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The coloured Tverberg theorem, extensions and new results

D. Jojić, G. Yu. Panina, and R. Živaljević

Abstract. We prove a *multiple coloured Tverberg theorem* and a *balanced coloured Tverberg theorem*, applying different methods, tools and ideas. The proof of the first theorem uses a multiple chessboard complex (as configuration space) and the Eilenberg–Krasnoselskii theory of degrees of equivariant maps for non-free group actions. The proof of the second result relies on the high connectivity of the configuration space, established by using discrete Morse theory.

Keywords: Tverberg theorem, chessboard complex, equivariant map.

§ 1. Introduction

We begin with two very early predecessors of the results discussed in this paper. It is well known that the sphere S^2 is non-embeddable in \mathbb{R}^2 .

The topological Radon theorem refines this result by stating that for any continuous map of a tetrahedron boundary to the plane, there are two disjoint faces of the tetrahedron whose images intersect. Here a *face* is defined as the intersection of the tetrahedron with a support plane. More precisely,

- 1) either the images of two opposite edges intersect,
- 2) or the image of a vertex belongs to the image of the opposite face.

The complete graph K_5 with five vertices (regarded as the 1-skeleton of the four-dimensional simplex) is non-embeddable in \mathbb{R}^2 .

The Van Kampen–Flores theorem strengthens this result by stating that for every continuous map $K_5 \rightarrow \mathbb{R}^2$, there are two disjoint edges whose images intersect.

For several decades the generalization of these results to higher dimensions and to intersections of different multiplicity has been one of the central research themes in topological combinatorics; see [1], [2] and § 2 for an introduction and an overview.

These generalizations are called *Tverberg type theorems* and *generalized Van Kampen–Flores theorems*. They are usually stated as assertions about continuous maps $f: \Delta^N \rightarrow \mathbb{R}^d$ or, more generally, about maps $f: K \rightarrow \mathbb{R}^d$, where $K \subseteq \Delta^N$ is a subcomplex. (Throughout the paper, we write Δ^N for the N -dimensional simplex with vertices $[N + 1] := \{1, \dots, N + 1\}$.)

For example, the *topological Tverberg theorem* [3]–[5] for the plane and intersection multiplicity 4 asserts that for every continuous map $\Delta^9 \rightarrow \mathbb{R}^2$, there are four

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pairwise disjoint faces of Δ^9 whose images have a common point. This result can be restated in terms of maps $K_{10} \rightarrow \mathbb{R}^2$, where K_{10} is the complete graph with 10 vertices; see [6], [7] and the survey [8], Theorem 2.3.2.

Another predecessor of our first new result (Theorem 1.1) is the following. Consider a 9-dimensional simplex Δ^9 whose vertex set $[10] = \{1, 2, \dots, 10\}$ is coloured by five colours: the vertices labeled by $2j - 1$ and $2j$ have the same colour. A coloured topological Tverberg-type theorem (see [9], [1], § 6.5, and [10]) asserts that for every continuous map from Δ^9 to the plane, there are four pairwise disjoint faces $\Delta_1, \dots, \Delta_4$ such that their images have a common point and each Δ_i is rainbow-like (that is, contains at most one vertex from each of the pairs $2j - 1, 2j$).

The following *multiple coloured Tverberg theorem* is our first main result. A new feature of Theorem 1.1 is that some vertices can appear twice as vertices of different faces Δ_i .

As above, it is instructive to visualize the set of vertices $[7] = \{1, 2, \dots, 7\}$ as coloured by four colours: $\{1, 2\}$ are red, $\{3, 4\}$ are blue, $\{5, 6\}$ are green, and the last vertex 7 is white.

Theorem 1.1. *Let Δ^6 be a 6-dimensional simplex with vertex set $[7] = \{1, 2, \dots, 7\}$.*

Then for every continuous map $f: \Delta^6 \rightarrow \mathbb{R}^2$, there are four faces $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ of Δ^6 such that

1. *their images intersect:*

$$f(\Delta_1) \cap f(\Delta_2) \cap f(\Delta_3) \cap f(\Delta_4) \neq \emptyset; \tag{1}$$

2. *every face Δ_i is rainbow, that is, contains at most one vertex in each pair $2j - 1, 2j$;*

3. *each vertex $k \in [7]$ of Δ^6 occurs in the faces Δ_i at most once when k is odd and at most twice when k is even.*

Our second main result is the *balanced coloured Tverberg theorem* (Theorem 1.2). It is an extension of the coloured Tverberg theorem of type B [11], [12] and a coloured analogue of the *balanced Van Kampen–Flores theorem*, Theorem 1.2 in [13] (see also [14] for a short proof).

Theorem 1.2. *Assume that $r = p^\nu$ is a prime power and $d \geq 1$. Define integers $k \geq 0$ and $0 < s \leq r$ by the condition*

$$r(k - 1) + s = (r - 1)d \tag{2}$$

or, more explicitly,

$$k := \lceil (r - 1)d/r \rceil \quad \text{and} \quad s := (r - 1)d - r(k - 1). \tag{3}$$

Put $N = (2r - 1)(k + 1) - 1$ and consider a simplex Δ^N , assuming that there is a partition of its set of vertices into colour classes: $[N + 1] = C_1 \sqcup \dots \sqcup C_{k+1}$, where $|C_j| = 2r - 1$ for each j . Then for every continuous map

$$f: \Delta^N \rightarrow \mathbb{R}^d,$$

there are r disjoint faces $\Delta_1, \dots, \Delta_r$ of Δ^N such that $f(\Delta_1) \cap \dots \cap f(\Delta_r) \neq \emptyset$ and

$$\text{these faces are rainbow-like: } |\Delta_i \cap C_j| \leq 1 \text{ for all } i \in [r] \text{ and } j \in [k + 1], \tag{4}$$

$$\dim(\Delta_i) \leq k \text{ for } 1 \leq i \leq s \text{ and } \dim(\Delta_i) \leq k - 1 \text{ for } s < i \leq r. \tag{5}$$

The paper is organized as follows. In §2 we describe the above two theorems in the context of recent progress concerning Tverberg-type results. This is followed by the proofs.

