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Blow-up and global solubility in the classical sense of the Cauchy problem for a formally hyperbolic equation with a non-coercive source

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Abstract. We consider an abstract Cauchy problem with non-linear operator coefficients and prove the existence of a unique non-extendable classical solution. Under certain sufficient close-to-necessary conditions, we obtain finite-time blow-up conditions and upper and lower bounds for the blow-up time. Moreover, under certain sufficient close-to-necessary conditions, we obtain a result on the existence of a global-in-time solution independently of the size of the initial functions.

Keywords: non-linear Sobolev-type equations, blow-up, local solubility, non-linear capacity, bounds for the blow-up time.

§ 1. Introduction

In the two classical papers [1] and [2], published in 1973 and 1974, Levine suggested a new energy method for studying the occurrence of blow-up in two Cauchy problems for the abstract formally parabolic and formally hyperbolic equations

$$Pu_t = -Au + F(u), \qquad u(0) = u_0, \tag{1.1}$$

$$Pu_{tt} = -Au + F(u), \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{1.2}$$

where the operators P and A are linear and the operator $F(u)$ is non-linear. Blow-up results wee obtained for classical as well as weak solutions. We also mention Straughan's paper [3].

In 1977 Kalantarov and Ladyzhenskaya published the paper [4], where the energy method was used to solve the following pair of abstract Cauchy problems:

$$Pu_t = -Au + B(u) + F(t,u), \qquad u(0) = u_0, \tag{1.3}$$

$$Pu_{tt} = -Au + B(u) - aPu_t + F(t,u), \qquad u(0) = u_0, \quad u'(0) = u_1 \tag{1.4}$$

with possibly non-linear operators $B(u)$. The following equation was considered in the paper [5] by Levine and Serrin, published in 1997:

$$(P(u_t))_t + A(u) + Q(t,u_t) = F(u), \qquad u(0) = u_0, \quad u'(0) = u_1. \tag{1.5}$$

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The following general equation of formally parabolic type was considered in 1998 (see [6]):

$$Q(t,u,u_t)+A(t,u)=F(t,u), \qquad u(0)=u_0. \tag{1.6}$$

The theme of proving blow-up of solutions of formally hyperbolic equations with positive energy began to develop at the same time. We mention the papers [7] and [8] by Pucci and Serrin in this connection. Recent results in this direction can be found in [9]–[12].

An important advance of the energy method was made in the paper [13] by Georgiev and Todorova on the first initial-boundary value problem for the equation

$$u_{tt}-\Delta u+au_t|u_t|^{m-1}=bu|u|^{p-1} \tag{1.7}$$

in a cylinder $[0,T]\times\Omega$. The result of Georgiev and Todorova was generalized by Messaoudi [14], [15] to the case of a non-linear non-local equation

$$u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)\,ds+|u_t|^{m-2}u_t=|u|^{p-2}u. \tag{1.8}$$

Our paper continues the investigations begun in [16]–[18]. The following abstract Cauchy problem was considered in [18]:

$$A\frac{d^2u}{dt^2}+\frac{d}{dt}\biggl(A_0u+\sum_{j=1}^{n}A_j(u)\biggr)+H'_f(u)=F'_f(u), \quad u(0)=u_0, \quad u'(0)=u_1, \tag{1.9}$$

where $H'_f(u)$ and $F'_f(u)$ are the Fréchet derivatives of non-linear operators. In that paper, we considered the classical and weak solutions of the Cauchy problem (1.9). We proved the existence of non-extendable solutions and obtained sufficient conditions for the finite-time blow-up of solutions. To prove finite-time blow-up, we used our modification (described in [19]) of Levine's energy method.

In the present paper we consider an abstract Cauchy problem of the form

$$\frac{d^2}{dt^2}\biggl(A_0u+\sum_{j=1}^{N}A_j(u)\biggr)+Lu=\frac{d}{dt}F(u), \qquad u(0)=u_0, \quad u'(0)=u_1, \tag{1.10}$$

where the operators A_0 and L are linear while the operators $A_j(u)$ and $F(u)$ are non-linear. Notice that the equation (1.10) contains a non-coercive source

$$\frac{d}{dt}F(u),$$

which considerably complicates the task of finding sufficient conditions for blow-up in the Cauchy problem (1.10). We shall prove the existence of a classical non-extendable (in time) solution of (1.10) under certain conditions on the operator coefficients. We shall obtain sufficient conditions for the finite-time blow-up of solutions, upper and lower bounds for the blow-up time, and sufficient conditions for the existence of a global solution of the problem for arbitrary initial data (not necessarily small).

Note that we have already considered a number of concrete examples of equations in the abstract form (1.10) and in more complicated forms. For example, the following equation arises in the blow-up instability theory of semiconductors [19]:

$$\frac{\partial^2}{\partial t^2}(\Delta u - \varepsilon u) + \Delta u + \frac{\partial |u|^p}{\partial t} = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad p > 1. \tag{1.11}$$

The following equations were suggested in [20] for a study of self-oscillations in systems with distributed parameters on the basis of a tunnel diode with non-linear characteristics:

$$\frac{\partial^2 \phi}{\partial t^2} - c_0^2 \frac{\partial^2 \phi}{\partial x^2} - \beta \frac{\partial^2}{\partial t^2}\frac{\partial^2 \phi}{\partial x^2} = \gamma \frac{\partial}{\partial t}(\phi^3 - \phi), \qquad \beta > 0, \quad \gamma > 0, \tag{1.12}$$

$$\frac{\partial^2 \phi}{\partial t^2} - c_0^2 \frac{\partial^2 \phi}{\partial x^2} - \beta \frac{\partial^2}{\partial t^2}\frac{\partial^2 \phi}{\partial x^2} + \gamma \frac{\partial^2}{\partial t\,\partial x}\biggl(\frac{\partial \phi}{\partial x} - \biggl(\frac{\partial \phi}{\partial x}\biggr)^2\biggr) = 0, \qquad \beta > 0, \quad \gamma > 0, \tag{1.13}$$

where $\phi = \phi(x,t)$ is the electric field potential (see also [21] and [22]). In [23] we obtained conditions for the occurrence of blow–up in the first boundary-value problems on an interval for the equations (1.12) and (1.13). In [24] we obtained rather delicate *a priori* bounds for solutions of the Cauchy problems for equations of Khokhlov–Zabolotskaya type:

$$\frac{\partial^2 u}{\partial x\,\partial t} - \Delta_{y,z} u = \frac{\partial^2 u^2}{\partial t^2}, \qquad \Delta_{y,z} := \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1.14}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u^2}{\partial t^2}, \tag{1.15}$$

$$\frac{\partial^2}{\partial t^2}(\Delta u - \varepsilon u) + \Delta u = \frac{\varepsilon^2}{2}\frac{\partial^2 u^2}{\partial t^2}. \tag{1.16}$$

These bounds enable one to obtain sufficient conditions for the finite-time blow-up of classical solutions along with an instantaneous blow-up result (see also [25]–[35]). Our derivation of *a priori* bounds uses the non-linear capacity method of Pokhozhaev and Mitidieri presented in [36].

§ 2. Conditions on the operator coefficients

Consider Banach spaces V_0, V_j, W_i, where $j = 1, \dots, n$ and $i = 1, 2$, with norms

$$\|\cdot\|_0, \quad \|\cdot\|_j, \quad |\cdot|_i$$

respectively. Let V_0^*, V_j^*, W_i^* be the conjugate Banach spaces with respect to the duality brackets

$$\langle\cdot\,,\cdot\rangle_0, \quad \langle\cdot\,,\cdot\rangle_j, \quad (\cdot\,,\cdot)_i$$

and with norms

$$\|\cdot\|_0^*, \quad \|\cdot\|_j^*, \quad |\cdot|_i^*$$

respectively. Assume that V_0, V_j, W_i are reflexive and separable for $j = 1, \dots, n$ and $i = 1, 2$. Assume also that

$$A_0\colon V_0 \to V_0^*, \qquad A_j\colon V_j \to V_j^*, \qquad L_1\colon W_1 \to W_1^*, \qquad F\colon W_2 \to W_2^*.$$

Conditions $\mathbf{A}_0$. (i) The operator $A_0\colon V_0 \to V_0^*$ is linear, continuous and symmetric. We have

$$\|A_0 u\|_0^* \leqslant M_0 \|u\|_0 \quad \text{for all} \quad u \in V_0.$$

(ii) The operator A_0 is coercive and we have

$$\langle A_0 u, u \rangle_0 \geqslant m_0 \|u\|_0^2 \quad \text{for all} \quad u \in V_0.$$

Remark 2.1. It follows from (i) and (ii) that the quantity $\langle A_0 u, u\rangle_0^{1/2}$ is an equivalent norm on V_0.

Conditions A. (i) The operator $A_j\colon V_j \to V_j^*$ is monotone and continuous.

(ii) The operator A_j is Fréchet differentiable. Its Fréchet derivative

$$A'_{jf}(u)\colon V_j \to \mathscr{L}(V_j, V_j^*)$$

is a continuous symmetric monotone non-negative definite operator for every fixed $u \in V_j$ and $A'_{jf}(0) = 0$.

(iii) The operator A_j is positive homogeneous:

$$A_j(ru) = r^{p_j - 1} A_j(u) \quad \text{for} \quad p_j > 2, \quad r \geqslant 0, \quad u \in V_j.$$

(iv) The following upper and lower bounds hold:

$$\|A_j(u)\|_j^* \leqslant M_j \|u\|_j^{p_j-1}, \qquad \langle A_j(u), u\rangle_j \geqslant m_j \|u\|_j^{p_j}, \quad M_j, m_j > 0.$$

Remark 2.2. It follows from (iv) that the quantity $\langle A_j(u), u\rangle_j^{1/p_j}$ is an equivalent norm on V_j.

Conditions F. (i) The operator $F\colon W_2 \to W_2^*$ is boundedly Lipschitz continuous, that is, we have

$$|F(u_1) - F(u_2)|_2 \leqslant \mu(R)|u_1 - u_2|_2 \quad \text{for all} \quad u_1, u_2 \in W_2,$$

where $R = \max\{|u_1|_2, |u_2|_2\}$ and $\mu = \mu(R)$ is a non-decreasing function bounded on every compact set.

