;

(2) *if $y \in F_0$, then $d(y_l, y) \to 0$ as $l \to \infty$*;

(3) *if $y \in \partial D'$ and the sequence $\{y_l \in D'\}$ is bounded, we have* $\operatorname{dist}(y_l, \partial D') \to 0$ *as $l \to \infty$*;

(4) *if $\{y\} = \overline{\mathbb{R}^n} \setminus \mathbb{R}^n$, either $y_l \to y$ as $l \to \infty$ in the topology of $\overline{\mathbb{R}^n}$, or $\underline{\lim}_{l\to\infty} d(y_l, 0) < \infty$ and $\lim_{k\to\infty} \operatorname{dist}(y_{l_k}, \partial D') = 0$ for every subsequence $\{y_{l_k} \in D'\}$ bounded in $\mathbb{R}^n$.*

Proof. Let us prove the claims of Proposition 2.11 one by one.

(1) Take a fundamental sequence $\{y_l \in D' \setminus F_0\}$, for $l \in \mathbb{N}$, with respect to the metric function ρ^{ω}_{p,F_0} and its subsequence $\{y_{l_k} \in D' \setminus F_0\}$, for $k \in \mathbb{N}$, converging in the topology of the Euclidean space $\mathbb{R}^n$ to some point $y \in D' \setminus F_0$ as $k \to \infty$. By (2.9), the sequence $\{f(y_l) \in D \setminus f(F_0)\}$, $l \in \mathbb{N}$, is also fundamental with respect to $\rho_{q,f(F_0)}$. In addition, since f is continuous at $y \in D' \setminus F_0$, we have the convergence $f(y_{l_k}) \to f(y)$ as $k \to \infty$. Lemma 2.1 implies the convergence $\rho_{q,f(F_0)}(f(y_{l_k}), f(y)) \to 0$ as $k \to \infty$. Since the sequence $\{f(y_l) \in D \setminus f(F_0)\}$, for $l \in \mathbb{N}$, is fundamental with respect to the metric function $\rho_{q,f(F_0)}$, we see that $\rho_{q,f(F_0)}(f(y_l), f(y)) \to 0$ as $l \to \infty$. Moreover, $f(y_l) \to f(y)$ as $l \to \infty$, again by Lemma 2.1. Since f^{-1} is continuous at $f(y)$, we infer that $y_l \to y$ as $l \to \infty$.

(2) Take a fundamental sequence $\{y_l \in D' \setminus F_0\}$, $l \in \mathbb{N}$, with respect to the metric function ρ^{ω}_{p,F_0} and its subsequence $\{y_{l_k} \in D' \setminus F_0\}$, $k \in \mathbb{N}$, converging in the topology of the Euclidean space $\mathbb{R}^n$ to some point $y \in F_0$ as $k \to \infty$. The second claim will be justified once we verify that the stated properties contradict the existence of a subsequence $\{y_{l_j}\}$, $j \in \mathbb{N}$, such that $d(y_{l_j}, y) \geqslant 1/\beta$ for all $j \in \mathbb{N}$, where $\beta > 1$ is some number. Indeed, if such a subsequence exists, then

$$d(f(y_{l_j}), f(y)) \geqslant \frac{1}{\beta'} \tag{2.20}$$

for all $j \in \mathbb{N}$, where $\beta' > 1$ is some number, whose existence is ensured by the locally uniform continuity of the homeomorphism f. On the other hand, the sequence $\{f(y_l) \in D \setminus f(F_0)\}$, for $l \in \mathbb{N}$, is fundamental with respect to the metric function $\rho_{q,f(F_0)}$. Applying the subordination principle, see Property 1.2, we infer that this sequence is also fundamental with respect to the metric function $\rho_{q,K}$ for an arbitrary compact set $K \subset \operatorname{int} f(F_0)$ with nonempty interior. By Lemma 2.1, the sequence $f(y_l)$ converges to $f(y) \notin K$ as $l \to \infty$. The latter contradicts (2.20).

(3) Take a partial limit $y = \lim_{j\to\infty} y_{l_j} \in \partial D'$ and assume on the contrary that there exists a subsequence $\{y_{l_k}\}$, for $k \in \mathbb{N}$, such that $\operatorname{dist}(y_{l_k}, \partial D') \geqslant \beta_0 > 0$ for all $k \in \mathbb{N}$, where β_0 is some number. By the latter property, since $\{y_l\}$ is bounded, we may assume that the subsequence $\{y_{l_k}\}$ converges to some $z \in D'$. Consequently, the hypotheses of the first claim are fulfilled, and so $y_l \to z$ as $l \to \infty$, which contradicts the property $\lim_{j\to\infty} y_{l_j} = y \in \partial D'$.

(4) If under the condition $\{y\} = \overline{\mathbb{R}^n} \setminus \mathbb{R}^n$ we have $\underline{\lim}_{l\to\infty} d(y_l, 0) = \infty$, then $y_l \to y$ as $l \to \infty$ in the topology of $\overline{\mathbb{R}^n}$.

Assume that if $\underline{\lim}_{l\to\infty} d(y_l, 0) < \infty$, then $\overline{\lim}_{k\to\infty} \operatorname{dist}(y_{l_k}, \partial D') > 0$ for some bounded subsequence $\{y_{l_k}\}$, for $k \in \mathbb{N}$. Then some subsequence $y_{l_{k_j}} \to z \in D'$ as $j \to \infty$. The first claim yields $y_l \to z \in D'$ as $l \to \infty$, which contradicts the hypotheses of claim (4). Proposition is proved.

Remark 2.12. Below we consider the fundamental sequences with respect to the metric function ρ^{ω}_{p,F_0} which satisfy just one of claims (1), (3), and (4) of Proposition 2.11.

2.2. Capacity metric and completion of the domain.

Definition 2.13. Denote by $D'_{\rho,p}$ the collection of points $\{y \in D' \setminus F_0\}$ with the capacity metric function ρ^{ω}_{p,F_0}.

Definition 2.14. Two *fundamental* sequences $\{y_l \in D'_{\rho,p}\}$ and $\{z_l \in D'_{\rho,p}\}$, $l \in \mathbb{N}$, with respect to the capacity metric function ρ^{ω}_{p,F_0} are called *equivalent* whenever $\rho^{\omega}_{p,F_0}(y_l, z_l) \to 0$ as $l \to \infty$.

Define a new metric space $(\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\omega}_{p,F_0})$:

(1) its elements are the *classes of equivalent fundamental sequences*, and

(2) the distance between two elements $X, Y \in \widetilde{D}'_{\rho,p}$ equals

$$\widetilde{\rho}^{\omega}_{p,F_0}(X, Y) = \lim_{l\to\infty} \rho^{\omega}_{p,F_0}(x_l, y_l), \tag{2.21}$$

where $\{x_l\}$ and $\{y_l\}$ are fundamental sequences in X and Y, respectively.

Assume henceforth that the metric space $(\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\omega}_{p,F_0})$ is nonempty.

By analogy with the Hausdorff completion theorem, see [57, Ch. 2, § 6] and [58, § 21.3], for instance, we can prove the following statement.

Proposition 2.15. *The following claims hold*:

(1) *the metric function* (2.21) *is independent of the choice of fundamental sequences* $\{x_l\}$ *in the class* X *and* $\{y_l\}$ *in the class* Y;

(2) *the metric function* (2.21) *in Definition* 2.14 *satisfies on* $\widetilde{D}'_{\rho,p}$ *the axioms of a metric space*;

(3) *the space* $(\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\,\omega}_{p,F_0})$ *includes a subset isometric to the metric space*

$$\{y \in D' \setminus F_0 \mid \rho^{\omega}_{p,F_0}(y,y) = 0\}$$

with the metric ρ^{ω}_{p,F_0}.

Proof. Recall how we identify the points of $\{y \in D' \setminus F_0 \mid \rho^{\omega}_{p,F_0}(y,y) = 0\}$ with the metric ρ^{ω}_{p,F_0} and those of some subset in $(\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\,\omega}_{p,F_0})$.

Associate to a point $y \in D'_{\rho,p}$ the equivalence class $i(y) \in \widetilde{D}'_{\rho,p}$ containing the constant sequence $\{y, y, \dots, y, \dots\}$. It is obvious that

$$\widetilde{\rho}^{\,\omega}_{p,F_0}(i(x), i(y)) = \rho^{\omega}_{p,F_0}(x,y),$$

so that the embedding

$$i\colon D'_{\rho,p} \to \widetilde{D}'_{\rho,p}$$

is an isometry. Proposition is proved.

Definition 2.16. Refer to the metric space $(D'_{\rho,p}, \rho^{\omega}_{p,F_0})$ to the subset $\{y \in D' \setminus F_0 \mid \rho^{\omega}_{p,F_0}(y,y) = 0\}$ with the metric ρ^{ω}_{p,F_0}.

Proposition 2.17. *Consider a homeomorphism* $f\colon D' \to D$ *that belongs to* $\mathcal{Q}_{p,q}(D',\omega;D)$, *where* $n-1 < q \leqslant p \leqslant n$ *for* $n \geqslant 3$ *and* $1 \leqslant q \leqslant p \leqslant 2$ *for* $n = 2$. *Fix an equivalence class* $h \in \widetilde{D}'_{\rho,p}$ *and take an arbitrary fundamental sequence* $\{y_l\}$ *in this class. Then the following behavior of* $\{y_l\}$ *is possible*:

(1) (a) $y_l \to y \in D' \setminus F_0$ *as* $l \to \infty$ *in the Euclidean metric and the limit* y *is unique*, *meaning that it is independent of the choice of sequence in* h;

(b) $y_l \to y \in F_0$ *as* $l \to \infty$ *in the Euclidean metric and the limit* y *is unique*;

(2) *otherwise*, *depending on the choice of fundamental sequence in* h, *the following cases are possible*:

(a) $\overline{\lim}_{l\to\infty} d(y_l, 0) < \infty$ *and then* $\operatorname{dist}(y_l, \partial D') \to 0$ *as* $l \to \infty$;

(b) $\overline{\lim}_{l\to\infty} d(y_l, 0) = \infty$ *and* $\underline{\lim}_{\,l\to\infty} d(y_l) < \infty$, *and then*

$$\lim_{l\to\infty} \operatorname{dist}(y_{l_k}, \partial D') = 0$$

for every bounded subsequence $\{y_{l_k} \in D'\}$ *of* $\mathbb{R}^n$;

(c) $\lim_{l\to\infty} d(y_l, 0) = \infty$.

Proof. The fundamental sequence $\{y_l\}$ of class $h \in \widetilde{D}'_{\rho,p}$ bounded in $\mathbb{R}^n$ satisfies the hypotheses of Proposition 2.11, and so its claims (1)–(4) can hold for it. It remains to verify that the same claims hold for every bounded sequence $\{z_l\}$ of class $h \in \widetilde{D}'_{\rho,p}$.

Indeed, the sequence $y_1, z_1, y_2, z_2, \dots, y_n, z_n, \dots$ is fundamental with respect to the metric function ρ^{ω}_{p,F_0}, bounded in $\mathbb{R}^n$, and has accumulation point y, which lies either in D' or in $\partial D'$.

In the first case by claim (1) of Proposition 2.11 some subsequence of the sequence

$$y_1,\ z_1,\ y_2,\ z_2,\ \dots,\ y_n,\ z_n,\ \dots \tag{2.22}$$

converges to $y \in D'$. Hence, both sequences (2.22) and $\{z_l\}$ converge to y as $l \to \infty$. In the second case no subsequence $\{z_{l_k}\}$ of the sequence $\{z_l\}$ can converge to any point $z \in D'$, because similar arguments would yield the impossible coincidence $y = z$. Then if the sequence $\{z_l\}$ is bounded, then claim (3) of Proposition 2.11 shows that $\operatorname{dist}(z_l, \partial D') \to 0$ as $l \to \infty$.

If some sequence $\{y_l\}$ of class $h \in \widetilde{D}'_{\rho,p}$ is not bounded, then we should apply claim (4) of Proposition 2.11 to justify claims (2)(b) and (2)(c) of Proposition 2.17.

Now take another fundamental sequence $\{z_l\}$, $l \in \mathbb{N}$, in the same class $h \in \widetilde{D}'_{\rho,p}$. Applying Proposition 2.11 to it, we conclude that z_l cannot converge to any point $z \in D'$, as otherwise y_l would also converge to $z \in D'$ as $l \to \infty$. Thus, for the sequence z_l, only claims (3) or (4) of Proposition 2.11 can hold, which proves Proposition 2.17.

The following example shows that each of the possibilities (a), (b), and (c) of part 2 of Proposition 2.17 can be realized in various sequences of the same class.

Example 2.18 (ridge domain). In [18], [26], and [45] there is an example of a simply-connected domain with nontrivial boundary elements, although the domain is locally connected at all boundary points of the Euclidean boundary. For $q = p = n = 3$ and $\omega \equiv 1$ consider the ridge domain

$$D' = \{x = (x_1, x_2, x_3)\colon |x_2| < x_1^{\alpha},\ \alpha > 2,\ 0 < x_1 < 1,\ 0 < x_3 < \infty\}.$$

Take the sequences

$$y_l^1 = \left(\frac{1}{l}, \frac{1}{2l^{\alpha}}, 1\right), \qquad y_l^3 = \left(\frac{1}{l}, \frac{1}{2l^{\alpha}}, l\right),$$

and define the sequence $\{y_l^2\}$ by alternating $\{y_l^1\}$ and $\{y_l^3\}$:

$$y_{2l}^2 = \left(\frac{1}{l}, \frac{1}{2l^{\alpha}}, 1\right) \quad \text{and} \quad y_{2l+1}^2 = \left(\frac{1}{l}, \frac{1}{2l^{\alpha}}, l\right).$$

Then $\{y_l^1\}$, $\{y_l^2\}$, and $\{y_l^3\}$ satisfy conditions (2)(a), (2)(b), and (2)(c) of Proposition 2.17, respectively, since $y_l^1, y_{2l}^1 \to (0,0,1)$ and $\lim_{l\to\infty} d(y_{2l+1}^2, 0) = \lim_{l\to\infty} d(y_l^3, 0) = \infty$. In addition, the chosen sequences lie in the same equivalence class $h \in \widetilde{D}'_{\rho,3}$. Here the metric ρ_{p,F_0}^{ω} is defined with respect to the Sobolev space $L_3^1(D')$ and $F_0 \subset D'$ is an arbitrary continuum with nonempty interior.