The proof of Theorem 1.1 (§3) is based on the Eilenberg–Krasnoselskii theory of degrees of equivariant maps for non-free group actions; see the monograph [15] for a detailed presentation of this theory.

The proof of Theorem 1.2 (§4) is based on the high connectivity of the configuration space (Proposition 4.2). This connectivity result is established by using discrete Morse theory and the methods from our papers [16]–[18].

For the reader’s convenience, we briefly outline the basic facts and ideas of discrete Morse theory in §5. A more detailed presentation can be found in [19]. The fundamental *comparison principle for equivariant maps* between spaces with non-free group actions is stated in §6.

§ 2. A brief overview of the Tverberg theorem and related results

The following result is known as the *topological Tverberg theorem*.

Theorem 2.1 (see [3], [4]). *Assume that r is a prime power. Then for every continuous map*

$$f: \Delta^{(r-1)(d+1)} \rightarrow \mathbb{R}^d,$$

there are disjoint faces $\Delta_1, \dots, \Delta_r \subseteq \Delta^{(r-1)(d+1)}$ such that

$$f(\Delta_1) \cap \dots \cap f(\Delta_r) \neq \emptyset.$$

Let K be a geometric realization of a finite simplicial complex. Following [20]–[24], we say that a continuous map $f: K \rightarrow \mathbb{R}^d$ is an *almost r -embedding* if $f(\Delta_1) \cap \dots \cap f(\Delta_r) = \emptyset$ for every r -tuple $\{\Delta_i\}_{i=1}^r$ of pairwise disjoint faces of K . If there are no almost r -embeddings of K in \mathbb{R}^d , we say that K is *not almost r -embeddable* in \mathbb{R}^d . In these terms, Theorem 2.1 asserts that

$$\Delta^{(r-1)(d+1)} \text{ is not almost } r\text{-embeddable in } \mathbb{R}^d. \tag{6}$$

The following four assertions illustrate results of coloured Tverberg type (see [2], [25] for more details and references):

$$K_{3,3} \text{ is not almost 2-embeddable in } \mathbb{R}^2, \tag{7}$$

$$K_{3,3,3} \text{ is not affinely almost 3-embeddable in } \mathbb{R}^2, \tag{8}$$

$$K_{5,5,5} \text{ is not almost 3-embeddable in } \mathbb{R}^3, \tag{9}$$

$$K_{4,4,4,4} \text{ is not almost 4-embeddable in } \mathbb{R}^3. \tag{10}$$

By definition, $K_{t_0, t_1, \dots, t_k} = [t_0] * [t_1] * \dots * [t_k]$ is the complete multipartite simplicial complex obtained as a joint of zero-dimensional complexes (finite sets).

For example, $K_{p,q} = [p] * [q]$ is the complete bipartite graph such that each of the p ‘red vertices’ is connected with each of the q ‘blue vertices’. In (7) we recognize a statement closely related to the non-planarity of $K_{3,3}$, while (8) says that the 2-dimensional complex $K_{3,3,3}$ admits no affine almost 3-embedding in \mathbb{R}^2 .

By a colouring of vertices of a simplex by $k+1$ colours we understand a partition $V = \text{Vert}(\Delta^N) = C_0 \sqcup C_1 \sqcup \dots \sqcup C_k$ into monochromatic subsets C_i . A subset $\Delta \subseteq V$ is called a *rainbow simplex* or a *rainbow face* if $|\Delta \cap C_i| \leq 1$ for all $i = 0, \dots, k$. If the cardinality of C_i is t_i , then K_{t_0, t_1, \dots, t_k} is precisely the subcomplex of all rainbow faces in Δ^N . A coloured Tverberg theorem is any statement of the form

$$K_{t_0, t_1, \dots, t_k} \text{ is not almost } r\text{-embeddable in } \mathbb{R}^d, \tag{11}$$

and the problem is to specify the values of r, d, k and t_i for which (11) holds.

We refer the reader to [11], [12], [26]–[28] and [1], [34], [8], [2] for more general results, proofs, history and applications of monochromatic and coloured Tverberg theorems.

Following [2], § 21.4, we classify coloured Tverberg theorems as theorems of *type A, B, or C* depending on whether $k = d, k < d, \text{ or } k > d$, where $k + 1$ is the number of colours and d is the dimension of the target space.

The main difference between types A and B is that r must satisfy the inequality $(r - 1)d/r \leq k$ in case B, while there are no *a priori* restrictions in case A.

In terms of this classification, (8) and (10) are topological Tverberg theorems of type A, while (7) and (9) are topological Tverberg theorems of type B.

The following results (Theorems 2.2 and 2.3) are the main representatives of these two classes of coloured Tverberg theorems. In particular, (7), (9) and (10) are their simple corollaries.

Note that when d is divisible by r , our second main result (Theorem 1.2) becomes a coloured Tverberg theorem of type B (Theorem 2.3).

Theorem 2.2 (type A; see [26]). *Suppose that $r \geq 2$ is a prime and $d \geq 1$. Then the complex $K_{r-1, r-1, \dots, r-1, 1} := [r - 1]^{*(d+1)} * [1]$, which is the join of $d + 1$ copies of the zero-dimensional complex $[r - 1]$ and a singleton, is not almost r -embeddable in \mathbb{R}^d .*

Theorem 2.3 (type B; see [12], [34]). *Suppose that $r = p^\nu$ is a prime power, $d \geq 1$, and k is an integer such that $(r - 1)d/r \leq k < d$. Then the complex $K_{2r-1, 2r-1, \dots, 2r-1} := [2r - 1]^{*(k+1)}$, which is the join of $k + 1$ copies of the zero-dimensional complex $[2r - 1]$, is not almost r -embeddable in \mathbb{R}^d .*

Remark 2.4. To simplify the notation and presentation, we do not distinguish between the N -dimensional (geometric) simplex Δ^N and the abstract simplicial complex $\Delta_{[m]} = 2^{[m]}$ spanned by m vertices ($m = N + 1$). Therefore, subsets $S \subset [m]$ are interpreted as faces of $\Delta_{[m]}$. For $S \subset [m]$, we have $\dim(S) = |S| - 1$, where $|S|$ is the cardinality of S .

