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Blow-up instability in non-linear wave models with distributed parameters

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Abstract. We consider two model non-linear equations describing electric oscillations in systems with distributed parameters on the basis of diodes with non-linear characteristics. We obtain equivalent integral equations for classical solutions of the Cauchy problem and the first and second initial-boundary value problems for the original equations in the half-space $x > 0$. Using the contraction mapping principle, we prove the local-in-time solubility of these problems. For one of these equations, we use the Pokhozhaev method of non-linear capacity to deduce *a priori* bounds giving rise to finite-time blow-up results and obtain upper bounds for the blow-up time. For the other, we use a modification of Levine's method to obtain sufficient conditions for blow-up in the case of sufficiently large initial data and give a lower bound for the order of growth of a functional with the meaning of energy. We also obtain an upper bound for the blow-up time.

Keywords: non-linear equations of Sobolev type, destruction, blow-up, local solubility, non-linear capacity, bounds for the blow-up time.

§ 1. Introduction

The so-called LC-chain transmission lines on the basis of semiconductor diodes (stabilitrons, varicaps with non-linear characteristics) are described by a differential-difference equation which reduces to the following equation [1] in the infinite transmission line limit:

$$\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\frac{\alpha}{c_0^2}\frac{\partial^2\phi^2}{\partial t^2}-\frac{\beta}{c_0^2}\frac{\partial^2}{\partial t^2}\frac{\partial^2\phi}{\partial x^2}=0, \qquad x\in\mathbb{R}^1, \quad t>0, \tag{1.1}$$

where $\phi=\phi(x,t)$ is the electric field potential, c_0 is the phase velocity of linear waves, α is the so-called coefficient of non-linearity and β is the coefficient of temporal dispersion. Note that analogous equations arise in the study of biological membranes and nerve fibres, which may be described as non-linear two-wire transmission lines with an active element similar to the $(p-n)$ junction [2]. Moreover, a similar equation arises in the study of non-linear ion-sound waves in the large Debye length limit [3]:

$$\frac{\partial^2}{\partial t^2}\biggl(\Delta\phi-\varepsilon\phi-\frac{\varepsilon^2}{2}\phi^2\biggr)+\Delta\phi=0, \qquad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}. \tag{1.2}$$

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Here $(x,y,z)\in\mathbb{R}^3$, $t>0$, $\phi=\phi(x,y,z,t)$ is the electric field potential and $\varepsilon=1/r_D^2$, where r_D is the Debye length.

Note that equations (1.1) and (1.2) contain the non-coercive non-linear term $\partial^2\phi^2/\partial t^2$. Such a non-linearity often arises in mathematical physics. The following equation was suggested in [4]:

$$\diamondsuit_{|\mathbf{A}|}E=0,\qquad \diamondsuit_{|\mathbf{A}|}:=\Delta-\frac{\partial}{\partial t}\left(|\mathbf{A}(E)|^2\frac{\partial}{\partial t}\right),\qquad (x,y,z)\in\mathbb{R}^3,\quad t>0.$$

Moreover, the following $(1+1)$-dimensional equation arises in electrodynamics:

$$c^2\Delta E=\frac{\partial^2}{\partial t^2}(-4\pi E^2+E),\qquad (x,y,z)\in\mathbb{R}^3,\quad t>0. \tag{1.3}$$

Sufficient conditions for the occurrence of blow-up regimes in the Cauchy problem and initial-boundary value problems for (1.3) were obtained in [3].

The following equation was suggested in [5] when considering self-oscillations in systems with distributed parameters on the basis of tunnel diodes with non-linear characteristics:

$$\frac{\partial^2\phi}{\partial t^2}-c_0^2\frac{\partial^2\phi}{\partial x^2}-\beta\frac{\partial^2}{\partial t^2}\frac{\partial^2\phi}{\partial x^2}=\gamma\frac{\partial}{\partial t}(\phi^3-\phi),\qquad x\in\mathbb{R}^1,\quad t>0,\quad \beta>0,\quad \gamma>0. \tag{1.4}$$