With the new notation and concepts, we can interpret Proposition 2.4 as follows.

Theorem 2.19 (extension of $\mathcal{Q}_{p,q}$-homeomorphisms). *Consider a homeomorphism $f\colon D' \to D$ of class $\mathcal{Q}_{p,q}(D', \omega; D)$, where $n - 1 < q \leqslant p \leqslant n$ for $n \geqslant 3$ and $1 \leqslant q \leqslant p \leqslant 2$ for $n = 2$. Then*

(1) *the mapping $f\colon D' \to D$ induces the Lipschitz mapping*

$$f\colon \big(D'_{\rho,p}, \widetilde{\rho}_{p,F_0}^{\omega}\big) \to \big(D_{\rho,q}, \widetilde{\rho}_{q,f(F_0)}\big)$$

of metric spaces, with the estimate for metric distances

$$\begin{cases} \widetilde{\rho}_{p,f(F_0)}\big(f(x), f(y)\big) \leqslant K_p \widetilde{\rho}_{p,F_0}^{\omega}(x,y) & \text{if } q = p, \\ \widetilde{\rho}_{q,f(F_0)}\big(f(x), f(y)\big) \leqslant \Psi_{p,q}(D' \setminus F_0)^{1/\sigma} \widetilde{\rho}_{p,F_0}^{\omega}(x,y) & \text{if } q < p, \end{cases} \tag{2.23}$$

for all points $x, y \in D'_{\rho,p}$, where $1/\sigma = 1/q - 1/p$;

(2) *the mapping* $f\colon D'\to D$ *induces the Lipschitz mapping*

$$\widetilde{f}\colon \big(\widetilde{D}'_{\rho,p},\widetilde{\rho}^{\,\omega}_{p,F_0}\big)\to\big(\widetilde{D}_{\rho,q},\widetilde{\rho}_{q,f(F_0)}\big)$$

of the "completed" metric spaces: to each element $X\in(\widetilde{D}'_{\rho,p},\widetilde{\rho}^{\,\omega}_{p,F_0})$ *associate the element* $\widetilde{f}(X)\in(\widetilde{D}_{\rho,q},\widetilde{\rho}_{q,f(F_0)})$ *containing the fundamental sequence* $\{f(x_l)\}$, *where* $\{x_l\}\in X$, *with the estimate for metric distances*

$$\begin{cases}\widetilde{\rho}_{p,f(F_0)}\big(\widetilde{f}(X),\widetilde{f}(Y)\big)\leqslant K_p\widetilde{\rho}^{\,\omega}_{p,F_0}(X,Y) & \text{if } q=p,\\ \widetilde{\rho}_{q,f(F_0)}\big(\widetilde{f}(X),\widetilde{f}(Y)\big)\leqslant \Psi_{p,q}(D'\setminus F_0)^{1/\sigma}\widetilde{\rho}^{\,\omega}_{p,F_0}(X,Y) & \text{if } q<p,\end{cases}\tag{2.24}$$

for $x,y\in\widetilde{D}'_{\rho,p}$.

Proof. Claim (1) and (2.23) follow directly from Proposition 2.4, while (2.24) follows from Definition (2.21) of the metric distance between the elements of "completed" spaces. Indeed, if a sequence $\{x_l\}$ belongs to $X\in\widetilde{D}'_{\rho,p}$, then by (2.23) the sequence $\{f(x_l)\}$ is fundamental with respect to the metric function $\widetilde{\rho}_{q,f(F_0)}$. We call the class of equivalent sequences containing $\{f(x_l)\}$ the image of the class X, and denote the resulting mapping by $\widetilde{f}$. Deducing that

$$\widetilde{\rho}_{p,f(F_0)}\big(\widetilde{f}(X),\widetilde{f}(Y)\big)=\lim_{l\to\infty}\widetilde{\rho}_{p,f(F_0)}\big(\widetilde{f}(x_l),\widetilde{f}(y_l)\big)$$

and using Definition (2.21), as well as (2.23), we obtain the claim.

Therefore, Proposition 2.19 determines the extended mapping $\widetilde{f}$.

Definition 2.20. Consider a homeomorphism $f\colon D'\to D$ of class $\mathcal{Q}_{p,q}(D',\omega;D)$, where $n-1<q\leqslant p\leqslant n$ for $n\geqslant 3$ and $1\leqslant q\leqslant p\leqslant 2$ for $n=2$. Denote by $\widetilde{f}\colon(\widetilde{D}'_{\rho,p},\widetilde{\rho}^{\,\omega}_{p,F_0})\to(\widetilde{D}_{\rho,q},\widetilde{\rho}_{q,f(F_0)})$ the extension of f to the "completed" metric spaces: to each $X\in(\widetilde{D}'_{\rho,p},\widetilde{\rho}^{\,\omega}_{p,F_0})$ we associate $\widetilde{f}(X)\in(\widetilde{D}_{\rho,q},\widetilde{\rho}_{q,f(F_0)})$ containing the fundamental sequence $\{f(x_l)\}$.

2.3. Capacity boundary. Boundary correspondence of mappings. By Proposition 2.17, in the topology of the extended space $\overline{\mathbb{R}}^n$ the limit points of the fundamental sequence $\{y_l\}$, $l\in\mathbb{N}$, of some class $h\in\widetilde{D}'_{\rho,p}$ can be

(1a) the points $y\in D'\setminus F_0$: in this case $y_l\to y\in D'\setminus F_0$ as $l\to\infty$ in the Euclidean metric;

(1b) the points $y\in F_0$: in this case $y_l\to y\in F_0$ as $l\to\infty$ in the Euclidean metric.

Otherwise, depending on the choice of fundamental sequence $\{y_l\}$, $l\in\mathbb{N}$, of class h, the possible variants are

(2a) the points $y\in\partial D'$;

(2b) the point $y=\infty$.

Clearly, in case (1a) we can identify the class $h\in\widetilde{D}'_{\rho,p}$ with some point $y\in D'\setminus F_0$, while in case (1b), with some point $y\in F_0$.

With this observation at hand, define the concept of the capacity boundary. By claim (3) of Proposition 2.15, the points of the metric space $(D'_{\rho,p},\rho^{\omega}_{p,F_0})$ are identified with those in some subset of $(\widetilde{D}'_{\rho,p},\widetilde{\rho}^{\,\omega}_{p,F_0})$ so that the embedding

$$i\colon D'_{\rho,p}\to\widetilde{D}'_{\rho,p}$$

is an isometry. Henceforth we identify $D'_{\rho,p}$ with the image $i(D'_{\rho,p})$ in $\widetilde{D}'_{\rho,p}$.

Definition 2.21. The complement

$$H^{\omega}_{\rho,p}(D') = \widetilde{D}'_{\rho,p} \setminus D' \quad \bigl(H_{\rho,q}(D) = \widetilde{D}_{\rho,q} \setminus D\bigr)$$

is called the *capacity boundary* of D' (respectively, D). The metric on the boundary is induced from the ambient space. The *capacity boundary elements* of the domain D' or D are the points of the capacity boundary $H^{\omega}_{\rho,p}(D')$ or $H_{\rho,q}(D)$.

Theorem 2.22 (boundary correspondence of $\mathcal{Q}_{p,q}$-homeomorphisms). *Consider a homeomorphism $f\colon D' \to D$ of class $\mathcal{Q}_{p,q}(D',\omega;D)$, where $n-1<q\leqslant p\leqslant n$ for $n\geqslant 3$ and $1\leqslant q\leqslant p\leqslant 2$ for $n=2$. Then the restriction $\widetilde{f}\,|_{H^{\omega}_{\rho,p}(D')}$ is a Lipschitz mapping*

$$\widetilde{f}\,|_{H^{\omega}_{\rho,p}(D')}\colon \bigl(H^{\omega}_{\rho,p}(D'), \widetilde{\rho}^{\omega}_{p,F_0}\bigr) \to \bigl(H_{\rho,q}(D), \widetilde{\rho}_{q,f(F_0)}\bigr) \tag{2.25}$$

of capacity boundaries.

Proof. Take the mapping $\widetilde{f}\colon (\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\omega}_{p,F_0}) \to (\widetilde{D}_{\rho,q}, \widetilde{\rho}_{q,f(F_0)})$ of Theorem 2.19. Then the restriction $\widetilde{f}\,|_{H^{\omega}_{\rho,p}(D')}$ is the Lipschitz mapping

$$\widetilde{f}\,|_{H^{\omega}_{\rho,p}(D')}\colon \bigl(H^{\omega}_{\rho,p}(D'), \widetilde{\rho}^{\omega}_{p,F_0}\bigr) \to \bigl(\widetilde{D}_{\rho,q}, \widetilde{\rho}_{q,f(F_0)}\bigr). \tag{2.26}$$

To prove the claim, it remains to verify that the image of this mapping lies in $(H_{\rho,q}(D), \widetilde{\rho}_{q,f(F_0)})$.

Assume on the contrary that there exists a boundary element $h\in(H^{\omega}_{\rho,p}(D'), \widetilde{\rho}^{\omega}_{p,F_0})$ such that $\widetilde{f}(h) = y \in (D, \widetilde{\rho}_{q,f(F_0)})$. Then there exists a sequence $\{x_l\}\in h$, where $h \in (\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\omega}_{p,F_0})$, such that $f(x_l)\to y$ in the metric space $(D_{\rho,q}, \widetilde{\rho}_{q,f(F_0)})$. By Proposition 2.10, the sequence $f(x_l)$ converges to $y\in D$ in the Euclidean metric as well. Therefore, $f^{-1}(f(x_l)) = x_l$ converges to $\varphi(y)\in D'$ in $\mathbb{R}^n$. Proposition 2.17 shows that every sequence $\{z_l\}\in h$ converges to $\varphi(y)\in D'$ in the Euclidean metric, and so in the metric space $(\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\omega}_{p,F_0})$ as well, see Proposition 2.10, which obviously contradicts the initial assumption. Theorem is proved.

2.4. Support of a boundary element. In this section, we fix an arbitrary number p satisfying $n-1<p\leqslant n$ for $n\geqslant 3$ and $1\leqslant p\leqslant 2$ for $n=2$.

Definition 2.23. Given a domain D' in $\mathbb{R}^n$, the *support* $\mathcal{S}_h$ of a boundary element $h\in H^{\omega}_{\rho,p}(D')$ is the set of all accumulation points in the topology of the extended space $\overline{\mathbb{R}^n}$ of all fundamental sequences with respect to the capacity metric lying in the equivalence class defining h.

Remark 2.24. Proposition 2.17 and Definition 2.21 show that no accumulation point of a sequence in $h\in H^{\omega}_{\rho,p}(D')$ fundamental with respect to the capacity metric belongs to D'. Therefore,

$$\mathcal{S}_h \subset \partial D' \cup \{\infty\}.$$

PROPOSITION 2.25. *If D' is a domain in $\mathbb{R}^n$, then*

(1) *the support $\mathcal{S}_h$ of a boundary element $h \in H^{\omega}_{\rho,p}(D')$ coincides with the intersection $\bigcap_{\varepsilon>0} \overline{B_\rho(h,\varepsilon) \cap D'}$:*

$$\mathcal{S}_h = \bigcap_{\varepsilon>0} \overline{B_\rho(h,\varepsilon) \cap D'}, \tag{2.27}$$

where the closure is taken in the topology of the extended space $\overline{\mathbb{R}^n}$;

(2) *if $\rho^{\omega}_{p,F_0}(h_1,h_2) = 0$ for two boundary elements $h_1, h_2 \in H^{\omega}_{\rho,p}(D')$, then $\mathcal{S}_{h_1} = \mathcal{S}_{h_2}$.*

PROOF. Split the proof into three stages.

(1) Fix a boundary element $h \in H^{\omega}_{\rho,p}(D')$. Let us verify the inclusion

$$\mathcal{S}_h \subset \bigcap_{\varepsilon>0} \overline{B_\rho(h,\varepsilon) \cap D'}. \tag{2.28}$$

By the definition of a boundary element $h \in H^{\omega}_{\rho,p}(D')$, there exists a fundamental sequence $\{y_l\} \in h$ with respect to the (ω,p)-metric function with $\rho^{\omega}_{p,F_0}(y_l,h) \to 0$ as $l \to \infty$. For the sequence $\{y_l \in D'_{\rho,p}\}$ and its subsequences only the behavior described in Proposition 2.17 is possible:

(a) $y_l \to y \in D' \setminus F_0$ or $y_l \to y \in F_0$ as $l \to \infty$ in the Euclidean metric and the limit y is unique, meaning independent of the choice of sequence in h;

(b) $\overline{\lim}_{l\to\infty} d(y_l,0) < \infty$ and then $\operatorname{dist}(y_l,\partial D') \to 0$ as $l \to \infty$;

(c) $\overline{\lim}_{l\to\infty} d(y_l,0) = \infty$ and $\underline{\lim}_{l\to\infty} d(y_l,0) < \infty$, and then

$$\lim_{l\to\infty} \operatorname{dist}(y_{l_k},\partial D') = 0$$

for every subsequence $\{y_{l_k} \in D'\}$ bounded in $\mathbb{R}^n$;

(d) if $d(y_l,0) \to \infty$, then $\infty \in \mathcal{S}_h$.

Definition 2.21 excludes case (a). In cases (b)–(d) we have

$$\mathcal{S}_h \subset \partial D' \cup \{\infty\}.$$

In these cases, for every $\varepsilon > 0$ the elements of the sequence $\{y_l \in D'\}$ starting with some index l_0 lie in $B_\rho(h,\varepsilon) \cap D'$ for all $l \geqslant l_0$. Thus the accumulation points of $\{y_l \in D'\}$ lie in the closure $\overline{B_\rho(h,\varepsilon) \cap D'}$ in the topology of the extended space $\overline{\mathbb{R}^n}$. Since we choose the fundamental sequence $\{y_l\} \in h$ for the boundary element h arbitrarily, it follows that $\mathcal{S}_h \subset \overline{B_\rho(h,\varepsilon) \cap D'}$. The inclusion (2.28) is established as $\varepsilon > 0$ is arbitrary.