2.1. The multiple Tverberg theorem. Assertion (8) was obtained by Bárány and Larman [29]. It says that every 9-tuple of points on the plane evenly coloured by three colours can be partitioned into three ‘rainbow triangles’ with non-empty intersection.

It is currently unknown whether or not the following non-linear (topological) version of (8) holds:

$$K_{3,3,3} \text{ is not almost } r\text{-embeddable in } \mathbb{R}^2. \tag{12}$$

Assertion (12) clearly follows from the stronger assertion

$$K_{3,3,3,1} \text{ is not almost 4-embeddable in } \mathbb{R}^2. \tag{13}$$

However, is also unknown whether or not (13) is true, and we suspect that this is not the case.

The following multiple coloured Tverberg theorem is a restatement of Theorem 1.1. It says that (13) holds for all continuous maps $f: K_{3,3,3,1} \rightarrow \mathbb{R}^2$ satisfying an additional (3-to-2)-constraint.

Definition 2.5. A function $[3] \rightarrow [2]$ that glues together the last two points of $[3]$ is called a (3-to-2)-map. More generally, a simplicial map $\alpha: K_{3,3,3,\dots} \rightarrow K_{2,2,2,\dots}$ is a (3-to-2)-map if it glues together two points in each copy of the 3-element set $[3]$.

Theorem 2.6. Let $K = K_{3,3,3,1} \cong [3] * [3] * [3] * [1]$ be a 3-dimensional simplicial complex with ten vertices divided into four colour classes, and let $f: K_{3,3,3,1} \rightarrow \mathbb{R}^2$ be a map admitting a factorization $f = \hat{f} \circ \alpha$ for some $\hat{f}: K_{2,2,2,1} \rightarrow \mathbb{R}^2$, where

$$\alpha: K_{3,3,3,1} \rightarrow K_{2,2,2,1}$$

is a (3-to-2)-map in the sense of Definition 2.5. Then there are four pairwise disjoint simplices (four pairwise disjoint rainbow simplices) $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ in K such that

$$f(\Delta_1) \cap f(\Delta_2) \cap f(\Delta_3) \cap f(\Delta_4) \neq \emptyset. \tag{14}$$

In other words, for any map $\hat{f}: K_{2,2,2,1} \rightarrow \mathbb{R}^2$, the composite $\hat{f} \circ \alpha$ is not an almost 4-embedding of $K_{3,3,3,1}$ in \mathbb{R}^2 .

Therefore (12) holds for a special class of non-linear maps.

Corollary 2.7. Assume that $f: K_{3,3,3} \rightarrow \mathbb{R}^2$ is a continuous map admitting a factorization

$$K_{3,3,3} \xrightarrow{\alpha} K_{2,2,2} \xrightarrow{\hat{f}} \mathbb{R}^2 \tag{15}$$

for some \hat{f} , where α is a (3-to-2)-map. Then there are three disjoint triangles $\Delta_1, \Delta_2, \Delta_3$ in $K_{3,3,3}$ such that

$$f(\Delta_1) \cap f(\Delta_2) \cap f(\Delta_3) \neq \emptyset.$$

2.2. The balanced coloured Tverberg theorem. Our balanced coloured Tverberg theorem (Theorem 1.2) can be regarded as an extension of the coloured Tverberg theorem of type B (Theorem 2.3) to the following theorem, which is referred to as the *balanced extension of the generalized Van Kampen–Flores theorem*.

Theorem 2.8 ([13], Theorem 1.2). *Suppose that $r \geq 2$ is a prime power, $d \geq 1$, $N \geq (r - 1)(d + 2)$ and $rk + s \geq (r - 1)d$ for some integers $k \geq 0$ and $s, 0 \leq s < r$. Then for every continuous map $f: \Delta^N \rightarrow \mathbb{R}^d$, there are r pairwise disjoint faces $\Delta_1, \dots, \Delta_r$ of Δ^N such that $f(\Delta_1) \cap \dots \cap f(\Delta_r) \neq \emptyset$, where $\dim \Delta_i \leq k + 1$ for $1 \leq i \leq s$ and $\dim \Delta_i \leq k$ for $s < i \leq r$.*

When d is divisible by r , that is, $s = 0$ and $\dim \Delta_i \leq k$ for all i , Theorem 2.8 becomes the generalized Van Kampen–Flores theorem; see [30]–[32].

The balanced coloured Tverberg theorem (Theorem 1.2) can now be described as a relative of Theorem 2.8 and a balanced extension of Theorem 2.3.

§ 3. Proof of the multiple coloured Tverberg theorem

In accordance with the *configuration space/test map scheme* [2], [1], [33], [34], the first step of the proof of Theorem 2.6 is a standard reduction to a problem of equivariant topology.

Beginning with a continuous map $f: K_{3,3,3,1} \rightarrow \mathbb{R}^2$, we define the associated configuration space as a deleted join

$$(K_{3,3,3,1})_{\Delta}^{*4} = ([3] * [3] * [3] * [1])_{\Delta}^{*4} \cong (\Delta_{3,4})^{*3} * [4],$$

where $\Delta_{3,4}$ is the standard chessboard complex of all arrangements of mutually non-attacking rooks on a (3×4) -chessboard.

The test map which tests whether or not a simplex $\tau = (\Delta_1, \Delta_2, \Delta_3, \Delta_4) \in (K_{3,3,3,1})_{\Delta}^{*4}$ satisfies (14) is defined as a Σ_4 -equivariant map

$$\Phi: (K_{3,3,3,1})_{\Delta}^{*4} \rightarrow (\mathbb{R}^2)^{*4}/D \hookrightarrow (W_4)^{\oplus 3}, \tag{16}$$

where $D \subset (\mathbb{R}^2)^{*4}$ is the diagonal (2-dimensional) subspace and W_4 is the standard 3-dimensional representation of Σ_4 . (Throughout the paper, Σ_4 denotes the symmetric group.)

Thus, the existence of a 4-tuple $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ satisfying (14) is equivalent to the existence of zeros of the Σ_4 -equivariant map (16).

For the next step, we need to use a multiple chessboard complex $\Delta_{2,4}^{1;\mathbf{L}}$ defined as the complex of all rook placements on a (2×4) -chessboard with at most two rooks in the second column and at most one rook in any row and in the first column. (Here we adopt Cartesian notation, that is, the (2×4) -chessboard is regarded as the Cartesian product $[2] \times [4]$ with two columns and four rows.)

