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On the critical exponent "instantaneous blow-up" versus "local solubility" in the Cauchy problem for a model equation of Sobolev type

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Abstract. We consider the Cauchy problem for a model partial differential equation of order three with a non-linearity of the form $|\nabla u|^q$. We prove that when $q \in (1, 3/2]$ the Cauchy problem in $\mathbb{R}^3$ has no local-in-time weak solution for a large class of initial functions, while when $q > 3/2$ there is a local weak solution.

Keywords: finite-time blow-up, non-linear waves, instantaneous blow-up.

§ 1. Introduction

The phenomenon of complete blow-up was first discovered for the equation

$$-\Delta u = |x|^{-2}u^2, \quad u \geqslant 0, \qquad x \in \Omega \setminus \{0\} \subset \mathbb{R}^N, \tag{1.1}$$

in the paper [1] by Brezis and Cabré. For a linear parabolic equation with a singular potential, instantaneous blow-up was obtained in [2]. For the non-linear singular parabolic equation

$$u_t - \Delta u = |x|^{-2}u^2, \quad u \geqslant 0, \qquad x \in \Omega \setminus \{0\} \subset \mathbb{R}^N, \quad t > 0, \tag{1.2}$$

the problem of instantaneous blow-up was considered for the first time in the paper [3] by Weissler. We note that the comparison method was used in these three papers, and the proof was technically rather complicated. In the papers of Pokhozhaev and Mitidieri (see the monograph [4] and the bibliography therein), results concerning complete and instantaneous blow-up were obtained in a much simpler and more efficient way, and also for equations of higher order, by the original method of non-linear capacity.

Later, instantaneous blow-up for non-linear parabolic and hyperbolic equations was considered in the papers of Galaktionov and Vázquez [5], Goldstein and Kombe [6], Giga and Umeda [7], Galakhov [8], [9] and others. In some papers, a method based on the comparison principle (for parabolic equations) was used, and the others used Pokhozhaev's method based on the method of non-linear capacity, which made it possible to obtain much more quickly and efficiently sufficient

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conditions for the absence of solutions of both parabolic and hyperbolic equations, including (non-Sobolev) equations of higher order.

The question of instantaneous blow-up in non-classical Sobolev equations was first studied in [10]. In particular, the following problem was considered there:

$$\frac{\partial}{\partial t}(u_{xx}+u)=u_{xx}, \qquad u(x,0)=u_0(x), \quad u(0,t)=u(l,t), \quad l>0. \tag{1.3}$$

As a corollary of Theorem 4.1 in [10], it was established that this problem has no bounded solution on an arbitrarily small interval of time provided that $l \in (0,\pi]$. This result can be explained by the presence of the operator $\partial_x^2 + I$ under the sign of differentiation with respect to time. Later such results appeared in the study of linear Sobolev-type equations of the form

$$\frac{\partial}{\partial t}(\Delta u+\lambda u)+\Delta u=0 \quad \text{for} \quad \lambda>0, \quad x\in\Omega\subset\mathbb{R}^N,$$

in the case when λ belongs to the spectrum of Δ in the bounded domain Ω (see the survey [11]). In particular, this survey describes the method of degenerate semigroups for studying linear Sobolev-type equations in which the coefficient of the leading derivative is a singular operator. The instantaneous blow-up effect for linear and non-linear Sobolev-type equations has not been studied subsequently since researchers have been interested in sufficient conditions for the existence of solutions.

Moreover, a new result obtained in the present paper is that the solution may be absent even when there are no singular coefficients of the form $|x|^{-\alpha}$ or $t^{-\beta}$ and the initial functions belong to $\mathbb{C}_0^\infty(\mathbb{R}^N)$.

In the problems under consideration, the effect of instantaneous blow-up occurs when the equation has a singularity (as in (1.2)) or when the initial function is subject to a non-standard growth condition (as in [7]). The equation

$$\frac{\partial}{\partial t}\Delta_3 u+\sigma_1\Delta_2 u+\sigma_2 u_{zz}=|\nabla u|^q, \qquad \sigma_1>0, \quad \sigma_2>0, \quad q>1, \tag{1.4}$$

where

$$\Delta_3=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \qquad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

has no explicit singularities, nor do we impose any specific growth conditions on the initial functions. We shall prove that when $1<q\leqslant 3/2$ the Cauchy problem has no local-in-time weak solutions, but when $q>3/2$ local weak solutions do exist. A possible reason is that the first summand is subordinate to the others when $1<q\leqslant 3/2$, so that, from the point of view of our analysis, the properties of the solutions of (1.4) become similar to those of the solutions of the stationary equation

$$\sigma_1\Delta_2 u+\sigma_2 u_{zz}=|\nabla u|^q, \qquad \sigma_1>0, \quad \sigma_2>0, \quad q>1, \qquad (x,y,z)\in\mathbb{R}^3, \tag{1.5}$$

for which the number $q_{kr}=3/2$ is a critical exponent [4] such that the only weak solution of (1.5) when $1<q\leqslant q_{kr}$ is an arbitrary constant, but when $q>q_{kr}$ there are non-trivial solutions on $\mathbb{R}^3$. Note that adding the term

$$-\frac{\partial u}{\partial t}$$

to the right-hand side of (1.4) drastically changes the situation. Although the term

$$\frac{\partial}{\partial t}\Delta_3 u$$

is again subordinate to the others when $1 < q \leqslant 3/2$, the properties of the solution of the Cauchy problem for (1.4) become similar to those of the solution of the Cauchy problem for

$$-\frac{\partial u}{\partial t} + \sigma_1 \Delta_2 u + \sigma_2 u_{zz} = |\nabla u|^q, \qquad \sigma_1 > 0, \quad \sigma_2 > 0, \quad q > 1, \qquad (x, y, z) \in \mathbb{R}^3, \tag{1.6}$$

and, in all cases, the solution of the Cauchy problem for the equation

$$\frac{\partial}{\partial t}(\Delta_3 u - u) + \sigma_1 \Delta_2 u + \sigma_2 u_{zz} = |\nabla u|^q, \qquad \sigma_1 > 0, \quad \sigma_2 > 0, \quad q > 1, \tag{1.7}$$

exists at least locally in time.

This paper continues the series of papers [12]–[14], which studied equations either isotropic in spatial variables or with a power-like non-linearity of the form

$$\frac{\partial}{\partial t}\Delta_3 u + \sigma_1 \Delta_2 u + \sigma_2 u_{zz} = |u|^q, \qquad \sigma_1 > 0, \quad \sigma_2 > 0, \quad q > 1. \tag{1.8}$$

In this paper we consider the Cauchy problem for the equation (1.4). We shall prove that it has no weak solutions for a large class of initial functions when $1 < q \leqslant 3/2$, but when $q > 3/2$ local weak solutions do exist.

Equations (1.6) and (1.7) belong to the class of non-linear equations of Sobolev type. We note that linear and non-linear equations of Sobolev type have been studied in many papers. In particular, initial boundary-value problems for equations of Sobolev type were considered in general form as well as in the form of examples in the papers [11], [15], [16] by Sviridyuk, Zagrebina and Zamyshlyaeva.

We also mention a numerical approach to the study of blow-up of solutions. It was suggested in [17]–[19] and successfully used by us for various equations in [20]–[25] and elsewhere.

§ 2. Derivation of the equation

We continue the study of non-linear processes in a semiconductor in an external constant magnetic field. Choose an orthogonal Cartesian coordinate system $Oxyz$ in such a way that the external magnetic field vector $\mathbf{B}_0$ is directed along the axis Oz. It is known from the classical paper [26] that the electroconductivity tensor $\{\sigma_{\alpha\beta}\}$ $(\alpha, \beta = x, y, z)$ is of the form

$$\sigma_{\alpha\beta} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ -\sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}, \qquad \sigma_{xx} = \sigma_{yy} > 0, \quad \sigma_{zz} > 0, \quad \sigma_{xy} > 0. \tag{2.1}$$

Moreover, $\sigma_{xx} \neq \sigma_{zz}$ when the external magnetic field is non-zero. We consider the electric part of the system of Maxwell equations in the quasi-stationary approximation:

$$\operatorname{div} \mathbf{D} = 4\pi e n, \qquad \mathbf{D} = \varepsilon \mathbf{E}, \qquad \operatorname{rot} \mathbf{E} = \mathbf{0}, \tag{2.2}$$

where $\mathbf{D}$ is the electric displacement field and $\mathbf{E}$ is the electric field. In the case when the first homology group of the domain $\Omega \subset \mathbb{R}^3$ is trivial, there is a potential ϕ of the electric field:

$$\mathbf{E} = -\nabla\phi, \qquad \Delta_3\phi = -\frac{4\pi e}{\varepsilon} n. \tag{2.3}$$

Moreover, the following equations hold:

$$\frac{\partial n}{\partial t} + \operatorname{div}\mathbf{J} = 0, \qquad \mathbf{J}_i = \sum_{j=1}^{3} \sigma_{ij}\mathbf{E}_j - \gamma\frac{\partial T}{\partial x_i}, \quad \gamma > 0, \tag{2.4}$$

where $\mathbf{J}$ is the vector of current density of free charges and n is the density of free charges. Here we take the heating of the semiconductor into account and T is its temperature. We use the following equation for the change of temperature in space and time:

$$\epsilon\frac{\partial T}{\partial t} = \Delta_3 T + Q(|\mathbf{E}|), \tag{2.5}$$

where the function $Q(|\mathbf{E}|)$ describes the dependence of the heat pumping on the modulus of the electric field $\mathbf{E}$, and where $\epsilon > 0$ is a small parameter. Therefore we replace (2.5) by the equation

$$\Delta_3 T + Q(|\mathbf{E}|) = 0. \tag{2.6}$$

We also adopt the following model dependence:

$$Q(|\mathbf{E}|) = q_0|\mathbf{E}|^q, \qquad q_0 > 0, \quad q > 1. \tag{2.7}$$

The system of equations (2.3), (2.4) and (2.6), (2.7) yields the following non-classical equation for the potential ϕ of the electric field:

$$\frac{\partial}{\partial t}\Delta_3\phi + \frac{4\pi e\sigma_{xx}}{\varepsilon}\Delta_2\phi + \frac{4\pi e\sigma_{zz}}{\varepsilon}\phi_{zz} = \frac{4\pi e\gamma q_0}{\varepsilon}|\nabla\phi|^q, \tag{2.8}$$

where we put

$$\Delta_3 \overset{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad \Delta_2 \overset{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

One can reduce the equation (2.8) to the form

$$\frac{\partial}{\partial t}\Delta_3 u + \sigma_1\Delta_2 u + \sigma_2 u_{zz} = |\nabla u|^q, \qquad \sigma_1 > 0, \quad \sigma_2 > 0, \quad q > 1. \tag{2.9}$$

Note that $\sigma_1 \neq \sigma_2$ when a non-zero external magnetic field is present.

§ 3. Notation

Here we define the weighted spaces of functions $\mathbb{C}([0,T];W_j)$, $j = 1,2$, which will be used throughout the paper.

Let W_1 be the Banach space of all functions in $\mathbb{C}_b^{(1)}(\mathbb{R}^3)$ with finite norm

$$\|v\|_{W_1} := \sup_{x\in\mathbb{R}^3}|v(x)| + \sum_{j=1}^{3}\sup_{x\in\mathbb{R}^3}(1+|x|^2)^{1/2}\left|\frac{\partial v(x)}{\partial x_j}\right|. \tag{3.1}$$

We write $\mathbb{C}([0,T];W_1)$ for the set of functions $v(t)\in W_1$ of $t\in[0,T]$ such that

$$\|v(t_1)-v(t_0)\|_{W_1}\to +0 \quad \text{for any} \quad t_0,t_1\in[0,T] \quad \text{as} \quad t_1\to t_0. \tag{3.2}$$

Then $\mathbb{C}([0,T];W_1)$ is a Banach space with respect to the norm

$$\|v\|_T=\sup_{t\in[0,T],\,x\in\mathbb{R}^3}|v(x,t)|+\sum_{j=1}^{3}\sup_{t\in[0,T],\,x\in\mathbb{R}^3}(1+|x|^2)^{1/2}\biggl|\frac{\partial v(x,t)}{\partial x_j}\biggr|.$$

We similarly define the Banach space

$$\mathbb{C}([0,T];W_2)$$

with respect to the norm

$$\|u\|_{1,T}=\sup_{x\in\mathbb{R}^3,\,t\in[0,T]}(1+|x|^2)^{1/2}|u(x,t)|+\sum_{j=1}^{3}\sup_{x\in\mathbb{R}^3,\,t\in[0,T]}(1+|x|^2)\biggl|\frac{\partial u(x,t)}{\partial x_j}\biggr|,$$

where $W_2\subset\mathbb{C}_b^{(1)}(\mathbb{R}^3)$ is the Banach space of functions with finite norm

$$\|u\|_{W_2}:=\sup_{x\in\mathbb{R}^3}(1+|x|^2)^{1/2}|u(x)|+\sum_{j=1}^{3}\sup_{x\in\mathbb{R}^3}(1+|x|^2)\biggl|\frac{\partial u(x)}{\partial x_j}\biggr|.$$

Let $\mathbb{C}^{(1)}([0,T];W_j)$, $j=1,2$, be the Banach space of differentiable functions $u(t)\colon[0,T]\to W_j$ such that $u(t)$, $u'(t)\in\mathbb{C}([0,T];W_j)$.

We write $\mathbb{C}_b((1+|x|^2)^{\alpha/2};\mathbb{R}^3)$ for the set of all functions $u(x)\in\mathbb{C}_b(\mathbb{R}^3)$ satisfying the inequality

$$|u(x)|\leqslant\frac{A}{(1+|x|^2)^{\alpha/2}},\qquad \alpha>0,$$

for some constant $A>0$ which depends on $u(x)$.