(2) In the case $\rho^{\omega}_{p,F_0}(h_1,h_2) = 0$ the equivalence classes of fundamental sequences for the boundary elements h_1 and h_2 coincide. Hence, we conclude that the supports of h_1 and h_2 coincide.

(3) To justify (2.27), it remains to verify the reverse inclusion to (2.28):

$$\bigcap_{\varepsilon>0} \overline{B_\rho(h,\varepsilon) \cap D'} \subset \mathcal{S}_h. \tag{2.29}$$

Indeed, if $x \in \bigcap_{\varepsilon>0} \overline{B_\rho(h,\varepsilon) \cap D'}$, then for each $l \in \mathbb{N}$ there exists $x_l \in B_\rho(h,1/l) \cap D'$ such that simultaneously $\rho^{\omega}_{p,F_0}(x_l,h) \to 0$ as $l \to \infty$ and (using Proposition 2.17

and extracting a subsequence if necessary) $x_l \to x$ in the topology of the extended space $\overline{\mathbb{R}^n}$. Therefore, the fundamental sequence $\{x_l\}$ with respect to the capacity metric determines a boundary element, which coincides with h. Thus, $x \in \mathcal{S}_h$ and (2.29) is established. The inclusions (2.28) and (2.29) are equivalent to (2.27). Proposition is proved.

PROPOSITION 2.26. *The support $\mathcal{S}_h$ of each boundary element $h \in H^{\omega}_{\rho,p}(D')$ is connected in the topology of the space $\overline{\mathbb{R}^n}$.*

PROOF. Assume on the contrary that for some boundary element $h \in H^{\omega}_{\rho,p}(D')$ there are two disjoint open sets $V, W \subset \overline{\mathbb{R}^n}$ with $\mathcal{S}_h \subset V \cup W$, while $\mathcal{S}_h \cap V \neq \varnothing$ and $\mathcal{S}_h \cap W \neq \varnothing$. Take two points $x \in \mathcal{S}_h \cap V$ and $y \in \mathcal{S}_h \cap W$ and fundamental sequences $\{x_m\}, \{y_m\} \in h$ with respect to the capacity metric such that $x_m \to x$ and $y_m \to y$ as $m \to \infty$. There is a curve $\gamma_m \subset D'$ with endpoints x_m and y_m such that $\operatorname{cap}((\gamma_m, F_0); L^1_p(D';\omega)) \to 0$ as $m \to \infty$. For all big enough m, starting with some there exists a point $z_m \in \gamma_m$ satisfying $z_m \notin V \cup W$. We emphasize that the sequence $\{z_m\}$, fundamental with respect to the capacity metric, belongs to the equivalence class h. Extracting a subsequence, we may assume that $z_m \to z_0$, where $z_0 \in \overline{D'} \setminus (V \cup W)$; here the closure is taken in the topology of the extended space $\overline{\mathbb{R}^n}$. Since $z_0 \notin \mathcal{S}_h$, we arrive at a contradiction with the definition of the support of a boundary element. Proposition is proved.

PROPOSITION 2.27. *Consider the support $\mathcal{S}_h$ of $h \in H^{\omega}_{\rho,p}(D')$. For every sequence $\{x_m\} \in h$ we have the convergence $x_m \to \mathcal{S}_h$ as $m \to \infty$ in the topology of the extended space $\overline{\mathbb{R}^n}$.*

PROOF. Proposition 2.25 excludes the possibility that $\mathcal{S}_h \cap D' \neq \varnothing$.

Suppose that $\mathcal{S}_h$ is bounded in $\mathbb{R}^n$ and $\mathcal{S}_h \subset \partial D'$. Suppose that there exists a subsequence $\{x_{m_k} \in D'\}$, for $k \in \mathbb{N}$, of some fundamental sequence $\{x_m\} \in h$ such that $d(x_{m_k}, \mathcal{S}_h) \geqslant \alpha > 0$ for all $k \in \mathbb{N}$, where α is some constant. Then the sequence $\{x_m\}$ has an accumulation point at some positive distance from $\mathcal{S}_h$. This point must lie in the support of the boundary element h, which contradicts the connectedness of $\mathcal{S}_h$.

However, if the support $\mathcal{S}_h$ is unbounded and the sequence x_m does not converge to $\mathcal{S}_h$ in the topology of the extended space $\overline{\mathbb{R}^n}$ then $\underline{\lim}_{m\to\infty} x_m < \infty$. Consequently, there exists a finite accumulation point at some positive distance from $\mathcal{S}_h$. As in the previous case, we arrive at a contradiction with the connectedness of $\mathcal{S}_h$. Proposition is proved.

PROPOSITION 2.28 (criterion for singleton support). *Given a boundary element $h \in H^{\omega}_{\rho,p}(D')$ of the domain D', the support $\mathcal{S}_h$ amounts to a single point if and only if for all fundamental sequences $\{x_m\}, \{y_m\} \in h$ with respect to the capacity metric there exist curves $\overline{x_m y_m}$, for $m \in \mathbb{N}$, with* $\operatorname{diam}(\overline{x_m y_m}) \to 0$ *as $m \to \infty$.*

PROOF. *Necessity*. Suppose that $\mathcal{S}_h = \{x_0\}$. Assume on the contrary that there exist fundamental sequences $\{x_m\}$ and $\{y_m\}$ of class h with respect to the capacity metric converging to x_0, curves $\gamma_m = \overline{x_m y_m}$ with

$$\operatorname{cap}^{1/p}\big((\gamma_m, F_0); L^1_p(D';\omega)\big) \to 0 \quad \text{as} \quad m \to \infty, \tag{2.30}$$

and a number $\alpha > 0$ such that

$$\operatorname{diam} \gamma_m \geqslant \alpha > 4d(x_m, y_m) \quad \text{for all} \quad m \in \mathbb{N}$$

because $x_m \to x_0$ and $y_m \to x_0$ as $m \to \infty$. Then, for each $m \in \mathbb{N}$, there exists a point $z_m \in \gamma_m$ such that, on the one hand,

$$d(x_m, z_m) > \frac{\alpha}{4}, \qquad d(y_m, z_m) > \frac{\alpha}{4} \tag{2.31}$$

and on the other hand, (2.7) and (2.30) yield $\rho^{\omega}_{p,F_0}(z_m, x_m) \to 0$ as $m \to \infty$. Hence, we infer that the sequence $\{z_m\}$, for $m \in \mathbb{N}$, is fundamental with respect to the capacity metric and belongs to the boundary element h. On the other hand, there exists a subsequence $\{z_{m_i}\}$, for $i \in \mathbb{N}$, converging to some point z_0; moreover, (2.31) implies that $z_0 \neq x_0$. Since $z_0 \in \mathcal{S}_h$ by the definition of support, we arrive at a contradiction with its being a singleton.

Sufficiency. By contradiction, suppose that there are two sequences $\{x_m\}$ and $\{y_m\} \in h$ fundamental with respect to the capacity metric and converging to distinct points x and y of the support $\mathcal{S}_h$. By the hypotheses, there exist curves $\gamma_m = \overline{x_m y_m}$ such that $\operatorname{diam} \gamma_m \to 0$ as $m \to \infty$. In particular, $\operatorname{diam} \gamma_m \geqslant d(x_m, y_m) \to d(x, y) > 0$ as $m \to \infty$, which, evidently, contradicts the property $\operatorname{diam} \gamma_m \to 0$ as $m \to \infty$ inferred from the assumption.

Proposition is proved.

2.5. Continuous extension of mappings of class $\mathcal{Q}_{p,q}(D', \omega; D)$ to the Euclidean boundary. In this section, we fix arbitrary numbers q and p satisfying $n - 1 < q \leqslant p \leqslant n$ for $n \geqslant 3$ and $1 \leqslant q \leqslant p \leqslant 2$ for $n = 2$.

In what follows, we define domains μ-connected at boundary points.

Definition 2.29 (connectedness properties [16], [18]). (1) A domain D' is called *locally connected* at $x \in \partial D'$ if for every neighborhood U of x there is a neighborhood $V \subset U$ of this point such that $V \cap D'$ is connected.

(2) An unbounded domain D' is called *locally connected* at ∞ if for every neighborhood U of ∞ there is a neighborhood $V \subset U$ of this point such that $V \cap D'$ is connected.

(3) A domain D' is called *locally μ-connected* at $x \in \partial D'$, where $\mu \in \mathbb{N}$, if for every neighborhood U of x there is a neighborhood $V \subset U$ of this point such that $V \cap D'$ consists of μ connected components, each of which is locally connected at x. Observe that a domain D' *locally* 1-*connected* at $x \in \partial D'$ is precisely the domain D' *locally connected* at $x \in \partial D'$.

(4) An unbounded domain D' is called *locally μ-connected* at ∞, where $\mu \in \mathbb{N}$, if for every neighborhood U of ∞ there is a neighborhood $V \subset U$ of this point such that $V \cap D'$ consists of μ connected components, each of which is locally connected at ∞. In the case $\mu = 1$ we obtain the domain D' *locally connected* at ∞.

(5) A domain D' is called *finitely connected* at $x \in \partial D'$ or $x = \infty$ whenever it is μ-connected at x for some $\mu \in \mathbb{N}$.

The following example demonstrates the appearance of domains which are multiply connected at boundary points.

EXAMPLE 2.30 (slit ball). Let $D' = B(0,1) \setminus (\{0\} \times [0,1)^{n-1})$. It is not difficult to see that D' is locally 2-connected at each point $x \in \{0\} \times (0,1)^{n-1}$. If $\omega = 1$ is the trivial weight and $p = n$, then condition (2.37) is met for every point $x \in \{0\} \times (0,1)^{n-1}$, and x lies in the support of two distinct boundary elements $h_+, h_- \in H_{\rho,n}(D')$.

Let us present the methods of [16, Theorem 1.10] for describing connectedness alternative to Definition 2.29 and useful below.

PROPOSITION 2.31. *Given a domain $D' \in \mathbb{R}^n$ and its boundary point $x \in \partial D'$, the following statements are equivalent*:

(1) *D' is locally μ-connected at x*;

(2) *for every neighborhood U of x there exists a neighborhood $V \subset U$ of this point such that $V \cap D'$ consists of μ connected components*, *the boundary of each of which contains x*;

(3) *μ is the smallest integer for which the following condition holds*: *given $\mu + 1$ sequences $\{x_{1,k}\}, \dots, \{x_{\mu+1,k}\}$ of points in D' converging to x*, *if V is some neighborhood of x*, *then there exists a connected component of $V \cap D'$ including subsequences of two distinct sequences.*

To obtain similar properties at ∞, we should use the stereographic projection to map the domain D' onto the unit sphere in $\mathbb{R}^{n+1}$ with the point ∞ going into the north pole, on which the property of local μ-connectedness at ∞ can be stated by analogy with the above.

EXAMPLE 2.32. On the plane $\mathbb{R}^2$ take the complement

$$B(0,4) \setminus \{x = (x_1, x_2) \in B(0,2) \mid x_1 \cdot x_2 = 0\}$$

as the domain D'. Fix two numbers $\alpha > -2$ and $p \in (1,2]$ with $p - 2 > \alpha$, as well as a continuum $F_0 \subset B(0,4) \setminus \overline{B(0,2)}$ with nonempty interior. As the weight function $\sigma\colon B(0,4) \to (0,\infty)$ take

$$D' \ni x \mapsto \sigma(x) = \begin{cases} \omega(x) & \text{if } x \in B(0,2) \cap D' \text{ and } x_1 \cdot x_2 > 0, \\ 2^\alpha & \text{otherwise,} \end{cases}$$

where ω is the weight function of example 2.7.

The domain D' is obviously 4-connected at 0: each intersection $B(0,r) \cap D'$, for $r \in (0,2)$, consists of 4 connected components. Denote them by V_1 and V_3 if $x_1 \cdot x_2 > 0$ and by V_2 and V_4 otherwise.

It is natural to define the weighted capacity of the condenser $\mathcal{E} = (\{0\}, F_0) \subset D'$ in the space $L^1_p(D'; \sigma)$ with respect to the connected component V_i as

$$\operatorname{cap}\bigl((\{0\}, F_0); L^1_p(V_i, D'; \omega)\bigr) = \inf_u \|u \mid L^1_p(D'; \omega)\|^p, \tag{2.32}$$

where the infimum is over all functions $u \in \operatorname{Lip}_{\mathrm{loc}}(D') \cap L^1_p(D'; \omega)$ such that $u|_{B(0,r) \cap V_i} \equiv 1$ for some $r > 0$, depending on u, and $u|_{F_0} \equiv 0$.

On account of Example 2.7, the capacity of the point 0 with respect to V_1 and V_3 is positive, and with respect to V_2 and V_4 it vanishes.

This example motivates the following definition.

Definition 2.33. Suppose that a domain D' is locally μ-connected at a boundary point $x \in \partial D'$ and denote by $V_1, V_2, \dots, V_\mu$ the distinct connected components of the intersection $B(x,r) \cap D'$, where $r \in (0, r_0)$ for sufficiently small $r_0 > 0$, whose boundaries contain x. Define the weighted capacity of the condenser $\mathcal{E} = (\{x\}, F_0) \subset D'$ in the space $L^1_p(D';\omega)$ with respect to the connected component V_i as

$$\operatorname{cap}\bigl((\{x\}, F_0); L^1_p(V_i, D';\omega)\bigr) = \inf_u \|u \mid L^1_p(D';\omega)\|^p, \tag{2.33}$$

where the infimum is over all functions $u \in \operatorname{Lip}_{\mathrm{loc}}(D') \cap L^1_p(D';\omega)$ such that $u|_{B(x,r)\cap V_i} \equiv 1$ for some $r \in (0, r_0)$, depending on u, and $u|_{F_0} \equiv 0$.