(ii) The operator F is positive homogeneous, that is,

$$F(ru) = r^{1+q} F(u) \quad \text{for} \quad q > 0, \quad r \geqslant 0, \quad u \in W_2.$$

(iii) The operator F has a symmetric Fréchet derivative

$$F'_f(\,\cdot\,)\colon W_2 \to \mathscr{L}(W_2, W_2^*).$$

(iv) The operator F satisfies the upper bound

$$|F(u)|_2^* \leqslant M |u|_2^{q+1} \quad \text{for all} \quad u \in W_2.$$

Conditions L. (i) The operator $L\colon W_1 \to W_1^*$ is linear, continuous and symmetric. We have

$$|Lu|_1^* \leqslant D_1 |u|_1 \quad \text{for all} \quad u \in W_1;$$

(ii) The operator L_1 is coercive and we have

$$(L_1 u, u)_1 \geqslant d_1 |u|_1^2 \quad \text{for all} \quad u \in W_1.$$

Remark 2.3. It follows from (i) and (ii) that the quantity $(L_1 u, u)_1^{1/2}$ is an equivalent norm on W_1.

We now consider the Banach spaces V_0, V_j, W_i, where $j = 1, \dots, n$ and $i = 1, 2$. Let H be a separable Hilbert space identified with its conjugate. Assume that the following conditions hold.

Conditions H. We have chains of dense continuous embeddings

$$V_0 \overset{ds}{\subset} V_j \overset{ds}{\subset} H \overset{ds}{\subset} V_j^* \overset{ds}{\subset} V_0^* \quad \text{for} \quad j = 1, \dots, n,$$
$$V_0 \overset{ds}{\subset} W_i \overset{ds}{\subset} H \overset{ds}{\subset} W_i^* \overset{ds}{\subset} V_0^* \quad \text{for} \quad i = 1, 2.$$

Note that the following properties hold in view of the conditions H:

$$\langle f, u \rangle_0 = \langle f, u \rangle_j \quad \text{for all} \quad f \in V_j^*, \ u \in V_0, \quad \text{and} \quad j = 1, \dots, n, \tag{2.1}$$
$$\langle f, u \rangle_0 = (f, u)_i \quad \text{for all} \quad f \in W_i^*, \ u \in V_0, \quad \text{and} \quad i = 1, 2. \tag{2.2}$$

§ 3. Auxiliary results

In this section we list a number of results which are needed in the main text. We shall prove them in the necessary general form.

Lemma 3.1. *Let $A\colon X \to X^*$ be a Fréchet differentiable operator with a symmetric Fréchet derivative*

$$A'_u(u)\colon X \to \mathcal{L}(X, X^*)$$

and with $A(su) = s^{p-1}A(u)$ for $s \geqslant 0$ and $p \geqslant 2$, where X is a Banach space with conjugate X^* with respect to the duality brackets $\langle \cdot, \cdot \rangle$. Then the functional

$$\psi(u) \equiv \langle A(u), u \rangle\colon X \to R^1$$

is continuously Fréchet differentiable and its Fréchet derivative is

$$\psi'_f(u) = pA(u) \quad \textit{for all} \quad u \in X.$$

Proof. We claim that the following equality of operators holds:

$$A'_u(u)u = (p-1)A(u).$$

Indeed, on the one hand (by the condition A, (iii)) we have

$$\frac{d}{ds}A(su) = \frac{d}{ds}(s^{p-1}A(u)) = (p-1)s^{p-2}A(u) = \frac{p-1}{s}s^{p-1}A(u) = \frac{p-1}{s}A(su),$$

and on the other hand, by the chain rule for Fréchet derivatives,

$$\frac{d}{ds}A(su) = A'_u(su)u.$$

Combining these equalities, we deduce that

$$\frac{p-1}{s}A(su) = A'_u(su)u.$$

Putting $s = 1$, we obtain the required equality

$$(p-1)A(u) = A'_u(u)u \quad \forall\, u \in X.$$

Consider the following chain of calculations:

$$\begin{aligned}\psi(u+h) - \psi(u) &= \langle A(u+h), u+h\rangle - \langle A(u), u\rangle \\ &= \langle A(u) + A'_u(u)h + \omega(u,h), u+h\rangle - \langle A(u), u\rangle \\ &= \langle A(u), h\rangle + \langle A'_u(u)h + \omega(u,h), u+h\rangle \\ &= \langle A(u), h\rangle + \langle A'_u(u)h, u\rangle + \langle A'_u(u)h, h\rangle + \langle \omega(u,h), u+h\rangle \\ &= \langle A(u) + A'_u(u)u, h\rangle + \overline{\omega}(u,h),\end{aligned}$$

where

$$\overline{\omega}(u,h) = \langle A'_u(u)h, h\rangle + \langle \omega(u,h), u+h\rangle.$$

Moreover,

$$|\overline{\omega}(u,h)| \leqslant \|A'_u(u)h\|_*\|h\| + \|\omega(u,h)\|_*[\|u\| + \|h\|] \leqslant c_1\|h\|^2 + \|\omega(u,h)\|_*[\|u\| + \|h\|].$$

Finally,

$$\lim_{\|h\|\to 0} \frac{|\overline{\omega}(u,h)|}{\|h\|} = 0.$$

Hence the Fréchet derivative of the functional $\psi(u)$ is

$$\psi'_f(u) = A(u) + A'_u(u)u = A(u) + (p-1)A(u) = pA(u). \tag{3.1}$$

It follows that $\psi(u)$ is continuously Fréchet differentiable. □

Lemma 3.2. *Suppose that all the hypotheses of Lemma 3.1 hold and we have $u(t) \in C^{(1)}([0,T];X)$ for some $T > 0$. Then the functional*

$$\psi(u)(t) \equiv \langle A(u), u\rangle \quad \textit{belongs to} \quad C^{(1)}([0,T]).$$

Proof. First of all, by Lemma 3.1 and the chain rule for Fréchet derivatives, we have

$$\frac{d\psi}{dt} = \langle \psi'_f(u), u'\rangle = p\langle A(u), u'\rangle.$$

Consider the function

$$f(t) \equiv \langle A(u), u'\rangle.$$

We claim that $f(t) \in C([0,T])$. Indeed, let $t \in [0,T]$ be fixed and $t + s \in [0,T]$. Then

$$\begin{aligned}f(t+s) - f(t) &= \langle A(u(t+s)), u'(t+s)\rangle - \langle A(u(t)), u'(t)\rangle \\ &= \langle A(u(t)) + A'_u(u(t))[u(t+s) - u(t)] + \omega(t,s), u'(t+s)\rangle \\ &\quad - \langle A(u(t)), u'(t)\rangle = \langle A(u(t)), u'(t+s) - u'(t)\rangle \\ &\quad + \langle A'_u(u(t))[u(t+s) - u(t)], u'(t+s)\rangle + \langle \omega(t,s), u'(t+s)\rangle.\end{aligned}$$

By this chain of equalities we arrive at the inequality

$$|f(t+s)-f(t)| \leqslant \|A(u(t))\|_* \, \|u'(t+s)-u'(t)\| + \|A'_u(u(t))\|_{\mathcal{L}(X;X^*)} \, \|u(t+s)-u(t)\| \, \|u'(t+s)\| + \|\omega(t,s)\|_* \, \|u'(t+s)\|.$$

Note that

$$\|u'(t+s)\| \leqslant \|u'(t)\| + \|u'(t+s)-u'(t)\| \leqslant c_1,$$

where $c_1 > 0$ is independent of $t, s \in [0,T]$. Thus we arrive at the bound

$$|f(t+s)-f(t)| \leqslant c_2\|u'(t+s)-u'(t)\| + c_3\|u(t+s)-u(t)\| + c_4\|\omega(t,s)\|_*,$$

where $c_2, c_3, c_4 \in (0,+\infty)$ depend only on $t \in [0,T]$. Moreover,

$$\lim_{\|u(t+s)-u(t)\|\to 0} \|\omega(t,s)\|_* = 0.$$

Thus we conclude that

$$\lim_{s\to 0} f(t+s) = f(t).$$

It follows that $f(t) \in C[0,T]$. □

Lemma 3.3. *Suppose that all the hypotheses of Lemma 3.1 hold and we have $u(t) \in C^{(1)}([0,T];X)$ for some $T > 0$. Then*

$$\langle (A(u))', u\rangle = \frac{p-1}{p}\frac{d}{dt}\langle A(u), u\rangle. \tag{3.2}$$

Proof. By Lemma 3.1 we find that

$$\frac{d}{dt}\langle A(u), u\rangle = p\langle A(u), u'\rangle. \tag{3.3}$$

This yields the equality

$$p\langle A(u), u'\rangle = \langle (A(u))', u\rangle + \langle A(u), u'\rangle,$$

which implies that

$$\langle (A(u))', u\rangle = (p-1)\langle A(u), u'\rangle.$$

Hence we have

$$\langle A(u), u'\rangle = \frac{1}{p-1}\langle (A(u))', u\rangle. \tag{3.4}$$

The desired equality (3.2) follows from (3.3) and (3.4). □

Lemma 3.4. *We have*

$$|(Lu,v)_1| \leqslant l\langle A_0u,u\rangle_0^{1/2}\langle A_0v,v\rangle_0^{1/2} \leqslant \frac{\varepsilon}{2}\langle A_0v,v\rangle_0 + \frac{l^2}{2\varepsilon}\langle A_0u,u\rangle_0 \quad \textit{for all} \quad u,v \in V_0, \quad \varepsilon > 0, \tag{3.5}$$

where $l > 0$ is a constant.