Multiple chessboard complexes were studied in [35], and our notation follows that paper. In particular, the vectors $\mathbf{1} = (1, 1, 1, 1)$ (resp. $\mathbf{L} = (1, 2)$) describe the restrictions on the number of rooks in the rows (resp. columns) of the (2×4) -chessboard.

Lemma 3.1. *Let $f: K_{3,3,3,1} \rightarrow \mathbb{R}^2$ be a map admitting a factorization $f = \hat{f} \circ \alpha$ for some map $\hat{f}: K_{2,2,2,1} \rightarrow \mathbb{R}^2$, where*

$$\alpha: K_{3,3,3,1} \rightarrow K_{2,2,2,1}$$

is a (3-to-2)-map in the sense of Definition 2.5. Then the equivariant map (16) admits a factorization $\Phi = \widehat{\Phi} \circ \pi$ into Σ_4 -equivariant maps, as shown in the following commutative diagram:

$$\begin{array}{ccc}
 (\Delta_{2,4}^{\mathbf{1};\mathbf{L}})^{*3} * [4] & \xrightarrow{\widehat{\Phi}} & (W_4)^{*3} \\
 \uparrow \pi & & \uparrow \cong \\
 (\Delta_{3,4})^{*3} * [4] & \xrightarrow{\Phi} & (W_4)^{*3},
 \end{array} \tag{17}$$

where $\Delta_{2,4}^{\mathbf{1};\mathbf{L}}$ is the multiple chessboard complex defined above and π is an epimorphism.

Proof. The proof is by elementary inspection. Note that the map $\widehat{\pi}: \Delta_{3,4} \rightarrow \Delta_{2,4}^{\mathbf{1};\mathbf{L}}$ which induces π in the diagram (17) can be described informally as the map that contracts two columns of the (3×4) -chessboard into one column of the (2×4) -chessboard. \square

Summarizing the first two steps, we observe that the proof of Theorem 2.6 will be complete if we show that the Σ_4 -equivariant map $\widehat{\Phi}$ always has a zero.

3.1. Equivariant maps from $(\Delta_{2,4}^{\mathbf{1};\mathbf{L}})^{*3}$. The Σ_4 -representation W_4 under consideration can be described as \mathbb{R}^3 with the action induced by the symmetries of a regular tetrahedron $\Delta_{[4]}$ centred at the origin. If the map $\widehat{\Phi}$ has no zeros, then there is a Σ_4 -equivariant map

$$g: (\Delta_{2,4}^{\mathbf{1};\mathbf{L}})^{*3} * [4] \rightarrow (\partial\Delta_{[4]})^{*3},$$

where $\partial\Delta_{[4]}$ is the boundary sphere of the simplex $\Delta_{[4]}$. However, this is ruled out by the following theorem.

Theorem 3.2. *Let $G = (\mathbb{Z}_2)^2 = \{1, \alpha, \beta, \gamma\}$ be the Klein four-group. Let $\Delta_{2,4}^{\mathbf{1};\mathbf{L}}$ be the multiple chessboard complex based on a (2×4) -chessboard, where $\mathbf{1} = (1, 1, 1, 1)$ and $\mathbf{L} = (1, 2)$, and let $\partial\Delta_{[4]} \cong S^2$ be the boundary of the simplex spanned by the vertices of $[4]$. Both $\Delta_{2,4}^{\mathbf{1};\mathbf{L}}$ and $\partial\Delta_{[4]} \cong S^2$ are G -spaces, where the group action in the first case permutes the rows of the chessboard $[2] \times [4]$, and in the second case it permutes the vertices of the simplex $\Delta_{[4]}$. Then there is no G -equivariant map*

$$f: (\Delta_{2,4}^{\mathbf{1};\mathbf{L}})^{*3} * [4] \rightarrow (\partial\Delta_{[4]})^{*3} \cong (S^2)^{*3} \cong S^8,$$

where the action of G on the join is diagonal.

Theorem 3.2 will be proved by arguments using the notion of degree of an equivariant map. These arguments can be traced back to Eilenberg and Krasnoselskii; see [15] for a thorough treatment and § 6 for a statement of one of the main theorems.

Before proving Theorem 3.2, we describe a convenient geometric model of the complex $\Delta_{2,4}^{\mathbf{1};\mathbf{L}}$. Recall that the Bier sphere $\text{Bier}(K)$ of a simplicial complex $K \subset 2^{[m]}$ is the deleted join $K *_\Delta K^\circ$ of K and its Alexander dual K° ; see [1] for more details.

Lemma 3.3. *The multiple chessboard complex $\Delta_{2,4}^{1;\mathbf{L}}$ is a triangulation of a 2-sphere. More explicitly, there is an isomorphism $\Delta_{2,4}^{1;\mathbf{L}} \cong \text{Bier}(\Delta_{[4]}^{(1)})$, where $\Delta_{[4]}^{(1)}$ is the 1-skeleton of the tetrahedron $\Delta_{[4]}$ and $\text{Bier}(K) = K *_\Delta K^\circ$ is the Bier sphere associated with a simplicial complex K .*

Proof. This follows directly from the observation that the subcomplexes of $\Delta_{2,4}^{1;\mathbf{L}}$ generated by the vertices in the second and first columns of the chessboard $[2] \times [4]$ are $K = \Delta_{[4]}^{(1)}$ and $K^\circ = (\Delta_{[4]}^{(1)})^\circ = \Delta_{[4]}^{(0)}$, respectively. \square

The following lemma describes the structure of $\Delta_{2,4}^{1;\mathbf{L}}$ as a G -space, where $G = (\mathbb{Z}_2)^2 = \{1, \alpha, \beta, \gamma\}$ is the Klein four-group.