If $\mu = 1$, then instead of notation (2.33) we will simply write

$$\operatorname{cap}\bigl((\{x\}, F_0); L^1_p(D';\omega)\bigr).$$

In the case $x = \infty$, the lower bound in (2.33) is taken over all functions $u \in \operatorname{Lip}_{\mathrm{loc}}(D') \cap L^1_p(D';\omega)$ such that $u|_{(\mathbb{R}^n \setminus B(x,r))\cap V_i} \equiv 1$ for some $r > 0$, depending on u, and $u|_{F_0} \equiv 0$, and denoted by

$$\operatorname{cap}\bigl((\{\infty\}, F_0); L^1_p(V_i, D';\omega)\bigr). \tag{2.34}$$

A boundary point $x \in \partial D'$ is called a *point of zero capacity with respect to the connected component* V_i whenever

$$\operatorname{cap}\bigl((\{x\}, F_0); L^1_p(V_i, D';\omega)\bigr) = 0. \tag{2.35}$$

If condition (2.35) is independent of the choice of continuum F_0, we simply write

$$\operatorname{cap}\bigl((\{x\}); L^1_p(V_i, D';\omega)\bigr) = 0. \tag{2.36}$$

Proposition 2.28 yields the following corollary.

Corollary 2.34. *The following claims hold.*

(1) *If the domain D' is locally connected at x_0 and the condition*

$$\operatorname{cap}\bigl((\{x_0\}, F_0); L^1_p(D';\omega)\bigr) = 0 \tag{2.37}$$

holds at x_0, then the boundary elements h_1 and $h_2 \in H^\omega_{\rho,p}(D')$ of the domain D' whose supports $\mathcal{S}_{h_1}$ and $\mathcal{S}_{h_2}$ meet at x_0 cannot be distinct: $h_1 = h_2$.

(2) *Suppose that the domain D' is locally μ-connected at x_0, and that at x_0 condition* (2.35)

$$\operatorname{cap}\bigl((\{x_0\}, F_0); L^1_p(V_i, D';\omega)\bigr) = 0$$

holds for all $i = 1, \dots, \mu$. Then the boundary elements $h_1, h_2, \dots, h_\mu, h_{\mu+1} \in H^\omega_{\rho,p}(D')$ of D' whose supports $\mathcal{S}_{h_1}, \mathcal{S}_{h_2}, \dots, \mathcal{S}_{h_\mu}, \mathcal{S}_{h_{\mu+1}}$ share the point x_0 cannot be distinct: at least two of them coincide.

PROOF. (1) Suppose that the supports $\mathcal{S}_{h_1}$ and $\mathcal{S}_{h_2}$ of two boundary elements $h_1, h_2 \in H^{\omega}_{\rho,p}(D')$ of D' meet at x_0. Take two arbitrary sequences $\{x_k\} \in h_1$ and $\{y_k\} \in h_2$ fundamental with respect to the metric ρ^{ω}_{p,F_0} such that $x_k \to x_0$ and $y_k \to x_0$ as $k \to \infty$. Since D' is locally connected at x_0, we can connect x_k and y_k with curves $\gamma_k = \overline{x_k y_k}$ such that $\operatorname{diam}\gamma_k \to 0$ as $k \to \infty$. Since D' is locally connected at x, condition (2.37) also yields

$$\operatorname{cap}\big((\gamma_k, F_0); L^1_p(D';\omega)\big) \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, we see that the sequence $\{x_k\}$ and $\{y_k\}$ are equivalent, which implies $h_1 = h_2$.

(2) Assume that the supports $\mathcal{S}_{h_1}, \mathcal{S}_{h_2}, \dots, \mathcal{S}_{h_{\mu+1}}$ of some boundary elements $h_1, h_2, \dots, h_{\mu+1} \in H^{\omega}_{\rho,p}(D')$, for $\mu \in \mathbb{N}$, of D' meet at x_0. Take an arbitrary fundamental sequence $\{x_{ik}\} \in h_i$ with respect to the metric ρ^{ω}_{p,F_0} such that $x_{ik} \to x_0$ as $k \to \infty$, for $i = 1, \dots, \mu + 1$. By claim (3) of Proposition 2.31, since D' is locally μ-connected at x_0, there exists a connected component V_{i_0}, for $1 \leqslant i_0 \leqslant \mu_0$, of the intersection $B(x_0, r) \cap D'$ containing subsequences, for instance, x_{1k_j} and x_{2l_j}, for $j \in \mathbb{N}$, of two distinct sequences x_{1k} and x_{2k}, for $k \in \mathbb{N}$. Since the connected component V_{i_0} is locally connected at x_0 and

$$\operatorname{cap}\big((\{x_0\}, F_0); L^1_p(V_{i_0}, D';\omega)\big) = 0,$$

the hypotheses of claim 1 hold, which yields $h_1 = h_2$. Corollary is proved.

DEFINITION 2.35 (associated support and connected components). Consider some boundary element $h \in H^{\omega}_{\rho,p}(D')$ whose support $\mathcal{S}_h$ contains $x \in \partial D'$ such that the domain D' is μ-connected at x, while $\{y_m\}$ is a fundamental sequence with respect to the metric ρ^{ω}_{p,F_0} belonging to the boundary element h and converging to x in the topology of $\overline{\mathbb{R}^n}$. Since D' is μ-connected at x, there exists at least one connected component V_i of the intersection $B(x, r) \cap D'$, where $r > 0$ is a sufficiently small number, which contains some subsequence $\{y_{m_k}\}$, for $k \in \mathbb{N}$. In this case, say that the support $\mathcal{S}_h$ of the boundary element h and the connected component V_i are *associated* with each other at $x \in \mathcal{S}_h$.

PROPOSITION 2.36. *The following claims hold.*

(1) *If D' is a locally μ-connected domain at x, the support $\mathcal{S}_h$ of some boundary element $h \in H^{\omega}_{\rho,p}(D')$ contains $x \in \partial D'$ and is associated with the connected component V_i at x, while the weighted capacity of x with respect to the connected component V_i vanishes,*

$$\operatorname{cap}\big((\{x\}, F_0); L^1_p(V_i, D';\omega)\big) = 0,$$

then, for every sequence $\{x_m \in V_i \cap D'\}$ of points, $d(x_m, x) \to 0$ implies that $\{x_m\} \in h$ and

$$\rho_{q,f(F_0)}\big(f(x_m), \widetilde{f}(h)\big) \to 0 \quad \text{as} \quad m \to \infty. \tag{2.38}$$

(2) *If D' is a locally μ-connected domain at ∞, the support $\mathcal{S}_h$ of some boundary element $h \in H^{\omega}_{\rho,p}(D')$ contains ∞ and is associated with the connected component V_i at ∞, while the weighted capacity of the point ∞ with respect to some connected component V_i vanishes,*

$$\operatorname{cap}\big((\{\infty\}, F_0); L^1_p(V_i, D';\omega)\big) = 0,$$

then, for every sequence $\{x_m \in V_i \cap D'\}$ *of points,* $d(x_m, 0) \to \infty$ *implies that* $\{x_m\} \in h$ *and* (2.38) *holds.*

PROOF. (1) Choose $x \in \partial D'$ and a sequence $\{x_m \in V_i \cap D'\}$ such that $d(x_m, x) \to 0$ as $m \to \infty$. Since $V_i \cap D'$ is locally connected at x, see claim (2) of Proposition 2.31, we infer the existence of curves $\overline{x_m x_{m+k}}$ with endpoints x_m and x_{m+k}, for $k \geqslant 1$, such that $\operatorname{diam} \overline{x_m x_{m+k}} \to 0$ as $m, k \to \infty$. Since $\operatorname{cap}((\{x\}, F_0); L^1_p(V_i, D'; \omega)) = 0$, Definition 2.33 yields $\rho^{\omega}_{p,F_0}(x_m, x_{m+k}) \to 0$ as $m, k \to \infty$. Thus, on the one hand the sequence $\{x_m\}$ is fundamental with respect to the metric ρ^{ω}_{p,F_0}, and on the other, $d(x_m, x) \to 0$ as $m \to \infty$.

Now take an arbitrary sequence $\{y_m \in V_i \cap D'\}$, for $m \in \mathbb{N}$, fundamental with respect to the metric ρ^{ω}_{p,F_0}, belonging to some boundary element h, and converging x in the Euclidean metric. Verify that every fundamental sequence $\{x_m\}$ with respect to the metric ρ^{ω}_{p,F_0} satisfies

$$\rho^{\omega}_{p,F_0}(x_m, y_m) \to 0 \quad \text{as} \quad m \to \infty. \tag{2.39}$$

As in the previous argument, we conclude that $\rho^{\omega}_{p,F_0}(x_m, y_m) \to 0$ as $m \to \infty$. Thus, property (2.39) and property $\{x_m\} \in h$ together with it are justified.

Applying (2.9), we deduce (2.38): indeed, the sequences $\{f(x_m)\}$ and $\{f(y_m)\}$ are equivalent with respect to the capacity metric function $\rho_{q,f(F_0)}$ in the domain D. Hence, $\{f(x_m)\} \in \widetilde{f}(h)$ and $\rho_{q,f(F_0)}(f(x_m), \widetilde{f}(h)) \to 0$ as $m \to \infty$.

(2) The second claim can be justified similarly.

Proposition is proved.

THEOREM 2.37 (boundary behavior of homeomorphisms). *Consider a homeomorphism* $f\colon D' \to D$ *of class* $\mathcal{Q}_{p,q}(D', \omega; D)$, *where* $n-1 < q \leqslant p \leqslant n$ *for* $n \geqslant 3$ *and* $1 \leqslant q \leqslant p \leqslant 2$ *for* $n = 2$, *as well as a weight function* $\omega \in L_{1,\mathrm{loc}}(D')$.

Suppose that the domain D'

(1) *is locally* μ*-connected at some boundary point* $y \in \partial D'$,

(2) *the support* $\mathcal{S}_h$ *of some boundary element* $h \in H^{\omega}_{\rho,p}(D')$ *contains* y,

(3) *we have* $\operatorname{cap}((\{y\}, F_0); L^1_p(V_i, D'; \omega)) = 0$, *where* V_i *is the connected component associated with the support* $\mathcal{S}_h$ *at* y.

Then the boundary behavior of the mapping $f\colon D' \to D$ *at* $x \in \partial D'$ *is*

$$f(z) \to \mathcal{S}_{\widetilde{f}(h)} \quad \text{as} \quad z \to y, \quad z \in V_i \cap D',$$

in the topology of the extended space $\mathbb{R}^n$.

PROOF. Take a sequence $\{y_m \in V_i \cap D'\}$ converging to $y \in \partial D'$ as $m \to \infty$. Proposition 2.36 shows that $\rho^{\omega}_{q,f(F_0)}(f(y_m), \widetilde{f}(h)) \to 0$ as $m \to \infty$. In addition, by Proposition 2.27 the sequence $\{f(y_m)\}$ converges to the support $\mathcal{S}_{\widetilde{f}(h)}$ in the topology of the extended space $\mathbb{R}^n$. The proof of Theorem 2.37 is complete.

COROLLARY 2.38 (continuous extension to boundary points). *Consider a homeomorphism* $f\colon D' \to D$ *of class* $\mathcal{Q}_{p,q}(D', \omega; D)$, *where* $n-1 < q \leqslant p \leqslant n$ *for* $n \geqslant 3$ *and* $1 \leqslant q \leqslant p \leqslant 2$ *for* $n = 2$, *as well as a weight function* $\omega \in L_{1,\mathrm{loc}}(D')$.

Suppose also that

(1) *the domain* D' *is locally* μ*-connected at some boundary point* $y \in \partial D'$;

(2) *the support* $\mathcal{S}_h$ *of a boundary element* $h \in H^{\omega}_{\rho,p}(D')$ *contains* y;

(3) *we have* $\operatorname{cap}((\{y\}, F_0); L^1_p(V_i, D'; \omega)) = 0$, *where* V_i *is the connected component associated with the support* $\mathcal{S}_h$ *at* y;

(4) *the support* $\mathcal{S}_{\widetilde{f}(h)}$ *of the boundary element* $\widetilde{f}(h)$ *amounts to a singleton*: $\mathcal{S}_{\widetilde{f}(h)} = \{x\} \in \partial D$.

Then the mapping $f\colon D' \to D$ *extends by continuity to* $y \in \partial D'$ *and*

$$\lim_{z\to y,\, z\in V_i\cap D'} f(z) = x.$$

Proof. Take a sequence $\{y_m \in V_i \cap D'\}$ converging to $y \in \partial D'$ as $m \to \infty$. Theorem 2.37 shows that

$$f(z) \to \mathcal{S}_{\widetilde{f}(h)} \quad \text{as} \quad z \to y, \quad z \in V_i \cap D'$$

in the topology of the extended space $\mathbb{R}^n$. Since by assumption the support $\mathcal{S}_{\widetilde{f}(h)}$ of the boundary element $\widetilde{f}(h)$ is a singleton, $\mathcal{S}_{\widetilde{f}(h)} = \{x\} \in \partial D$, the above implies that the sequence $\{f(y_m)\}$ converges to $x \in \partial D$. The proof of Corollary 2.38 is complete.

Corollary 2.38 yields the next one.