We also put

$$O(x,R):=\{y\in\mathbb{R}^3\colon |y-x|<R\}.$$

§ 4. Instantaneous blow-up of weak solutions of the Cauchy problem

Here is the definition of a weak solution of the Cauchy problem classically posed in the following form:

$$\mathfrak{M}_{x,t}[u](x,t)\stackrel{\text{def}}{=}\Delta_3\frac{\partial u}{\partial t}+\sigma_1\Delta_2 u+\sigma_2 u_{x_3x_3}=|\nabla u|^q,\qquad q>1,\quad \sigma_1,\sigma_2>0, \tag{4.1}$$

$$u(x,0)=u_0(x). \tag{4.2}$$

Definition 1. A function $u(x,t)\in L^q(0,T;W^{1,q}_{\mathrm{loc}}(\mathbb{R}^3))$ satisfying the equality

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}\bigl[(\nabla u(x,t),\nabla\phi'(x,t))-\sigma_1 u_{x_1}(x,t)\phi_{x_1}(x,t)\\ -\sigma_1 u_{x_2}(x,t)\phi_{x_2}(x,t)-\sigma_2 u_{x_3}(x,t)\phi_{x_3}(x,t)\bigr]\,dx\,dt\\ +\int_{\mathbb{R}^3}(\nabla u_0(x),\nabla\phi(x,0))\,dx=\int_0^T\int_{\mathbb{R}^3}|\nabla u(x,t)|^q\phi(x,t)\,dx\,dt\end{aligned} \tag{4.3}$$

for all functions $\phi(x,t) \in \mathbb{C}^{\infty,1}_{x,t}(\mathbb{R}^3 \times [0,T])$, is called a local weak solution of the Cauchy problem (4.1) and (4.2), where

$$\phi(x,T)=0 \quad \text{for all} \quad x \in \mathbb{R}^3, \qquad \operatorname{supp}_x \phi(x,t) \subset O(0,R) \quad \text{for all} \quad t \in [0,T],$$
$$R = R(\phi) > 0, \qquad u_0(x) \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^3).$$

We define the class U of initial functions $u_0(x)$ for which we shall prove instantaneous blow-up of local weak solutions of the Cauchy problem in the sense of Definition 1.

Definition 2. We say that $u_0(x) \in U$ if $u_0(x) \in W^{1,q}(\mathbb{R}^3)$ and there are $x_0 \in \mathbb{R}^3$ and $R_0 > 0$ such that $u_0(x) \in H^2(O(x_0,R_0))$ and

$$\mu\{x \in O(x_0,R_0) \colon \Delta_3 u_0(x) \neq 0\} > 0,$$

where μ is the standard Lebesgue measure in $\mathbb{R}^3$.

Theorem 1. *If $u_0(x) \in U$ and $q \in (1, 3/2]$, then there is no local weak solution of the Cauchy problem for any $T > 0$, that is, instantaneous blow-up of local weak solutions of the Cauchy problem occurs.*

Proof. The proof uses the method of non-linear capacity of Pokhozhaev and Mitidieri [4] and a special choice of the test function $\phi(x,t)$ in the equation (4.3) of Definition 1. Namely, we take

$$\phi(x,t) = \phi_T(t)\phi_R(x), \qquad \phi_T(t) = \left(1 - \frac{t}{T}\right)^{\lambda}, \quad \lambda > q',$$
$$\phi_R(x) = \phi_0\left(\frac{|x|^2}{R^2}\right), \quad \phi_0(s) = \begin{cases} 1 & \text{if } s \in [0,1/2], \\ 0 & \text{if } s \geqslant 1, \end{cases} \quad \phi_0(s) \in \mathbb{C}^{\infty}_0[0,+\infty),$$

where $\phi_0(s)$ is a monotone decreasing function. We have the following estimates based on using Hölder's inequality with appropriate exponents:

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{R}^3} (\nabla u(x,t), \nabla \phi'(x,t))\, dx\, dt \right| \\ &\quad \leqslant \frac{\lambda}{T} \int_0^T \int_{\mathbb{R}^3} \left(1 - \frac{t}{T}\right)^{\lambda-1} |\nabla u(x,t)||\nabla \phi_R(x)|\, dx\, dt \\ &\quad = \frac{\lambda}{T} \int_0^T \int_{\mathbb{R}^3} \left(1 - \frac{t}{T}\right)^{\lambda/q} |\nabla u(x,t)| \phi_R^{1/q}(x) \left(1 - \frac{t}{T}\right)^{\lambda/q' - 1} \frac{|\nabla \phi_R(x)|}{\phi_R^{1/q}(x)}\, dx\, dt \\ &\quad \leqslant \frac{\lambda}{T} c_1(R,T) I_R^{1/q}, \end{aligned} \tag{4.4}$$

where

$$I_R := \int_0^T \int_{\mathbb{R}^3} \phi_T(t)\phi_R(x)|\nabla u|^q \, dx\, dt, \tag{4.5}$$

$$\begin{aligned} c_1(R,T) &:= \left(\int_0^T \int_{\mathbb{R}^3} \left(1-\frac{t}{T}\right)^{\lambda-q'} \frac{|\nabla\phi_R(x)|^{q'}}{\phi_R^{q'/q}(x)}\, dx\, dt\right)^{1/q'} \\ &= \left(\frac{T}{\lambda-q'+1}\right)^{1/q'} c_2 R^{(3-q')/q'}, \qquad c_3>0, \end{aligned} \tag{4.6}$$

$$\left|\int_0^T \int_{\mathbb{R}^3} u_{x_j}(x,t)\phi_{x_j}(x,t)\, dx\, dt\right| \leqslant \int_0^T \int_{\mathbb{R}^3} |\nabla u(x,t)||\nabla\phi(x,t)|\, dx\, dt \leqslant I_R^{1/q} c_3(R,T), \tag{4.7}$$

with

$$c_3(R,T) := \left(\int_0^T \int_{\mathbb{R}^3} \left(1-\frac{t}{T}\right)^{\lambda} \frac{|\nabla\phi_R(x)|^{q'}}{\phi_R^{q'/q}(x)}\, dx\, dt\right)^{1/q'} = \left(\frac{T}{\lambda+1}\right)^{1/q'} c_2 R^{(3-q')/q'}, \tag{4.8}$$

$$\begin{aligned} \left|\int_{\mathbb{R}^3} (\nabla u_0(x), \nabla\phi(x,0))\, dx\right| &\leqslant \int_{\mathbb{R}^3} |\nabla u_0(x)|\, |\nabla\phi_R(x)|\, dx \\ &\leqslant \| |\nabla u_0| \|_{L^q(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} |\nabla\phi_R(x)|^{q'}\, dx\right)^{1/q'} = \| |\nabla u_0| \|_{L^q(\mathbb{R}^3)} c_4 R^{(3-q')/q'}. \end{aligned} \tag{4.9}$$

We now apply the bounds (4.4)–(4.9) to (4.3) and obtain the inequality

$$\frac{\lambda}{T} c_1(R,T) I_R^{1/q} + (2\sigma_1+\sigma_2) c_3(R,T) I_R^{1/q} + \| |\nabla u_0| \|_{L^q(\mathbb{R}^3)} c_4 R^{(3-q')/q'} \geqslant I_R. \tag{4.10}$$

Using Hölder's inequality with parameter $\varepsilon = 1/4$,

$$ab \leqslant \frac{1}{4}a^2 + b^2,$$

we deduce from (4.10) that

$$2\frac{\lambda^2}{T^2} c_1^2(R,T) + 2(2\sigma_1+\sigma_2)^2 c_3^2(R,T) + 2\| |\nabla u_0| \|_{L^q(\mathbb{R}^3)} c_4 R^{(3-q')/q'} \geqslant I_R. \tag{4.11}$$

Put $R = N \in \mathbb{N}$ and consider the sequence of functions

$$H_N(x,t) := |\nabla u(x,t)|^q \phi_N(x)\phi_T(t), \qquad H_{N+1}(x,t) \geqslant H_N(x,t), \tag{4.12}$$

for almost all $(x,t) \in \mathbb{R}^3 \times [0,T]$. We require that the following inequality should hold:

$$3 - q' \leqslant 0 \quad \Longrightarrow \quad 1 < q \leqslant \frac{3}{2}. \tag{4.13}$$

Then it follows from (4.6)–(4.9) that the right-hand side of (4.11) is bounded by a constant $K > 0$ and, therefore,

$$\int_0^T \int_{\mathbb{R}^3} H_N(x,t)\, dx\, dt \leqslant K < +\infty. \tag{4.14}$$

Hence we conclude from the monotone convergence theorem that

$$\lim_{N\to+\infty}\int_0^T\int_{\mathbb{R}^3}H_N(x,t)\,dx\,dt=\int_0^T\int_{\mathbb{R}^3}|\nabla u(x,t)|^q\,dx\,dt\leqslant K<+\infty. \tag{4.15}$$

Consider the cases $1<q<3/2$ and $q=3/2$ separately. When $1<q<3/2$, we use (4.11) and the bounds (4.6)–(4.9) to conclude that

$$I_N:=\int_0^T\int_{\mathbb{R}^3}\phi_T(t)\phi_N(x)|\nabla u|^q\,dx\,dt\to+0\quad\text{as}\quad N\to+\infty. \tag{4.16}$$

The case $q=3/2$ is critical. It can be considered in the same way as all the critical cases in [4].

Thus, when $q\in(1,3/2]$ we arrive at the equality

$$\int_0^T\int_{\mathbb{R}^3}|\nabla u(x,t)|^q\biggl(1-\frac{t}{T}\biggr)^\lambda\,dx\,dt=0$$
$$\Longrightarrow\quad u(x,t)=F(t)\quad\text{for almost all}\quad(x,t)\in\mathbb{R}^3\times[0,T].$$

Substituting the resulting equality $u(x,t)=F(t)$ into (4.3), we have

$$\int_{\mathbb{R}^3}(\nabla u_0(x),\nabla\phi(x,0))\,dx=0$$

for all functions $\phi(x,t)$ satisfying the conditions of Definition 1. Therefore, for an arbitrary function $\phi(x,t)$ of the form

$$\phi(x,t)=\phi_1(x)\biggl(1-\frac{t}{T}\biggr),\qquad\phi_1(x)\in\mathbb{C}_0^\infty(\mathbb{R}^3),\quad\operatorname{supp}\phi_1(x)\subset O(x_0,R_0),$$

and for $u_0(x)\in U$, integration by parts yields that

$$\int_{O(x_0,R_0)}\Delta u_0(x)\phi_1(x)\,dx=0\quad\text{for all}\quad\phi_1(x)\in\mathbb{C}_0^\infty(O(x_0,R_0)).$$

By the fundamental lemma of the calculus of variations, we can conclude that

$$\Delta u_0(x)=0\quad\text{for almost all}\quad x\in O(x_0,R_0),$$

contrary to the definition of the class $U\ni u_0(x)$. □

§ 5. The existence of an inextensible solution of the auxiliary integral equation for $q>3/2$

In this section we consider the auxiliary integral equation

$$u(x,t)=\int_{\mathbb{R}^3}\mathscr{E}(x-y,t)\Delta_3u_0(y)\,dy+\int_0^t\int_{\mathbb{R}^3}\mathscr{E}(x-y,t-\tau)|\nabla u|^q\,dy\,d\tau, \tag{5.1}$$

where the function

$$\mathscr{E}(x,t)=-\frac{\theta(t)}{4\pi|x|}\exp\biggl(-\frac{\sigma_1+\beta(x)}{2}t\biggr)I_0\biggl(\frac{\sigma_1-\beta(x)}{2}t\biggr) \tag{5.2}$$

is a fundamental solution of the operator

$$\mathfrak{M}_{x,t}[w](x,t) := \Delta_{3x}\frac{\partial w}{\partial t} + \sigma_1\Delta_{2x}w(x,t) + \sigma_2 w_{x_3x_3} \tag{5.3}$$

with

$$\beta(x) = \frac{\sigma_2(x_1^2+x_2^2)+\sigma_1 x_3^2}{x_1^2+x_2^2+x_3^2}, \qquad \sigma_j \geqslant 0, \quad j=1,2.$$

Some properties of the fundamental solution $\mathscr{E}(x,t)$ are collected in the following lemma.

Lemma 1. 1) *For* $x \neq 0$,

$$\mathscr{E}(x,0) = -\frac{1}{4\pi|x|}. \tag{5.4}$$

2) $\mathscr{E}(x,t) \in \mathbb{C}^\infty((\mathbb{R}^3 \setminus \{0\}) \times [0,+\infty))$.
3) *If* $x \in \mathbb{R}^3 \setminus \{0\}$ *and* $t \in [0,T]$, *then*

$$\biggl|\frac{\partial^k \mathscr{E}(x,t)}{\partial t^k}\biggr| \leqslant \frac{A_1(T)}{|x|}, \qquad \biggl|\frac{\partial^{k+1}\mathscr{E}(x,t)}{\partial t^k\,\partial x_j}\biggr| \leqslant \frac{A_2(T)}{|x|^2}, \quad j=1,2,3, \tag{5.5}$$

$$\biggl|\frac{\partial^{k+2}\mathscr{E}(x,t)}{\partial t^k\,\partial x_j\,\partial x_l}\biggr| \leqslant \frac{A_3(T)}{|x|^3}, \qquad j,l=1,2,3, \quad k\in\mathbb{N}, \tag{5.6}$$

with constants $0 < A_n(T) < +\infty$ *for* $n=1,2,3$.