Proof. On the one hand, by the condition L , (ii) we arrive at the Schwarz inequality

$$|(Lu,v)_1| \leqslant (Lu,u)_1^{1/2}(Lv,v)_1^{1/2}.$$

On the other hand, by the conditions H we have a continuous embedding $V_0 \subset W_1$ of Banach spaces. Finally, by the condition A_0(ii), the quantity $\langle A_0 u,u\rangle_0^{1/2}$ is a norm on V_0. Hence we have

$$(Lu,u)_1^{1/2} \leqslant l^{1/2}\langle A_0 u,u\rangle_0^{1/2} \quad \text{for all} \quad u \in V_0.$$

It remains to use the Cauchy–Bunyakovskii inequality with some ε. □

§ 4. Solution of a differential inequality

In this section we obtain a lower bound for the functional $\Phi(t) \in \mathbb{C}^{(2)}[0,T]$ satisfying the integro-differential inequality

$$\Phi\Phi'' - \alpha(\Phi')^2 + \beta\Phi^2 + \gamma_1\Phi(t) + \gamma_2 T\int_0^t \Phi(s)\,ds\,\Phi(t) \geqslant 0 \quad \text{for} \quad t \in [0,T] \tag{4.1}$$

where $\alpha > 1$ and $\beta \geqslant 0$, $\gamma_1 \geqslant 0$, $\gamma_2 \geqslant 0$. Suppose that

$$\Phi'(0) > 0, \qquad \Phi(0) > 0. \tag{4.2}$$

Then there is a $t_1 \in (0,T)$ such that

$$\Phi(t) > 0, \quad \Phi'(t) \geqslant 0 \quad \text{for all} \quad t \in [0,t_1]. \tag{4.3}$$

The following relations hold:

$$\int_0^t \Phi(s)\,ds = s\Phi(s)\Big|_{s=0}^{s=t} - \int_0^t s\Phi'(s)\,ds \leqslant T\Phi(t) \quad \text{for all} \quad t \in [0,t_1]. \tag{4.4}$$

Thus, in view of (4.4), we can deduce the following differential inequality from (4.1):

$$\Phi\Phi'' - \alpha(\Phi')^2 + [\beta + \gamma_2 T^2]\Phi^2 + \gamma_1\Phi \geqslant 0 \quad \text{for} \quad t \in [0,t_1]. \tag{4.5}$$

Dividing both sides of (4.5) by $\Phi^{1+\alpha}(t)$, we obtain the inequality

$$\frac{\Phi''}{\Phi^\alpha} - \alpha\frac{(\Phi')^2}{\Phi^{1+\alpha}} + [\beta + \gamma_2 T^2]\Phi^{1-\alpha} + \gamma_1\Phi^{-\alpha} \geqslant 0 \quad \text{for all} \quad t \in [0,t_1]. \tag{4.6}$$

We introduce a new function

$$Z(t) := \Phi^{1-\alpha}(t), \qquad \alpha > 1. \tag{4.7}$$

Then (4.6) yields that

$$Z''(t) \leqslant (\alpha-1)(\beta + \gamma_2 T^2)Z(t) + (\alpha-1)\gamma_1 Z^{\alpha/(\alpha-1)}(t) \quad \text{for} \quad t \in [0,t_1]. \tag{4.8}$$

Note that

$$Z'(t) = (1-\alpha)\Phi^{-\alpha}(t)\Phi'(t) \leqslant 0 \quad \text{for} \quad t \in [0,t_1]. \tag{4.9}$$

Multiplying both sides of (4.8) by $Z'(t)$, we obtain the inequality

$$Z'(t)Z''(t) \geqslant (\alpha-1)(\beta+\gamma_2 T^2)Z(t)Z'(t) + (\alpha-1)\gamma_1 Z^{\alpha/(\alpha-1)}(t)Z'(t) \tag{4.10}$$

for $t \in [0, t_1]$, which may be rewritten in the form

$$\frac{1}{2}\frac{d}{dt}(Z')^2 \geqslant \frac{(\alpha-1)(\beta+\gamma_2 T^2)}{2}\frac{d}{dt}Z^2(t) + \frac{(\alpha-1)^2\gamma_1}{2\alpha-1}\frac{d}{dt}Z^{(2\alpha-1)/(\alpha-1)}(t) \tag{4.11}$$

for $t \in [0, t_1]$. Integrating this inequality with respect to time, we have

$$(Z'(t))^2 \geqslant A + (\alpha-1)(\beta+\gamma_2 T^2)Z^2(t) + \frac{2(\alpha-1)^2\gamma_1}{2\alpha-1}Z^{(2\alpha-1)/(\alpha-1)}(t) \tag{4.12}$$

for $t \in [0, t_1]$, where

$$A := (Z'(0))^2 - (\alpha-1)(\beta+\gamma_2 T^2)Z^2(0) - \frac{2(\alpha-1)^2\gamma_1}{2\alpha-1}Z^{(2\alpha-1)/(\alpha-1)}(0). \tag{4.13}$$

To go further, we will need the inequality

$$A > 0. \tag{4.14}$$

In view of (4.7), (4.9) and (4.13), it is equivalent to the inequality

$$(\Phi'(0))^2 > \frac{\beta+\gamma_2 T^2}{\alpha-1}\Phi^2(0) + \frac{2\gamma_1}{2\alpha-1}\Phi(0). \tag{4.15}$$

Suppose that (4.15) holds and, therefore, $A > 0$. It follows from (4.12) that

$$(Z'(t))^2 > A > 0 \quad \text{for} \quad t \in [0, t_1]. \tag{4.16}$$

The inequalities (4.16) and (4.9) yield the following chain of inequalities for $t \in [0, t_1]$:

$$|Z'(t)| \geqslant A^{1/2} \;\Rightarrow\; Z'(t) \leqslant -A^{1/2} < 0 \;\Rightarrow\; (1-\alpha)\Phi^{-\alpha}(t)\Phi'(t) \leqslant -A^{1/2} < 0$$
$$\Rightarrow\; \Phi'(t) \geqslant \frac{A^{1/2}}{\alpha-1}\Phi^{\alpha}(t). \tag{4.17}$$

Since $\Phi'(t) \geqslant 0$ for $t \in [0, t_1]$, we have

$$\Phi(t) \geqslant \Phi(0) > 0 \quad \text{for} \quad t \in [0, t_1].$$

Using this and (4.17), we arrive at the inequality

$$\Phi'(t) \geqslant \frac{A^{1/2}}{\alpha-1}\Phi^{\alpha}(t) \geqslant \frac{A^{1/2}}{\alpha-1}\Phi^{\alpha}(0) > 0. \tag{4.18}$$

Thus, in particular, $\Phi'(t_1) > 0$. Therefore, repeating the arguments, we obtain that

$$\Phi'(t) \geqslant \frac{A^{1/2}}{\alpha-1}\Phi^{\alpha}(0) > 0 \quad \text{for} \quad t \in [0, T]. \tag{4.19}$$

Using this and (4.9), we conclude that

$$Z'(t) < 0 \quad \text{for} \quad t \in [0, T].$$

Therefore from the inequality (4.16), which holds for $t \in [0, T]$, we arrive at the inequality

$$\begin{aligned} Z(t) \leqslant Z(0) - A^{1/2}t \ \Rightarrow \ \Phi^{1-\alpha}(t) \leqslant \Phi^{1-\alpha}(0) - A^{1/2} \\ \Rightarrow \ \Phi(t) \geqslant \frac{1}{[\Phi^{1-\alpha}(0) - A^{1/2}t]^{1/(\alpha-1)}}, \end{aligned} \tag{4.20}$$

which is obtained under the conditions (4.2), (4.15) and (4.16). We now require that the following equality holds:

$$A^{1/2}T = \Phi^{1-\alpha}(0). \tag{4.21}$$

This equality can be rewritten in the form

$$(\Phi'(0))^2 = \frac{1}{T^2(\alpha-1)^2}(\Phi(0))^2 + \frac{\beta + \gamma_2 T^2}{\alpha - 1}(\Phi(0))^2 + \frac{2\gamma_1}{2\alpha - 1}\Phi(0). \tag{4.22}$$

Generally speaking, this equation may have four roots. We are interested only in the smallest positive root $T = T_1 > 0$. Thus we have proved the following assertion.

Theorem 4.1. *Suppose that $\Phi(t) \in \mathbb{C}^{(2)}[0, T_0)$ satisfies the differential inequality (4.1) and*

$$\Phi(0) > 0, \qquad \Phi'(0) > 0, \qquad \alpha > 1, \tag{4.23}$$

where the initial conditions $\Phi(0)$ and $\Phi'(0)$ are such that there exists a smallest positive root T_1 of the equation

$$(\Phi'(0))^2 = \frac{1}{T_1^2(\alpha-1)^2}(\Phi(0))^2 + \frac{\beta + \gamma_2 T_1^2}{\alpha - 1}(\Phi(0))^2 + \frac{2\gamma_1}{2\alpha - 1}\Phi(0). \tag{4.24}$$

Then $\Phi(t)$ satisfies the inequality

$$\Phi(t) \geqslant \frac{1}{[\Phi^{1-\alpha}(0) - A^{1/2}t]^{1/(\alpha-1)}} \tag{4.25}$$

for all $t \in [0, T_0)$ and $T_0 \leqslant T_1 < +\infty$, where

$$A := (\alpha-1)^2\Phi^{-2\alpha}(0)\biggl[(\Phi'(0))^2 - \frac{\beta + \gamma_2 T_1^2}{\alpha - 1}(\Phi(0))^2 - \frac{2\gamma_1}{2\alpha - 1}\Phi(0)\biggr] > 0. \tag{4.26}$$

§ 5. Statement of the problem

Suppose that all the conditions stated in § 2 hold. Consider the following Cauchy problem for a second-order abstract differential equation:

$$\frac{d^2}{dt^2}\biggl(A_0 u + \sum_{j=1}^{n} A_j(u)\biggr) + Lu = \frac{d}{dt}F(u), \qquad u(0) = u_1, \quad u'(0) = u_1. \tag{5.1}$$

We give a definition of a classical solution of this abstract Cauchy problem.