Corollary 2.39 (continuous extension to the Euclidean boundary). *Consider a homeomorphism* $f\colon D' \to D$ *of class* $\mathcal{Q}_{p,q}(D', \omega; D)$, *where* $n-1 < q \leqslant p \leqslant n$ *for* $n \geqslant 3$ *and* $1 \leqslant q \leqslant p \leqslant 2$ *for* $n = 2$, *as well as a weight function* $\omega \in L_{1,\mathrm{loc}}(D')$. *The following claims hold*:

(1) *if* D' *is locally connected at* $y \in \partial D'$ *and* $\operatorname{cap}((\{y\}, F_0); L^1_p(D'; \omega)) = 0$, *then* y *lies in the support* $\mathcal{S}_h$ *of some boundary element* $h \in H^{\omega}_{\rho,p}(D')$;

(2) *if the support* $\mathcal{S}_{\widetilde{f}(h)}$ *of the boundary element* $\widetilde{f}(h)$ *is a singleton*, $\mathcal{S}_{\widetilde{f}(h)} = \{x\} \in \partial D$, *then the mapping* $f\colon D' \to D$ *extends by continuity to* $y \in \mathcal{S}_h$ *of the boundary element* $h \in H^{\omega}_{\rho,p}(D')$, *and*

$$\lim_{z\to y,\, z\in D'} f(z) = x \quad \textit{for every point} \quad y \in \mathcal{S}_h. \tag{2.40}$$

Proof. All hypotheses of Proposition 2.36 are obviously met, and so y lies in some boundary element $h \in H^{\omega}_{\rho,p}(D')$. The argument above and the hypotheses of the corollary ensure the fulfillment of the conditions of Corollary 2.38 for $\mu = 1$. It shows that the mapping $f\colon D' \to D$ extends by continuity to $y \in \mathcal{S}_h$, and the limit equals (2.40). Corollary is proved.

Example 2.40 (domain with nontrivial boundary elements). Consider $D = (0,1)^2 \setminus \bigcup_{k\in\mathbb{N}} I_k \subset \mathbb{R}^2$, where $I_k = [1/2, 1) \times \{1/2^k\}$ determine the cuts. It is not difficult to see that $I = [1/2, 1) \times \{0\}$ is the support of a boundary element for $p = 2$ and $\omega \equiv 1$.

Example 2.41. For the domain from Example 2.18, the edge of the ridge

$$E = \{x = (x_1, x_2, x_3) \colon x_1 = x_2 = 0,\, 0 \leqslant x_3 \leqslant \infty\}$$

is indeed the support of a boundary element.

REMARK 2.42. For the weight ω and the domain D' such that the collection $H^{\omega}_{\rho,p}(D')$ of boundary elements is independent of the choice of the continuum F_0, the support $\mathcal{S}_h$ of an arbitrary boundary element $h \in H^{\omega}_{\rho,p}(D')$ is independent of the choice of F_0, and consequently, all statements of this section are absolute.

§ 3. Moduli of curve families and homeomorphisms of class $\mathcal{Q}_{p,q}(D',\omega)$

Consider a domain D' in $\mathbb{R}^n$, where $n \geqslant 2$, a weight function $\omega\colon D' \to (0,\infty)$ of class $L_{1,\mathrm{loc}}$, and a family Γ of (continuous) curves or paths $\gamma\colon [a,b] \to D'$.

Recall that, given a curve family Γ in D' and a real number $p \geqslant 1$, the weighted *p-modulus* of Γ is defined as

$$\operatorname{mod}^{\omega}_p(\Gamma) = \inf_{\rho} \int_{D'} \rho^p(x)\omega(x)\,dx,$$

where the infimum is over all nonnegative Borel functions $\rho\colon D' \to [0,\infty]$ with

$$\int_{\gamma} \rho\,ds \geqslant 1 \tag{3.1}$$

for all (locally) rectifiable curves $\gamma \in \Gamma$. In the case of trivial weight $\omega \equiv 1$ we write $\operatorname{mod}_p(\Gamma)$ instead of $\operatorname{mod}^1_p(\Gamma)$. Recall that the integral in (3.1) for a rectifiable curve $\gamma\colon [a,b] \to D'$ is defined as

$$\int_{\gamma} \rho\,ds = \int_0^{l(\gamma)} \rho(\widetilde{\gamma}(t))\,dt,$$

where $l(\gamma)$ is the length of $\gamma\colon [a,b] \to D'$, while $\widetilde{\gamma}\colon [0,l(\gamma)] \to D'$ is its natural parametrization, that is, the unique continuous mapping with $\gamma = \widetilde{\gamma} \circ S_{\gamma}$, where $S_{\gamma}\colon [a,b] \to [0,l(\gamma)]$ is the length function, defined at $t \in [a,b]$ as $S_{\gamma}(t) = l(\gamma|_{[a,t]})$. If γ is only a locally rectifiable curve, then we put

$$\int_{\gamma} \rho\,ds = \sup \int_{\gamma'} \rho\,ds$$

with the least upper bound taken over all rectifiable subcurves $\gamma'\colon [a',b'] \to D'$ of γ, where $[a',b'] \subset (a,b)$ and $\gamma' = \gamma_{[a',b']}$.

The functions ρ satisfying (3.1) are called *admissible functions*, or *metrics*, for the family Γ.

An equivalent description of the mappings of classes $\mathcal{Q}_{p,q}(D',\omega;D)$ is obtained in [33] in the modular language: to this end, we should replace capacity in the definition of $\mathcal{Q}_{p,q}(D',\omega;D)$ by the modulus of the curve family whose endpoints lie on the plates of the condenser.

REMARK 3.1. It is observed in [32, Section 4.4] that in the case $q = p = n$ ($n-1 < q = p < n$) the class of homeomorphisms $\mathcal{Q}_{n,n}(D',\omega;D)$ ($\mathcal{Q}_{p,p}(D',\omega;D)$) is included into the class of ω-homeomorphisms ((p,ω)-homeomorphisms)[4] [21] ([59]), defined via a controlled variation of the modulus of the curve family.

[4]Note that [21] ([59]) used the term Q-homeomorphism ((p,Q)-homeomorphism), where the letter Q stands for the weight function, while in this article the same letter in the term "$\mathcal{Q}_{p,q}(D',\omega;D)$-homeomorphism" is the first letter of the word "quasiconformal".

We will verify that, actually, the class $\mathcal{Q}_{n,n}(D',\omega;D)$ coincides with the family of ω-homeomorphisms of [21, § 4.1]. Consider two domains D' and D in $\mathbb{R}^n$, where $n \geqslant 2$, and a function $\omega\colon D' \to [1,\infty)$ of class $L_{1,\mathrm{loc}}$. Recall that a homeomorphism $f\colon D' \to D$ is called an *ω-homeomorphism* whenever

$$\operatorname{mod}_n(f\Gamma) \leqslant \int_{D'} \omega(x)\cdot \rho^n(x)\,dx \tag{3.2}$$

for each family Γ of paths in D' and every admissible function ρ for Γ. By [33, Theorem 19], the homeomorphisms satisfying (3.2) coincide with the homeomorphisms $f\colon D' \to D$ of class $\mathcal{Q}_{n,n}(D',\omega;D)$.

Some properties of the homeomorphisms of class $\mathcal{Q}_{p,q}(D',\omega)$ were studied in [27] (for $n-1<q<p=n$, the value $\Psi_{q,n}(U)$ instead of $\Psi_{q,n}(U\setminus F)$, and $\omega\equiv 1$), [21], [60]–[64] (all for $q=p=n$ and $\omega = Q$), [65], [66] (for $1<q=p<n$ and $\omega=Q$), and many others. In all articles mentioned except [27] the distortion of the geometry of condensers is stated in the language of moduli of curve families, which in a series of cases is a more restrictive characteristic than capacity as far as meaningful applications are concerned.

§ 4. Geometry the boundary

In this section, we consider geometric concepts and the main results of other approaches to the boundary behavior problem.

Definition 4.1. The boundary $\partial D'$ of a domain D' is called (p,ω)-*weakly flat* at $x_0\in\partial D'$, where $p>1$, if for every neighborhood U of x_0 and every number $\lambda>0$, there is a neighborhood $V\subset U$ of x_0 such that for all continua[5] F_0 and F_1 in D', intersecting ∂U and ∂V, the capacity of the condenser $\mathcal{E}=(F_1,F_0)$ satisfies $\operatorname{cap}(\mathcal{E};L_p(D',\omega))\geqslant\lambda$. The boundary $\partial D'$ is called (p,ω)-*weakly flat* whenever it is (p,ω)-weakly flat at each of its points.

A point $x_0\in\partial D'$ is called (p,ω)-*strongly accessible*, where $p>1$, if for every neighborhood U of x_0, there exist a neighborhood $V\subset U$ of this point, a compact set $F_0\subset D'$, and a number $\delta>0$, such that for all continua F_1 in D' intersecting ∂U and ∂V the capacity of the condenser $\mathcal{E}=(F_1,F_0)$ is bounded from below: $\operatorname{cap}(\mathcal{E};L_p(D',\omega))\geqslant\delta$. The boundary $\partial D'$ is called (p,ω)-*strongly accessible* whenever each of its points is (p,ω)-strongly accessible.

In the unweighted case for $p=n$ the properties of the boundary to be weakly flat and strongly accessible are introduced in [21, § 3.8] in terms of moduli of curve families. These conditions generalize properties $P1$ and $P2$ of [18, § 17] and the properties of the boundary to be quasiconformally flat and quasiconformally accessible [16]. The case of arbitrary $p>n-1$ is considered, for instance, in [67].

Proposition 4.2. *Suppose that* $1\leqslant p<\infty$. *If a domain* $D'\subset\mathbb{R}^n$, *where* $n\geqslant 2$, *has* (p,ω)-*weakly flat boundary and* $\omega\in L_{1,\mathrm{loc}}(D')$ *then*

(1) *the boundary* $\partial D'$ *is* (p,ω)-*strongly accessible*;

(2) D' *is locally connected at the boundary points.*

[5] In this definition the interior of F_0 can be empty.

PROOF. The proof follows the scheme of the proof of Proposition 3.1 and Lemma 3.15 of [21] with obvious adjustments.

REMARK 4.3. Since in the unweighted case the modulus and capacity coincide [68]–[70], the properties of the boundary to be weakly flat and strongly accessible of [21] precisely coincide with the case of trivial weight and $p = n$ in Definition 4.1 of $(n, 1)$-weakly flat and $(n, 1)$-strongly accessible boundary.

Moreover, a point $x_0 \in \partial D'$ is $(n, 1)$-strongly accessible whenever it is *quasiconformally accessible* [16, Definition 1.7]: given a neighborhood U of x_0, there are a continuum $F_0 \subset D'$ and a number $\delta > 0$ such that $\operatorname{cap}((F_1, F_0); L^1_p(D', \omega)) \geqslant \delta$ for all connected sets F_1 in D' satisfying $x_0 \in \overline{F}_1$ and $F_1 \cap \partial U \neq \varnothing$.

Note the following connection between the singleton support of a boundary element and the above conditions on the geometry of the boundary.

PROPOSITION 4.4. *Given a weight ω and a domain D' satisfying Remark 2.42, take a boundary element $h \in H^\omega_{\rho,p}(D')$ and a point $x_0 \in S_h$ which is (p, ω)-strongly accessible in the sense of Definition 4.1. Then $\mathcal{S}_h = \{x_0\}$.*

PROOF. Assume on the contrary that x_0 is (p, ω)-strongly accessible and there exists a point $y_0 \in \mathcal{S}_h$ with $d(x_0, y_0) \geqslant \alpha > 0$. By the definition of the support of a boundary element, there exist fundamental sequences $\{x_m \in D'_{\rho,p}\}$ and $\{y_m \in D'_{\rho,p}\}$ with respect to the metric ρ^ω_{p,F_0} such that $x_m \to x_0$ and $y_m \to y_0$ in the topology of the extended Euclidean space. Fix a neighborhood $V \subset U = B(x_0, \alpha/3)$ of x_0, a compact set $F_0 \subset D'$, and a number $\delta > 0$ according to Definition 4.1. Find a number m_0 such that $x_m \in V$ and $y_m \in B(y_0, \alpha/3)$ for all $m \geqslant m_0$. It is obvious that for $m \geqslant m_0$ every curve $\overline{x_m y_m}$ crosses ∂V and ∂U, and so, since the image of the curve is a continuum, the definition of strong accessibility yields $\operatorname{cap}((\overline{x_m y_m}, F_0); L_p(D', \omega)) \geqslant \delta$.

By the definition of the capacity metric (2.7), among the mentioned continua with endpoints $x_m \in V$ and $y_m \in B(y_0, \alpha/3)$ there is $\gamma_m = \overline{x_m y_m}$ such that

$$\rho^\omega_{p,F_0}(x_m, y_m) \geqslant \operatorname{cap}\bigl((\gamma_m, F_0); L_p(D', \omega)\bigr) - \frac{\delta}{2^m} \geqslant \delta\biggl(1 - \frac{1}{2^m}\biggr). \tag{4.1}$$

On the other hand, x_0, $y_0 \in \mathcal{S}_h$ implies that the sequences $\{x_m \in D'_{\rho,p}\}$ and $\{y_m \in D'_{\rho,p}\}$ are equivalent. Therefore, $\rho^\omega_{p,F_0}(x_m, y_m) \to 0$, which contradicts (4.1). Proposition is proved.

COROLLARY 4.5 OF THEOREM 2.19 ([25]; [26, Ch. 5, Theorem 1.3]; [17, Theorem 10.4]). *Consider two domains D and D' in $\mathbb{R}^n$, where $n \geqslant 2$. Every quasiconformal mapping $f\colon D' \to D$ admits a homeomorphic extension to the capacity boundary*

$$\widetilde{f}\,|_{H_{\rho,n}(D')}\colon \bigl(H_{\rho,n}(D'), \widetilde{\rho}_{n,F_0}\bigr) \to \bigl(H_{\rho,n}(D), \widetilde{\rho}_{n,f(F_0)}\bigr).$$

PROOF. By Definition 1.4, the quasiconformal mapping belongs to the class $\mathcal{Q}_{n,n}(D', 1; D)$. The claim follows directly from Theorem 2.22.

COROLLARY 4.6 OF THEOREM 2.38. *Consider two domains D and D' in $\mathbb{R}^n$, where $n \geqslant 2$, and a homeomorphism $f\colon D' \to D$ satisfying one of the following conditions:*

(1) *f is quasiconformal, D' is locally connected on the boundary, and ∂D is quasiconformally accessible* [16, Theorem 2.4].

(2) *$f \in \mathcal{Q}_{n,n}(D',\omega;D)$, in particular, f is an ω-homeomorphism in the sense of Remark 3.1, for*[6] *$\omega \in \mathrm{BMO}(\overline{D'})$, D' is locally connected on the boundary, and ∂D is $(n,1)$-strongly accessible* [21, Lemma 5.3].