Lemma 3.4. *As a G -space, the sphere $\Delta_{2,4}^{1;\mathbf{L}}$ is homeomorphic to the regular octahedral sphere centred at the origin and the generators α, β, γ are the rotations by 180° around the axes connecting the pairs of opposite vertices of the octahedron.*

*More explicitly, let \mathbb{R}_α^1 be the one-dimensional G -representation characterized by the conditions $\alpha x = x$ and $\beta x = \gamma x = -x$ (we also define \mathbb{R}_β^1 and \mathbb{R}_γ^1 in a similar way) and let $S_\alpha^0, S_\beta^0, S_\gamma^0$ be the corresponding zero-dimensional G -spheres. Then the complex $\Delta_{2,4}^{1;\mathbf{L}}$ is G -isomorphic to the 2-sphere $S(\mathbb{R}_\alpha^1 \oplus \mathbb{R}_\beta^1 \oplus \mathbb{R}_\gamma^1) \cong S_\alpha^0 * S_\beta^0 * S_\gamma^0$ with induced G -action.*

Remark 3.5. Here is a geometric interpretation (visualization) of the G -isomorphism $\Delta_{2,4}^{1;\mathbf{L}} \cong \text{Bier}(\Delta_{[4]}^{(1)})$. The complex $K = \Delta_{[4]}^{(1)}$ and its dual $K^\circ = \Delta_{[4]}^{(0)}$ can be geometrically realized as the tetrahedron $\Delta_{[4]}$ and its polar body $\Delta_{[4]}^\circ$. If both tetrahedra are inscribed in the cube I^3 , the geometric realization of $\text{Bier}(K)$ can be regarded as a triangulation of the boundary $\partial(I^3)$ of the cube.

Lemma 3.6. *As a G -space, the boundary sphere $\partial\Delta_{[4]}$ of the tetrahedron is isomorphic to the octahedral sphere described in Lemma 3.4. Moreover, there is a radial G -isomorphism $\rho: \partial(I^3) \rightarrow \partial\Delta_{[4]}$.*

Summarizing, we can see that the G -spheres studied in this section have two combinatorial interpretations ($\Delta_{2,4}^{1;\mathbf{L}}$ and $\partial\Delta_{[4]} = 2^{[4]} \setminus \{[4]\}$) and three equivalent geometric incarnations (the boundary $\partial(I^3)$ of the cube, the boundary $\partial\Delta_{[4]}$ of the tetrahedron and the boundary $S_\alpha^0 * S_\beta^0 * S_\gamma^0$ of the octahedron).

3.2. Completion of the proof of Theorem 3.2.

Proposition 3.7. *Let $\phi: (\Delta_{2,4}^{1;\mathbf{L}})^{(3)} \rightarrow (\partial\Delta_{[4]})^{(3)}$ be an arbitrary G -equivariant map. Then*

$$\deg(\phi) \equiv 1 \pmod{2}.$$

Proof. It follows from Theorem 6.1 that $\deg(\phi) \equiv \deg(\psi) \pmod{2}$ for any equivariant maps ϕ, ψ between these spaces. Here we use the fact that $(\Delta_{2,4}^{1;\mathbf{L}})^{(3)} \cong (S^2)^{(3)} \cong S^8$ is a topological manifold. Note that the inequality (20), which is necessary for Theorem 6.1 to be applicable, becomes an equality in view of the decomposition (19).

Hence it suffices to produce a map ψ of odd degree. We know that $(\Delta_{2,4}^{1;\mathbf{L}})^{*(3)}$ and $(\partial\Delta_{[4]})^{*(3)}$ are G -isomorphic 8-dimensional spheres. Taking $\psi: (\Delta_{2,4}^{1;\mathbf{L}})^{*(3)} \rightarrow (\partial\Delta_{[4]})^{*(3)}$ as the G -isomorphism, we obtain $\deg(\psi) = \pm 1$. \square

Proof of Theorem 3.2. We have

$$\begin{CD}
 (\Delta_{2,4}^{1;\mathbf{L}})^{*(3)} * [4] @>f>> (\partial\Delta_{[4]})^{*(3)} \\
 @VVe @VVV \\
 (\Delta_{2,4}^{1;\mathbf{L}})^{*(3)} @>\phi>> (\partial\Delta_{[4]})^{*(3)}.
 \end{CD}
 \tag{18}$$

Suppose that there is a G -equivariant map f . Let e be the inclusion map and let $\phi = f \circ e$ be the composite of these maps.

The map e is homotopically trivial since $\text{Im}(e) \subset \text{Cone}(v)$ for every $v \in [4]$. However, the degree of ϕ is odd by Proposition 3.7. Contradiction. \square

Remark 3.8. It has been pointed out by a referee that an alternative and somewhat shorter proof of Theorem 3.2 can be obtained by using Volovikov’s theorem [5], [31] instead of Theorem 6.1 as in the proof of Theorem 1.2 (see § 4). Indeed, the 8-connectivity of $(\Delta_{2,4}^{1;\mathbf{L}})^{*(3)} * [4]$ is an immediate consequence of the homeomorphism $\Delta_{2,4}^{1;\mathbf{L}} \cong S^2$ (Lemma 3.3). Moreover, the action of $G = (\mathbb{Z}_2)^2$ on $(\Delta_{2,4}^{1;\mathbf{L}})^{*(3)} * [4]$ and $(\partial\Delta_{[4]})^{*(3)}$ has no fixed points since, by Lemmas 3.4 and 3.6, there are $(\mathbb{Z}_2)^2$ -homeomorphisms

$$(\Delta_{2,4}^{1;\mathbf{L}})^{*(3)} \cong (S_\alpha^0)^{*3} * (S_\beta^0)^{*3} * (S_\gamma^0)^{*3} \cong (\partial\Delta_{[4]})^{*(3)}.
 \tag{19}$$

§ 4. Proof of the balanced coloured Tverberg theorem

Following the *configuration space/test map scheme* [1], [2], we describe the configuration space $\mathfrak{C} \subseteq \Delta_{[m]}^{*(r)}$ used in the proof of Theorem 1.2.

Definition 4.1. Put $m = N + 1$ and $\Delta_{[m]} = \Delta^N$. The configuration space \mathfrak{C} of all r -tuples of disjoint rainbow simplices satisfying the restrictions listed in Theorem 1.2 is the simplicial complex whose simplices are labelled by $(A_1, \dots, A_r; B)$, where

- $[m] = A_1 \sqcup \dots \sqcup A_r \sqcup B$ is a partition such that $B \neq [m]$;
- each A_i is a rainbow set (a rainbow simplex) and, in particular, $|A_i| \leq k + 1$ for every $i \in [r]$;
- the number of simplices A_i with $|A_i| = k + 1$ does not exceed s .