Here $\phi=\phi(x,t)$ is the electric field potential. Conditions for the occurrence of blow-up regimes in the first boundary-value problem on an interval were obtained in [6]. The following equation arises in the study of quasi-stationary processes in semiconductors (see [7]):

$$\begin{gathered}\frac{\partial^2\phi}{\partial t^2}-c_0^2\frac{\partial^2\phi}{\partial x^2}-\beta\frac{\partial^2}{\partial t^2}\frac{\partial^2\phi}{\partial x^2}=\gamma\frac{\partial}{\partial t}|\phi|^q\phi,\\ x\in\mathbb{R}^1,\quad t>0,\quad q>0,\quad \beta>0,\quad \gamma>0.\end{gathered} \tag{1.5}$$

We use the term 'distributed parameters' in the title of the present paper because the equations originally arising for finite electric chains (or electric chains of nerve cells in the biological case) are differential-difference equations, and partial differential equations arise in the infinite chain limit.

In this paper we study the Cauchy problem ($x\in\mathbb{R}^1$, $t>0$) and the first and second boundary-value problems in a half-space ($x>0$, $t>0$) for the equations (1.1) and (1.5). We prove their unique solubility in the classical sense and use the non-linear capacity method of Pokhozhaev [8] to obtain *a priori* bounds which give rise to conditions for the occurrence of blow-up regimes in the Cauchy problem and the second mixed boundary-value problem for (1.1). In the Cauchy problem for (1.5), we use the method of Levine [9], [10] (see also [7]) to obtain sufficient conditions for the destruction of solutions with sufficiently large initial data. We consider the equations (1.1) and (1.5) in a single paper because they have a common linear part. We use the potential theory for this linear part to obtain equivalent integral equations. They differ only in the volume potential that depends on the corresponding non-linearities. The solubility of the resulting non-linear integral equations can be studied using the contraction mapping principle. We first prove their solubility (in certain Banach spaces to be defined in the next section) on a small time interval $[0,T]$. Nothing more can be said about the equation (1.1) since the order of the derivative in time of the non-linear term

$$\frac{\partial^2 u^2}{\partial t^2}$$

coincides with the leading order of the derivative in time in the linear part of the equation. But for (1.5), the order of the derivative of the non-linear term

$$\frac{\partial |u|^q u}{\partial t}$$

is equal to 1 and the leading order in the linear part is equal to 2. Hence we prove the existence of so-called non-extendable solutions of the corresponding problems. This means that we prove the existence of a $T_0>0$ such that the solution $u(x,t)$ of the integral equation exists in the class $\mathbb{C}([0,T_0);\mathbb{B})$ and either $T_0=+\infty$ or $T_0<+\infty$, and the following limit property holds in the latter case:

$$\lim_{t\uparrow T_0} \|u(t)\|_{\mathbb{B}} = +\infty.$$

We mention the papers [11]–[23] devoted to investigation of sufficient conditions for the occurrence of blow-up regimes and local-in-time solubility of non-linear equations of mathematical physics. They contain analytical and numerical studies of the occurrence of blow-up regimes in mathematical models of plasma physics and the physics of semiconductors. Conditions for instantaneous blow-up have been found for some equations ([3], [12]).

Equations (1.1), (1.4) and (1.5) belong to the class of non-linear equations of Sobolev type. We note that linear and non-linear equations of Sobolev type have been studied in many papers. For example, initial-boundary value problems for equations of Sobolev type have been considered in a general form and as examples in the papers of Sviridyuk, Zagrebina and Zamyshlyaeva [24]–[26].

§ 2. Notation

Given any $a,b,d\in\mathbb{R}^1\cup\{-\infty\}\cup\{+\infty\}$, we write $\lfloor a,b\rceil$ for an interval of any of the four types