Then f admits a continuous extension $\overline{f}\colon \overline{D'} \to \overline{D}$ to the boundary.

Proof. Verify that the hypotheses of Corollary 2.39 hold in both cases, and so $f\colon D' \to D$ extends by continuity to the closure $\overline{D'}$.

In case (1) for every point $x \in \overline{D'}$ we have $\operatorname{cap}((\{x\},F_0);L^1_n(D')) = 0$. Since every quasiconformal mapping is of class $\mathcal{Q}_{n,n}(D',1;D)$, it remains to verify that if $x \in \mathcal{S}_h$ and $h \in H_{\rho,n}(D')$, then the support $\mathcal{S}_{\widetilde{f}(h)}$ of the boundary element $\widetilde{f}(h)$ is a singleton, where $\widetilde{f}$ is the extension of f of Theorem 2.19. The latter follows from the quasiconformal accessibility of ∂D, Proposition 4.4, and Remark 4.3. The possibility of extending the mapping f by continuity to $\partial D'$ follows from Corollary 2.39.

In case (2) observe first of all that Example 2.9 yields $\operatorname{cap}((\{x\},F_0);L^1_p(D';\omega))=0$ for every boundary point $x \in \partial D'$, and this property is local. Hence, it is independent of the continuum F_0. Moreover, by Remark 3.1, the ω-homeomorphism f belongs to $\mathcal{Q}_{n,n}(D',\omega;D)$. As above, Proposition 4.4 shows that the support $\mathcal{S}_{\widetilde{f}(h)}$ of the boundary element $\widetilde{f}(h)$ is a singleton, and Corollary 2.39 guarantees the required result.

Corollary is proved.

Remark 4.7. In the planar case, $n = 2$, the capacity boundary $H_{\rho,2}$ with respect to the Sobolev class L^1_2 is homeomorphic to the boundary of prime ends, see [71], for instance. In the space $\mathbb{R}^n$, where $n \geqslant 3$, it is known that for the domains quasiconformally equivalent to a domain with locally quasiconformal boundary, called *regular domains*, the completion in the prime ends topology is equivalent to the completion in the modular [17] and capacity [26] metrics.

Example 4.8. Take the domain $D' = [0,1]^3 \subset \mathbb{R}^3$, the weight $\omega(y) = y_1^\beta$ with $\beta > -3$, and the ridge domain from Example 2.18:

$$D = \{x = (x_1,x_2,x_3)\colon |x_2| < x_1^\alpha,\ 0 < x_1,x_3 < 1\} \subset \mathbb{R}^3, \qquad \alpha > 2.$$

Consider the mapping f whose inverse $\varphi(x) = f^{-1}(x)$ is defined as

$$\varphi(x) = \begin{pmatrix} x_1 \\ x_2 x_1^\alpha \\ x_3 \end{pmatrix} : D \to D'.$$

It is not difficult to verify that

$$|D\varphi(x)| \approx \max\{1, \alpha x_2 x_1^{\alpha-1}, x_1^\alpha\} \approx 1 \quad \text{and} \quad \det J(x,f) = x_1^\alpha,$$
$$K^{1,\omega}_{3,3}(x,\varphi) \approx x_1^{-(\beta+\alpha)/3} \in L_\infty(D) \quad \text{for} \quad \beta+\alpha \leqslant 0.$$

[6] That is, ω is the restriction to D' of some function $\overline{\omega} \in \mathrm{BMO}(U)$, where U is an open set with $U \supset \overline{D'}$.

Then Theorem 1.6 shows that f is of class $\mathcal{Q}_{3,3}(D',\omega;D)$ and Theorem 2.19 can be applied to it: there exists a continuous extension $f\colon(\widetilde{D}'_{\rho,3},\widetilde{\rho}^{\,\omega}_{3,F_0})\to(\widetilde{D}_{\rho,3},\widetilde{\rho}_{3,f(F_0)})$.

As far as the authors are aware, this example cannot be handled in the framework of other articles concerning boundary correspondence. For instance, [13], [22] require that the boundary of the domain D be $(n,1)$-strongly accessible. In the case of D under consideration, the ridge is neither $(n,1)$-weakly flat nor $(n,1)$-strongly accessible for $\alpha>2$. Indeed, [16, Example 5.5] shows that the points on the ridge are quasiconformally accessible if and only if $1<\alpha<2$ and are not quasiconformally flat for any $\alpha>1$. In addition, it is not difficult to verify that necessary conditions for the ridge to be quasiconformally flat and quasiconformally accessible are also necessary for the ridge to be $(n,1)$-weakly flat and $(n,1)$-strongly accessible, see [16, Theorems 5.3, 5.4].

§ 5. Applications

In this section, we apply the results on boundary behavior to the homeomorphisms of certain classes $\mathcal{Q}_{p,q}(D',\omega;D)$ considered in the examples of this article.

5.1. The homeomorphism of Example 1.13. The following mapping is considered in [31].

For $n-1<s<\infty$, take a homeomorphism $f\colon D'\to D$ of open domains D', $D\subset\mathbb{R}^n$, where $n\geqslant 2$, such that

(1) $f\in W^1_{n-1,\mathrm{loc}}(D')$;

(2) the mapping f has finite distortion;

(3) the outer distortion function

$$D'\ni y\mapsto K^{1,1}_{n-1,s}(y,f)=\begin{cases}\dfrac{|Df(y)|}{|\det Df(y)|^{1/s}} & \text{if } \det Df(y)\neq 0,\\ 0 & \text{if } \det Df(y)=0\end{cases}\tag{5.1}$$

belongs to $L_\sigma(D)$, where $\sigma=(n-1)p$ and $p=s/(s-(n-1))$.

Then by [28, Theorem 4] the inverse homeomorphism $\varphi=f^{-1}\colon D\to D'$ has the following properties:

(4) $\varphi\in W^1_{p,\mathrm{loc}}(D)$, $p=s/(s-(n-1))$;

(5) φ has finite distortion.

The original homeomorphism $f\colon D'\to D$ has the following properties:

(6) it is of class $\mathcal{Q}_{p,p}(D',\omega;D)$ with the constant $K_p=1$ [31, Corollary 26] and the weight function $\omega\in L_{1,\mathrm{loc}}(D')$ defined as

$$\omega(y)=\begin{cases}\dfrac{|\mathrm{adj}\,Df(y)|^p}{|\det Df(y)|^{p-1}} & \text{if } y\in D'\setminus Z',\\ 1 & \text{otherwise},\end{cases}\tag{5.2}$$

see [31, formula (37)], where $Z'=\{y\in D'\colon Df(y)=0\}$;

(7) if $p>n-1$ (which corresponds to $s<n+1/(n-2)$), then the composition operator

$$f^*\colon L^1_{p'}(D)\cap\mathrm{Lip}_{\mathrm{loc}}(D)\to L^1_{p'}(D';\theta)$$

is bounded, where $p'=p/(p-(n-1))$ and $\theta(y)=\omega^{-(n-1)/(p-(n-1))}(y)$.

Proposition 5.1. *The results of this article concerning the boundary behavior of homeomorphisms, namely, Theorems 2.19 and 2.37, Corollaries 2.38 and 2.39, are applicable to the mapping f of Subsection 5.1.*

Explicitly, for $n \leqslant s < n + 1/(n-2)$ the homeomorphism f introduced above has the following properties:

(1) *the mapping f induces a Lipschitz mapping $f\colon (D'_{\rho,p}, \widetilde{\rho}^{\omega}_{p,F_0}) \to (D_{\rho,p}, \widetilde{\rho}_{p,f(F_0)})$ of metric spaces: $\widetilde{\rho}_{p,f(F_0)}(f(x), f(y)) \leqslant \widetilde{\rho}^{\omega}_{p,F_0}(x,y)$ for all points $x, y \in D'_{\rho,p}$;*

(2) *the mapping f induces a Lipschitz mapping $\widetilde{f}\colon (\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\omega}_{p,F_0}) \to (\widetilde{D}_{\rho,p}, \widetilde{\rho}_{p,f(F_0)})$ of "completed" metric spaces:*

to $X \in (\widetilde{D}'_{\rho,p}, \widetilde{\rho}^{\omega}_{p,F_0})$ associate $\widetilde{f}(X) \in (\widetilde{D}_{\rho,q}, \widetilde{\rho}_{q,f(F_0)})$, which contains the fundamental sequence $\{f(x_l)\}$, where $\{x_l\} \in X$:

$$\widetilde{\rho}_{p,f(F_0)}\big(\widetilde{f}(X), \widetilde{f}(Y)\big) \leqslant \widetilde{\rho}^{\omega}_{p,F_0}(X,Y)$$

for $X, Y \in \widetilde{D}'_{\rho,p}$;

(3) *the restriction $\widetilde{f}\,|_{H^{\omega}_{\rho,p}(D')}\colon (H^{\omega}_{\rho,p}(D'), \widetilde{\rho}^{\omega}_{p,F_0}) \to (H_{\rho,p}(D), \widetilde{\rho}_{p,f(F_0)})$ is a Lipschitz mapping of capacity boundaries;*

(4) *if the domain D' is locally μ-connected at a boundary point $y \in \partial D'$, the support $\mathcal{S}_h$ of the boundary element $h \in H^{\omega}_{\rho,p}(D')$ contains y, and*

$$\operatorname{cap}\big((\{y\}, F_0); L^1_p(V_i, D'; \omega)\big) = 0,$$

where V_i is the connected component associated with $\mathcal{S}_h$ at y, then $f(z) \to \mathcal{S}_{\widetilde{f}(h)}$ as $z \to y$ with $z \in V_i \cap D'$ in the topology of the extended space $\mathbb{R}^n$;

(5) *if the domain D' is locally μ-connected at a boundary point $y \in \partial D'$, the support $\mathcal{S}_h$ of the boundary element $h \in H^{\omega}_{\rho,p}(D')$ contains y and*

$$\operatorname{cap}\big((\{y\}, F_0); L^1_p(V_i, D'; \omega)\big) = 0,$$

where V_i is the connected component associated with $\mathcal{S}_h$ at y and $\mathcal{S}_{\widetilde{f}(h)} = \{x\} \in \partial D$, then the mapping $f\colon D' \to D$ extends by continuity to $y \in \partial D'$ and

$$\lim_{z \to y,\, z \in V_i \cap D'} f(z) = x;$$

(6) *if the domain D' is locally connected at $y \in \partial D'$ and*

$$\operatorname{cap}\big((\{y\}, F_0); L^1_p(D'; \omega)\big) = 0,$$

then y lies in the support $\mathcal{S}_h$ of some boundary element $h \in H^{\omega}_{\rho,p}(D')$;

(7) *if $\mathcal{S}_{\widetilde{f}(h)} = \{x\} \in \partial D$, then the mapping $f\colon D' \to D$ extends by continuity to $y \in \mathcal{S}_h$ of the boundary element $h \in H^{\omega}_{\rho,p}(D')$ and*

$$\lim_{z \to y,\, z \in D'} f(z) = x \quad \text{for every points} \quad y \in \mathcal{S}_h.$$

Let us compare the above example with the mapping of [72], which considers a $W^1_{1,\mathrm{loc}}$-homeomorphism $f\colon D' \to D$ with finite distortion, whose *outer distortion function*

$$K^{1,1}_{n,n}(y,f) = \begin{cases} \dfrac{|Df(y)|}{|\det Df(y)|^{1/n}} & \text{if } \det Df(y) \neq 0, \\ 0 & \text{if } \det Df(y) = 0 \end{cases} \tag{5.3}$$

belongs to $L_{(n-1)n,\mathrm{loc}}(D')$.

Verify that this mapping is a particular case for $s = n$ of the scale mapping considered above: $f \in W^1_{n-1,\mathrm{loc}}(D')$ with the distortion function (5.1). To this end, we have to show that the $W^1_{1,\mathrm{loc}}$-homeomorphism $f\colon D' \to D$ is of class $f \in W^1_{n-1,\mathrm{loc}}(D')$. To verify the last property, observe that f induces the composition operator

$$f^*\colon L^1_n(D) \cap \mathrm{Lip}_{\mathrm{loc}}(D) \to L^1_{n-1,\mathrm{loc}}(D')$$

in the sense that $u \circ f \in L^1_{n-1,\mathrm{loc}}(D')$ for every function $u \in L^1_n(D) \cap \mathrm{Lip}_{\mathrm{loc}}(D)$.

Indeed, consider a compactly embedded domain $U \Subset D'$. Take $u \in L^1_n(f(U)) \cap \mathrm{Lip}_{\mathrm{loc}}(f(U))$. The composition $u \circ f$ clearly lies in $\mathrm{ACL}(U)$. Let us show that the derivatives of the composition are integrable. We can find the derivative of the composition as

$$\frac{\partial(u \circ f)}{\partial y_i}(y) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(f(y))\, \frac{\partial f_j}{\partial y_i}(y)$$

provided that $f(y)$ is a point of differentiability of u and $\partial(u \circ f)(y)/\partial y_i = 0$ otherwise because in this case $y \in Z'$ and $Df(y) = 0$ a. e. Since the distortion function (5.3) is of class $L_{(n-1)n}(U)$, we have

$$\begin{aligned}
&\int_U |\nabla(u \circ f)(y)|^{n-1}\, dy \\
&\quad\leqslant \int_{U\setminus(Z'\cup\Sigma')} |\nabla u(f(y))|^{n-1} \det Df(y)^{(n-1)/n} \cdot \frac{|Df(y)|^{n-1}}{\det Df(y)^{(n-1)/n}}\, dy \qquad (5.4)\\
&\quad\leqslant \Big(\int_{U\setminus(Z'\cup\Sigma')} |\nabla u(f(y))|^{n} \det Df(y)\, dy\Big)^{(n-1)/n} \\
&\qquad\times \Big(\int_{U\setminus(Z'\cup\Sigma')} \Big(\frac{|Df(y)|}{|\det Df(y)|^{1/n}}\Big)^{(n-1)n} dy\Big)^{1/n} \qquad (5.5)\\
&\quad= \|K^{1,1}_{n,n}(\,\cdot\,, f) \mid L_{(n-1)n}(U)\|^{n-1} \Big(\int_{f(U)} |\nabla u(x)|^n\, dx\Big)^{(n-1)/n}.
\end{aligned}$$

To go from (5.4) to (5.5), we use Hölder's inequality with the summability exponents $n/(n-1)$ and n.