Proof. This follows from the properties of the Infeld function $I_0(x)$ and the explicit formula (5.2) for the function $\mathscr{E}(x,t)$. □

It is convenient to pass from the function $u(x,t)$ in the integral equation (5.1) to a new function

$$v(x,t) = (1+|x|^2)^{1/2}u(x,t). \tag{5.7}$$

In view of the equality

$$\begin{aligned}|\nabla u|^q &= \biggl|\nabla\frac{v(x,t)}{(1+|x|^2)^{1/2}}\biggr|^q = \biggl|\frac{1}{(1+|x|^2)^{1/2}}\nabla v - \frac{x}{(1+|x|^2)^{3/2}}v(x,t)\biggr|^q \\ &= \frac{1}{(1+|x|^2)^q}\biggl|(1+|x|^2)^{1/2}\nabla v - \frac{x}{(1+|x|^2)^{1/2}}v\biggr|^q\end{aligned} \tag{5.8}$$

in the class of differentiable functions, this yields the integral equation

$$\begin{aligned}v(x,t) &= \int_{\mathbb{R}^3} G_\alpha(x,y,t)(1+|y|^2)^\alpha\Delta_3 u_0(y)\,dy \\ &+ \int_0^t\int_{\mathbb{R}^3} G_q(x,y,t-\tau)\biggl|(1+|y|^2)^{1/2}\nabla v(y,\tau) - \frac{y}{(1+|y|^2)^{1/2}}v\biggr|^q\,dy\,d\tau,\end{aligned} \tag{5.9}$$

where

$$G_\gamma(x,y,t) := \frac{(1+|x|^2)^{1/2}}{(1+|y|^2)^\gamma}\mathscr{E}(x-y,t), \qquad \gamma > 0. \tag{5.10}$$

The theorem on inextensible solutions of (5.9) will be proved in the Banach space $\mathbb{C}([0,T];W_1)$, which was defined in § 3, with respect to the norm $\|\cdot\|_T$:

$$\|v\|_T := \sup_{t\in[0,T],\,x\in\mathbb{R}^3} |v(x,t)| + \sum_{j=1}^{3} \sup_{t\in[0,T],\,x\in\mathbb{R}^3} (1+|x|^2)^{1/2} \left|\frac{\partial v(x,t)}{\partial x_j}\right|. \tag{5.11}$$

Theorem 2. *Suppose that* $q > 3/2$. *Then for every function* $u_0(x) \in \mathbb{C}^2(\mathbb{R}^3)$ *satisfying the condition*

$$|\Delta_3 u_0(x)| \leqslant \frac{A_4}{(1+|x|^2)^{\alpha}}, \qquad \alpha > \frac{3}{2}, \tag{5.12}$$

one can find a $T_0 = T_0(u_0) > 0$ *such that for every* $T \in (0, T_0)$ *there is a unique solution*

$$v(x,t) \in \mathbb{C}([0,T];W_1) \tag{5.13}$$

of the integral equation (5.9). *Moreover, either* $T_0 = +\infty$, *or* $T_0 < +\infty$, *and in the latter case the following limit property holds*:

$$\lim_{T\uparrow T_0} \|v\|_T = +\infty. \tag{5.14}$$

Proof. We begin with the following lemma on the properties of the function $G_\gamma(x, y,t)$ defined in (5.10).

Lemma 2. *Suppose that* $\gamma > 3/2$. *Then for* $t \in [0,T]$ *one has*

$$\begin{aligned}
&\sup_{(x,t)\in\mathbb{R}^3\times(0,+\infty)} \int_{\mathbb{R}^3} \left|\frac{\partial^k G_\gamma(x,y,t)}{\partial t^k}\right| dy \\
&\leqslant A_1(T) \sup_{(x,t)\in\mathbb{R}^3\times(0,+\infty)} \int_{\mathbb{R}^3} \frac{(1+|x|^2)^{1/2}}{(1+|y|^2)^{\gamma}|x-y|}\,dy \leqslant B_1(T) < +\infty,
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
&\sup_{(x,t)\in\mathbb{R}^3\times(0,+\infty)} (1+|x|^2)^{1/2} \int_{\mathbb{R}^3} \left|\frac{\partial^{k+1} G_\gamma(x,y,t)}{\partial x_j\,\partial t^k}\right| dy \\
&\leqslant A_1(T) \sup_{(x,t)\in\mathbb{R}^3\times(0,+\infty)} \int_{\mathbb{R}^3} \frac{(1+|x|^2)^{1/2}}{(1+|y|^2)^{\gamma}|x-y|}\,dy \\
&\quad + A_2(T) \sup_{(x,t)\in\mathbb{R}^3\times(0,+\infty)} \int_{\mathbb{R}^3} \frac{1+|x|^2}{(1+|y|^2)^{\gamma}|x-y|^2}\,dy \leqslant B_2(T) < +\infty, \qquad j=1,2,3,
\end{aligned} \tag{5.16}$$

for $k = 0, 1, 2$.

Proof. Note that if $x \neq y$ and $t \geqslant 0$, then

$$\begin{aligned}
\frac{\partial^{k+1} G_\gamma(x,y,t)}{\partial x_j \partial t^k} &= \frac{x_j}{(1+|x|^2)^{1/2}} \frac{1}{(1+|y|^2)^{\gamma}} \frac{\partial^k \mathscr{E}(x-y,t)}{\partial t^k} \\
&\quad + \frac{(1+|x|^2)^{1/2}}{(1+|y|^2)^{\gamma}} \frac{\partial^{k+1} \mathscr{E}(x-y,t)}{\partial x_j\,\partial t^k}, \qquad j = 1,2,3.
\end{aligned}$$

We shall use the bounds (5.5) for the fundamental solution $\mathscr{E}(x,t)$.

Step 1. *Estimation of the integral* (5.15). Passing to the spherical coordinate system, we obtain the following expression:

$$I := \int_{\mathbb{R}^3} dy \, \frac{1}{|y|(1+|x-y|^2)^\gamma} = 2\pi \int_0^{+\infty} dr \int_0^\pi d\theta \, \frac{r\sin\theta}{(1+|x|^2+r^2-2|x|r\cos\theta)^\gamma}.$$

Integrating with respect to $\theta \in (0,\pi)$, we obtain

$$I = \frac{2\pi}{\gamma-1}\frac{1}{|x|}\int_0^{+\infty} dr \left[\frac{1}{(1+(r-|x|)^2)^{\gamma-1}} - \frac{1}{(1+(r+|x|)^2)^{\gamma-1}}\right] =: \frac{1}{|x|}(I_1+I_2).$$

Suppose that $|x|>1$. Then when $\gamma > 3/2$ we have

$$I_1 = \frac{2\pi}{\gamma-1}\int_0^{+\infty} dr \, \frac{1}{(1+(r-|x|)^2)^{\gamma-1}} = \frac{2\pi}{\gamma-1}\int_{-|x|}^{+\infty} dz \, \frac{1}{(1+z^2)^{\gamma-1}} < +\infty,$$
$$I_2 = \frac{2\pi}{\gamma-1}\int_0^{+\infty} dr \, \frac{1}{(1+(r+|x|)^2)^{\gamma-1}} \leqslant \frac{2\pi}{\gamma-1}\int_0^{+\infty} dr \, \frac{1}{(1+r^2)^{\gamma-1}} < +\infty.$$

Suppose that $|x| \leqslant 1$. Then the expression for I can be reduced by changes of variables to the form

$$I = \frac{2\pi}{\gamma-1}\frac{1}{|x|}\int_{-|x|}^{|x|} dz \, \frac{1}{(1+z^2)^{\gamma-1}} \leqslant \frac{2\pi}{\gamma-1}\frac{1}{|x|}2|x| \leqslant \frac{4\pi}{\gamma-1}.$$

Step 2. *Estimation of the integral* (5.16). In fact, we need only estimate the integral

$$I = \int_{\mathbb{R}^3} \frac{1}{|x-y|^2}\frac{1}{(1+|y|^2)^\gamma}\, dy \quad \text{for} \quad \gamma > \frac{3}{2}.$$

We first consider the case when $|x|>1$. Pass to a spherical coordinate system whose axis Oz coincides with Ox. Then we have

$$I = 2\pi \int_0^{+\infty} dr \int_0^\pi d\theta \, \frac{r^2\sin\theta}{(1+r^2)^\gamma}\,\frac{1}{|x|^2+r^2-2|x|r\cos\theta}. \tag{5.17}$$

Put

$$a = |x|^2 + r^2, \qquad b = 2|x|r.$$

We separately calculate

$$\int_0^\pi d\theta \, \frac{\sin\theta}{a-b\cos\theta} = -\frac{1}{b}\ln\left(\frac{a-b}{a+b}\right) = -\frac{1}{2|x|r}\ln\left(\frac{|x|-r}{|x|+r}\right)^2.$$

Therefore,

$$I = -\frac{\pi}{|x|}\int_0^{+\infty} \frac{r}{(1+r^2)^\gamma}\ln\left(\frac{|x|-r}{|x|+r}\right)^2 dr.$$

Suppose that $\varepsilon \in (0,1)$. Then

$$I = I_1 + I_2 + I_3, \tag{5.18}$$

where

$$I_1 = -\frac{\pi}{|x|} \int_0^{\varepsilon|x|} \frac{r}{(1+r^2)^\gamma} \ln\biggl(\frac{|x|-r}{|x|+r}\biggr)^2 dr, \tag{5.19}$$

$$I_2 = -\frac{\pi}{|x|} \int_{\varepsilon|x|}^{|x|/\varepsilon} \frac{r}{(1+r^2)^\gamma} \ln\biggl(\frac{|x|-r}{|x|+r}\biggr)^2 dr, \tag{5.20}$$

$$I_3 = -\frac{\pi}{|x|} \int_{|x|/\varepsilon}^{+\infty} \frac{r}{(1+r^2)^\gamma} \ln\biggl(\frac{|x|-r}{|x|+r}\biggr)^2 dr. \tag{5.21}$$

Consider the integral I_1. By Lagrange's formula,

$$\ln(1-t) = -\frac{1}{1-t_{1\varepsilon}} t, \quad \ln(1+t) = \frac{1}{1+t_{2\varepsilon}} t, \qquad t, t_{1\varepsilon}, t_{2\varepsilon} \in (0,\varepsilon).$$

Hence the following estimate holds:

$$\biggl|\ln\biggl(1-\frac{r}{|x|}\biggr) - \ln\biggl(1+\frac{r}{|x|}\biggr)\biggr| \leqslant c_1(\varepsilon)\frac{r}{|x|}, \qquad r \in [0, \varepsilon|x|]. \tag{5.22}$$

Therefore we have a chain of relations

$$\begin{aligned} |I_1| &\leqslant \frac{2\pi}{|x|} \int_0^{\varepsilon|x|} \frac{r}{(1+r^2)^\gamma} \biggl|\ln\biggl(1-\frac{r}{|x|}\biggr) - \ln\biggl(1+\frac{r}{|x|}\biggr)\biggr| dr \\ &\leqslant \frac{2\pi c_1(\varepsilon)}{|x|^2} \int_0^{+\infty} \frac{r^2}{(1+r^2)^\gamma} dr \leqslant \frac{A_5(\varepsilon)}{|x|^2} \quad \text{for} \quad \gamma > \frac{3}{2}. \end{aligned} \tag{5.23}$$

Consider the integral I_2:

$$\begin{aligned} |I_2| &\leqslant \frac{\pi}{|x|} \int_{\varepsilon|x|}^{|x|/\varepsilon} \frac{r}{(1+r^2)^\gamma} \biggl|\ln\biggl(\frac{|x|-r}{|x|+r}\biggr)^2\biggr| dr \\ &\overset{r=t|x|}{=} \frac{\pi}{|x|}|x|^2 \int_\varepsilon^{1/\varepsilon} \frac{t}{(1+t^2|x|^2)^\gamma} \biggl|\ln\biggl(\frac{1-t}{1+t}\biggr)^2\biggr| dt \\ &\leqslant \frac{\pi}{|x|^{2\gamma-1}} \int_\varepsilon^{1/\varepsilon} \frac{1}{t^{2\gamma-1}} \biggl|\ln\biggl(\frac{1-t}{1+t}\biggr)^2\biggr| dt \leqslant \frac{A_6(\varepsilon)}{|x|^{2\gamma-1}}, \qquad \gamma > \frac{3}{2}. \end{aligned} \tag{5.24}$$

Finally, consider the integral I_3. By Lagrange's formula, we have a chain of relations

$$\begin{aligned} |I_3| &\leqslant \frac{2\pi}{|x|} \int_{|x|/\varepsilon}^{+\infty} \frac{r}{(1+r^2)^\gamma} \biggl|\ln\biggl(1-\frac{|x|}{r}\biggr) - \ln\biggl(1+\frac{|x|}{r}\biggr)\biggr| dr \\ &\leqslant c_1(\varepsilon)2\pi \int_{|x|/\varepsilon}^{+\infty} \frac{1}{(1+r^2)^\gamma} dr \leqslant c_1(\varepsilon)2\pi \int_{|x|/\varepsilon}^{+\infty} \frac{1}{r^{2\gamma}} dr \\ &= c_1(\varepsilon)2\pi \frac{1}{2\gamma-1}\biggl(\frac{\varepsilon}{|x|}\biggr)^{2\gamma-1} = \frac{A_7(\varepsilon)}{|x|^{2\gamma-1}}, \qquad \gamma > \frac{3}{2}. \end{aligned} \tag{5.25}$$

Thus we conclude that there is a constant $A > 0$ such that the following bound holds for $|x| > 1$:

$$|I| \leqslant \frac{A_8}{|x|^2} \quad \text{for} \quad \gamma > \frac{3}{2}. \tag{5.26}$$

We now consider the case when $|x| \leqslant 1$. For convenience we rewrite the original integral in the form

$$I = \int_{\mathbb{R}^3} \frac{1}{|y|^2} \frac{1}{(1+|x-y|^2)^\gamma}\, dy. \tag{5.27}$$

Again passing to the spherical coordinate system and using the bounds $|\sin\theta| \leqslant 1$ and $\cos\theta \leqslant 1$, we obtain the inequalities

$$\begin{aligned} I &= 2\pi \int_0^{+\infty} dr \int_0^\pi d\theta\, \frac{\sin\theta}{(1+|x|^2+r^2-2|x|r\cos\theta)^\gamma} \\ &\leqslant 2\pi^2 \int_0^{+\infty} dr\, \frac{1}{(1+|x|^2+r^2-2|x|r)^\gamma} = 2\pi^2 \int_0^{+\infty} dr \frac{1}{(1+(|x|-r)^2)^\gamma} \\ &= 2\pi^2 \int_{-|x|}^{+\infty} \frac{dt}{(1+t^2)^\gamma} \leqslant 2\pi^2 \int_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^\gamma} := A_9 < +\infty. \end{aligned} \tag{5.28}$$

Then we arrive at the estimate

$$|I| \leqslant \frac{A_{10}}{1+|x|^2} \quad \text{for all} \quad x \in \mathbb{R}^3. \qquad \square \tag{5.29}$$

We introduce the potentials

$$U_0(x,t) := U_0[\rho_0](x) := \int_{\mathbb{R}^3} G_\gamma(x,y,t)\rho_0(y)\, dy, \tag{5.30}$$

$$U_1(x,t) := U_1[\rho](x,t) := \int_0^t \int_{\mathbb{R}^3} G_\gamma(x,y,t-\tau)\rho(y,\tau)\, dy\, d\tau. \tag{5.31}$$

Their properties are collected in the following lemma.