$$(a,b), \quad [a,b], \quad (a,b], \quad [a,b).$$

We write $\lfloor a,d]$ for the intervals

$$(a,d], \quad [a,d],$$

and $[d,b\rceil$ for the two intervals

$$[d,b), \quad [d,b].$$

By the function space $\mathbb{C}^{(m)}([0,d];\mathbb{C}^{(n)}\lfloor a,b\rceil)$ we mean the set of functions $u(x,t)$ of $(x,t)\in\lfloor a,b\rceil\times[0,d]$ such that u and all its partial derivatives of order at most m in the variable $t\in[0,d]$ and at most n in the variable $x\in\lfloor a,b\rceil$ (with the derivatives in x and t being taken in any order) belong to $\mathbb{C}(\lfloor a,b\rceil\times[0,d])$. Here the derivatives at the boundary points of $[0,d] \times \lfloor a,b \rceil$ are understood as the corresponding one-sided derivatives. Then all these mixed partial differentiations in $x \in \lfloor a,b \rceil$ and $t \in [0,d]$ commute. The class of functions $\mathbb{C}^{(m)}([0,T];\mathbb{C}_b^{(n)}\lfloor a,b \rceil)$ is defined in a similar way. The subscript b means that the function and all its partial derivatives of total order less than or equal to $m+n$ belong to $\mathbb{C}([0,T];\mathbb{C}_b\lfloor a,b \rceil)$, that is, they are continuous bounded functions of $(x,t) \in \lfloor a,b \rceil \times [0,T]$.

By the class of functions $\mathbb{C}^{(m)}([0,T];\mathbb{C}_b^{(n)}((1+x^2)^{\alpha/2};[0,+\infty)))$ for $\alpha > 0$ we understand the set of functions $u(x,t) \in \mathbb{C}^{(m)}([0,T];\mathbb{C}_b^{(n)}[0,+\infty))$ satisfying the following bound for $(x,t) \in [0,+\infty) \times [0,T]$:

$$(1+x^2)^{\alpha/2}\left|\frac{\partial^{k+l}u(x,t)}{\partial x^k\,\partial t^l}\right| \leqslant c(T,k,l,\alpha) < +\infty,$$

where $k \in \{0,1,\dots,n\}$, $l \in \{0,1,\dots,m\}$, $m,n \in \mathbb{Z}_+$. The class of functions $\mathbb{C}^{(m)}([0,T];\mathbb{C}_b^{(n)}((1+x^2)^{\alpha/2};\mathbb{R}^1))$ is defined in a similar way. By $\mathbb{C}^{(m)}([0,T_0);\mathbb{C}_b^{(n)}((1+x^2)^{\alpha/2};[0,+\infty)))$ and $\mathbb{C}^{(m)}([0,T_0);\mathbb{C}_b^{(n)}((1+x^2)^{\alpha/2};\mathbb{R}^1))$ we mean the sets of functions $u(x,t)$ such that, for every $T \in (0,T_0)$, one has $u(x,t) \in \mathbb{C}^{(m)}([0,T];\mathbb{C}_b^{(n)}((1+x^2)^{\alpha/2};[0,+\infty)))$ and $u(x,t) \in \mathbb{C}^{(m)}([0,T];\mathbb{C}_b^{(n)}((1+x^2)^{\alpha/2};\mathbb{R}^1))$ respectively. Moreover,

$$\mathbb{C}_b((1+x^2)^{\alpha/2};\mathbb{R}^1) = \left\{v(x) \in \mathbb{C}^{(2)}(\mathbb{R}^1)\colon \sup_{x\in\mathbb{R}^1}(1+x^2)^{\alpha/2}\left|\frac{\partial^k v(x)}{\partial x^k}\right| < +\infty\right\},$$

$$\mathbb{C}_b((1+x^2)^{\alpha/2};[0,+\infty)) = \left\{v(x) \in \mathbb{C}^{(2)}(\mathbb{R}^1)\colon \sup_{x\in[0,+\infty)}(1+x^2)^{\alpha/2}\left|\frac{\partial^k v(x)}{\partial x^k}\right| < +\infty\right\}$$

for $k = 0,1,2$. We write

$$\|v\|_T := \sup_{(x,t)\in\mathbb{R}^1\times[0,T]}\sum_{j=0}^{2}\left|\frac{\partial^j v(x,t)}{\partial t^j}\right|$$

for the norm on the Banach space $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1))$ and

$$\|v\|_T^+ := \sup_{(x,t)\in[0,+\infty)\times[0,T]}\sum_{j=0}^{2}\left|\frac{\partial^j v(x,t)}{\partial t^j}\right|$$

for the norm on the Banach space $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b[0,+\infty))$. We also write