Note that the dimension of the simplex $(A_1, \dots, A_r; B)$ is $|A_1| + \dots + |A_r| - 1$. Moreover, the facets of $(A_1, \dots, A_r; B)$ can be formally obtained by deleting an element of one of the sets A_i and adding this element to B .

Proposition 4.2. *The configuration space \mathfrak{C} is $(rk + s - 2)$ -connected.*

Let us explain briefly how Theorem 1.2 can be deduced from Proposition 4.2. This standard argument was used, for example, in the proof of the topological Tverberg theorem; see [1], §6, or [2], [5].

Suppose that Theorem 1.2 is false. Then there is a $(\mathbb{Z}/p)^\nu$ -equivariant map

$$\Psi_f: \mathfrak{C} \rightarrow \mathbb{R}^{(d+1)r}$$

whose image is disjoint from the diagonal $D = \{(y, y, \dots, y) : y \in \mathbb{R}^{d+1}\}$. This contradicts Volovikov’s theorem [5], [31] since $\mathbb{R}^{(d+1)r} \setminus D$ is $(\mathbb{Z}/p)^\nu$ -homotopy equivalent to a sphere of dimension $(r - 1)(d + 1) - 1 = rk + s - 2$ while the configuration space \mathfrak{C} is $(rk + s - 2)$ -connected.

Proof of Proposition 4.2. We begin by introducing some useful abbreviations.

A set $A \subset [m]$ is said to be C_i -full if it contains a vertex of colour C_i . A simplex $(A_1, \dots, A_r; B)$ is said to be C_i -full if each A_i is C_i -full or, equivalently, if $|\bigcup_{i=1}^r A_i \cap C_i| = r$. A simplex $(A_1, \dots, A_r; B)$ is said to be $(k + 1)$ -full if it contains (the maximal allowed number) s of $(k + 1)$ -sets among the A_i . A simplex $(A_1, \dots, A_r; B)$ is said to be saturated if it is $(k + 1)$ -full and $|A_i| \geq k$ for every i .

Saturated simplices are maximal faces of the configuration space \mathfrak{C} . Their dimension is $rk + s - 1$.

Following discrete Morse theory and Theorem 5.1, we shall define a matching for \mathfrak{C} . Given any simplex $(A_1, \dots, A_r; B)$, we shall either describe a simplex paired with it, or recognize it as a critical (that is, unmatched) simplex.

This will be done stepwise. We shall have r ‘big’ steps, each of which splits into $k + 1$ successive small steps. The big steps treat the sets A_i one-by-one, and the small steps treat the colours one-by-one.

Step 1.

Step 1.1. Assume that the vertices of each colour are enumerated as $\{1, 2, \dots, 2r - 1\}$.

We put

$$a_1^1 = \min[(A_1 \cup B) \cap C_1]$$

and match $(A_1 \cup a_1^1, A_2, \dots, A_r; B)$ with $(A_1, A_2, \dots, A_r; B \cup a_1^1)$ whenever both of these simplices belong to \mathfrak{C} .

A simplex of type $(A_1 \cup a_1^1, A_2, \dots, A_r; B) \in \mathfrak{C}$ is not matched if and only if it is equal to

$$(\{a_1^1\}, \emptyset, \dots, \emptyset; [m] \setminus \{a_1^1\}).$$

This is a zero-dimensional simplex. It will stay unmatched till the end of the matching process.

If a simplex of type $(A_1, A_2, \dots, A_r; B \cup a_1^1)$ is unmatched, then A_1 is either C_1 -full, or $|A_1| = k$ and $(A_1, A_2, \dots, A_r; B \cup a_1^1)$ is $(k + 1)$ -full.

Step 1.2. Put

$$a_1^2 = \min[(A_1 \cup B) \cap C_2]$$

and match $(A_1 \cup a_1^2, A_2, \dots, A_r; B)$ with $(A_1, A_2, \dots, A_r; B \cup a_1^2)$ whenever both of these simplices belong to \mathfrak{C} and were not matched at Step 1.1.

- If a simplex of type $(A_1, A_2, \dots, A_r; B \cup a_1^2)$ is unmatched, then A_1 is either C_2 -full, or $|A_1| = k$ and $(A_1, A_2, \dots, A_r; B \cup a_1^2)$ is $(k + 1)$ -full.

Such simplices are said to be ‘*Step 1.2–Type 1’-unmatched*’.

- If a simplex of type $(A_1 \cup a_1^2, A_2, \dots, A_r; B)$ is unmatched, then $|A_1 \cup a_1^2| = k$ and $(A_1 \cup a_1^2, A_2, \dots, A_r; B)$ is $(k + 1)$ -full (these conditions are necessary but not sufficient). The reason is that in this case $(A_1, A_2, \dots, A_r; B \cup a_1^2)$ belongs to \mathfrak{C} , but may have been matched at Step 1.1.

Such simplices are said to be ‘Step 1.2–Type 2’-unmatched.

In what follows, we use similar abbreviations. ‘Step $i.j$ –Type 1’ means that one cannot move an element coloured by j from B to A_i . ‘Step $i.j$ –Type 2’ means that one cannot move an element coloured by j from A_i to B .

Step 1.3 and subsequent steps (up to Step $1.k + 1$) follow by analogy.

Summarizing, we can make the following conclusion.

Lemma 4.3. *Except for the unique zero-dimensional unmatched simplex, if a simplex $(A_1, \dots, A_r; B)$ is unmatched after Step 1, then*

- 1) either $|A_1| = k + 1$,
- 2) or $|A_1| = k$ and $(A_1, \dots, A_r; B)$ is $(k + 1)$ -full.

Proof. This follows directly by analysing the matching algorithm at small steps. \square

Step 2. We now treat A_2 for the simplices that remain unmatched after Step 1.

Step 2.1. We put

$$a_2^1 = \min[(A_2 \cup B) \setminus [1, a_1^1] \cap C_1]$$

and match $(A_1, A_2 \cup a_2^1, \dots, A_r; B)$ with $(A_1, A_2, \dots, A_r; B \cup a_2^1)$ whenever both of these simplices belong to \mathfrak{C} and have not been matched at Step 1.