Lemma 3. *For any $\rho_0(x) \in \mathbb{C}_b(\mathbb{R}^3)$ and $\rho(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b(\mathbb{R}^3))$ one has $U_0(x,t)$, $U_1(x,t) \in \mathbb{C}([0,T];W_1)$ when $\gamma > 3/2$.*

Proof. Step 1. We claim that

$$U_0(x,t), U_1(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b(\mathbb{R}^3)). \tag{5.32}$$

Indeed, note that $U_0(x,t), U_1(x,t) \in \mathbb{C}(\mathbb{R}^1)$ for every $t \in [0,T]$. Below, we shall prove the stronger inclusion $U_0(x,t), U_1(x,t) \in \mathbb{C}^{(1)}(\mathbb{R}^1)$ for every $t \in [0,T]$.

By (5.15),

$$\begin{aligned} |U_0(x,t_2) - U_0(x,t_1)| &\leqslant \int_{\mathbb{R}^3} |\rho_0(y)||G_\gamma(x,y,t_2) - G_\gamma(x,y,t_1)|\, dy \\ &= \int_{\mathbb{R}^3} |\rho_0(y)| \left| \int_{t_2}^{t_1} \frac{\partial}{\partial s} G_\gamma(x,y,s)\, ds \right| dy \\ &\leqslant \sup_{y\in\mathbb{R}^3} |\rho_0(y)||t_2-t_1| \sup_{x\in\mathbb{R}^3,\, s\in[t_1,t_2]} \int_{\mathbb{R}^3} \left| \frac{\partial G_\gamma(x,y,s)}{\partial s} \right| dy \\ &\leqslant B_1(T) \sup_{y\in\mathbb{R}^3} |\rho_0(y)||t_2-t_1|. \end{aligned} \tag{5.33}$$

Thus, for all $t_1, t_2 \in [0,T]$ one has

$$\sup_{x\in\mathbb{R}^3} |U_0(x,t_2) - U_0(x,t_1)| \leqslant B_1(T) \sup_{y\in\mathbb{R}^3} |\rho_0(y)||t_2 - t_1|. \tag{5.34}$$

Moreover, the following expression holds in view of (5.15):

$$\sup_{x\in\mathbb{R}^3,\, t\in[0,T]} |U_0(x,t)| \leqslant B_1(T) \sup_{y\in\mathbb{R}^3} |\rho_0(y)|. \tag{5.35}$$

Hence $U_0(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b(\mathbb{R}^3))$.

We now claim that $U_1(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b(\mathbb{R}^3))$. Indeed, for all $t_1, t_2 \in [0,T]$ we have a chain of inequalities

$$\begin{aligned}
&|U_1(x,t_2) - U_1(x,t_1)| \\
&\quad\leqslant \left| \int_0^{t_2} \int_{\mathbb{R}^3} G_\gamma(x,y,t_2-\tau)\rho(y,\tau)\,dy\,d\tau - \int_0^{t_1}\int_{\mathbb{R}^3} G_\gamma(x,y,t_1-\tau)\rho(y,\tau)\,dy\,d\tau \right| \\
&\quad\leqslant \int_{t_1}^{t_2}\int_{\mathbb{R}^3} |G_\gamma(x,y,t_2-\tau)||\rho(y,\tau)|\,dy\,d\tau \\
&\qquad + \int_0^{t_1}\int_{\mathbb{R}^3} |G_\gamma(x,y,t_2-\tau) - G_\gamma(x,y,t_1-\tau)||\rho(y,\tau)|\,dy\,d\tau \\
&\quad =: I_{11}(x,t_2,t_1) + I_{12}(x,t_2,t_1).
\end{aligned} \tag{5.36}$$

In view of (5.15), the integral I_{12} satisfies the following bound similar to (5.33):

$$\begin{aligned}
I_{12} &\leqslant \int_0^{t_1}\int_{\mathbb{R}^3}\int_{t_1-\tau}^{t_2-\tau} \left|\frac{\partial G_\gamma(x,y,s)}{\partial s}\right| ds\,|\rho(y,\tau)|\,dy\,d\tau \\
&\leqslant B_1(T)T|t_2-t_1| \sup_{\tau\in[0,T],\, y\in\mathbb{R}^3} |\rho(y,\tau)|
\end{aligned} \tag{5.37}$$

and I_{11} satisfies the inequality

$$I_{11} \leqslant B_1(T)|t_2 - t_1| \sup_{\tau\in[0,T],\, y\in\mathbb{R}^3} |\rho(y,\tau)|. \tag{5.38}$$

Moreover, we have

$$|U_1(x,t)| \leqslant TB_1(T) \sup_{\tau\in[0,T],\, y\in\mathbb{R}^3} |\rho(y,\tau)|. \tag{5.39}$$

It follows from (5.36)–(5.39) that $U_1(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b(\mathbb{R}^3))$.

Step 2. We claim that $U_0(x,t), U_1(x,t) \in \mathbb{C}([0,T];W_1)$. Indeed, consider the potential $U_0(x,t)$:

$$U_0(x,t) = U_{01}(x,t) + U_{02}(x,t), \tag{5.40}$$

where U_{01} and U_{02} are of the form

$$U_{01}(x,t) = \int_{O(x_{00},R)} G_\gamma(x,y,t)\rho_0(y)\,dy$$
$$= (1+|x|^2)^{1/2}\int_{O(x_{00},R)} \mathscr{E}(x-y,t)\frac{\rho_0(y)}{(1+|y|^2)^\gamma}\,dy, \tag{5.41}$$
$$U_{02}(x,t) = \int_{\mathbb{R}^3\setminus O(x_{00},R)} G_\gamma(x,y,t)\rho_0(y)\,dy$$
$$= (1+|x|^2)^{1/2}\int_{\mathbb{R}^3\setminus O(x_{00},R)} \mathscr{E}(x-y,t)\frac{\rho_0(y)}{(1+|y|^2)^\gamma}\,dy. \tag{5.42}$$

By (5.5), when $x \neq y$ and $t \in [0,T]$ one has

$$|\mathscr{E}(x-y,t)| \leqslant \frac{A_1(T)}{|x-y|}, \qquad \left|\frac{\partial \mathscr{E}(x-y,t)}{\partial x_j}\right| \leqslant \frac{A_2(T)}{|x-y|^2}. \tag{5.43}$$

Note that the result of Lemma 4.1 in [27] was obtained from bounds of the form (5.43) for the fundamental solution of the Laplace operator and not from an explicit formula for this solution. Arguing in a similar way, we establish that $U_{01}(x,t) \in \mathbb{C}^{(1)}(\mathbb{R}^3)$ for every $t \in [0,T]$ and, moreover,

$$\frac{\partial U_{01}(x,t)}{\partial x_j} = \int_{O(x_{00},R)} \frac{\partial G_\gamma(x,y,t)}{\partial x_j}\rho_0(y)\,dy. \tag{5.44}$$

Since the integrand in $U_{02}(x,t)$ has no singularities and $q > 3/2$, we also conclude that $U_{02}(x,t) \in \mathbb{C}^{(1)}(\mathbb{R}^3)$ for every $t \in [0,T]$ and, moreover,

$$\frac{\partial U_{02}(x,t)}{\partial x_j} = \int_{\mathbb{R}^3\setminus O(x_{00},R)} \frac{\partial G_\gamma(x,y,t)}{\partial x_j}\rho_0(y)\,dy. \tag{5.45}$$

Thus, it follows from (5.44) and (5.45) that $U_0(x,t) \in \mathbb{C}^{(1)}(\mathbb{R}^3)$ for every $t \in [0,T]$ and one has

$$\frac{\partial U_0(x,t)}{\partial x_j} = \int_{\mathbb{R}^3} \frac{\partial G_\gamma(x,y,t)}{\partial x_j}\rho_0(y)\,dy. \tag{5.46}$$

In view of (5.16), we have a chain of inequalities

$$(1+|x|^2)^{1/2}\left|\frac{\partial U_0(x,t_2)}{\partial x_j} - \frac{\partial U_0(x,t_1)}{\partial x_j}\right|$$
$$\leqslant (1+|x|^2)^{1/2}\int_{\mathbb{R}^3}\left|\frac{\partial G_\gamma(x,y,t_2)}{\partial x_j} - \frac{\partial G_\gamma(x,y,t_1)}{\partial x_j}\right||\rho_0(y)|\,dy$$
$$\leqslant (1+|x|^2)^{1/2}\int_{\mathbb{R}^3}\int_{t_1}^{t_2}\left|\frac{\partial^2 G_\gamma(x,y,s)}{\partial s\,\partial x_j}\right|ds\,|\rho_0(y)|\,dy$$
$$\leqslant |t_2-t_1|\sup_{y\in\mathbb{R}^3}|\rho_0(y)|\sup_{s\in[0,T],\,x\in\mathbb{R}^3}(1+|x|^2)^{1/2}\int_{\mathbb{R}^3}\left|\frac{\partial^2 G_\gamma(x,y,s)}{\partial s\,\partial x_j}\right|dy$$
$$\leqslant B_2(T)|t_2-t_1|\sup_{y\in\mathbb{R}^3}|\rho_0(y)|. \tag{5.47}$$

Moreover,

$$(1+|x|^2)^{1/2}\left|\frac{\partial U_0(x,t)}{\partial x_j}\right| \leqslant B_2(T)\sup_{y\in\mathbb{R}^3}|\rho_0(y)|. \tag{5.48}$$

In view of (5.34) and (5.35), we find from (5.47) and (5.48) that

$$U_0(x,t)\in\mathbb{C}([0,T];W_1). \tag{5.49}$$

Our next aim is to prove that $U_1(x,t)\in\mathbb{C}([0,T];W_1)$. In the same way, we conclude from (5.43) that $U_1(x,t)\in\mathbb{C}^{(1)}(\mathbb{R}^3)$ for every $t\in[0,T]$ and, moreover, the following equality holds (compare with (5.46)):

$$\frac{\partial U_1(x,t)}{\partial x_j}=\int_0^t\int_{\mathbb{R}^3}\frac{\partial G_\gamma(x,y,t-\tau)}{\partial x_j}\rho(y,\tau)\,dy\,d\tau. \tag{5.50}$$

In view of (5.16), the following bounds hold for all $t_1,t_2\in[0,T]$:

$$\begin{aligned}
&(1+|x|^2)^{1/2}\left|\frac{\partial U_1(x,t_2)}{\partial x_j}-\frac{\partial U_1(x,t_1)}{\partial x_j}\right|\\
&\leqslant(1+|x|^2)^{1/2}\int_0^{t_1}\int_{\mathbb{R}^3}\left|\frac{\partial G_\gamma(x,y,t_2-\tau)}{\partial x_j}-\frac{\partial G_\gamma(x,y,t_1-\tau)}{\partial x_j}\right||\rho(y,\tau)|\,dy\,d\tau\\
&\quad+(1+|x|^2)^{1/2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left|\frac{\partial G_\gamma(x,y,t_2-\tau)}{\partial x_j}\right||\rho(y,\tau)|\,dy\,d\tau\\
&\leqslant(1+|x|^2)^{1/2}\int_0^{t_1}\int_{\mathbb{R}^3}\int_{t_1-\tau}^{t_2-\tau}\left|\frac{\partial^2 G_\gamma(x,y,s)}{\partial x_j\,\partial s}\right|ds\,dy\,d\tau\\
&\quad+(1+|x|^2)^{1/2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left|\frac{\partial G_\gamma(x,y,t_2-\tau)}{\partial x_j}\right||\rho(y,\tau)|\,dy\,d\tau\\
&\leqslant T\sup_{y\in\mathbb{R}^3,\,\tau\in[0,T]}|\rho(y,\tau)||t_2-t_1|\sup_{x\in\mathbb{R}^3,\,s\in[0,T]}(1+|x|^2)^{1/2}\int_{\mathbb{R}^3}\left|\frac{\partial^2 G_\gamma(x,y,s)}{\partial x_j\,\partial s}\right|dy\\
&\quad+\sup_{y\in\mathbb{R}^3,\,\tau\in[0,T]}|\rho(y,\tau)||t_2-t_1|\sup_{x\in\mathbb{R}^3,\,\tau\in[0,T]}(1+|x|^2)^{1/2}\int_{\mathbb{R}^3}\left|\frac{\partial G_\gamma(x,y,\tau)}{\partial x_j}\right|dy\\
&\leqslant[TB_2(T)+B_2(T)]\sup_{y\in\mathbb{R}^3,\,\tau\in[0,T]}|\rho(y,\tau)||t_2-t_1|.
\end{aligned} \tag{5.51}$$

Moreover,

$$\begin{aligned}
&(1+|x|^2)^{1/2}\left|\frac{\partial U_1(x,t)}{\partial x_j}\right|\\
&\qquad\leqslant T\sup_{y\in\mathbb{R}^3,\,\tau\in[0,T]}|\rho(y,\tau)|\sup_{x\in\mathbb{R}^3,\,\tau\in[0,T]}(1+|x|^2)^{1/2}\int_{\mathbb{R}^3}\left|\frac{\partial G_\gamma(x,y,\tau)}{\partial x_j}\right|dy\\
&\qquad\leqslant TB_2(T)\sup_{y\in\mathbb{R}^3,\,\tau\in[0,T]}|\rho(y,\tau)|.
\end{aligned} \tag{5.52}$$

Thus, in view of (5.36)–(5.39) and the bounds (5.51), (5.52), we conclude that $U_1(x,t)\in\mathbb{C}([0,T];W_1)$. □

Our task is to study the integral equation (5.9) in the weighted Banach space $\mathbb{C}([0,T];W_1)$, which was defined in § 3, with respect to the norm (5.11).