Definition 5.1. A function $u(t) \in \mathbb{C}^{(2)}([0,T];V_0)$ is called a classical solution of the Cauchy problem (5.1) if

$$\frac{d^2}{dt^2}A_j(u) \in \mathbb{C}([0,T];V_0^*) \quad \text{for all} \quad j=1,\dots,n, \tag{5.2}$$

the equality (5.1) holds for every $t \in [0,T]$ and is understood in the sense of the Banach space V_0^*, and

$$u_0 \in V_0, \qquad u_1 \in V_0. \tag{5.3}$$

Let $u(t) \in \mathbb{C}^{(2)}([0,T];V_0)$ be a classical solution of (5.1). Let $\phi(t) \in \mathbb{C}[0,T]$ be an arbitrary function. Consider the function

$$\psi(t) := \int_t^T \phi(s)\,ds \in \mathbb{C}^{(1)}[0,T], \qquad t \in [0,T]. \tag{5.4}$$

Note that $\psi(T)=0$ and $\psi'(t) = -\phi(t)$. One has the following integration-by-parts formulae for Bochner integrals in V_0^*:

$$\begin{aligned}
&\int_0^T \frac{d^2}{dt^2}\biggl(A_0u(t)+\sum_{j=1}^n A_j(u)(t)\biggr)\psi(t)\,dt \\
&= \frac{d}{dt}\biggl(A_0u(t)+\sum_{j=1}^n A_j(u)(t)\biggr)\psi(t)\bigg|_{t=0}^{t=T} + \int_0^T \frac{d}{dt}\biggl(A_0u(t)+\sum_{j=1}^n A_j(u)(t)\biggr)\phi(t)\,dt \\
&= -\biggl(A_0u_1+\sum_{j=1}^n A_j'(u_0)u_1\biggr)\int_0^T \phi(t)\,dt + \int_0^T \frac{d}{dt}\biggl(A_0u(t)+\sum_{j=1}^n A_j(u)(t)\biggr)\phi(t)\,dt,
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
\int_0^T Lu(t)\psi(t)\,dt &= \int_0^t Lu(s)\,ds\,\psi(t)\bigg|_{t=0}^{t=T} + \int_0^T \phi(t)\int_0^t Lu(s)\,ds\,dt \\
&= \int_0^T \phi(t)\int_0^t Lu(s)\,ds\,dt,
\end{aligned} \tag{5.6}$$

$$\int_0^T \frac{d}{dt}F(u)(t)\psi(t)\,dt = -F(u_0)\int_0^T \phi(t)\,dt + \int_0^T F(u)(t)\phi(t)\,dt. \tag{5.7}$$

Multiplying both sides of (5.1) by $\psi(t)$ and using (5.5)–(5.7), we obtain that

$$\int_0^T \biggl[\frac{d}{dt}\biggl(A_0u(t)+\sum_{j=1}^n A_j(u)(t)\biggr) + \int_0^t Lu(s)\,ds - F(u) - f\biggr]\phi(t)\,dt = 0 \tag{5.8}$$

for all $\phi(t) \in \mathbb{C}[0,T]$, where

$$f := -F(u_0) + A_0u_1 + \sum_{j=1}^n A_{jf}'(u_0)u_1. \tag{5.9}$$

The resulting equality (5.8) enables us to define strong solutions of the Cauchy problem (5.1).

Definition 5.2. A function $u(t) \in \mathbb{C}^{(1)}([0,T];V_0)$ is called a strong solution of the Cauchy problem (5.1) if the equality (5.8) holds for any function $\phi(t) \in \mathbb{C}[0,T]$ and one has $u(0) = u_0 \in V_0$, $u_1 \in V_0$.

Let the Banach space V_0 be separable (as required in § 2) and let $\{w_j\}_{j=1}^{+\infty}$ be a Galerkin basis in V_0. By Theorem 1.3 in [37], $\mathbb{C}([0,T];V_0)$ contains an everywhere-dense vector subspace

$$\biggl\{\phi_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j \colon c_{mj}(t) \in \mathbb{C}[0,T],\ m \in \mathbb{N}\biggr\}.$$

Since the function $\phi(t) \in \mathbb{C}[0,T]$ in (5.8) is arbitrary, we have

$$\int_0^T \biggl[\frac{d}{dt}\biggl(A_0 u(t) + \sum_{j=1}^{n} A_j(u)(t)\biggr) + \int_0^t Lu(s)\,ds - F(u) - f\biggr] c_{mj}(t)\,dt = 0 \quad (5.10)$$

for $j = 1, \dots, m$. Taking the scalar product of both sides of (5.10) and w_j in the sense of the duality brackets $\langle\,\cdot\,,\cdot\,\rangle_0$ and summing the results over $j = 1, \dots, m$, we have

$$\int_0^T \biggl\langle \frac{d}{dt}\biggl(A_0 u(t) + \sum_{j=1}^{n} A_j(u)(t)\biggr) + \int_0^t Lu(s)\,ds - F(u) - f,\, p_m(t)\biggr\rangle_0 dt = 0, \quad (5.11)$$

where

$$p_m(t) := \sum_{j=1}^{m} c_{mj}(t) w_j.$$

Since the functions $p_m(t)$ form an everywhere-dense subset of $\mathbb{C}([0,T];V_0)$, we deduce from (5.10) that

$$\int_0^T \biggl\langle \frac{d}{dt}\biggl(A_0 u(t) + \sum_{j=1}^{n} A_j(u)(t)\biggr) + \int_0^t Lu(s)\,ds - F(u) - f,\, v(t)\biggr\rangle_0 dt = 0 \quad (5.12)$$

for all $v(t) \in \mathbb{C}([0,T];V_0)$. Thus the following assertion holds.

Theorem 5.3. *Let the Banach space V_0 be separable. Then, in the class of strong solutions of the Cauchy problem* (5.1), *the equality* (5.8) *holds for any $\phi(t) \in \mathbb{C}[0,T]$ if and only if the equality* (5.12) *holds for any $v(t) \in \mathbb{C}([0,T];V_0)$.*

Remark 5.4. In view of the equality of the duality brackets (2.1) and (2.2), one can rewrite (5.12) in the following equivalent form:

$$\int_0^T \biggl[\biggl\langle \frac{d}{dt} A_0 u(t), v(t)\biggr\rangle_0 + \sum_{j=1}^{n} \biggl\langle \frac{d}{dt} A_j(u)(t), v(t)\biggr\rangle_j + \int_0^t (Lu(s), v(t))_1\,ds - (F(u), v(t))_2 - \langle f, v(t)\rangle_0\biggr] dt = 0 \quad (5.13)$$

for all $v(t) \in \mathbb{C}([0,T];V_0)$. Taking $v(t)=\phi(t)w$ in (5.13), where $\phi(t)\in\mathbb{C}[0,T]$ and $w\in V_0$, we obtain that

$$\int_0^T \phi(t)\biggl[\biggl\langle \frac{d}{dt}A_0u(t),w\biggr\rangle_0+\sum_{j=1}^n\biggl\langle\frac{d}{dt}A_j(u)(t),w\biggr\rangle_j \\ +\int_0^t (Lu(s),w)_1\,ds-(F(u),w)_2-\langle f,w\rangle_0\biggr]\,dt=0 \tag{5.14}$$

for all $\phi(t)\in\mathbb{C}[0,T]$ and $w\in V_0$. By the conditions on the operator coefficients, we have

$$\biggl\langle \frac{d}{dt}A_0u(t),w\biggr\rangle_0\in\mathbb{C}[0,T],\qquad \biggl\langle\frac{d}{dt}A_j(u)(t),w\biggr\rangle_j\in\mathbb{C}[0,T], \tag{5.15}$$

$$\int_0^t (Lu(s),w)_1\,ds\in\mathbb{C}[0,T],\qquad (F(u),w)_2\in\mathbb{C}[0,T]. \tag{5.16}$$

Hence, by the fundamental lemma of calculus of variations, it follows from the equality (5.14) and the properties (5.15), (5.16) that

$$\biggl\langle \frac{d}{dt}A_0u(t),w\biggr\rangle_0+\sum_{j=1}^n\biggl\langle\frac{d}{dt}A_j(u)(t),w\biggr\rangle_j \\ +\int_0^t (Lu(s),w)_1\,ds-(F(u),w)_2-\langle f,w\rangle_0=0 \tag{5.17}$$

for all $w\in V_0$ and all $t\in[0,T]$.

We now consider the abstract Cauchy problem

$$\frac{d}{dt}\biggl(A_0u(t)+\sum_{j=1}^n A_j(u)(t)\biggr)+\int_0^t Lu(s)\,ds=F(u)+f,\qquad u(0)=u_0. \tag{5.18}$$

We give a definition of a classical solution of this problem.

Definition 5.5. A classical solution of the Cauchy problem (5.18) is a function $u(t)\in\mathbb{C}^{(1)}([0,T];V_0)$ satisfying the equality (5.18) for every $t\in[0,T]$ in the sense of V_0^*, where $u_0\in V_0$ and $f\in V_0^*$.

It is clear that every classical solution of the Cauchy problem (5.18) is a strong solution of the Cauchy problem (5.1).

§ 6. The existence of a non-extendable classical solution of the Cauchy problem (5.1)

First of all we need to prove that the operator

$$A(u)\equiv A_0u+\sum_{j=1}^n A_j(u)\colon V_0\to V_0^* \tag{6.1}$$

is invertible and the inverse operator is Lipschitz continuous. To do this, we shall prove that all the hypotheses of the Browder–Minty theorem hold for $A(u)$. Thus,

(I) *the operator $A(u)$ is radially continuous.*

This follows from the continuity of the operators A_0 and $A_j(\,\cdot\,)$.

(II) *The operator $A(u)$ is strongly monotone.*

Indeed, the following chain of inequalities holds:

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle_0 = \langle A_0 u_1 - A_0 u_2, u_1 - u_2\rangle_0 \\ + \sum_{j=1}^{n} \langle A_j(u_1) - A_j(u_2), u_1 - u_2\rangle_j \geqslant \langle A_0 u_1 - A_0 u_2, u_1 - u_2\rangle_0 \geqslant m_0 \|u_1 - u_2\|_0^2.$$

(III) *The operator $A(u)$ is coercive.*

Indeed, the following chain of inequalities holds:

$$\langle A(u), u\rangle_0 = \langle A_0 u, u\rangle_0 + \sum_{j=1}^{n} \langle A_j(u), u\rangle_j \geqslant m_0 \|u\|_0^2 + \sum_{j=1}^{n} m_j \|u\|_j^{p_j} \geqslant m_0 \|u\|_0^2.$$

Thus, by the Browder–Minty theorem, the operator

$$A(u)\colon V_0 \to V_0^*$$

has an inverse operator

$$A^{-1}(v)\colon V_0^* \to V_0.$$

We claim that the operator $A^{-1}(\,\cdot\,)$ is Lipschitz continuous. Indeed, since $A(u)$ is strongly monotone, we have the chain of inequalities

$$m_0 \|u_1 - u_2\|_0^2 \leqslant \langle A(u_1) - A(u_2), u_1 - u_2\rangle_0 \leqslant \|A(u_1) - A(u_2)\|_0^* \|u_1 - u_2\|_0 \\ \Rightarrow\ m_0 \|u_1 - u_2\|_0 \leqslant \|A(u_1) - A(u_2)\|_0^*,$$

which yields the desired inequality

$$\|A^{-1}(w_1) - A^{-1}(w_2)\|_0 \leqslant \frac{1}{m_0} \|w_1 - w_2\|_0^* \quad \text{for all} \quad w_1, w_2 \in V_0^*. \tag{6.2}$$

Thus, if we introduce the notation

$$A(u) = v, \tag{6.3}$$

then the abstract Cauchy problem (5.18) can be rewritten in the class of functions $v(t) \in \mathbb{C}^{(1)}([0, T]; V_0^*)$ in the following equivalent form:

$$\frac{dv}{dt} = -\int_0^t ds\, L A^{-1}(v)(s) + F(A^{-1}(v)) + f, \tag{6.4}$$

$$v(0) = v_0 = A(u_0) \in V_0^*, \qquad f \in V_0^*. \tag{6.5}$$

Remark 6.1. The initial condition (6.5) is the only thing to be verified in order to show that this statement is correct. First of all, note that the following chain of inequalities holds in the class $v(t) \in \mathbb{C}([0,T];V_0^*)$:

$$\|u(t_1)-u(t_2)\|_0 = \|A^{-1}(v(t_1)) - A^{-1}(v(t_2))\|_0 \\ \leqslant \|v(t_1)-v(t_2)\|_0^* \to 0 \quad \text{as} \quad t_1 \to t_2.$$

Hence $u(t) \in \mathbb{C}([0,T];V_0)$. Since A_0 and A_j are continuous, we conclude that

$$v_0 = v(0) = \lim_{t\downarrow 0} v(t) = \lim_{t\downarrow 0} A(u(t)) = A(u(0)) = A(u_0).$$

Thus the initial condition (6.5) is correct.