- If a simplex of type $(A_1, A_2, \dots, A_r; B \cup a_2^1)$ is not matched now, then either $|A_2| = k$ and $(A_1, A_2, \dots, A_r; B \cup a_2^1)$ is $(k + 1)$ -full, or A_2 is C_1 -full.

Such simplices are called ‘Step 2.1–Type 1’-simplices.

- If a simplex of type $(A_1, A_2 \cup a_2^1, \dots, A_r; B)$ is unmatched, then it is $(k + 1)$ -full and $|A_2| = k + 1$.

Such simplices are called ‘Step 2.1–Type 2’-simplices.

Step 2.2. We put

$$a_2^2 = \min[(A_2 \cup B) \setminus [1, a_1^1] \cap C_2]$$

and match $(A_1, A_2 \cup a_2^2, \dots, A_r; B)$ with $(A_1, A_2, \dots, A_r; B \cup a_2^2)$ whenever both of these simplices belong to \mathfrak{C} and were not matched earlier, that is, at Step 1 or Step 2.1.

Step 2.3 and subsequent steps (up to Step $2.k + 1$) follow by analogy.

Summarizing, we reach the following conclusion.

Lemma 4.4. *Except for the unique zero-dimensional unmatched simplex, if a simplex $(A_1, \dots, A_r; B)$ remains unmatched after Step 2, then it is also unmatched after Step 1 (and satisfies Lemma 4.3). Moreover,*

- 1) either $|A_2| = k + 1$,
- 2) or $|A_2| = k$ and $(A_1, \dots, A_r; B)$ is $(k + 1)$ -full.

Steps 3, 4, ... and $r - 1$ follow by analogy.

Lemma 4.5. *The numbers a_j^i are well defined for all the steps $j = 1, 2, \dots, r - 1$.*

Proof. Indeed, for $(A_1, \dots, A_r; B) \in \mathfrak{C}$, the set $B \cap C_i$ contains at least $r - 1$ elements. (Here we use the fact that $|C_i| = 2r - 1$ and $|A_j \cap C_i| \leq 1$ for each j .) The entries $a_1^i, a_2^i, \dots, a_{j-1}^i$ are either not in $B \cap C_i$, or (by construction) they are the smallest consecutive entries in $B \cap C_i$. Their total number is strictly less than $r - 2$. \square

Special attention should be paid to the last Step r .

First of all, we observe that the following is already known (by construction).

Lemma 4.6. *Except for the unique zero-dimensional unmatched simplex, if a simplex $(A_1, \dots, A_r; B)$ is unmatched after Step $r - 1$, then*

- 1) either $|A_1| = |A_2| = \dots = |A_{r-1}| = k + 1$,
- 2) or $|A_i| = k$ for some i , and $(A_1, \dots, A_r; B)$ is $(k + 1)$ -full.

Proof. This follows from Lemma 4.4 and its analogues for Steps $1, \dots, r - 1$. \square

Step r . We now turn our attention to A_r .

Step $r.1$. We put

$$a_r^1 = \min [((A_r \cup B) \setminus [1, a_{r-1}^1]) \cap C_1].$$

The set $[((A_r \cup B) \setminus [1, a_{r-1}^1]) \cap C_1]$ may be empty for $(A_1, \dots, A_r; B)$, so that a_r^1 is undefined.

This means that $(A_1, \dots, A_r; B)$ is C_1 -full. Such simplices remain unmatched and are said to be ‘Step $r.1$ –Type 3’-unmatched.

If a_r^1 is well defined, we proceed in the standard way: match $(A_1, A_2, \dots, A_r \cup a_r^1; B)$ and $(A_1, A_2, \dots, A_r; B \cup a_r^1)$ if these two simplices belong to \mathfrak{C} and have not been matched before.

Step $r.2$. We put

$$a_r^2 = \min [((A_r \cup B) \setminus [1, a_{r-1}^2]) \cap C_2].$$

Once again, if this number is undefined, then $(A_1, \dots, A_r; B)$ is a C_2 -full simplex and we leave it to be ‘Step $r.2$ –Type 3’ unmatched.

Otherwise we proceed in the standard way.

Step $r.3$ and subsequent steps (up to Step $r.k + 1$) follow by analogy.

Summarizing, we make the following conclusion.

Lemma 4.7. *Except for the unique zero-dimensional unmatched simplex, if a simplex $(A_1, \dots, A_r; B)$ remains unmatched after Step r , then it is saturated.*

Proof. We have $|A_i| \geq k$ for all $i = 1, \dots, r - 1$ by Lemma 4.6.

If a simplex $(A_1, \dots, A_r; B)$ is such that $|A_i| < k$ for some i , then some colour does not occur in A_i . Let j be the smallest index of a missing colour. It follows that this simplex was matched at Step $i.j$ since a_j^i is well defined and can be added to A_i .

At every Step $i.j$, the simplex $(A_1, \dots, A_r; B)$ is either of Type 1, or of Type 2, or (this can occur only at Step $r.j$) of Type 3. If it was of Type 2 at least once (it does not matter at which step), then the same lemma implies that it is $(k + 1)$ -full, hence saturated.

If the simplex was always of Type 1 at Steps $1, \dots, r - 1$ and is not saturated, then $|A_i| = k + 1$ for all $i = 1, \dots, r - 1$. Since $s < r$, it is saturated. \square

It remains to prove that the matching is acyclic.

Assume that we have a gradient path

$$\alpha_0^p \nearrow \beta_0^{p+1} \searrow \alpha_1^p \nearrow \beta_1^{p+1} \searrow \alpha_2^p \nearrow \beta_2^{p+1} \searrow \dots \searrow \alpha_m^p \nearrow \beta_m^{p+1} \searrow \alpha_{m+1}^p.$$

For every simplex α , we consider the sequence of numbers

$$\Pi(\alpha) := (a_1^1, a_1^2, \dots, a_1^{k+1}, a_2^1, \dots, a_2^{k+1}, \dots, a_r^1, \dots, a_r^{k+1}).$$

These are all the numbers a_j^i listed in the same order as they appear in the matching algorithm. When a_r^i is undefined, we let it be ∞ .