Furthermore, observe that $f(U)$ is a bounded open set, so that the coordinate function $u_j(x) \mapsto x_j$ lies in $L^1_n(f(U))$. By (5.4), (5.5) the composition $(u_j \circ f)(y) = f_j(y)$ for $y \in D'$ is of class $f_j \in L^1_{n-1,\mathrm{loc}}(D')$, for $j = 1, \dots, n$, while the mapping $f\colon D' \to D$ is of class $W^1_{n-1,\mathrm{loc}}(D')$.

Therefore, the mapping of [72] satisfies all hypotheses of Example 1.13 with $s = n$, and thus, the claim of Proposition 5.1 holds for it.

5.2. The homeomorphism of Example 1.16. Consider the mapping of Example 1.16 in the case that it is a homeomorphism. Then we have some homeomorphism $f\colon D' \to D$ of class $\mathcal{OD}(D'; s, r; \theta, 1)$, where $n - 1 < s \leqslant r < \infty$, with *outer bounded θ-weighted (s, r)-distortion*, meaning that

(1) $f \in W^1_{n-1,\mathrm{loc}}(D')$;
(2) f has finite distortion;

(3) the distortion function

$$D' \ni x \mapsto K^{\theta,1}_{s,r}(x,f) = \begin{cases} \dfrac{\theta^{1/s}(x)|Df(x)|}{|\det Df(x)|^{1/r}} & \text{if } \det Df(x) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is of class $L_\rho(D')$, where ρ is found from the condition $1/\rho = 1/s - 1/r$ and $\rho = \infty$ for $s = r$.

Proposition 5.2. *On assuming that* $\omega(x) = \theta^{-(n-1)/(s-(n-1))}(x)$ *is locally integrable, the homeomorphism* $f\colon D' \to D$ *of class* $\mathcal{OD}(D'; s, r; \theta, 1)$, *where* $n \leqslant s \leqslant r < n + 1/(n+2)$, *belongs to the family* $\mathcal{Q}_{p,q}(D', \omega; D)$, *where* $q = r/(r-(n-1))$ *and* $p = s/(s-(n-1))$ *with* $n - 1 < q \leqslant p \leqslant n$. *Furthermore, the factors in the right-hand side of* (1.8) *are equal to* $K_p = \|K^{\theta,1}_{r,r}(\,\cdot\,, f) \mid L_\infty(D')\|^{n-1}$ *for* $q = p$ *and*

$$\Psi_{p,q}\big(Q(x,R) \setminus \overline{Q(x,r)}\big)^{1/\sigma} = \big\|K^{\theta,1}_{s,r}(\,\cdot\,, f) \bigm| L_\rho(Q(x,R) \setminus \overline{Q(x,r)})\big\|^{n-1} \quad \textit{for} \quad q < p,$$

where $1/\sigma = 1/q - 1/p = (n-1)/\varrho$.

Therefore, Theorems 2.22 and 2.37 concerning boundary behavior and their Corollaries 2.38 and 2.39 apply to the mapping $f\colon D' \to D$. In particular, applying Corollary 2.39, we obtain the following proposition.

Proposition 5.3. *Under the hypotheses of Proposition* 5.2, *assume that*

(1) *the domain* D' *is locally connected at every point* $y \in \partial D'$ *and*

$$\operatorname{cap}\big((\{y\}, F_0); L^1_p(D'; \omega)\big) = 0,$$

(2) *the support* $\mathcal{S}_{\widetilde{f}(h)}$ *of the boundary element* $\widetilde{f}(h)$ *is a singleton*: $\mathcal{S}_{\widetilde{f}(h)} = \{x\} \in \partial D$, *where* $h \in H^\omega_{\rho,p}(D')$ *is the boundary element containing* $\{y\}$.

Then we obtain an extension by continuity of the homeomorphism $f\colon D' \to D$ *at the point* y *of the support* $\mathcal{S}_h$ *of the boundary element* $h \in H^\omega_{\rho,p}(D')$ *such that*

$$\lim_{z \to y,\, z \in D'} f(z) = x \quad \textit{for every point} \quad y \in \mathcal{S}_h.$$

A similar result is obtained in [67, Theorem 2] under stronger restrictions: $f \in W^1_{s,\mathrm{loc}}(D')$, and so $n - 1 < s$, condition (1) holds, but instead of condition (2) it is assumed that the points $x \in \partial D$ are q-strongly accessible for $q = r/(r-(n-1))$. Recall that under this condition the support $\mathcal{S}_h$ of $x \in h$ is a singleton, see Proposition 4.4. Therefore, the fulfillment of the hypotheses of Theorem [67, Theorem 2] ensures that conditions (1) and (2) above hold. Then, there exists a continuous extension of the mapping $f\colon D' \to D$ to the Euclidean boundary.

Proposition 5.4. *Assume the hypotheses of Proposition* 5.2. *If the domain* D' *is locally connected at the boundary, while the boundary* ∂D *is* q*-weakly flat for* $q = r/(r-(n-1))$, *then the mapping* f^{-1} *admits a continuous extension* $\widetilde{f}^{-1}\colon \overline{D} \to \overline{\mathbb{R}^n}$.

Proof. Assume on the contrary that the mapping f^{-1} has no limit at some point $x_0 \in \partial D$. Then there exist two distinct points $y_1, y_2 \in \partial D'$ and two sequences $\{x_{1,k} \in D\}$, $\{x_{2,k} \in D\}$ such that

$$\lim_{x_{1,k} \to x_0} f^{-1}(x_{1,k}) = y_1 \neq y_2 = \lim_{x_{2,k} \to x_0} f^{-1}(x_{2,k}).$$

Choose two balls $B_i = B(y_i, r_i)$, for $i = 1, 2$, satisfying $\overline{B}_1 \cap \overline{B}_2 = \varnothing$. Since the domain D' is locally connected at the boundary, for the ball B_i there is a connected component of $B_i \cap D'$ which includes $U_i = B(y_i, \widetilde{r}_i) \cap D'$ for some $\widetilde{r}_i \in (0, r_i)$, for $i = 1, 2$.

Take a positive number $h < \operatorname{dist}(B_1, B_2)$. By the subordination principle, Property 1.2, the piecewise linear function u defined as

$$u(y) = \begin{cases} 1 & \text{for } y \in B(y_1, r_1) \cap D', \\ 0 & \text{for } y \in \mathbb{R}^n \setminus (B(y_1, r_1 + h) \cap D') \end{cases}$$

is admissible for the condenser $E' = (F_1', F_2')$ for every continuum $F_i' \Subset B_i \cap D'$. Take a number P such that $P > C\|u \mid L^1_p(D', \omega)\|$, where C is the constant in (1.9).

By construction, $x_0 \in \overline{f(U_1)} \cap \overline{f(U_2)}$. Suppose that V is a neighborhood of x_0 so small that

$$f(U_i) \setminus V \neq \varnothing, \qquad i = 1, 2.$$

Since ∂D is q-weakly flat, for some neighborhood $W \subset V$ of x_0 and some continuum $F_i \subset f(U_i)$, for $i = 1, 2$, intersecting ∂V and ∂W, we have $\operatorname{cap}^{1/q}((F_1, F_2); L_q(D)) \geqslant P$. Choose F_i' so that $F_i' = f(F_i)$. Then the relations

$$\begin{aligned} P &\leqslant \operatorname{cap}^{1/q}\bigl((F_1, F_2); L^1_q(D)\bigr) = \operatorname{cap}^{1/q}\bigl(f^{-1}(E'); L^1_q(D)\bigr) \\ &\leqslant C \operatorname{cap}^{1/p}(E'; L^1_p(D', \omega)) < P \end{aligned}$$

lead to a contradiction. Proposition is proved.

Some results similar to Propositions 5.2–5.4 were obtained in [67, Theorem 1] under stronger restrictions: $f \in W^1_{s,\mathrm{loc}}(D')$ and $s > n - 1$.

References

1. C. Carathéodory, “Über die Begrenzung einfach zusammenhängender Gebiete”, *Math. Ann.*, **73**:3 (1913), 323–370.
2. W. F. Osgood, E. H. Taylor, “Conformal transformations on the boundaries of their regions of definition”, *Trans. Amer. Math. Soc.*, **14**:2 (1913), 277–298.
3. Г. Д. Суворов, “Простые концы последовательности плоских областей, сходящейся к ядру”, *Матем. сб.*, **33(75)**:1 (1953), 73–100; англ. пер.: G. D. Suvorov, “On the prime ends of a sequence of plane regions converging to a nucleus”, Amer. Math. Soc. Transl. Ser. 2, **1**, Amer. Math. Soc., Providence, R.I., 1955, 67–93.
4. D. B. A. Epstein, “Prime ends”, *Proc. London Math. Soc.* (3), **42**:3 (1981), 385–414.
5. В. А. Зорич, “Соответствие границ при Q-квазиконформных отображениях шара”, *Докл. АН СССР*, **145**:6 (1962), 1209–1212; англ. пер.: V. A. Zorich, “Correspondence of the boundaries in Q-quasiconformal mapping of a sphere”, *Soviet Math. Dokl.*, **3** (1962), 1183–1186.
6. В. Зорич, “Определение граничных элементов посредством сечений”, *Докл. АН СССР*, **164**:4 (1965), 736–739; англ. пер.: V. A. Zorič, “Determination of boundary elements by means of sections”, *Soviet Math. Dokl.*, **6** (1965), 1284–1287.

7. A. Björn, J. Björn, N. Shanmugalingam, "The Dirichlet problem for p-harmonic functions with respect to the Mazurkiewicz boundary, and new capacities", *J. Differential Equations*, **259**:7 (2015), 3078–3114.
8. J. Milnor, *Dynamics in one complex variable*, 3rd ed., Ann. of Math. Stud., **160**, Princeton Univ. Press, Princeton, NJ, 2006, viii+304 pp.
9. L. Rempe, "On prime ends and local connectivity", *Bull. Lond. Math. Soc.*, **40**:5 (2008), 817–826.
10. Г. Д. Суворов, *Простые концы и последовательности плоских отображений*, Наукова думка, Киев, 1986, 160 с.
11. T. Adamowicz, A. Björn, J. Björn, N. Shanmugalingam, "Prime ends for domains in metric spaces", *Adv. Math.*, **238** (2013), 459–505.
12. T. Adamowicz, "Prime ends in metric spaces and quasiconformal-type mappings", *Anal. Math. Phys.*, **9**:4 (2019), 1941–1975.
13. Д. А. Ковтонюк, В. И. Рязанов, "Простые концы и классы Орлича–Соболева", *Алгебра и анализ*, **27**:5 (2015), 81–116; англ. пер.: D. A. Kovtonyuk, V. I. Ryazanov, "Prime ends and Orlicz–Sobolev classes", *St. Petersburg Math. J.*, **27**:5 (2016), 765–788.
14. T. Kuusalo, "Quasiconformal mappings without boundary extensions", *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **10** (1985), 331–338.
15. E. C. Schlesinger, "Conformal invariants and prime ends", *Amer. J. Math.*, **80** (1958), 83–102.
16. R. Näkki, *Boundary behavior of quasiconformal mappings in n-space*, Ann. Acad. Sci. Fenn. Ser. A I, **484**, Suomalainen Tiedeakademia, Helsinki, 1970, 50 pp.
17. R. Näkki, "Prime ends and quasiconformal mappings", *J. Anal. Math.*, **35** (1979), 13–40.
18. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math., **229**, Springer-Verlag, Berlin–New York, 1971, xiv+144 pp.
19. M. Vuorinen, *Exceptional sets and boundary behavior of quasiregular mappings in n-space*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, **11**, Suomalainen Tiedeakatemia, Helsinki, 1976, 44 pp.
20. M. Vuorinen, "On the boundary behavior of locally K-quasiconformal mappings in space", *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **5**:1 (1980), 79–95.
21. O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, *Moduli in modern mapping theory*, Springer Monogr. Math., Springer, New York, 2009, xii+367 pp.
22. Е. А. Севостьянов, "О граничном продолжении и равностепенной непрерывности семейств отображений в терминах простых концов", *Алгебра и анализ*, **30**:6 (2018), 97–146; англ. пер.: E. A. Sevost'yanov, "Boundary behavior and equicontinuity for families of mappings in terms of prime ends", *St. Petersburg Math. J.*, **30**:6 (2019), 973–1005.
23. M. Cristea, "Boundary behaviour of the mappings satisfying generalized inverse modular inequalities", *Complex Var. Elliptic Equ.*, **60**:4 (2015), 437–469.
24. С. К. Водопьянов, "О граничном соответствии при квазиконформных отображениях пространственных областей", *Сиб. матем. журн.*, **16**:3 (1975), 630–633; англ. пер.: S. K. Vodop'yanov, "On a boundary correspondence for quasiconformal mappings of three-dimensional domains", *Siberian Math. J.*, **16**:3 (1975), 487–490.
25. В. М. Гольдштейн, С. К. Водопьянов, "Метрическое пополнение области при помощи конформной емкости, инвариантное при квазиконформных отображениях", *Докл. АН СССР*, **238**:5 (1978), 1040–1042; англ. пер.: V. M. Gol'dšteĭn, S. K. Vodop'janov, "Metric completion of a domain by using a conformal capacity invariant under quasi-conformal mappings", *Soviet Math. Dokl.*, **19**:1 (1978), 158–161.