To prove the existence of a solution of (5.9), we choose a closed bounded convex subset $D_{R,T}$ in $\mathbb{C}([0,T];W_1)$ of the form

$$D_{R,T} := \{v(x,t) \in \mathbb{C}([0,T];W_1)\colon \|v\|_T \leqslant R\}. \tag{5.53}$$

Rewrite (5.9) in the form

$$v(x,t) = H(v)(x,t), \tag{5.54}$$

where

$$H(v)(x,t) = h_\alpha(x,t) + H_1(v)(x,t), \tag{5.55}$$

$$h_\alpha(x,t) = \int_{\mathbb{R}^3} G_\alpha(x,y,t)(1+|y|^2)^\alpha \Delta_3 u_0(y)\,dy, \tag{5.56}$$

$$H_1(v)(x,t) = \int_0^t \int_{\mathbb{R}^3} G_q(x,y,t-\tau) \times \biggl|(1+|y|^2)^{1/2}\nabla v(y,\tau) - \frac{y}{(1+|y|^2)^{1/2}} v(y,\tau)\biggr|^q dy\,d\tau. \tag{5.57}$$

Lemma 4. *Suppose that $u_0(x) \in \mathbb{C}^{(2)}(\mathbb{R}^3)$ and the bound (5.12) holds. Then the operator $H(\,\cdot\,)$ defined in (5.55) for $q > 3/2$ acts as*

$$H(\,\cdot\,)\colon \mathbb{C}([0,T];W_1) \to \mathbb{C}([0,T];W_1). \tag{5.58}$$

Proof. Step 1. We claim that the function $h_\alpha(x,t)$ given by the explicit formula (5.56) belongs to

$$\mathbb{C}([0,T];W_1) \quad \text{for every} \quad T > 0.$$

Indeed, note that under the condition (5.12) on $u_0(x) \in \mathbb{C}^{(2)}(\mathbb{R}^3)$ one has

$$\rho_0(y) = (1+|y|^2)^\alpha \Delta_3 u_0(y) \in \mathbb{C}_b(\mathbb{R}^3),$$

whence, by Lemma 3,

$$U_0[\rho_0](x,t) \in \mathbb{C}([0,T];W_1).$$

Step 2. Consider the function

$$\rho(x,t) = \biggl|(1+|x|^2)^{1/2}\nabla v(x,t) - \frac{x}{(1+|x|^2)^{1/2}} v(x,t)\biggr|^q, \tag{5.59}$$

where $v(x,t) \in \mathbb{C}([0,T];W_1)$. Note that $\rho(x,t) \in \mathbb{C}(\mathbb{R}^3)$ for $t \in [0,T]$.

On the one hand,

$$\sup_{x\in\mathbb{R}^3,\, t\in[0,T]} |\rho(x,t)| \leqslant c(q)\Bigl(\sup_{x\in\mathbb{R}^3,\, t\in[0,T]} (1+|x|^2)^{1/2}|\nabla v(x,t)|\Bigr)^q + c(q)\Bigl(\sup_{x\in\mathbb{R}^3,\, t\in[0,T]} |v(x,t)|\Bigr)^q < +\infty, \tag{5.60}$$

where $c(q)$ is a positive constant. On the other hand, we have the inequality

$$|\rho(x,t_2)-\rho(x,t_1)| \\ \leqslant q\max\{J_1^{q-1},J_2^{q-1}\}\bigl[(1+|x|^2)^{1/2}|\nabla v(x,t_2)-\nabla v(x,t_1)|+|v(x,t_2)-v(x,t_1)|\bigr], \tag{5.61}$$

where

$$J_k := \left|(1+|x|^2)^{1/2}\nabla v(x,t_k)-\frac{x}{(1+|x|^2)^{1/2}}v(x,t_k)\right|, \qquad k=1,2.$$

By (5.60),

$$\sup_{x\in\mathbb{R}^3,\,t_k\in[0,T]} J_k = A < +\infty \quad \text{for} \quad k=1,2. \tag{5.62}$$

Since $v(x,t)\in\mathbb{C}([0,T];W_1)$, it follows from (5.61) and (5.62) that

$$\sup_{x\in\mathbb{R}^3}|\rho(x,t_2)-\rho(x,t_1)| \leqslant qA^{q-1}\Bigl[\sup_{x\in\mathbb{R}^3}(1+|x|^2)^{1/2}|\nabla v(x,t_2)-\nabla v(x,t_1)| \\ +\sup_{x\in\mathbb{R}^3}|v(x,t_2)-v(x,t_1)|\Bigr]\to +0 \tag{5.63}$$

as $|t_2-t_1|\to +0$ for any $t_1,t_2\in[0,T]$. Hence it follows from (5.60) and (5.63) that $\rho(x,t)\in\mathbb{C}([0,T];\mathbb{C}_b(\mathbb{R}^3))$. Using the result of Lemma 3 about the potential $U_1(x,t)$, we conclude that $U_1(x,t)\in\mathbb{C}([0,T];W_1)$.

Hence it follows from (5.55) that

$$H(v)(x,t)=U_0[\rho_0](x,t)+U_1[\rho](x,t)\in\mathbb{C}([0,T];W_1)$$

for all $u_0(x)\in\mathbb{C}^{(2)}(\mathbb{R}^3)$ possessing the property (5.12) and for an arbitrary function

$$v(x,t)\in\mathbb{C}([0,T];W_1). \qquad \square$$

Fix any function $u_0(x)\in\mathbb{C}^2(\mathbb{R}^3)$ satisfying the condition (5.12). Choose a large $R>0$ such that the concluding inequality in the following chain holds:

$$\|h_\alpha\|_T \leqslant \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}|G_\alpha(x,y,t)|(1+|y|^2)^\alpha|\Delta_3 u_0(y)|\,dy \\ +\sum_{j=1}^{3}\sup_{x\in\mathbb{R}^3}(1+|x|^2)^{1/2}\int_{\mathbb{R}^3}\left|\frac{\partial G_\alpha(x,y,t)}{\partial x_j}\right|(1+|y|^2)^\alpha|\Delta_3 u_0(y)|\,dy \\ \leqslant A_4B_1(T)+3B_2(T)A\leqslant \frac{R}{2}. \tag{5.64}$$

The corresponding inequalities hold in view of (5.12), (5.15) and (5.16).

Lemma 5. *For an arbitrary $R>0$ and $q>3/2$ there is a small $T>0$ such that*

$$H_1(v)\colon D_{R,T}\to D_{R/2,T}. \tag{5.65}$$

Proof. Let $R > 0$ be arbitrary. It was proved in the proof of Lemma 4 that

$$H_1(\,\cdot\,)\colon \mathbb{C}([0,T];W_1) \to \mathbb{C}([0,T];W_1)$$

for every $T > 0$. We put

$$\rho(y,\tau) := \left|(1+|y|^2)^{1/2}\nabla v(y,\tau) - \frac{y}{(1+|y|^2)^{1/2}} v(y,\tau)\right|^q. \tag{5.66}$$

Then the function

$$H_1(x,t) := H_1(v)(x,t) = \int_0^t \int_{\mathbb{R}^3} G_q(x,y,t-\tau)\rho(y,\tau)\,dy\,d\tau \tag{5.67}$$

satisfies the following chain of inequalities:

$$\begin{aligned}\|H_1(x,t)\|_T \leqslant{} & \sup_{x\in\mathbb{R}^3,\, t\in[0,T]} \int_0^t \int_{\mathbb{R}^3} |G_q(x,y,t-\tau)||\rho(y,\tau)|\,dy\,d\tau \\ & + \sum_{j=1}^3 \sup_{x\in\mathbb{R}^3,\, t\in[0,T]} \int_0^t \int_{\mathbb{R}^3} (1+|x|^2)^{1/2}\left|\frac{\partial G_q(x,y,t-\tau)}{\partial x_j}\right||\rho(y,\tau)|\,dy\,d\tau \\ \leqslant{} & T[B_1(T)+3B_2(T)] \sup_{y\in\mathbb{R}^3,\, \tau\in[0,T]} |\rho(y,\tau)|. \end{aligned} \tag{5.68}$$

Note that

$$\begin{aligned}\sup_{y\in\mathbb{R}^3,\, \tau\in[0,T]} |\rho(y,\tau)| &\leqslant \sup_{y\in\mathbb{R}^3,\, \tau\in[0,T]} [(1+|y|^2)^{1/2}|\nabla v(y,\tau)| + |v(y,\tau)|]^q \\ &\leqslant \sup_{y\in\mathbb{R}^3,\, \tau\in[0,T]} \left[\sum_{j=1}^3 (1+|y|^2)^{1/2}\left|\frac{\partial v(y,\tau)}{\partial y_j}\right| + |v(y,\tau)|\right]^q \leqslant R^q \end{aligned} \tag{5.69}$$

if $v(x,t) \in D_{R,T}$. It follows from (5.68) and (5.69) that

$$\|H_1(x,t)\|_T \leqslant T[B_1(T)+3B_2(T)]R^q, \qquad q > \frac{3}{2}. \tag{5.70}$$

Choose a small $T > 0$ such that

$$T[B_1(T)+3B_2(T)]R^{q-1} \leqslant \frac{1}{2}. \tag{5.71}$$

Then we deduce from (5.70) that

$$\|H_1(x,t)\|_T \leqslant \frac{R}{2}, \tag{5.72}$$

as required. □

Choosing a large $R > 0$ such that the resulting inequality (5.64) holds, we can deduce the following assertion from Lemma 5.

Lemma 6. *Suppose that $q > 3/2$. Then for every $u_0(x) \in \mathbb{C}^{(2)}(\mathbb{R}^3)$ satisfying (5.12), one can find a sufficiently large $R > 0$ and a sufficiently small $T > 0$ such that*

$$H(\,\cdot\,)\colon D_{R,T} \to D_{R,T}, \tag{5.73}$$

where $D_{R,T} \subset \mathbb{C}([0,T];W_1)$ is the closed ball defined in (5.53).

We proceed to prove that $H(v)(x,t)$ is a contraction on the closed ball $D_{R,T}$ for sufficiently small $T > 0$.

Lemma 7. *If*

$$qT(B_1+3B_2)R^{q-1} \leqslant \frac{1}{2}, \tag{5.74}$$

then $H(v)(x,t)$ is a contraction on $D_{R,T}$.

Proof. Suppose that $v_1(x,t), v_2(x,t) \in D_{R,T}$. We have

$$\begin{aligned}
&\biggl|\biggl|(1+|x|^2)^{1/2}\nabla v_1 - \frac{x}{(1+|x|^2)^{1/2}}v_1\biggr|^q - \biggl|(1+|x|^2)^{1/2}\nabla v_2 - \frac{x}{(1+|x|^2)^{1/2}}v_2\biggr|^q\biggr| \\
&\leqslant q\max\biggl\{\biggl|(1+|x|^2)^{1/2}\nabla v_1 - \frac{x}{(1+|x|^2)^{1/2}}v_1\biggr|^{q-1}, \\
&\quad \biggl|(1+|x|^2)^{1/2}\nabla v_2 - \frac{x}{(1+|x|^2)^{1/2}}v_2\biggr|^{q-1}\biggr\}\bigl[(1+|x|^2)^{1/2}|\nabla v_1 - \nabla v_2| + |v_1 - v_2|\bigr] \\
&\leqslant q\max\bigl\{|(1+|x|^2)^{1/2}|\nabla v_1| + |v_1||^{q-1},\ |(1+|x|^2)^{1/2}|\nabla v_2| + |v_2||^{q-1}\bigr\} \\
&\quad\times \bigl[(1+|x|^2)^{1/2}|\nabla v_1 - \nabla v_2| + |v_1 - v_2|\bigr] \\
&\leqslant q\max\biggl\{\biggl|(1+|x|^2)^{1/2}\sum_{j=1}^{3}\biggl|\frac{\partial v_1}{\partial x_j}\biggr| + |v_1|\biggr|^{q-1}, \biggl|(1+|x|^2)^{1/2}\sum_{j=1}^{3}\biggl|\frac{\partial v_1}{\partial x_j}\biggr| + |v_1|\biggr|^{q-1}\biggr\} \\
&\quad\times\biggl[(1+|x|^2)^{1/2}\sum_{j=1}^{3}\biggl|\frac{\partial v_1}{\partial x_j} - \frac{\partial v_2}{\partial x_j}\biggr| + |v_1 - v_2|\biggr] \leqslant qR^{q-1}\|v_1 - v_2\|_T.
\end{aligned} \tag{5.75}$$

Put

$$\rho_j(y,\tau) = \biggl|(1+|x|^2)^{1/2}\nabla v_j - \frac{x}{(1+|x|^2)^{1/2}}v_j\biggr|^q, \qquad j = 1,2. \tag{5.76}$$

By (5.75), we arrive at the bound

$$\|\rho_1 - \rho_2\|_T \leqslant qR^{q-1}\|v_1 - v_2\|_T. \tag{5.77}$$

We have

$$\begin{aligned}
\|H(v_1) - H(v_2)\|_T &\leqslant \sup_{x\in\mathbb{R}^3,\, t\in[0,T]} \int_0^t \int_{\mathbb{R}^3} |G_q(x,y,t-\tau)||\rho_1(y,\tau) - \rho_2(y,\tau)|\,dy\,d\tau \\
&\quad + \sum_{j=1}^{3} \sup_{x\in\mathbb{R}^3,\, t\in[0,T]} \int_0^t \int_{\mathbb{R}^3} (1+|x|^2)^{1/2}\biggl|\frac{\partial G_q(x,y,t-\tau)}{\partial x_j}\biggr||\rho_1(y,\tau) - \rho_2(y,\tau)|\,dy\,d\tau \\
&\leqslant T[B_1(T) + 3B_2(T)]\|\rho_1 - \rho_2\|_T.
\end{aligned} \tag{5.78}$$