$$\|v\|_{\alpha,T} := \sup_{(x,t)\in\mathbb{R}^1\times[0,T]}(1+x^2)^{\alpha/2}|v(x,t)|$$

for the norm on the Banach space $\mathbb{C}([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};\mathbb{R}^1))$, $\alpha > 0$, and

$$\|v\|_{\alpha,T}^+ := \sup_{(x,t)\in[0,+\infty)\times[0,T]}(1+x^2)^{\alpha/2}|v(x,t)|$$

for the norm on the Banach space $\mathbb{C}([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};[0,+\infty)))$, $\alpha > 0$.

We also use the notation

$$\left\{\frac{\partial^j f(x-\xi,t)}{\partial x^j}\right\}, \qquad j=1,2,$$

for $f(x-\xi,t)\in\mathbb{C}^{(2)}\lfloor a,\xi]\cap\mathbb{C}^{(2)}[\xi,b\rceil$, where $\xi\in(a,b)$ for all $t\geqslant 0$ and the derivatives at ξ are understood as one-sided derivatives. This function is equal to

$$\frac{\partial^j f(x-\xi,t)}{\partial x^j} \quad \text{for} \quad x\neq\xi, \quad t\geqslant 0,$$

and is defined in an arbitrary way at $x=\xi$. Let $\|\cdot\|_p$ be the standard norm on the Lebesgue space $L^p(\mathbb{R}^1)$, where $p\geqslant 1$. We write $\mathbb{C}^{(n)}([0,T];H^s(\mathbb{R}^1))$ for the space of $H^s(\mathbb{R}^1)$-valued functions $f(t)\colon[0,T]\to H^s(\mathbb{R}^1)$ (where $H^s(\mathbb{R}^1)$ is the familiar Sobolev space) such that the strong derivatives $f^{(k)}(t)$, $t\in[0,T]$, belong to $\mathbb{C}([0,T];H^s(\mathbb{R}^1))$ for all $k=0,\dots,n$. Here the strong derivative $f'(t_0)$ of a function $f(t)$ at a point $t_0\in(0,T)$ is understood in the following sense:

$$\lim_{t\to t_0}\left\|\frac{f(t)-f(t_0)}{t-t_0}-f'(t_0)\right\|_{H^s(\mathbb{R}^1)}=0,$$

and the strong derivatives at $t_0=0$ and $t_0=T$ are understood as one-sided limits. The space $\mathbb{C}([0,T];H^s(\mathbb{R}^1))$ is the set of functions $f(t)\in H^s(\mathbb{R}^1)$ such that

$$\|f(t_1)-f(t_2)\|_{H^s(\mathbb{R}^1)}\to+0 \quad \text{as} \quad |t_1-t_2|\to+0 \quad \text{for all} \quad t_1,t_2\in[0,T].$$

§ 3. Definitions of the classical solutions of the problems

In this paper we shall consider the two equations

$$\frac{\partial^2}{\partial t^2}(u_{xx}-u)+u_{xx}-\frac{\partial^2 u^2}{\partial t^2}=0, \tag{3.1}$$

$$\frac{\partial^2}{\partial t^2}(u_{xx}-u)+u_{xx}+\frac{\partial |u|^q u}{\partial t}=0, \qquad q>0. \tag{3.2}$$

Definition 1. A classical solution of the Cauchy problem for (3.1) or (3.2) is a function

$$u(x,t)\in\mathbb{C}^{(2)}\bigl([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1)\bigr)$$

in the case of (3.1) or a function

$$u(x,t)\in\mathbb{C}^{(2)}\bigl([0,T];\mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2};\mathbb{R}^1)\bigr) \quad \text{with} \quad \alpha>0$$

in the case of (3.2) such that the equations (3.1), (3.2) hold pointwise for all $(x,t)\in\mathbb{R}^1\times[0,T]$ for some $T>0$, and the initial conditions

$$u(x,0)=u_0(x), \quad \frac{\partial u}{\partial t}(x,0)=u_1(x), \qquad x\in\mathbb{R}^1,$$

hold, where $u_0(x),u_1(x)\in\mathbb{C}_b^{(2)}(\mathbb{R}^1)$ in the case of the equation (3.1) or $u_0(x),u_1(x)\in\mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2};\mathbb{R}^1)$ in the case of (3.2).