26. С. К. Водопьянов, В. М. Гольдштейн, Ю. Г. Решетняк, “О геометрических свойствах функций с первыми обобщенными производными”, *УМН*, **34**:1(205) (1979), 17–65; англ. пер.: S. K. Vodop’yanov, V. M. Gol’dshtein, Yu. G. Reshetnyak, “On geometric properties of functions with generalized first derivatives”, *Russian Math. Surveys*, **34**:1 (1979), 19–74.
27. В. И. Кругликов, “Простые концы пространственных областей с переменными границами”, *Докл. АН СССР*, **297**:5 (1987), 1047–1050; англ. пер.: V. I. Kruglikov, “Prime ends of spatial domains with variable boundaries”, *Soviet Math. Dokl.*, **36**:3 (1988), 565–568.
28. С. К. Водопьянов, “О регулярности отображений, обратных к соболевским”, *Матем. сб.*, **203**:10 (2012), 3–32; англ. пер.: S. K. Vodop’yanov, “Regularity of mappings inverse to Sobolev mappings”, *Sb. Math.*, **203**:10 (2012), 1383–1410.
29. С. К. Водопьянов, “Операторы композиции весовых пространства Соболева и теория $\mathcal{Q}_p$-гомеоморфизмов”, *Докл. РАН. Матем., информ., проц. упр.*, **494** (2020), 21–25; англ. пер.: S. K. Vodopyanov, “Composition operators on weighted Sobolev spaces and the theory of $\mathcal{Q}_p$-homeomorphisms”, *Dokl. Math.*, **102**:2 (2020), 371–375.
30. С. К. Водопьянов, “Об аналитических и геометрических свойствах отображений в теории $\mathcal{Q}_{q,p}$-гомеоморфизмов”, *Матем. заметки*, **108**:6 (2020), 925–929; англ. пер.: S. K. Vodop’yanov, “On the analytic and geometric properties of mappings in the theory of $\mathcal{Q}_{q,p}$-homeomorphisms”, *Math. Notes*, **108**:6 (2020), 889–894.
31. С. К. Водопьянов, “О регулярности отображений, обратных к соболевским, и теория $\mathcal{Q}_{q,p}$-гомеоморфизмов”, *Сиб. матем. журн.*, **61**:6 (2020), 1257–1299; англ. пер.: S. K. Vodopyanov, “The regularity of inverses to Sobolev mappings and the theory of $\mathcal{Q}_{q,p}$-homeomorphisms”, *Siberian Math. J.*, **61**:6 (2020), 1002–1038.
32. С. К. Водопьянов, А. О. Томилов, “Функциональные и аналитические свойства одного класса отображений квазиконформного анализа”, *Изв. РАН. Сер. матем.*, **85**:5 (2021), 58–109; англ. пер.: S. K. Vodopyanov, A. O. Tomilov, “Functional and analytic properties of a class of mappings in quasi-conformal analysis”, *Izv. Math.*, **85**:5 (2021), 883–931.
33. С. К. Водопьянов, “Об эквивалентности двух подходов к задачам квазиконформного анализа”, *Сиб. матем. журн.*, **62**:6 (2021), 1252–1270; англ. пер.: S. K. Vodopyanov, “On the equivalence of two approaches to problems of quasiconformal analysis”, *Siberian Math. J.*, **62**:6 (2021), 1010–1025.
34. С. К. Водопьянов, “О совпадении функций множества в квазиконформном анализе”, *Матем. сб.*, **213**:9 (2022), 3–33; англ. пер.: S. K. Vodopyanov, “Coincidence of set functions in quasiconformal analysis”, *Sb. Math.*, **213**:9 (2022), 1157–1186.
35. С. К. Водопьянов, “Основы квазиконформного анализа двухиндексной шкалы пространственных отображений”, *Сиб. матем. журн.*, **59**:5 (2018), 1020–1056; англ. пер.: S. K. Vodopyanov, “Basics of the quasiconformal analysis of a two-index scale of spatial mappings”, *Siberian Math. J.*, **59**:5 (2018), 805–834.
36. С. К. Водопьянов, “О дифференцируемости отображений класса Соболева W^1_{n-1} с условиями на функцию искажения”, *Сиб. матем. журн.*, **59**:6 (2018), 1240–1267; англ. пер.: S. K. Vodopyanov, “Differentiability of mappings of the Sobolev space W^1_{n-1} with conditions on the distortion function”, *Siberian Math. J.*, **59**:6 (2018), 983–1005.
37. А. Д. Ухлов, “Отображения, порождающие вложения пространств Соболева”, *Сиб. матем. журн.*, **34**:1 (1993), 185–192; англ. пер.: A. D. Ukhlov, “On mappings generating the embeddings of Sobolev spaces”, *Siberian Math. J.*, **34**:1 (1993), 165–171.
38. С. К. Водопьянов, А. Д. Ухлов, “Пространства Соболева и (P,Q)-квазиконформные отображения групп Карно”, *Сиб. матем. журн.*, **39**:4 (1998), 776–795; англ. пер.: S. K. Vodop’yanov, A. D. Ukhlov, “Sobolev spaces and (P,Q)-quasiconformal mappings of Carnot groups”, *Siberian Math. J.*, **39**:4 (1998), 665–682.

39. С. К. Водопьянов, А. Д. Ухлов, “Операторы суперпозиции в пространствах Соболева”, *Изв. вузов. Матем.*, 2002, № 10, 11–33; англ. пер.: S. K. Vodop’yanov, A. D. Ukhlov, “Superposition operators in Sobolev spaces”, *Russian Math.* (*Iz. VUZ*), **46**:10 (2002), 9–31.
40. С. Л. Соболев, *Некоторые применения функционального анализа в математической физике*, Изд-во Ленингр. ун-та, Л., 1950, 255 с.; англ. пер.: S. L. Sobolev, *Applications of functional analysis in mathematical physics*, Transl. Math. Monogr., **7**, Amer. Math. Soc., Providence, R.I., 1963, vii+239 pp.
41. В. Г. Мазья, *Пространства С. Л. Соболева*, Изд-во Ленингр. ун-та, Л., 1985, 416 с.; англ. пер.: V. G. Maz’ja, *Sobolev spaces*, Springer Ser. Soviet Math., Springer-Verlag, Berlin, 1985, xix+486 pp.; *Sobolev spaces. With applications to elliptic partial differential equations*, 2nd rev. and augm. ed., Grundlehren Math. Wiss., **342**, Springer, Heidelberg, 2011, xxviii+866 pp.
42. Г. Д. Мостов, “Квазиконформные отображения в n-мерном пространстве и жесткость гиперболических пространственных форм”, *Математика*, **16**:5 (1972), 105–157; пер. с англ.: G. D. Mostow, “Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms”, *Inst. Hautes Études Sci. Publ. Math.*, **34** (1968), 53–104.
43. F. W. Gehring, “Lipschitz mappings and p-capacity of rings in n-space”, *Advances in the theory of Riemann surfaces* (Stony Brook, NY, 1969), Ann. of Math. Stud., **66**, Princeton Univ. Press, Princeton, NJ, 1971, 175–193.
44. Ю. Г. Решетняк, *Пространственные отображения с ограниченным искажением*, Наука, Новосибирск, 1982, 286 с.; англ. пер.: Yu. G. Reshetnyak, *Space mappings with bounded distortion*, Transl. Math. Monogr., **73**, Amer. Math. Soc., Providence, RI, 1989, xvi+362 pp.
45. F. W. Gehring, J. Väisälä, “The coefficients of quasiconformality of domains in space”, *Acta Math.*, **114** (1965), 1–70.
46. H. M. Reimann, “Über harmonische Kapazität und quasikonforme Abbildungen im Raum”, *Comment. Math. Helv.*, **44** (1969), 284–307.
47. J. Lelong-Ferrand, “Étude d’une classe d’applications liées à des homomorphismes d’algébres de fonctions, et généralisant les quasi-conformes”, *Duke Math. J.*, **40** (1973), 163–186.
48. С. К. Водопьянов, “Допустимые замены переменных для функций классов Соболева на (суб)римановых многообразиях”, *Матем. сб.*, **210**:1 (2019), 63–112; англ. пер.: S. K. Vodopyanov, “Admissible changes of variables for Sobolev functions on (sub-)Riemannian manifolds”, *Sb. Math.*, **210**:1 (2019), 59–104.
49. A. Molchanova, S. Vodopyanov, “Injectivity almost everywhere and mappings with finite distortion in nonlinear elasticity”, *Calc. Var. Partial Differential Equations*, **59**:1 (2020), 17, 25 pp.
50. J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Math. Monogr., The Clarendon Press, Oxford Univ. Press, New York, 1993, vi+363 pp.
51. T. Rado, P. V. Reichelderfer, *Continuous transformations in analysis. With an introduction to algebraic topology*, Grundlehren Math. Wiss., **LXXV**, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1955, vii+442 pp.
52. М. Гусман, *Дифференцирование интегралов в* $\mathbb{R}^n$, Математика: новое в зарубежной науке, **9**, Мир, М., 1978, 200 с.; пер. с англ.: M. de Guzmán, *Differentiation of integrals in* $\mathbb{R}^n$, Lecture Notes in Math., **481**, Springer-Verlag, Berlin–New York, 1975, xii+266 pp.
53. С. К. Водопьянов, А. Д. Ухлов, “Функции множества и их приложения в теории пространств Лебега и Соболева. I”, *Матем. тр.*, **6**:2 (2003), 14–65; англ. пер.:

S. K. Vodop'yanov, A. D. Ukhlov, "Set functions and their applications in the theory of Lebesgue and Sobolev spaces. I", *Siberian Adv. Math.*, **14**:4 (2004), 78–125.

54. S. K. Vodopyanov, "On Poletsky-type modulus inequalities for some classes of mappings", *Владикавк. матем. журн.*, **24**:4 (2022), 58–69.

55. D. V. Isangulova, S. K. Vodopyanov, "Coercive estimates and integral representation formulas on Carnot groups", *Eurasian Math. J.*, **1**:3 (2010), 58–96.

56. Ю. Г. Решетняк, "О понятии емкости в теории функций с обобщенными производными", *Сиб. матем. журн.*, **10**:5 (1969), 1109–1138; англ. пер.: Yu. G. Reshetnyak, "The concept of capacity in the theory of functions with generalized derivatives", *Siberian Math. J.*, **10**:5 (1969), 818–842.

57. Г. Е. Шилов, *Математический анализ. Специальный курс*, Физматгиз, М., 1961, 436 с.; англ. пер.: G. Ye. Shilov, *Mathematical analysis. A special course*, Pergamon Press, Oxford–New York–Paris, 1965, xii+481 pp.

58. F. Hausdorff, *Set theory*, Transl. from the German, 2nd ed., Chelsea Publishing Co., New York, 1962, 352 pp.

59. R. R. Salimov, E. A. Sevost'yanov, "ACL and differentiability of open discrete ring (p, Q)-mappings", *Mat. Stud.*, **35**:1 (2011), 28–36.

60. В. И. Рязанов, Е. А. Севостьянов, "Равностепенная непрерывность квазиконформных в среднем отображений", *Сиб. матем. журн.*, **52**:3 (2011), 665–679; англ. пер.: V. I. Ryazanov, E. A. Sevost'yanov, "Equicontinuity of mean quasiconformal mappings", *Siberian Math. J.*, **52**:3 (2011), 524–536.

61. Р. Р. Салимов, "Абсолютная непрерывность на линиях и дифференцируемость одного обобщения квазиконформных отображений", *Изв. РАН. Сер. матем.*, **72**:5 (2008), 141–148; англ. пер.: R. R. Salimov, "ACL and differentiability of a generalization of quasi-conformal maps", *Izv. Math.*, **72**:5 (2008), 977–984.

62. R. Salimov, "ACL and differentiability of Q-homeomorphisms", *Ann. Acad. Sci. Fenn. Math.*, **33**:1 (2008), 295–301.

63. Р. Р. Салимов, Е. А. Севостьянов, "Теория кольцевых Q-отображений в геометрической теории функций", *Матем. сб.*, **201**:6 (2010), 131–158; англ. пер.: R. R. Salimov, E. A. Sevost'yanov, "The theory of shell-based Q-mappings in geometric function theory", *Sb. Math.*, **201**:6 (2010), 909–934.

64. E. Sevost'yanov, S. Skvortsov, *On behavior of homeomorphisms with inverse modulus conditions*, 2018, arXiv: 1801.01808v9.

65. Р. Р. Салимов, Е. А. Севостьянов, "О некоторых локальных свойствах пространственных обобщенных квазиизометрий", *Матем. заметки*, **101**:4 (2017), 594–610; англ. пер.: R. R. Salimov, E. A. Sevost'yanov, "On local properties of spatial generalized quasi-isometries", *Math. Notes*, **101**:4 (2017), 704–717.

66. R. Salimov, "On Q-homeomorphisms with respect to p-modulus", *Ann. Univ. Buchar. Math. Ser.*, **2(LX)**:2 (2011), 207–213.

67. М. В. Трямкин, "Граничное соответствие для гомеоморфизмов с весовым ограниченным (p, q)-искажением", *Матем. заметки*, **102**:4 (2017), 632–636; англ. пер.: M. V. Tryamkin, "Boundary correspondence for homeomorphisms with weighted bounded (p, q)-distortion", *Math. Notes*, **102**:4 (2017), 591–595.

68. J. Hesse, "A p-extremal length and p-capacity equality", *Ark. Mat.*, **13**:1-2 (1975), 131–144.

69. В. А. Шлык, "О равенстве p-емкости и p-модуля", *Сиб. матем. журн.*, **34**:6 (1993), 216–221; англ. пер.: V. A. Shlyk, "The equality between p-capacity and p-modulus", *Siberian Math. J.*, **34**:6 (1993), 1196–1200.

70. H. Aikawa, M. Ohtsuka, "Extremal length of vector measures", *Ann. Acad. Sci. Fenn. Math.*, **24**:1 (1999), 61–88.
71. V. Gol'dshtein, A. Ukhlov, *Boundary values of functions of Dirichlet spases L^1_2 on capacitary boundaries*, 2014, arXiv: 1405.3472.
72. E. Afanas'eva, V. Ryazanov, R. Salimov, E. Sevost'yanov, "On boundary extension of Sobolev classes with critical exponent by prime ends", *Lobachevskii J. Math.*, **41**:11 (2020), 2091–2102.

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