Using (5.77) and (5.78), we arrive at the desired inequality

$$\|H(v_1) - H(v_2)\|_T \leqslant T[B_1(T) + 3B_2(T)]qR^{q-1}\|v_1 - v_2\|_T. \qquad \square \tag{5.79}$$

We now use the standard algorithm for extending solutions in time. This algorithm was described in [28] for Volterra integral equations in $\mathbb{C}([0,T];\mathbb{B})$, where $\mathbb{B}$ is a Banach space. In our case, $\mathbb{B} = W_1$. In outline, the scheme of extension in time is as follows. Having already proved the existence of a small $T_1 > 0$ such that the integral equation (5.9) has a unique solution $v(x,t) \in \mathbb{C}([0,T_1];W_1)$, we can rewrite (5.9) in the following form for $t \in [T_1,T]$, $T > T_1$:

$$v(x,t) = v(x,T_1) + \int_{T_1}^{t}\int_{\mathbb{R}^3} G_q(x,y,t-\tau)\left|(1+|y|^2)^{1/2}\nabla v(y,\tau) - \frac{y}{(1+|y|^2)^{1/2}}v\right|^q dy\,d\tau, \tag{5.80}$$

where

$$\begin{aligned} v(x,T_1) &= \int_{\mathbb{R}^3} G_\alpha(x,y,T_1)(1+|y|^2)^\alpha \Delta_3 u_0(y)\,dy \\ &+ \int_0^{T_1}\int_{\mathbb{R}^3} G_q(x,y,T_1-\tau)\left|(1+|y|^2)^{1/2}\nabla v(y,\tau) - \frac{y}{(1+|y|^2)^{1/2}}v\right|^q dy\,d\tau. \end{aligned} \tag{5.81}$$

We have $v(x,T_1) \in W_1$ by Lemma 4. Choose a large $R > 0$ in such a way that

$$\|v(x,T_1)\| := \sup_{x\in\mathbb{R}^1}|v(x,T_1)| + \sum_{j=1}^{3}\sup_{x\in\mathbb{R}^1}(1+|x|^2)^{1/2}\left|\frac{\partial v(x,T_1)}{\partial x_j}\right| \leqslant \frac{R}{2}. \tag{5.82}$$

Then repeat the proofs of Lemmas 5–7 to show that the integral equation (5.80) has a solution on the interval $t \in [T_1,T_2]$ for some $T_2 > T_1$. Continuing this algorithm, we conclude that either the solution extends unrestrictedly to the whole time axis, or there is a moment of time $T_0 = T_0(u_0) > 0$ such that

$$\lim_{T\uparrow T_0}\|v\|_T = +\infty.$$

Thus we arrive at the conclusion of the theorem. $\square$

We now need to state and prove a result on the solution $u(x,t)$ of the integral equation (5.1).

Theorem 3. *For every $u_0(x) \in \mathbb{C}^{(2)}(\mathbb{R}^3)$ satisfying the condition (5.12), there is a maximum number $T_0 = T_0(u_0) > 0$ such that for every $T \in (0,T_0)$ the integral equation (5.1) has the unique solution*

$$u(x,t) \in \mathbb{C}([0,T];W_2).$$

Moreover, either $T_0 = +\infty$, or $T_0 < +\infty$, and in the latter case we have

$$\lim_{T\uparrow T_0}\|u\|_{1,T} = +\infty, \tag{5.83}$$

where

$$\|u\|_{1,T} := \sup_{x\in\mathbb{R}^3,\, t\in[0,T]}(1+|x|^2)^{1/2}|u(x,t)| + \sum_{j=1}^{3}\sup_{x\in\mathbb{R}^3,\, t\in[0,T]}(1+|x|^2)\left|\frac{\partial u(x,t)}{\partial x_j}\right|. \tag{5.84}$$

Proof. Note that the solution $u(x,t)$ of the integral equation (5.1) and the solution $v(x,t)$ of the integral equation (5.9) are related by the equality

$$v(x,t) = (1+|x|^2)^{1/2} u(x,t). \tag{5.85}$$

Moreover, $u(x,t)$ is a solution of (5.1) if and only if $v(x,t)$ is a solution of (5.9).

Lemma 8. *We have the double inequality*

$$\frac{1}{2}\|v\|_T \leqslant \|u\|_{1,T} \leqslant 4\|v\|_T. \tag{5.86}$$

Proof. Note that

$$\frac{\partial v}{\partial x_j} = (1+|x|^2)^{1/2}\,\frac{\partial u}{\partial x_j} + \frac{x_j}{(1+|x|^2)^{1/2}} u. \tag{5.87}$$

We have $v(x,t) \in \mathbb{C}([0,T];W_1)$ for every $T \in (0,T_0)$. Hence the following chains of inequalities hold:

$$\begin{aligned}
\|v\|_T &= \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} |v(x,t)| + \sum_{j=1}^{3} \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}\left|\frac{\partial v(x,t)}{\partial x_j}\right| \\
&= \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}|u(x,t)| \\
&\quad + \sum_{j=1}^{3} \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}\left|(1+|x|^2)^{1/2}\frac{\partial u}{\partial x_j} + \frac{x_j}{(1+|x|^2)^{1/2}} u\right| \\
&\leqslant 2 \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}|u(x,t)| \\
&\quad + \sum_{j=1}^{3} \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)\left|\frac{\partial u}{\partial x_j}\right| \leqslant 2\|u\|_{1,T},
\end{aligned} \tag{5.88}$$

$$\begin{aligned}
\|u\|_{1,T} &= \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}|u(x,t)| + \sum_{j=1}^{3} \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)\left|\frac{\partial u}{\partial x_j}\right| \\
&\leqslant \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}|u(x,t)| \\
&\quad + \sum_{j=1}^{3} \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}\left|(1+|x|^2)^{1/2}\frac{\partial u}{\partial x_j} + \frac{x_j}{(1+|x|^2)^{1/2}} u\right| \\
&\quad + \sum_{j=1}^{3} \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}\left|\frac{x_j}{(1+|x|^2)^{1/2}} u\right| \\
&\leqslant 4 \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} |v(x,t)| + \sum_{j=1}^{3} \sup_{x\in\mathbb{R}^3,\,t\in[0,T]} (1+|x|^2)^{1/2}\left|\frac{\partial v(x,t)}{\partial x_j}\right| \leqslant 4\|v\|_T.
\end{aligned} \tag{5.89}$$

This proves Lemma 8. □

Let $t_1, t_2 \in [0,T]$ be arbitrary numbers. Then

$$v(x,t_2) - v(x,t_1) = (1+|x|^2)^{1/2}[u(x,t_2) - u(x,t_1)].$$

In our derivation of (5.89) we actually proved that

$$\begin{aligned}\sup_{x\in\mathbb{R}^3}(1+|x|^2)^{1/2}|u(x,t_2)-u(x,t_1)| + \sum_{j=1}^{3}\sup_{x\in\mathbb{R}^3}(1+|x|^2)\left|\frac{\partial u(x,t_2)}{\partial x_j} - \frac{\partial u(x,t_1)}{\partial x_j}\right| \\ \leqslant 4\sup_{x\in\mathbb{R}^3}|v(x,t_2)-v(x,t_1)| + 4\sum_{j=1}^{3}\sup_{x\in\mathbb{R}^3}(1+|x|^2)^{1/2}\left|\frac{\partial v(x,t_2)}{\partial x_j} - \frac{\partial v(x,t_1)}{\partial x_j}\right| \to +0\end{aligned} \tag{5.90}$$

as $|t_2 - t_1| \to +0$ for any $t_1, t_2 \in [0,T]$. Combining this with (5.89), we see that $u(x,t) \in \mathbb{C}([0,T]; W_2)$ for every $T \in (0, T_0)$. The double inequality (5.86) implies that if $T_0 < +\infty$, then

$$\lim_{T\uparrow T_0}\|u\|_{1,T} = +\infty. \qquad \square$$

§ 6. Solubility of the Cauchy problem in the weak sense (4.3) for $q > 3/2$

The following main assertion holds.

Theorem 4. *If $q > 3/2$, then for every function $u_0(x) \in \mathbb{C}^2(\mathbb{R}^3)$ satisfying the conditions*

$$|u_0(x)| \leqslant \frac{D_1}{(1+|x|^2)^{1/2}}, \qquad |\nabla u_0(x)| \leqslant \frac{D_2}{1+|x|^2}, \tag{6.1}$$

$$|\Delta_3 u_0(x)| \leqslant \frac{D_3}{(1+|x|^2)^{\alpha}}, \qquad \alpha > \frac{3}{2}, \tag{6.2}$$

the Cauchy problem has a local-in-time weak solution in the sense of Definition 1.

Proof. Step 1. *Properties of non-classical heat potentials.* Our current task is to study some properties of the following non-classical volume heat potentials:

$$V_0(x,t) := V_0[\rho_0](x,t) := \int_{\mathbb{R}^3}\mathscr{E}(x-y,t)\rho_0(y)\,dy, \tag{6.3}$$

$$V(x,t) := V[\rho](x,t) := \int_0^t\int_{\mathbb{R}^3}\mathscr{E}(x-y,t-\tau)\rho(y,\tau)\,dy\,d\tau \tag{6.4}$$

under certain conditions on the densities $\rho_0(x)$ and $\rho(x,t)$. We first state a classical result which follows directly from [29].

Lemma 9. *Suppose that $\rho_0(x) \in \mathbb{C}_b((1+|x|^2)^{\alpha}; \mathbb{R}^3)$ for $\alpha > 3/2$. Then the classical Newtonian volume potential*

$$W_0(x) := W_0[\rho_0](x) := -\int_{\mathbb{R}^3}\frac{1}{4\pi|x-y|}\rho_0(y)\,dy$$

satisfies the equality

$$\langle \Delta_x W_0(x), \phi(x)\rangle = \langle \rho_0(x), \phi(x)\rangle$$

for all $\phi(x) \in \mathscr{D}(\mathbb{R}^3)$, *where* $\langle\,\cdot\,,\cdot\,\rangle$ *is the duality bracket between* $\mathscr{D}(\mathbb{R}^3)$ *and* $\mathscr{D}'(\mathbb{R}^3)$, *and the operator* Δ_x *is understood in the sense of distributional derivatives.*

Proof. Although this result is "classical", we give a proof of it since a similar technique will be used in a more complicated case to prove the equality (6.6).

Fix an arbitrary test function $\phi(x) \in \mathscr{D}(\mathbb{R}^3)$. Suppose that

$$\operatorname{supp}\phi(x) \subset O(0,R) \quad \text{for some} \quad R > 0.$$

Then, clearly,

$$\operatorname{supp}\Delta_x\phi(x) \subset O(0,R) \subset O(0,nR) \quad \text{for all} \quad n \geqslant 2.$$

The following chain of equalities holds:

$$\begin{aligned}
\langle \Delta_x W_0(x), \phi(x)\rangle &= \langle W_0(x), \Delta_x\phi(x)\rangle \\
&= \int_{\mathbb{R}^3} W_0(x)\Delta_x\phi(x)\,dx = \int_{O(0,R)} W_0(x)\Delta_x\phi(x)\,dx \\
&= -\frac{1}{4\pi}\int_{O(0,R)} \Delta_x\phi(x)\biggl[\int_{O(0,2R)} \frac{\rho_0(y)}{|x-y|}\,dy + \int_{\mathbb{R}^3\setminus O(0,2R)} \frac{\rho_0(y)}{|x-y|}\,dy\biggr]\,dx \\
&= -\frac{1}{4\pi}\int_{O(0,R)} \Delta_x\phi(x)\int_{O(0,2R)} \frac{\rho_0(y)}{|x-y|}\,dy\,dx. \qquad (6.5)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{O(0,R)} \Delta_x\phi(x)\int_{\mathbb{R}^3\setminus O(0,2R)} \frac{\rho_0(y)}{|x-y|}\,dy\,dx \\
&\qquad = \int_{O(x,R)} \phi(x)\Delta_x\int_{\mathbb{R}^3\setminus O(0,2R)} \frac{\rho_0(y)}{|x-y|}\,dy\,dx = 0
\end{aligned}$$

since, in the classical sense,

$$\Delta_x\int_{\mathbb{R}^3\setminus O(0,2R)} \frac{\rho_0(y)}{|x-y|}\,dy = \int_{\mathbb{R}^3\setminus O(0,2R)} \rho_0(y)\Delta_x\frac{1}{|x-y|}\,dy = 0 \quad \text{for} \quad x \in O(0,R).$$

We continue the chain (6.5)

$$\begin{aligned}
\langle \Delta_x W_0(x), \phi(x)\rangle &= -\int_{O(0,R)} \Delta_x\phi(x)\int_{O(0,2R)} \frac{\rho_0(y)}{4\pi|x-y|}\,dy\,dx \\
&= -\int_{O(0,2R)} \Delta_x\phi(x)\int_{O(0,2R)} \frac{\rho_0(y)}{4\pi|x-y|}\,dy\,dx \\
&= -\int_{O(0,2R)} \rho_0(y)\int_{O(0,3R)} \frac{1}{4\pi|x-y|}\Delta_x\phi(x)\,dx\,dy \\
&= \int_{O(0,2R)} \rho_0(y)\phi(y)\,dy = \int_{\mathbb{R}^3} \rho_0(y)\phi(y)\,dy = \langle \rho_0, \phi\rangle,
\end{aligned}$$

where we have used the well-known equality

$$\int_{O(0,3R)} \frac{1}{4\pi|x-y|}\Delta_x \phi(x)\,dx\,dy = -\phi(y) \quad \text{for} \quad y \in O(0,2R),$$

which holds, in particular, for any function $\phi(x) \in \mathbb{C}_0^\infty(O(0,3R))$ with $\operatorname{supp}\phi \subset O(0,R)$ (see, for example, [30]). □

We can now study the non-classical volume heat potential $V(x,t) = V[\rho](x,t)$ defined in (6.4). The following lemma is essentially an analogue of Lemma 9.