Definition 2. A classical solution of the first boundary-value problem for (3.1) or (3.2) is a function

$$u(x,t) \in \mathbb{C}^{(2)}\big([0,T];\mathbb{C}_b^{(2)}[0,+\infty)\big)$$

in the case of (3.1) or a function

$$u(x,t) \in \mathbb{C}^{(2)}\big([0,T];\mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2};[0,+\infty))\big) \quad \text{with} \quad \alpha > 0$$

in the case of (3.2) such that the equations (3.1), (3.2) hold at every point $(x,t) \in [0,+\infty) \times [0,T]$ for some $T > 0$ and the initial and boundary conditions

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \qquad x \in [0,+\infty),$$
$$u(0,t) = \nu(t), \qquad t \in [0,T],$$

hold, where $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}[0,+\infty)$ in the case of the equation (3.1) or $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2};[0,+\infty))$ in the case of (3.2), and $\nu(t) \in \mathbb{C}^{(2)}[0,T]$ for both.

Definition 3. A classical solution of the second boundary-value problem for (3.1) or (3.2) is a function

$$u(x,t) \in \mathbb{C}^{(2)}\big([0,T];\mathbb{C}_b^{(2)}[0,+\infty)\big)$$

in the case of (3.1) or a function

$$u(x,t) \in \mathbb{C}^{(2)}\big([0,T];\mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2};[0,+\infty))\big) \quad \text{with} \quad \alpha > 0$$

in the case of (3.2) such that the equations (3.1), (3.2) hold at every point $(x,t) \in [0,+\infty) \times [0,T]$ for some $T > 0$ and the initial and boundary conditions

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \qquad x \in [0,+\infty),$$
$$u_x(0,t) = \mu(t), \qquad t \in [0,T],$$

hold, where $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}[0,+\infty)$ in the case of the equation (3.1) or $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2};[0,+\infty))$ in the case of (3.2), and $\mu(t) \in \mathbb{C}^{(2)}[0,T]$ for both.

§ 4. Potential theory

To study the questions of local-in-time solubility of the problems posed above, we use the potential theory developed in [27]. We note that potential theory for non-classical equations of Sobolev type was first considered by Kapitonov [28]. It was then studied by Gabov and Sveshnikov [29], [30] and their students (for example, Pletner [31]).

Consider the operator

$$\mathfrak{M}_{x,t}u(x,t) := \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x^2} - u\right) + \frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t). \tag{4.1}$$

The potential theory for this operator was developed in [27]. Its fundamental solution is of the form

$$\mathscr{E}(x,t) = -\frac{\theta(t)}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\mu x}}{\mu\sqrt{\mu^2+1}}\sin\biggl(\frac{\mu t}{\sqrt{\mu^2+1}}\biggr)\,d\mu. \tag{4.2}$$

It follows directly from the formula (4.2) that $\mathscr{E}(x,t)$ is an even function of x. Therefore one can write

$$\mathscr{E}(x,t) = -\frac{\theta(t)}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\mu|x|}}{\mu\sqrt{\mu^2+1}}\sin\biggl(\frac{\mu t}{\sqrt{\mu^2+1}}\biggr)\,d\mu.$$

Using the analytic extension

$$F(z) = \frac{e^{izx}}{z\sqrt{z^2+1}}\sin\biggl(\frac{zt}{\sqrt{z^2+1}}\biggr),$$

where $\sqrt{z}$ is the principal branch. of the integrand in (4.2) to the upper half-plane $\operatorname{Im} z > 0$ for $x > 0$ and to the lower half-plane $\operatorname{Im} z < 0$ for $x < 0$ except the singular points $z = \pm i$, one can obtain the representations

$$\mathscr{E}(x,t) = -\frac{\theta(t)}{2\pi}\int_{C_\varepsilon^+(i)}\frac{e^{izx}}{z\sqrt{z^2+1}}\sin\biggl(\frac{zt}{\sqrt{z^2+1}}\biggr)\,dz, \tag{4.3}$$

$$\mathscr{E}(x,t) = -\frac{\theta(t)}{2\pi}\int_{C_\varepsilon^+(-i)}\frac{e^{izx}}{z\sqrt{z^2+1}}\sin\biggl(\frac{zt}{\sqrt{z^2+1}}\biggr)\,dz, \tag{4.4}$$

where $C_\varepsilon(\pm i) := \{z\in\mathbb{C}\colon |z\mp i| = \varepsilon\}$ with $\varepsilon\in(0,1)$.