Note that the problem (6.4) is equivalent in the class $v(t) \in \mathbb{C}^{(1)}([0,T];V_0^*)$ to the integral equation

$$v(t) = v_0 + \int_0^t ds\, G(v)(s), \tag{6.6}$$

where

$$G(v)(t) = -\int_0^t ds\, LA^{-1}(v)(s) + F(A^{-1}(v)) + f. \tag{6.7}$$

We seek a solution $v(t)$ of (6.6) in the class $\mathbb{C}([0,T];V_0^*)$. To do this, we rewrite (6.6) in the operator form

$$v(t) = H(v)(t), \tag{6.8}$$

where

$$H(v)(t) = v_0 + \int_0^t ds\, G(v)(s).$$

Define a closed bounded convex subset of the Banach space $\mathbb{C}([0,T];V_0^*)$ by putting

$$B_r \equiv \left\{ v(t) \in \mathbb{C}([0,T];V_0^*) \;\middle|\; \|v\| \equiv \sup_{t\in[0,T]} \|v\|_0^*(t) \leqslant r \right\}$$

for some $T > 0$ and $r > 0$. Using the property (6.2), the conditions L and F, and the results in [38], one can prove the following assertion.

Theorem 6.2. *For every $v_0 \in V_0^*$ there is a $T_0 = T_0(v_0) > 0$ such that the equation (6.8) has a unique solution $v(t) \in \mathbb{C}([0,T_0);V_0^*)$ and either $T_0 = +\infty$, or $T_0 < +\infty$ and the following limit property holds in the latter case*:

$$\lim_{t\uparrow T_0} \|v\|_0^*(t) = +\infty. \tag{6.9}$$

Note that

$$G\colon \mathbb{C}([0,T_0);V_0^*) \to \mathbb{C}([0,T_0);V_0^*)$$

and, therefore,

$$\int_0^t G(v)(s)\, ds \in \mathbb{C}^{(1)}([0,T_0);V_0^*).$$

Hence the solution $v(t)$ of the equation (6.8) belongs to $\mathbb{C}^{(1)}([0,T_0);V_0^*)$.

Thus we arrive at the following equation with known right-hand side:

$$A_0u+\sum_{j=1}^{n}A_j(u)=v(t)\in\mathbb{C}^{(1)}([0,T_0);V_0^*)\quad\text{for some}\quad T_0>0. \tag{6.10}$$

We need the following theorem (see [39], Theorem 12.3.3, or [40], Theorem 4.2.1).

Theorem 6.3. *Let P be a continuously differentiable map from an open ball $U=B_r(x_0)$ in a Banach space X to a Banach space Y. Suppose that the operator $\Lambda:=P'_f(x_0)$ maps X bijectively onto Y. Then P maps some neighbourhood V of the point x_0 bijectively onto a neighbourhood W of $P(x_0)$. Moreover, the map $R:=P^{-1}\colon W\to V$ is continuously differentiable and one has*

$$R'_f(y)=\bigl(P'_f(P^{-1}(y))\bigr)^{-1},\qquad y\in W.$$

We define an operator $P(u):=A_0u+\sum_{j=1}^{n}A_j(u)$, $U=X=V_0$, $Y=V_0^*$. Consider its Fréchet derivative

$$P'_f(u_0)=A_0+\sum_{j=1}^{n}A'_j(u_0)\colon V_0\to V_0^*,\qquad u_0\in V_0. \tag{6.11}$$

We claim that the operator $P'_f(u_0)$ has an inverse for every $u_0\in V_0$. To prove this, we again use the monotonicity of the operators considered. Thus,

(I) *the operator $P'_f(u_0)$ is radially continuous.*

This follows from the continuity of A_0 since $A'_{jf}(u_0)\in\mathscr{L}(V_0;V_0^*)$ for a fixed $u_0\in V_0$.

(II) *The operator $P'_f(u_0)$ is strongly monotone.*

Indeed, the following chain of inequalities holds:

$$\Bigl\langle\Bigl[A_0+\sum_{j=1}^{n}A'_{jf}(u_0)\Bigr]v_1-\Bigl[A_0+\sum_{j=1}^{n}A'_{jf}(u_0)\Bigr]v_2,v_1-v_2\Bigr\rangle_0$$
$$=\langle A_0v_1-A_0v_2,v_1-v_2\rangle_0+\sum_{j=1}^{n}\langle A'_{jf}(u_0)v_1-A'_{jf}(u_0)v_2,\,v_1-v_2\rangle_j\geqslant m_0\|v_1-v_2\|_0^2.$$

(III) *The operator $P'_f(u_0)$ is coercive.*

This follows from part (II) and the linearity of this operator for a fixed function $u(t)\in\mathbb{C}([0,T_0);V_0)$. Thus, the operator

$$P'_f(u_0):=A_0+\sum_{j=1}^{n}A'_{jf}(u_0)\colon V_0\to V_0^*$$

is invertible. Hence, by Theorem 6.3, it follows from (6.10) that

$$u(t)=R(v(t))\in\mathbb{C}^{(1)}([0,T_0);V_0). \tag{6.12}$$

Thus Theorems 6.2, 6.3 and (6.10) yield the following assertion.

Theorem 6.4. *For any functions $u_0 \in V_0$ and $f \in V_0^*$ there is a $T_0 = T_0(u_0, f) > 0$ such that for every $T \in (0, T_0)$ the problem* (5.18) *has a unique classical solution $u(t)$ of the class $\mathbb{C}^{(1)}([0,T]; V_0)$ and either $T_0 = +\infty$, or $T_0 < +\infty$ and the following limit property holds in the latter case*:

$$\lim_{t\uparrow T_0} \left\| A_0 u + \sum_{j=1}^{n} A_j(u) \right\|_0^* (t) = +\infty. \tag{6.13}$$

We claim that the classical solution of (5.18) actually possesses a greater smoothness: $u(t) \in \mathbb{C}^{(2)}([0,T_0); V_0)$. Indeed, on the one hand, it follows from (6.4) that $v(t) \in \mathbb{C}^{(2)}([0,T_0); V_0^*)$. Suppose that the operators $A_j(u)$, $j = 1, \ldots, n$, are twice continuously Fréchet differentiable for all $u \in V_j$. Then the operator

$$P(u) = A_0 u + \sum_{j=1}^{n} A_j(u)$$

is also twice Fréchet differentiable for every $u \in V_0$. We use Theorem 5.4.4 in [40].

Theorem 6.5. *Suppose that E and F are Banach spaces*, *$V \subset E$ and $W \subset F$ are open subsets*, *and*

$$f \colon V \to W$$

is a C^1-diffeomorphism. If f belongs to the class C^n, *then the inverse homeomorphism $g = f^{-1}$ is also a map of class C^n.*

Thus the following assertion holds.

Theorem 6.6. *Suppose that all the conditions in* § 2 *on the operator coefficients A_0*, *A*, *L and F hold. Assume also that the operators $A_j(u)$ are twice continuously Fréchet differentiable for all $u \in V_j$. Then for any u_0 and u_1 in V_0 there is a $T_0 = T_0(u_0, u_1) > 0$ such that the Cauchy problem* (5.1) *has a unique classical solution $u(t)$ of the class $\mathbb{C}^{(2)}([0,T_0); V_0)$ and either $T_0 = +\infty$, or $T_0 < +\infty$ and the limit property* (6.13) *holds in the latter case.*

Proof. By Theorems 6.4 and 6.5 we conclude that the unique solution $u(t)$ of the equation

$$A_0 u + \sum_{j=1}^{n} A_j(u) = v(t) \in \mathbb{C}^{(2)}([0,T_0); V_0^*) \tag{6.14}$$

belongs to the class $\mathbb{C}^{(2)}([0,T_0); V_0)$. Since the operator A_0 is linear and the non-linear operators $A_j(u)$ are twice continuously Fréchet differentiable for all $u \in V_j$, one can differentiate the equality (6.14) twice with respect to $t \in [0,T_0)$ and obtain the following equality in view of (6.6):

$$\begin{aligned} \frac{d^2}{dt^2}\left(A_0 u + \sum_{j=1}^{n} A_j(u) \right) &= \frac{d^2 v(t)}{dt^2} \\ &= -LA^{-1}(v) + \frac{dF(A^{-1}(v))(t)}{dt} = -Lu + \frac{dF(u)(t)}{dt}. \end{aligned} \tag{6.15}$$

It follows from the equality (5.18) that

$$\frac{du(t)}{dt} = \left(A_0 + \sum_{j=1}^{n} A'_{jf}(u)\right)^{-1}\left[-\int_0^t Lu(s)\,ds + F(u)(t) + f\right], \tag{6.16}$$

where

$$f = -F(u_0) + A_0 u_1 + \sum_{j=1}^{n} A'_{jf}(u_0)u_1.$$

Therefore it follows from (6.16) that

$$\left.\frac{du(t)}{dt}\right|_{t=0} = u_1 \in V_0.$$

Thus, in the class of classical solutions in the sense of Definition 5.1, the Cauchy problem (5.1) is equivalent to the Cauchy problem (5.18). □