Lemma 10. *Suppose that $\rho(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b((1+|x|^2)^\alpha;\mathbb{R}^3))$ for $\alpha > 3/2$. Then*

$$V(x,t) \in \mathbb{C}^{(1)}([0,T];W_2),$$

where W_2 is the Banach space defined in § 3. Moreover,

$$\langle \mathfrak{M}_{x,t}[V](x,t),\phi(x)\rangle = \langle \rho(x,t),\phi(x)\rangle \tag{6.6}$$

for all $\phi(x) \in \mathscr{D}(\mathbb{R}^3)$ and all $t \in [0,T]$, where $\langle\cdot,\cdot\rangle$ is the duality bracket between $\mathscr{D}(\mathbb{R}^3)$ and $\mathscr{D}'(\mathbb{R}^3)$ and

$$\mathfrak{M}_{x,t}[w](x,t) := \Delta_3\frac{\partial w(x,t)}{\partial t} + \sigma_1\Delta_2 w(x,t) + \sigma_2 w_{x_3x_3}(x,t).$$

Proof. Part 1. Since

$$\rho(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b((1+|x|^2)^\alpha;\mathbb{R}^3)),$$

we have

$$(1+|x|^2)^\alpha\rho(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b(\mathbb{R}^3)). \tag{6.7}$$

Therefore, exactly as in the proof of Lemma 3 for $\alpha > 3/2$, one can prove in view of (5.15) and (5.16) that

$$V(x,t) \in \mathbb{C}([0,T];W_2). \tag{6.8}$$

Note that the following pointwise equality holds for all $(x,t) \in \mathbb{R}^3 \times [0,T]$:

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} &= -\int_{\mathbb{R}^3}\frac{1}{4\pi|x-y|}\rho(y,t)\,dy + \int_0^t\int_{\mathbb{R}^3}\mathscr{E}_1(x-y,t-\tau)\rho(y,\tau)\,dy\,d\tau \\ &= W_0[\rho](x,t) + W_1[\rho](x,t),\end{aligned} \tag{6.9}$$

where

$$\mathscr{E}_1(x-y,t-\tau) := \frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial t}, \tag{6.10}$$

$$W_0(x,t) := W_0[\rho](x,t) = -\int_{\mathbb{R}^3}\frac{1}{4\pi|x-y|}\rho(y,t)\,dy, \tag{6.11}$$

$$W_1(x,t) := W_1[\rho](x,t) := \int_0^t\int_{\mathbb{R}^3}\mathscr{E}_1(x-y,t-\tau)\rho(y,\tau)\,dy\,d\tau. \tag{6.12}$$

Since $\rho(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b((1+|x|^2)^\alpha;\mathbb{R}^3))$, we can use Lemma 4.1 of [27] in exactly the same way as in the proof of Lemma 3 to show that

$$W_0(x,t) \in \mathbb{C}([0,T];W_2). \tag{6.13}$$

The function $W_1(x,t)$ can be studied in the same way as the function $U_1(x,t)$ in Lemma 3. In view of (5.15) and (5.16), one can prove that

$$W_1(x,t) \in \mathbb{C}([0,T];W_2). \tag{6.14}$$

Hence we conclude from (6.9), (6.13) and (6.14) that

$$\frac{\partial V(x,t)}{\partial t} \in \mathbb{C}([0,T];W_2).$$

Thus, $V(x,t) \in \mathbb{C}^{(1)}([0,T];W_2)$.

Part 2. By Lemma 9,

$$\langle \Delta_{3x} W_0(x,t), \phi(x)\rangle = \langle \rho(x,t), \phi(x)\rangle \quad \text{for all} \quad t \in [0,T] \tag{6.15}$$

and for any test function $\phi(x) \in \mathscr{D}(\mathbb{R}^3)$. We have

$$\begin{aligned} &\langle \Delta_{3x} W_1(x,t) + \sigma_1 \Delta_{2x} V(x,t) + \sigma_2 V_{x_3x_3}(x,t), \phi(x)\rangle \\ &\qquad = \langle W_1(x,t), \Delta_{3x}\phi(x)\rangle + \langle V(x,t), \sigma_1 \Delta_{2x}\phi(x) + \sigma_2 \phi_{x_3x_3}\rangle =: J_1 + J_2 \end{aligned} \tag{6.16}$$

for any test function $\phi(x) \in \mathscr{D}(\mathbb{R}^3)$. Hence there is an $R = R(\phi) > 0$ such that $\operatorname{supp}\phi(x) \subset O(0,R)$. We consider J_1 and J_2 separately. The following chain of equalities holds:

$$\begin{aligned} J_1 &= \int_{O(0,R)} dx\, \Delta_{3x}\phi(x) \int_0^t d\tau \int_{\mathbb{R}^3} dy\, \mathscr{E}_1(x-y,t-\tau)\rho(y,\tau) \\ &= \int_0^t d\tau \int_{O(0,R)} dx\, \Delta_{3x}\phi(x) \biggl[\int_{O(0,2R)} \mathscr{E}_1(x-y,t-\tau)\rho(y,\tau)\,dy \\ &\qquad + \int_{\mathbb{R}^3\setminus O(0,2R)} \mathscr{E}_1(x-y,t-\tau)\rho(y,\tau)\,dy\biggr] =: J_{11} + J_{12}, \end{aligned} \tag{6.17}$$

where

$$J_{11} := \int_0^t d\tau \int_{O(0,R)} dx\, \Delta_{3x}\phi(x) \int_{O(0,2R)} \mathscr{E}_1(x-y,t-\tau)\rho(y,\tau)\,dy, \tag{6.18}$$

$$J_{12} := \int_0^t d\tau \int_{O(0,R)} dx\, \Delta_{3x}\phi(x) \int_{\mathbb{R}^3\setminus O(0,2R)} \mathscr{E}_1(x-y,t-\tau)\rho(y,\tau)\,dy. \tag{6.19}$$

Note that integration by parts yields the equality

$$J_{12} = \int_0^t d\tau \int_{O(0,R)} dx\, \phi(x) \int_{\mathbb{R}^3\setminus O(0,2R)} \Delta_{3x}\mathscr{E}_1(x-y,t-\tau)\rho(y,\tau)\,dy. \tag{6.20}$$

Consider J_2. We have

$$\begin{aligned}
J_2 &= \langle V(x,t), \sigma_1\Delta_{2x}\phi(x) + \sigma_2\phi_{x_3x_3}\rangle \\
&= \int_{O(0,R)} dx\,[\sigma_1\Delta_{2x}\phi(x) + \sigma_2\phi_{x_3x_3}(x)] \int_0^t d\tau \int_{\mathbb{R}^3} \mathscr{E}(x-y,t-\tau)\rho(y,\tau)\,dy \\
&= \int_0^t d\tau \int_{O(0,R)} dx\,[\sigma_1\Delta_{2x}\phi(x) + \sigma_2\phi_{x_3x_3}(x)] \\
&\qquad \times \left[\int_{O(0,2R)} \mathscr{E}(x-y,t-\tau)\rho(y,\tau)\,dy + \int_{\mathbb{R}^3\setminus O(0,2R)} \mathscr{E}(x-y,t-\tau)\rho(y,\tau)\,dy\right] \\
&=: J_{21} + J_{22}, \qquad (6.21)
\end{aligned}$$

where

$$J_{21} := \int_0^t d\tau \int_{O(0,R)} dx\,[\sigma_1\Delta_{2x}\phi(x) + \sigma_2\phi_{x_3x_3}(x)] \times \int_{O(0,2R)} \mathscr{E}(x-y,t-\tau)\rho(y,\tau)\,dy, \qquad (6.22)$$

$$J_{22} := \int_0^t d\tau \int_{O(0,R)} dx\,[\sigma_1\Delta_{2x}\phi(x) + \sigma_2\phi_{x_3x_3}(x)] \times \int_{\mathbb{R}^3\setminus O(0,2R)} \mathscr{E}(x-y,t-\tau)\rho(y,\tau)\,dy. \qquad (6.23)$$

Integration by parts yields that

$$J_{22} = \int_0^t d\tau \int_{O(0,R)} dx\,\phi(x) \int_{\mathbb{R}^3\setminus O(0,2R)} [\sigma_1\Delta_{2x}\mathscr{E}(x-y,t-\tau) + \sigma_2\mathscr{E}_{x_3x_3}(x-y,t-\tau)]\rho(y,\tau)\,dy. \qquad (6.24)$$

It follows from the expressions (6.20) and (6.24) that

$$J_{12} + J_{22} = \int_0^t d\tau \int_{O(0,R)} dx\,\phi(x) \int_{\mathbb{R}^3\setminus O(0,2R)} \left[\Delta_{3x}\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial t} + \sigma_1\Delta_{2x}\mathscr{E}(x-y,t-\tau) + \sigma_2\mathscr{E}_{x_3x_3}(x-y,t-\tau)\right]\rho(y,\tau)\,dy = 0 \qquad (6.25)$$

by the definition of the fundamental solution $\mathscr{E}(x,t)$. In view of (6.18) and (6.22), we have

$$\begin{aligned}
J_{11} + J_{21} &= \int_0^t d\tau \int_{O(0,R)} dx\,\Delta_{3x}\phi(x) \int_{O(0,2R)} \mathscr{E}_1(x-y,t-\tau)\rho(y,\tau)\,dy \\
&\quad + \int_0^t d\tau \int_{O(0,R)} dx\,[\sigma_1\Delta_{2x}\phi(x) + \sigma_2\phi_{x_3x_3}(x)] \int_{O(0,2R)} \mathscr{E}(x-y,t-\tau)\rho(y,\tau)\,dy \\
&= \int_0^t d\tau \int_{O(0,2R)} dy\,\rho(y,\tau) \int_{O(0,3R)} dx \left[\frac{\mathscr{E}(x-y,t-\tau)}{\partial t}\Delta_{3x}\phi(x) \right.\\
&\quad \left. + \sigma_1\mathscr{E}(x-y,t-\tau)\Delta_{2x}\phi(x) + \sigma_2\mathscr{E}(x-y,t-\tau)\phi_{x_3x_3}(x)\right]. \qquad (6.26)
\end{aligned}$$

We consider the expression

$$K := \int_{O(0,3R)} \left[\frac{\mathscr{E}(x-y,t-\tau)}{\partial t} \Delta_{3x}\phi(x) + \sigma_1 \mathscr{E}(x-y,t-\tau)\Delta_{2x}\phi(x) + \sigma_2 \mathscr{E}(x-y,t-\tau)\phi_{x_3x_3}(x) \right] dx \qquad (6.27)$$

separately. Note that there is a limit equality

$$K = \lim_{\varepsilon \to +0} K^\varepsilon, \qquad (6.28)$$

where

$$K^\varepsilon := \int_{O(0,3R)\setminus O(y,\varepsilon)} \left[\frac{\mathscr{E}(x-y,t-\tau)}{\partial t} \Delta_{3x}\phi(x) + \sigma_1 \mathscr{E}(x-y,t-\tau)\Delta_{2x}\phi(x) + \sigma_2 \mathscr{E}(x-y,t-\tau)\phi_{x_3x_3}(x) \right] dx \qquad (6.29)$$

for any $y \in O(0,2R)$ and $\varepsilon \in (0, R/2)$. Integrating by parts in the integral (6.29), we obtain the equality

$$K^\varepsilon = K_1^\varepsilon + K_2^\varepsilon + K_3^\varepsilon, \qquad (6.30)$$

where

$$K_1^\varepsilon = \int_{\partial O(0,3R)\cup \partial O(y,\varepsilon)} \left\{ \frac{\partial \mathscr{E}(x-y,t-\tau)}{\partial t} \, \frac{\partial \phi(x)}{\partial n_x} + \sigma_1 \mathscr{E}(x-y,t-\tau) \left[\frac{\partial \phi(x)}{\partial x_1} \cos(n_x, e_1) + \frac{\partial \phi(x)}{\partial x_2} \cos(n_x, e_2) \right] + \sigma_2 \mathscr{E}(x-y,t-\tau) \, \frac{\partial \phi(x)}{\partial x_3} \cos(n_x, e_3) \right\} dS_x, \qquad (6.31)$$

$$K_2^\varepsilon = -\int_{\partial O(0,3R)\cup \partial O(y,\varepsilon)} \left[\frac{\partial^2 \mathscr{E}(x-y,t-\tau)}{\partial t \, \partial n_x} + \sigma_1 \, \frac{\partial \mathscr{E}(x-y,t-\tau)}{\partial x_1} \cos(n_x, e_1) + \sigma_1 \, \frac{\partial \mathscr{E}(x-y,t-\tau)}{\partial x_2} \cos(n_x, e_2) + \sigma_2 \, \frac{\partial \mathscr{E}(x-y,t-\tau)}{\partial x_3} \cos(n_x, e_3) \right] \phi(x) \, dS_x, \qquad (6.32)$$

$$K_3^\varepsilon = \int_{O(0,3R)\setminus O(y,\varepsilon)} \phi(x) \mathfrak{M}_{x,t}[\mathscr{E}](x-y,t-\tau) \, dx = 0, \qquad (6.33)$$

since it follows from the definition of the fundamental solution $\mathscr{E}(x-y,t-\tau)$ that

$$\mathfrak{M}_{x,t}[\mathscr{E}](x-y,t) = 0 \quad \text{for all} \quad (x,t) \in (O(0,3R) \setminus O(y,\varepsilon)) \times [0,T].$$