Lemma 4.8. *Along the path $\Pi(\alpha)$ is strictly decreasing with respect to the lexicographic order. Hence the matching is acyclic.*

Proof. This follows from a case-by-case analysis.

First of all, it suffices to consider only three-step paths:

$$\alpha_0^p \nearrow \beta_0^{p+1} \searrow \alpha_1^p \nearrow \beta_1^{p+1}.$$

1. Suppose that $\alpha_0^p \nearrow \beta_0^{p+1}$ means adding a colour i to A_j and $\beta_0^{p+1} \searrow \alpha_1^p$ means removing a colour $i' > i$ from A_j . Then

- 1) either α_1^p is matched with some $(p - 1)$ -dimensional simplex obtained by removing colour i from A_j and the path terminates here,
- 2) or α_1^p is matched before Step $j.i$.

2. Suppose that $\alpha_0^p \nearrow \beta_0^{p+1}$ means adding a colour i to A_j and $\beta_0^{p+1} \searrow \alpha_1^p$ means removing a colour $i' < i$ from A_j . Then α_1^p is matched before Step $j.i$.

3. Suppose that $\alpha_0^p \nearrow \beta_0^{p+1}$ means adding a colour i to A_j and $\beta_0^{p+1} \searrow \alpha_1^p$ means removing a colour i' from $A_{j'}$ with $j' < j$. Then

- 1) either α_1^p is matched by adding colour i' to $A_{j'}$,
- 2) or α_1^p is matched before Step $j'.i'$.

4. Suppose that $\alpha_0^p \nearrow \beta_0^{p+1}$ means adding a colour i to A_j and $\beta_0^{p+1} \searrow \alpha_1^p$ means removing a colour i' from $A_{j'}$ with $j' > j$. Then

- 1) either α_1^p is matched with some $(p - 1)$ -dimensional simplex obtained by removing colour i from A_j and the path terminates here,
- 2) or α_1^p is matched before Step $j'.i'$. \square

This completes the proof of Proposition 4.2 and that of Theorem 1.2.

§ 5. Appendix 1. Discrete Morse theory

By definition [19], a discrete Morse function on a simplicial complex $K \subseteq 2^V$ is an acyclic matching on the Hasse diagram of the partially ordered set (K, \subseteq) .

Here are some details. The p -dimensional simplices (p -simplices for brevity) of a simplicial complex K are denoted by $\alpha^p, \alpha_j^p, \beta^p, \sigma^p, \dots$. A *discrete vector field* is a set of pairs $D = \{\dots, (\alpha^p, \beta^{p+1}), \dots\}$ (called a *matching*) such that

- (a) each simplex of the complex occurs in at most one pair;
- (b) in each pair $(\alpha^p, \beta^{p+1}) \in D$, the simplex α^p is a facet of β^{p+1} ;
- (c) the empty set $\emptyset \in K$ is not matched, that is, $(\alpha^p, \beta^{p+1}) \in D$ implies that $p \geq 0$.

The pair (α^p, β^{p+1}) can be informally thought of as a vector in the vector field D . Therefore it is often denoted by $\alpha^p \rightarrow \beta^{p+1}$ or $\alpha^p \nearrow \beta^{p+1}$ (in this case α^p and β^{p+1} are informally referred to as the *beginning* and the *end* of the arrow $\alpha^p \rightarrow \beta^{p+1}$).

Let D be a discrete vector field. A *gradient path* in D is a sequence of simplices

$$\alpha_0^p \nearrow \beta_0^{p+1} \searrow \alpha_1^p \nearrow \beta_1^{p+1} \searrow \alpha_2^p \nearrow \beta_2^{p+1} \searrow \cdots \searrow \alpha_m^p \nearrow \beta_m^{p+1} \searrow \alpha_{m+1}^p$$

satisfying the following conditions:

- 1) $(\alpha_i^p, \beta_i^{p+1})$ belongs to D for each i ;
- 2) α_{i+1}^p is a facet of β_i^{p+1} for each $i = 0, \dots, m$;
- 3) $\alpha_i \neq \alpha_{i+1}$ for each $i = 0, \dots, m - 1$.

A path is *closed* if $\alpha_{m+1}^p = \alpha_0^p$. A *discrete Morse function* (DMF for brevity) is a discrete vector field without closed paths.

The *critical simplices* of a discrete Morse function are those simplices in the complex that are not matched. The Morse inequality [19] implies that critical simplices cannot be completely avoided.

In the present paper we use the following theorem.

Theorem 5.1 [19]. *Assume that a discrete Morse function on a simplicial complex K has a single zero-dimensional critical simplex σ^0 and that all other critical simplices have the same dimension $N > 1$. Then K is homotopy equivalent to a wedge of N -dimensional spheres.*

If all critical simplices, except for σ^0 , are of dimension at least N , then the complex K is $(N - 1)$ -connected.

§ 6. Appendix 2. Comparison principle for equivariant maps

The following theorem was proved in [15], § 2, Theorem 2.1. Note that the hypothesis that the H_i -fixed point sets S^{H_i} are locally k -connected for $k \leq \dim(M^{H_i}) - 1$ holds automatically when S is a representation sphere. Therefore, in this case it suffices to show that S^{H_i} is globally $(\dim(M^{H_i}) - 1)$ -connected or, equivalently,

$$\dim(M^{H_i}) \leq \dim(S^{H_i}), \quad i = 1, \dots, m. \tag{20}$$

Theorem 6.1. *Let G be a finite group acting on a compact topological manifold $M = M^n$ and on a sphere $S \cong S^n$ of the same dimension, let $N \subset M$ be a closed invariant subset, and let $(H_1), (H_2), \dots, (H_k)$ be the orbit types in $M \setminus N$. Assume that S^{H_i} is a globally and locally k -connected set for all $k = 0, 1, \dots, \dim(M^{H_i}) - 1$, where $i = 1, \dots, k$. Then the following relation holds for every pair of G -equivariant maps $\Phi, \Psi: M \rightarrow S$ that are equivariantly homotopic on N :*

$$\deg(\Psi) \equiv \deg(\Phi) \pmod{\text{GCD}\{|G/H_1|, \dots, |G/H_k|\}}. \tag{21}$$

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