§ 7. Blow-up of a strong solution of (5.1) for $q + 2 > \overline{p}$

Let $u(t) \in \mathbb{C}^{(1)}([0,T_0);V_0)$ be a classical solution of the problem (5.18). First of all we put

$$\Phi(t) = \frac{1}{2}\langle A_0 u, u\rangle_0 + \sum_{j=1}^{n} \frac{p_j - 1}{p_j}\langle A_j(u), u\rangle_j, \tag{7.1}$$

$$J(t) = \langle A_0 u', u'\rangle_0 + \sum_{j=1}^{n}(p_j - 1)\langle A'_{jf}(u)u', u'\rangle_j. \tag{7.2}$$

Lemma 7.1. *We have*

$$(\Phi'(t))^2 \leqslant \overline{p}\, J(t)\Phi(t) \quad \textit{for} \quad \overline{p} = \max_{j=1,\dots,n} p_j, \quad t \in [0,T_0). \tag{7.3}$$

Proof. By the conditions A and A_0, Schwarz' inequality holds for the Fréchet derivatives $A'_{j,u}\colon V_j \to \mathcal{L}(V_j;V_j^*)$ of the operators $A_j\colon V_j \to V_j^*$:

$$\begin{aligned}|\langle (A_j(u))', u\rangle_j| = |\langle A'_{jf}(u)u', u\rangle_j| &\leqslant \langle A'_{jf}(u)u', u'\rangle_j^{1/2}\langle A'_{jf}(u)u, u\rangle_j^{1/2} \\ &= \langle (A_j(u))', u'\rangle_j^{1/2}(p_j - 1)^{1/2}\langle A_j(u), u\rangle_j^{1/2},\end{aligned} \tag{7.4}$$

$$|\langle A_0 u', u\rangle_0| \leqslant \langle A_0 u', u'\rangle_0^{1/2}\langle A_0 u, u\rangle_0^{1/2}. \tag{7.5}$$

Here we have used the equality

$$A'_{jf}(v)v = (p_j - 1)A_j(v), \tag{7.6}$$

which was proved at the beginning of Lemma 3.1. It follows from (7.4)–(7.6) that

$$
\begin{aligned}
\left|\frac{d}{dt}\Phi\right|^2 &\leqslant \left||\langle A_0u', u\rangle_0| + \sum_{j=1}^N |\langle (A_j(u))', u\rangle_j|\right|^2 \\
&\leqslant \left(\langle A_0u', u'\rangle_0 + \sum_{j=1}^N \langle (A_j(u))', u'\rangle_j\right)\left(\langle A_0u, u\rangle_0 + \sum_{j=1}^N (p_j-1)\langle A_j(u), u\rangle_j\right) \\
&\leqslant \overline{p}\left(\langle A_0u', u'\rangle_0 + \sum_{j=1}^N \langle (A_j(u))', u'\rangle_j\right)\left(\frac{1}{\overline{p}}\langle A_0u, u\rangle_0 + \sum_{j=1}^N \frac{p_j-1}{\overline{p}}\langle A_j(u), u\rangle_j\right) \\
&\leqslant \overline{p}\left(\langle A_0u', u'\rangle_0 + \sum_{j=1}^N \langle (A_j(u))', u'\rangle_j\right)\left(\frac{1}{2}\langle A_0u, u\rangle_0 + \sum_{j=1}^N \frac{p_j-1}{p_j}\langle A_j(u), u\rangle_j\right) \\
&= \overline{p}\, J(t)\Phi(t),
\end{aligned}
\tag{7.7}
$$

where $\overline{p} = \max_{j=1,\dots,n} p_j > 2$. □

Note that Definition 5.2 of a strong solution of the Cauchy problem (5.1) is equivalent to the equality (5.17). We first put $w = u(t) \in \mathbb{C}^{(1)}([0, T_0); V_0)$ in (5.17) and, in view of (3.2) and the definition (7.1) of the functional $\Phi(t) \in \mathbb{C}[0, T_0)$, obtain the first energy equality

$$
\frac{d\Phi}{dt} + \int_0^t ds\,(Lu(s), u(t))_1 = (F(u), u)_2 + \langle f, u\rangle_0. \tag{7.8}
$$

Then we put $w = u'(t) \in \mathbb{C}([0, T_0); V_0)$ in (5.17) and, in view of the definition (7.2) of the functional $J(t)$, obtain the second energy equality

$$
J(t) + \int_0^t ds\,(Lu(s), u'(t))_1 = \frac{1}{q+2}\frac{d}{dt}(F(u), u)_2 + \frac{d}{dt}\langle f, u\rangle_0, \tag{7.9}
$$

where we have used the equality

$$
(F(u), u')_2 = \frac{1}{q+2}\,\frac{d}{dt}(F(u), u)_2,
$$

which follows from the equality (3.1) in the proof of Lemma 3.1. Expressing the quantity $(F(u), u)_2$ using (7.8), substituting it into (7.9) and making elementary transformations, we obtain the following expression for the functional $J(t)$:

$$
J(t) = \frac{1}{q+2}\frac{d^2\Phi(t)}{dt^2} + \frac{1}{q+2}(Lu, u)_1 - \frac{q+1}{q+2}\int_0^t (Lu(s), u'(t))\,ds + \frac{q+1}{q+2}\langle f, u'\rangle_0. \tag{7.10}
$$

In what follows we will use the Cauchy–Bunyakovskii inequality with an arbitrary $\varepsilon_1 > 0$:

$$
a \cdot b \leqslant \varepsilon_1 a^2 + \frac{1}{4\varepsilon_1} b^2, \qquad a,\, b \geqslant 0.
$$

The following relations hold:

$$
\frac{1}{q+2}|(Lu, u)_1| \leqslant \frac{l}{q+2}\langle A_0u, u\rangle_0 = \frac{2l}{q+2}\Phi(t), \tag{7.11}
$$

where we have used the inequaity (3.5):

$$\frac{q+1}{q+2}\left|\int_0^t (Lu(s),u'(t))_1\,ds\right| \leqslant \frac{q+1}{q+2}l\int_0^t \langle A_0u(s),u(s)\rangle_0^{1/2}\langle A_0u'(t),u'(t)\rangle_0^{1/2}\,ds$$
$$\leqslant \varepsilon\langle A_0u'(t),u'(t)\rangle_0 + \left(\frac{q+1}{q+2}\right)^2\frac{T}{4\varepsilon}\int_0^t \langle A_0u(s),u(s)\rangle_0\,ds$$
$$\leqslant \varepsilon J(t) + \left(\frac{q+1}{q+2}\right)^2\frac{T}{2\varepsilon}\int_0^t \Phi(s)\,ds, \tag{7.12}$$

$$\frac{q+1}{q+2}|\langle f,u'\rangle_0| \leqslant \frac{q+1}{q+2}\|f\|_0^*\,\|u'\|_0 \leqslant \frac{q+1}{q+2}\|f\|_0^*\frac{1}{m_0^{1/2}}\langle A_0u',u'\rangle_0^{1/2}$$
$$\leqslant \varepsilon J(t) + \left(\frac{q+1}{q+2}\right)^2\frac{\|f\|_0^{*2}}{4m_0\varepsilon}. \tag{7.13}$$

Thus we obtain the following bound from (7.10) in view of (7.11)–(7.13):

$$(1-2\varepsilon)J(t) \leqslant \frac{1}{q+2}\frac{d^2\Phi(t)}{dt^2} + \frac{2l}{q+2}\Phi(t) + \left(\frac{q+1}{q+2}\right)^2\frac{T}{2\varepsilon}\int_0^t \Phi(s)\,ds + \left(\frac{q+1}{q+2}\right)^2\frac{\|f\|_0^{*2}}{4m_0\varepsilon}. \tag{7.14}$$

Suppose that $\varepsilon \in (0,1/2)$. Then (7.3) and (7.14) yield the following second-order ordinary differential inequality

$$\Phi\Phi'' - \frac{q+2}{\overline{p}}(1-2\varepsilon)(\Phi')^2 + 2l\Phi^2 + \frac{(q+1)^2}{q+2}\,\frac{T}{2\varepsilon}\int_0^t \Phi(s)\,ds\,\Phi(t) + \frac{(q+1)^2}{q+2}\,\frac{\|f\|_0^{*2}}{4m_0\varepsilon}\Phi(t) \geqslant 0. \tag{7.15}$$

Making the change $2\varepsilon \mapsto \varepsilon$, we can rewrite this inequality in the general form

$$\Phi\Phi'' - \alpha(\Phi')^2 + \beta\Phi^2 + \gamma_1\Phi(t) + \gamma_2 T\int_0^t \Phi(s)\,ds\,\Phi(t) \geqslant 0, \tag{7.16}$$

where

$$\alpha = \frac{q+2}{\overline{p}}(1-\varepsilon), \qquad \beta = 2l, \qquad \gamma_1 = \frac{(q+1)^2}{q+2}\,\frac{\|f\|_0^{*2}}{2m_0\varepsilon}, \qquad \gamma_2 = \frac{(q+1)^2}{q+2}\,\frac{1}{\varepsilon}. \tag{7.17}$$

We require that $\alpha > 1$. This leads to the inequalities

$$0 < \varepsilon < \frac{q+2-\overline{p}}{q+2}, \qquad q+2 > \overline{p}. \tag{7.18}$$

Moreover, we have

$$2\alpha - 1 = \frac{\alpha_1 - \alpha_2\varepsilon}{\overline{p}}, \qquad \alpha_1 = 2(q+2) - \overline{p}, \qquad \alpha_2 = 2(q+2). \tag{7.19}$$

Consider the auxiliary function

$$h(x) = x(\alpha_1 - \alpha_2 x) \geqslant 0 \quad \text{for} \quad x \in \left[0, \frac{\alpha_1}{\alpha_2}\right]. \tag{7.20}$$

Its maximum is attained at the point

$$x_0 = \frac{\alpha_1}{2\alpha_2} = \frac{2(q+2) - \overline{p}}{4(q+2)}. \tag{7.21}$$

Suppose that the following inequality holds:

$$q + 2 > \overline{p}. \tag{7.22}$$

We consider two cases separately:

$$q + 2 \leqslant \frac{3}{2}\overline{p} \quad \text{and} \quad \frac{3}{2}\overline{p} < q + 2. \tag{7.23}$$

In the second case we have

$$x_0 = \frac{\alpha_1}{2\alpha_2} = \frac{2(q+2) - \overline{p}}{4(q+2)} < \frac{q + 2 - \overline{p}}{q+2}, \tag{7.24}$$

and in the first case we have

$$x_0 = \frac{\alpha_1}{2\alpha_2} = \frac{2(q+2) - \overline{p}}{4(q+2)} \geqslant \frac{q + 2 - \overline{p}}{q+2}. \tag{7.25}$$

Choose the parameter $\varepsilon > 0$ in the coefficients (7.17) in such a way that the coefficient

$$\frac{2\gamma_1}{2\alpha - 1}$$

in (4.24) attains its minimum value.