Moreover,

$$\int_{\partial O(0,3R)} \left\{ \frac{\partial \mathscr{E}(x-y,t-\tau)}{\partial t} \, \frac{\partial \phi(x)}{\partial n_x} + \sigma_1 \mathscr{E}(x-y,t-\tau) \left[\frac{\partial \phi(x)}{\partial x_1} \cos(n_x, e_1) + \frac{\partial \phi(x)}{\partial x_2} \cos(n_x, e_2) \right] \right.$$

$$+\,\sigma_2\mathscr{E}(x-y,t-\tau)\,\frac{\partial\phi(x)}{\partial x_3}\cos(n_x,e_3)\biggr\}\,dS_x=0$$

since $\operatorname{supp}\phi(x)\subset O(0,R)$. We also have

$$\begin{aligned}\int_{\partial O(y,\varepsilon)}\biggl\{&\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial t}\,\frac{\partial\phi(x)}{\partial n_x}\\&+\sigma_1\mathscr{E}(x-y,t-\tau)\biggl[\frac{\partial\phi(x)}{\partial x_1}\cos(n_x,e_1)+\frac{\partial\phi(x)}{\partial x_2}\cos(n_x,e_2)\biggr]\\&+\sigma_2\mathscr{E}(x-y,t-\tau)\,\frac{\partial\phi(x)}{\partial x_3}\cos(n_x,e_3)\biggr\}\,dS_x\to 0\end{aligned}$$

as $\varepsilon\to+0$ for every fixed $y\in O(0,2R)$ since $\phi(x)\in\mathbb{C}_0^\infty(O(0,3R))$ and the fundamental solution $\mathscr{E}(x,t)$ satisfies the bounds (5.5) while the surface area of the sphere $\partial O(y,\varepsilon)$ is equal to $2\pi\varepsilon^2$. Hence we have

$$\lim_{\varepsilon\to+0}K_1^\varepsilon=0.\tag{6.34}$$

Finally, since $\phi(x)=0$ on $\partial O(0,3R)$, the expression for K_2^ε reduces to the integral

$$\begin{aligned}K_2^\varepsilon=-\int_{\partial O(y,\varepsilon)}\biggl[&\frac{\partial^2\mathscr{E}(x-y,t-\tau)}{\partial t\,\partial n_x}+\sigma_1\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_1}\cos(n_x,e_1)\\&+\sigma_1\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_2}\cos(n_x,e_2)+\sigma_2\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_3}\cos(n_x,e_3)\biggr]\phi(x)\,dS_x,\end{aligned}$$

which can be rewritten in the form

$$\begin{aligned}K_2^\varepsilon=-\phi(y)\int_{\partial O(y,\varepsilon)}\biggl[&\frac{\partial^2\mathscr{E}(x-y,t-\tau)}{\partial t\,\partial n_x}+\sigma_1\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_1}\cos(n_x,e_1)\\&+\sigma_1\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_2}\cos(n_x,e_2)+\sigma_2\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_3}\cos(n_x,e_3)\biggr]\,dS_x\\+\int_{\partial O(y,\varepsilon)}\biggl[&\frac{\partial^2\mathscr{E}(x-y,t-\tau)}{\partial t\,\partial n_x}+\sigma_1\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_1}\cos(n_x,e_1)\\&+\sigma_1\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_2}\cos(n_x,e_2)+\sigma_2\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_3}\cos(n_x,e_3)\biggr]\\&\times[\phi(y)-\phi(x)]\,dS_x\\=:{}&K_{21}^\varepsilon+K_{22}^\varepsilon.\end{aligned}$$

Note that

$$|\phi(x)-\phi(y)|\leqslant a(y,\varepsilon)|x-y|\quad\text{for all}\quad x\in O(y,\varepsilon).$$

Hence, in view of the bounds (5.5) for the fundamental solution $\mathscr{E}(x,t)$, we arrive at the limit property

$$\lim_{\varepsilon\to+0}K_{22}^\varepsilon=0.$$

Notice that

$$\int_{\partial O(y,\varepsilon)}\biggl[\frac{\partial^2\mathscr{E}(x-y,t-\tau)}{\partial t\,\partial n_x}+\sigma_1\,\frac{\partial\mathscr{E}(x-y,t-\tau)}{\partial x_1}\cos(n_x,e_1)$$

$$+ \sigma_1 \frac{\partial \mathscr{E}(x-y,t-\tau)}{\partial x_2} \cos(n_x, e_2) + \sigma_2 \frac{\partial \mathscr{E}(x-y,t-\tau)}{\partial x_3} \cos(n_x, e_3) \Big] \, dS_x = 0,$$

which can be verified using the Laplace transform. Therefore,

$$\lim_{\varepsilon \to +0} K_2^\varepsilon = 0. \tag{6.35}$$

Thus, in view of the limit properties (6.33), (6.35) and the equality (6.33), we conclude from (6.30) that

$$\lim_{\varepsilon \to +0} K^\varepsilon = 0 \tag{6.36}$$

and, therefore, $K = 0$ by (6.28). Hence it follows from (6.26) that

$$J_{11} + J_{21} = 0.$$

Therefore, by (6.16), we have

$$\langle \Delta_{3x} W_1(x,t) + \sigma_1 \Delta_{2x} W_2(x,t) + \sigma_2 W_{2x_3x_3}(x,t), \phi(x) \rangle = 0$$

for all $\phi(x) \in \mathscr{D}(\mathbb{R}^3)$. In view of (6.15), we arrive at (6.6). □

Lemma 11. *For any density $\rho_0(x) \in \mathbb{C}([0,T]; \mathbb{C}((1+|x|^2)^\alpha; \mathbb{R}^3))$ with $\alpha > 3/2$, the non-classical volume potential $V_0(x,t)$ (defined in (6.3)) belongs to $\mathbb{C}^{(1)}([0,T]; W_2)$ for every $T > 0$. Moreover, we have*

$$\langle \mathfrak{M}_{x,t} V_0(x,t), \phi(x) \rangle = 0 \quad \textit{for} \quad t \in [0, +\infty) \tag{6.37}$$

for all $\phi(x) \in \mathscr{D}(\mathbb{R}^3)$.

Proof. Repeat *verbatim* the corresponding part of the proof of Lemma 10. □

Lemma 12. *Suppose that $u_0(x) \in \mathbb{C}^2(\mathbb{R}^3)$ has the following properties:*

$$|u_0(x)| \leqslant \frac{A_1}{(1+|x|^2)^{1/2}}, \qquad |\nabla u_0(x)| \leqslant \frac{A_2}{1+|x|^2},$$
$$|\Delta_3 u_0(x)| \leqslant \frac{A_3}{(1+|x|^2)^\alpha}, \quad \alpha > \frac{3}{2}.$$

Then

$$-\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \Delta_3 u_0(y) \, dy = u_0(x). \tag{6.38}$$

Proof. This can be proved by applying to $u_0(x)$ Green's third formula for the Laplace operator in $O(0,R)$ and then letting $R \to +\infty$ and using the inequalities in the hypotheses of the lemma. □

We can now prove the following assertion.

Lemma 13. *For every function $u_0(x)$ satisfying the hypotheses of Lemma 12 and for every point $x \in \mathbb{R}^3$ we have*

$$V_0[\Delta_3 u_0(x)](x,0) = u_0(x). \tag{6.39}$$

Proof. Note that the following representation holds for every point $x \in \mathbb{R}^3$:

$$V_0[\Delta_3 u_0](x,0) = -\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \Delta_3 u_0(y)\, dy.$$

Now use Lemma 12. □

We finally conclude from Lemmas 10 and 11 that, by the integral equation (5.1),

$$u(x,t) = V[|\nabla u|^q](x,t) + V_0[\Delta_3 u_0](x,t) \in \mathbb{C}^{(1)}([0,T]; W_2) \quad \text{for every} \quad T \in (0,T_0).$$

Therefore the following assertion holds.

Lemma 14. *For any function $u_0(x) \in \mathbb{C}^{(2)}(\mathbb{R}^3)$ satisfying the inequalities (6.1) and (6.2), the solution of the integral equation (5.1) belongs to*

$$\mathbb{C}^{(1)}([0,T]; W_2) \quad \textit{for all} \quad T \in (0,T_0). \tag{6.40}$$

Step 2. Relation of the constructed solution to local weak solutions of the Cauchy problem. Note that $u(x,t) \in \mathbb{C}([0,T]; W_2)$ for every $t \in (0,T_0)$ by Theorem 3. Hence,

$$\rho(x,t) := |\nabla u|^q \in \mathbb{C}([0,T]; \mathbb{C}_b((1+|x|^2)^q; \mathbb{R}^3)), \qquad q > \frac{3}{2}. \tag{6.41}$$

In view of Lemmas 10 and 11 and the explicit form of the integral equation (5.1) we have

$$\langle \mathfrak{M}_{x,t}[u](x,t), \phi(x) \rangle = \langle |\nabla u(x,t)|^q, \phi(x) \rangle \quad \text{for all} \quad \phi(x) \in \mathscr{D}(\mathbb{R}^3),$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket between $\mathscr{D}(\mathbb{R}^3)$ and $\mathscr{D}'(\mathbb{R}^3)$. Moreover, note that the test function $\phi(x)$ may depend on $t \in [0,T]$ as a parameter. Thus we have actually proved that

$$\langle \mathfrak{M}_{x,t}[u](x,t), \phi(x,t) \rangle = \langle |\nabla u(x,t)|^q, \phi(x,t) \rangle \quad \text{for} \quad t \in [0,T] \tag{6.42}$$

and for all $\phi(x,t) \in \mathbb{C}^{\infty;1}_{x,t}(\mathbb{R}^3 \times [0,T])$ satisfying the conditions in the definition (4.3) of a weak solution. Observe that since $|\nabla u(x,t)|^q \in \mathbb{C}([0,T]; \mathbb{C}_b((1+|x|^2)^q; \mathbb{R}^3))$, we have

$$\langle |\nabla u(x,t)|^q, \phi(x,t) \rangle = \int_{\mathbb{R}^3} |\nabla u(x,t)|^q \phi(x,t)\, dx \quad \text{for all} \quad t \in [0,T]. \tag{6.43}$$

Moreover,

$$\begin{aligned}
\langle \mathfrak{M}_{x,t}[u](x,t), \phi(x,t) \rangle &= \left\langle \Delta_{3x} \frac{\partial u(x,t)}{\partial t} + \sigma_1 \Delta_{2x} u(x,t) + \sigma_2 u_{x_3 x_3}(x,t), \phi(x,t) \right\rangle \\
&= -\sum_{j=1}^{3} \left\langle \frac{\partial^2 u(x,t)}{\partial x_j\, \partial t}, \frac{\partial \phi(x,t)}{\partial x_j} \right\rangle - \sigma_1 \sum_{j=1}^{2} \left\langle \frac{\partial u(x,t)}{\partial x_j}, \frac{\partial \phi(x,t)}{\partial x_j} \right\rangle \\
&\quad - \sigma_2 \left\langle \frac{\partial u(x,t)}{\partial x_3}, \frac{\partial \phi(x,t)}{\partial x_3} \right\rangle.
\end{aligned} \tag{6.44}$$

We also have

$$\sum_{j=1}^{3}\left\langle \frac{\partial^2 u(x,t)}{\partial x_j\,\partial t}, \frac{\partial \phi(x,t)}{\partial x_j}\right\rangle = \int_{\mathbb{R}^3}\left(\nabla\frac{\partial u(x,t)}{\partial t}, \nabla\phi(x,t)\right)dx$$
$$= \int_{\mathbb{R}^3}\left(\frac{\partial \nabla u(x,t)}{\partial t}, \nabla\phi(x,t)\right)dx$$
$$= \int_{\mathbb{R}^3}\frac{\partial}{\partial t}(\nabla u(x,t), \nabla\phi(x,t))\,dx - \int_{\mathbb{R}^3}(\nabla u(x,t), \nabla\phi'_t(x,t))\,dx, \qquad (6.45)$$

$$\sum_{j=1}^{2}\left\langle \frac{\partial u(x,t)}{\partial x_j}, \frac{\partial \phi(x,t)}{\partial x_j}\right\rangle = \int_{\mathbb{R}^3}[u_{x_1}(x,t)\phi_{x_1}(x,t) + u_{x_2}(x,t)\phi_{x_2}(x,t)]\,dx, \qquad (6.46)$$

$$\left\langle \frac{\partial u(x,t)}{\partial x_3}, \frac{\partial \phi(x,t)}{\partial x_3}\right\rangle = \int_{\mathbb{R}^3} u_{x_3}(x,t)\phi_{x_3}(x,t)\,dx \quad \text{for} \quad t\in[0,T] \qquad (6.47)$$

and for all $\phi(x,t)\in \mathbb{C}^{\infty;1}_{x,t}(\mathbb{R}^3\times[0,T])$ satisfying the conditions in the definition (4.3) of a weak solution.

Integrating both sides of (6.45) with respect to $t\in[0,T]$, we find that

$$\int_0^T \sum_{j=1}^{3}\left\langle \frac{\partial^2 u(x,t)}{\partial x_j\,\partial t}, \frac{\partial \phi(x,t)}{\partial x_j}\right\rangle dt$$
$$= -\int_{\mathbb{R}^3}(\nabla u_0(x), \nabla\phi(x,0))\,dx - \int_0^T\int_{\mathbb{R}^3}(\nabla u(x,t), \nabla\phi'_t(x,t))\,dx\,dt \qquad (6.48)$$

for the test functions $\phi(x,t)$ in the definition (4.3) of a weak solution. In particular, $\phi(x,T)=0$.

Integrating both sides of (6.42) with respect to $t\in[0,T]$, we obtain (4.3) in view of (6.45)–(6.48). Thus, for $q>3/2$ and for arbitrary initial functions $u_0(x)$ satisfying the hypotheses of the theorem, the Cauchy problem has at least one local weak solution in the sense of Definition 1. □

Remark 1. The question of the uniqueness of a local weak solution of the Cauchy problem for $q>3/2$ is still open.

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