Remark 7.2. Minimizing this coefficient is necessary in order to include as many elements $f \in V_0^*$ as possible in the blow-up effect.

Note that the coefficient considered is of the form

$$\frac{2\gamma_1}{2\alpha - 1} = \frac{1}{\varepsilon(\alpha_1 - \alpha_2 \varepsilon)} \frac{\overline{p}(q+1)^2}{q+2} \frac{\|f\|_0^{*2}}{m_0}. \tag{7.26}$$

This function of $\varepsilon > 0$ attains its minimum value at the point

$$\varepsilon_0 = \begin{cases} \dfrac{q + 2 - \overline{p}}{q+2} & \text{if } q + 2 \leqslant \dfrac{3}{2}\overline{p}, \\ \dfrac{2(q+2) - \overline{p}}{4(q+2)} & \text{if } \dfrac{3}{2}\overline{p} < q + 2. \end{cases}$$

However, when $q + 2 \leqslant 3\overline{p}/2$, we have $\alpha = 1$, which is inappropriate. Therefore we choose ε in the following way:

$$\varepsilon = \varepsilon_0 = \begin{cases} \dfrac{q + 2 - \overline{p}}{q+2} - \delta & \text{if } q + 2 \leqslant \dfrac{3}{2}\overline{p}, \\ \dfrac{2(q+2) - \overline{p}}{4(q+2)} & \text{if } \dfrac{3}{2}\overline{p} < q + 2, \end{cases} \tag{7.27}$$

for any small $\delta > 0$. Substitute this value of $\varepsilon = \varepsilon_0$ in the coefficients (7.17). We claim that the hypotheses of Theorem 4.1 hold.

Indeed, fix arbitrary $u_0 \in V_0$ and $f \in V_0^*$ and let $u_1 \in V_0$ be the unique solution of the following equation in V_0^*:

$$A_0 u_1 + \sum_{j=1}^{n} A'_{jf}(u_0)u_1 = F(u_0) + f \in V_0^*. \tag{7.28}$$

Note that a solution $u_1 \in V_0$ of this equation does indeed exist by the Browder–Minty theorem. In our case, the functional $\Phi(t) \in \mathbb{C}^{(2)}[0, T_0)$ is of the form (7.1). Therefore for $t = 0$ we have

$$\Phi(0) = \frac{1}{2}\langle A_0 u_0, u_0\rangle_0 + \sum_{j=1}^{n} \frac{p_j - 1}{p_j}\langle A_j(u_0), u_0\rangle_j \tag{7.29}$$

and the Fréchet derivative of $\Phi(t)$ is of the following form by Lemma 3.3:

$$\Phi'(t) = \langle A_0 u', u\rangle_0 + \sum_{j=1}^{n}\langle (A_j(u))', u\rangle_j = \langle A_0 u', u\rangle_0 + \sum_{j=1}^{n}\langle A'_{jf}(u)u', u\rangle_j. \tag{7.30}$$

Hence we obtain that

$$\Phi'(0) = \langle A_0 u_1, u_0\rangle_0 + \sum_{j=1}^{n}\langle A'_{jf}(u_0)u_1, u_0\rangle_j. \tag{7.31}$$

In view of (7.28), this yields the following expression:

$$\Phi'(0) = (F(u_0), u_0)_2 + \langle f, u_0\rangle_0. \tag{7.32}$$

We rewrite the equation (4.24) in the following equivalent form:

$$K_1 T_1^4 + K_2 T_1^2 + K_3 = 0, \tag{7.33}$$

where

$$K_1 = \frac{\gamma_2}{\alpha - 1}(\Phi(0))^2, \qquad K_3 = \frac{1}{(\alpha - 1)^2}(\Phi(0))^2, \tag{7.34}$$

$$K_2 = \frac{\beta}{\alpha - 1}(\Phi(0))^2 + \frac{2\gamma_1}{2\alpha - 1}\Phi(0) - (\Phi'(0))^2. \tag{7.35}$$

Define the following functions:

$$\begin{aligned} I_1(R) &= \left(\Phi'(0)\big|_{Ru_0}\right)^2 = \left((F(Ru_0), Ru_0)_2 + \langle f, Ru_0\rangle_0\right)^2 \\ &= \left(R^{q+2}(F(u_0), u_0)_2 + R\langle f, u_0\rangle_0\right)^2, \end{aligned} \tag{7.36}$$

$$I_2(R) = \Phi(0)\big|_{Ru_0} = R^2 \frac{1}{2}\langle Au_0, u_0\rangle_0 + \sum_{j=1}^{n} R^{p_j}\frac{p_j - 1}{p_j}\langle A_j(u_0), u_0\rangle_j. \tag{7.37}$$

Substitute Ru_0, $R \geqslant 0$, instead of u_0 into the right-hand sides of (7.34) and (7.35). Moreover, put $x = T_1^2$. Then the biquadratic equation takes the form

$$K_1 x^2 + K_2 x + K_3 = 0. \tag{7.38}$$

First of all, we see from the condition $q+2 > \overline{p} = \max_{j=1,\dots,n} p_j$ and (7.36), (7.37) that the coefficient K_2 will be negative for sufficiently large $R > 0$ and, under the condition $(F(u_0), u_0)_2 \neq 0$, the discriminant

$$\mathscr{D} = K_2^2 - 4K_1K_3$$

is positive for sufficiently large $R > 0$. Thus, for sufficiently large $R > 0$ the equation (7.38) has a positive root

$$T_1^2 = x = \frac{-K_2 + \sqrt{K_2^2 - 4K_1K_2}}{2K_1} > 0.$$

This proves the following assertion.

Theorem 7.3. *Suppose that $u_0 \in V_0$, $f \in V_0^*$, $u_1 \in V_0$ is a solution of (7.28), and*

$$(F(u_0), u_0) \neq 0.$$

Then, for sufficiently large $R > 0$, the functional $\Phi(t)$ defined by the formula (7.1) with initial function Ru_0 satisfies the inequality (4.25).

Lemma 7.4. *We have the two-sided inequality*

$$M_1 \Phi^{1/2}(t) \leqslant \|A(u)\|_0^* \leqslant M_2 \Phi^{1/2} + \sum_{j=1}^{n} B_j \Phi^{(p_j-1)/p_j}(t), \tag{7.39}$$

where the constants M_1, M_2 and B_j are positive and independent of $u(t)$, and

$$A(u) := A_0 u + \sum_{j=1}^{n} A_j(u).$$

Proof. We first prove the lower bound. On the one hand, we have

$$\langle A(u), u\rangle_0 = \langle A_0 u, u\rangle + \sum_{j=1}^{n} \langle A_j(u), u\rangle_j \geqslant \Phi(t). \tag{7.40}$$

On the other hand, we have

$$\langle A(u), u\rangle_0 \leqslant \|A(u)\|_0^* \|u\|_0 \leqslant \|A(u)\|_0^* \frac{1}{m_0^{1/2}} \langle A_0 u, u\rangle_0^{1/2} \leqslant \|A(u)\|_0^* \frac{2^{1/2}}{m_0^{1/2}} \Phi^{1/2}(t). \tag{7.41}$$

The lower bound follows from (7.40) and (7.41):

$$\|A(u)\|_0^* \geqslant \left(\frac{m_0}{2}\right)^{1/2} \Phi^{1/2}(t) = M_1 \Phi^{1/2}(t). \tag{7.42}$$

We now prove the upper bound. The following chain of relationsd holds by the definition of the norm $\|\cdot\|_0^*$:

$$\begin{aligned}\|A(u)\|_0^* &= \sup_{\|h\|_0\leqslant 1}|\langle A(u),h\rangle_0| = \sup_{\|h\|_0\leqslant 1}\left|\langle A_0u,h\rangle_0+\sum_{j=1}^n\langle A_j(u),h\rangle_j\right|\\ &\leqslant \sup_{\|h\|_0\leqslant 1}\langle A_0u,u\rangle_0^{1/2}\langle A_0h,h\rangle_0^{1/2}+\sup_{\|h\|_0\leqslant 1}\sum_{j=1}^n\|A_j(u)\|_j^*\|h\|_j\\ &\leqslant c_3\Phi^{1/2}(t)+c_4\sum_{j=1}^n\|u\|_j^{p_j-1}\leqslant c_3\Phi^{1/2}(t)+c_5\sum_{j=1}^n\langle A_j(u),u\rangle^{(p_j-1)/p_j}\\ &\leqslant c_3\Phi^{1/2}(t)+c_6\sum_{j=1}^n\Phi^{(p_j-1)/p_j}(t)=M_2\Phi^{1/2}+\sum_{j=1}^n B_j\Phi^{(p_j-1)/p_j}(t).\qquad\square\end{aligned}\tag{7.43}$$

This lemma yields the following assertion.

Theorem 7.5. *Suppose that the initial function $u_0\in V_0$ is replaced by Ru_0, and let $u_1\in V_0$ be a solution of the equation (7.28) with $f\in V_0^*$, where u_0 is replaced by Ru_0. Then, for all sufficiently large $R>0$, the existence time $T_0>0$ of the classical solution of the problem (5.1) is finite and the following limit property holds*:

$$\lim_{t\uparrow T_0}\Phi(t)=+\infty.\tag{7.44}$$

We also have an upper bound $T_0\leqslant T_1$ for the blow-up time, where T_1 is the positive solution of the biquadratic equation (4.24).

We now obtain a lower bound for the blow-up time T_0. Note that the following bounds hold by the conditions in § 2:

$$|(F(u),u)_2|\leqslant M|u|_2^{q+2}\leqslant c_1\|u\|_0^{q+2}\leqslant c_2\langle A_0u,u\rangle_0^{(q+2)/2}\leqslant c_3\Phi^{(q+2)/2}.\tag{7.45}$$

By Lemma 3.4 we have

$$\begin{aligned}\left|\int_0^t(Lu(s),u(t))_1\,ds\right| &\leqslant \frac{1}{2}\langle A_0u,u\rangle_0+\frac{Tl^2}{2}\int_0^t\langle A_0u(s),u(s)\rangle_0\,ds\\ &\leqslant \Phi(t)+Tl^2\int_0^t\Phi(s)\,ds,\end{aligned}\tag{7.46}$$

$$|\langle f,u\rangle_0|\leqslant\|f\|_0^*\,\|u\|_0\leqslant\frac{\|f\|_0^*}{m^{1/2}}\langle A_0u,u\rangle_0^{1/2}\leqslant\left(\frac{2}{m}\right)^{1/2}\|f\|_0^*\Phi^{1/2}(t).\tag{7.47}$$

Note that the following relations hold:

$$\begin{aligned}\int_0^t\left[\Phi(\tau)+Tl^2\int_0^\tau\Phi(s)\,ds\right]d\tau &= \int_0^t\Phi(\tau)\,d\tau+Tl^2\int_0^t\int_0^\tau\Phi(s)\,ds\,d\tau\\ &=\int_0^t\Phi(\tau)\,d\tau+Tl^2\int_0^t(t-s)\Phi(s)\,ds\leqslant(1+T^2l^2)\int_0^t\Phi(s)\,ds,\end{aligned}\tag{7.48}$$

$$\left(\frac{2}{m}\right)^{1/2}\|f\|_0^*\int_0^t\Phi^{1/2}(s)\,ds\leqslant\frac{1}{m}\|f\|_0^{*2}+\frac{T}{2}\int_0^t\Phi(s)\,ds.\tag{7.49}$$

Thus the equality (7.8) and the inequalities (7.45)–(7.49) yield that

$$\Phi(t) \leqslant \Phi(0) + \frac{\|f\|_0^{*2}}{m} + c_3 \int_0^t \Phi^{1+q/2}(s)\,ds + \left(1 + \frac{T}{2} + l^2T^2\right) \int_0^t \Phi(s)\,ds. \quad (7.50)$$

We now use a corollary of the three-parameter Young inequality

$$ab \leqslant a^{q_1} + gb^{q_2}, \qquad g = \frac{1}{q_2(q_1)^{q_2/q_1}}, \qquad \frac{1}{q_1} + \frac{1}{q_2} = 1 \quad (7.51)$$

with

$$q_1 = \frac{q+2}{2}, \qquad q_2 = \frac{q+2}{q}, \qquad g = \frac{q}{q+2}\left(\frac{2}{q+2}\right)^{1/q}.$$

The following bound holds:

$$\Phi(s)\left(1 + \frac{T}{2} + l^2T^2\right) \leqslant \Phi^{(q+2)/2}(s) + g\left(1 + \frac{T}{2} + l^2T^2\right)^{(q+2)/q}. \quad (7.52)$$

This and the inequality (7.50) yield a bound of the form

$$\Phi(t) \leqslant d_1 + d_2 \int_0^t \Phi^{1+q/2}(s)\,ds, \quad (7.53)$$

where

$$d_1 = \Phi(0) + \frac{\|f\|_0^{*2}}{m} + T\frac{q}{q+2}\left(\frac{2}{q+2}\right)^{1/q}\left(1 + \frac{T}{2} + l^2T^2\right)^{(q+2)/q}, \qquad d_2 = c_3 + 1.$$

By the Gronwall–Bellman–Bihari theorem (see [41], p. 112) we have

$$\Phi(t) \leqslant \frac{d_1}{[1 - qd_1^{q/2}d_2t/2]^{2/q}}. \quad (7.54)$$

An easy analysis of the formula (7.54) leads to the following assertion.

Lemma 7.6. *The existence time $T_0 > 0$ of a classical solution of the Cauchy problem (5.1) satisfies the lower bound*

$$T_0 \geqslant T_2, \quad (7.55)$$

where $T_2 > 0$ is the root of an equation

$$\frac{q}{2}\left[\Phi(0) + \frac{\|f\|_0^{*2}}{m} + T_2\frac{q}{q+2}\left(\frac{2}{q+2}\right)^{1/q}\left(1 + \frac{T_2}{2} + l^2T_2^2\right)^{(q+2)/q}\right]^{q/2}(1 + c_3)T_2 = 1. \quad (7.56)$$

§ 8. Global-in-time solubility of the Cauchy problem (5.1) for $q+2\leqslant\overline{p}$

We make a number of assumptions. Suppose that the following inequality holds:

$$q+2\leqslant\overline{p}=\max_{j=1,\dots,n}p_j \tag{8.1}$$

and there is a V_j with a continuous embedding

$$V_j\subset W_2\quad\text{and}\quad p_j=\overline{p}. \tag{8.2}$$

Then the following chain of relations holds:

$$\begin{aligned}|(F(u),u)_2|&\leqslant M_1|u|_2^{q+2}\leqslant M_{2j}\|u\|_j^{q+2}=\frac{M_{2j}}{m_j^{(q+2)/p_j}}(m_j\|u\|_j^{p_j})^{(q+2)/p_j}\\&\leqslant M_{3j}\langle A_j(u),u\rangle_j^{(q+2)/p_j}\leqslant M_{4j}\Phi^{(q+2)/p_j}.\end{aligned} \tag{8.3}$$

In view of (7.46)–(7.49) and (8.3) we obtain from the equality (7.8) that

$$\Phi(t)\leqslant\Phi(0)+\frac{\|f\|_0^{*2}}{m}+M_{4j}\int_0^t\Phi^{(q+2)/p_j}(s)\,ds+\biggl(1+\frac{T}{2}+l^2T^2\biggr)\int_0^t\Phi(s)\,ds. \tag{8.4}$$

Consider two cases, $q+2=p_j$ and $q+2<p_j$. In the first case we arrive at the inequality

$$\Phi(t)\leqslant\Phi(0)+\frac{\|f\|_0^{*2}}{m}+\biggl(M_{4j}+1+\frac{T}{2}+l^2T^2\biggr)\int_0^t\Phi(s)\,ds. \tag{8.5}$$

In the second case we again use the three-parameter Young inequality (7.51) with

$$q_1=\frac{p_j}{q+2},\qquad q_2=\frac{p_j}{p_j-q-2}$$

and obtain that

$$M_{4j}\Phi^{(q+2)/p_j}\leqslant\Phi+M_j^{p_j/(p_j-q-2)}g_j,\qquad g_j=\frac{p_j-q-2}{p_j}\biggl(\frac{q+2}{p_j}\biggr)^{(q+2)/(p_j-q-2)}. \tag{8.6}$$

Using this and (8.4), we arrive at the following bound:

$$\Phi(t)\leqslant\Phi(0)+\frac{\|f\|_0^{*2}}{m}+M_j^{p_j/(p_j-q-2)}g_jT+\biggl(2+\frac{T}{2}+l^2T^2\biggr)\int_0^t\Phi(s)\,ds. \tag{8.7}$$

By the Gronwall–Bellman theorem, it follows from [41] that a solution of the inequalities (8.5) and (8.7) is a function $\Phi(t)$ bounded on every interval $t\in[0,T]$. Therefore, by Theorem 6.6 and Lemma 7.4 we arrive at the following assertion.

Theorem 8.1. *Suppose that $q+2\leqslant\overline{p}=\max_{j=1,\dots,n}p_j$ and there is a Banach space V_j which can be embedded continuously in W_2 with $p_j=\overline{p}$. Then the existence time T_0 of the solution is equal to $+\infty$.*

Remark 8.2. Note that the hypotheses of Theorem 8.1 impose no restrictions on the size of the initial functions u_0 and u_1.

§ 9. Examples

We give examples of initial-boundary value problems for which the results obtained above hold. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$.

Example 9.1. Consider the following initial-boundary value problem:

$$\frac{\partial^2}{\partial t^2}\left(\Delta u - u + \sum_{j=1}^{n} |u|^{p_j-2}u\right) - u - \frac{\partial |u|^q u}{\partial t} = 0, \tag{9.1}$$

$$u(0) = u_0 \in H_0^1(\Omega), \qquad u'(0) = u_1 \in H_0^1(\Omega), \tag{9.2}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{9.3}$$

where $p_j > 2$, $q > 0$. Here we consider the following Banach spaces:

$$V_0 = H_0^1(\Omega), \qquad V_j = L^{p_j}(\Omega), \qquad W_1 = H = L^2(\Omega), \qquad W_2 = L^{q+2}(\Omega). \tag{9.4}$$

Example 9.2. Consider the following initial-boundary value problem:

$$\frac{\partial^2}{\partial t^2}\left(\Delta u - u + \sum_{j=1}^{n} |u|^{p_j-2}u\right) + a_1\Delta u - a_2 u - \frac{\partial |u|^q u}{\partial t} = 0, \tag{9.5}$$

$$u(0) = u_0 \in H_0^1(\Omega), \qquad u'(0) = u_1 \in H_0^1(\Omega), \tag{9.6}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{9.7}$$

where $p_j > 2$, $q > 0$, $a_1 > 0$ and $a_2 > 0$. Here we consider the following Banach spaces:

$$V_0 = H_0^1(\Omega), \qquad V_j = L^{p_j}(\Omega), \qquad W_1 = H_0^1(\Omega), \tag{9.8}$$

$$H = L^2(\Omega), \qquad W_2 = L^{q+2}(\Omega). \tag{9.9}$$

Example 9.3. Consider the following initial-boundary value problem:

$$\frac{\partial^2}{\partial t^2}\left(-\Delta^2 u + \Delta u + \sum_{j=1}^{n} \operatorname{div}(|\nabla u|^{p_j-2}\nabla u)\right) + \Delta u = \frac{\partial}{\partial t}\operatorname{div}(|\nabla u|^q \nabla u), \tag{9.10}$$

$$u(0) = u_0 \in H_0^2(\Omega), \qquad u'(0) = u_1 \in H_0^2(\Omega), \tag{9.11}$$

$$u(x,t) = \frac{\partial u(x,t)}{\partial n_x} = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{9.12}$$

where $p_j > 2$, $q > 0$. Here we consider the following Banach spaces:

$$V_0 = H_0^2(\Omega), \qquad V_j = W_0^{1,p_j}(\Omega), \qquad W_1 = H_0^1(\Omega), \tag{9.13}$$

$$H = L^2(\Omega), \qquad W_2 = W_0^{1,q+2}(\Omega). \tag{9.14}$$

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