The integral representations (4.2)–(4.4) of the fundamental solution $\mathscr{E}(x,t)$ give rise to the properties collected in the following lemma.

Lemma 1. 1) *For all $m,n\in\mathbb{Z}_+$ we have*

$$\mathscr{E}(x,t)\in\mathbb{C}^{(m)}\bigl([0,+\infty);\mathbb{C}^{(n)}(-\infty,0]\bigr)\cap\mathbb{C}^{(m)}\bigl([0,+\infty);\mathbb{C}^{(n)}[0,+\infty)\bigr).$$

2) *The function*

$$F(x,t) := \frac{\partial^2\mathscr{E}(x,t)}{\partial t^2} + \mathscr{E}(x,t)$$

belongs to $\mathbb{C}^{(n)}\bigl([0,+\infty);\mathbb{C}^{(2)}(\mathbb{R}^1)\bigr)$ for every $n\in\mathbb{Z}_+$.

3) *The fundamental solution $\mathscr{E}(x,t)$ satisfies the following equalities for $t\geqslant 0$, $x\in\mathbb{R}^1$:*

$$\mathscr{E}(x,0)=0,\qquad \mathscr{E}(0,t) = -\frac{1}{2}\int_0^t J_0(s)\,ds,\qquad \frac{\partial\mathscr{E}}{\partial t}(x,0) = -\frac{1}{2}e^{-|x|},$$
$$\frac{\partial^2\mathscr{E}}{\partial t^2}(x,0)=0,\qquad \mathfrak{N}_{x,t}\mathscr{E}(0,t)=0,\qquad \mathfrak{N}_{x,t}\frac{\partial\mathscr{E}}{\partial t}(x,0) = \frac{x}{4}e^{-|x|}, \tag{4.5}$$

where $J_0(s)$ is the Bessel function of order zero and the operator $\mathfrak{N}_{x,t}$ acts on a function $u(x,t)$ by the rule

$$\mathfrak{N}_{x,t}u(x,t) := \frac{\partial}{\partial x}\biggl(\frac{\partial^2 u(x,t)}{\partial t^2} + u(x,t)\biggr). \tag{4.6}$$

4) *For all* $x \neq 0$, $t \geqslant 0$ *and* $k, l \in \mathbb{Z}_+$ *we have the following bounds for the fundamental solution*:

$$\left|\frac{\partial^{k+l}\mathscr{E}(x,t)}{\partial x^k \partial t^l}\right| \leqslant a_0(\varepsilon,k,l)\exp\bigl(-\varepsilon|x| + a_1(\varepsilon)t\bigr) \tag{4.7}$$

for every $\varepsilon \in (0,1)$, *where* $a_0(\varepsilon,k,l) > 0$ *and* $a_1(\varepsilon) > 0$ *are constants.*

Proof. See [27]. □

Definition 4. Given any $\varepsilon \in (0,1)$ and $T > 0$, we write $M_\varepsilon(T)$ for the set of functions $u(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}^{(2)}(\mathbb{R}^1))$ satisfying the inequalities

$$\begin{gathered} |\mathfrak{M}_{x,t}u(x,t)| \leqslant c_1(T)\exp(\varepsilon|x|), \\ |u(x,t)| \leqslant c_2(T)\exp(\varepsilon|x|), \qquad |u_x(x,t)| \leqslant c_3(T)\exp(\varepsilon|x|), \\ |u_{xx}(x,0) - u(x,0)| \leqslant c_4(T)\exp(\varepsilon|x|), \qquad |u_{txx}(x,0) - u_t(x,0)| \leqslant c_5(T)\exp(\varepsilon|x|) \end{gathered}$$

for all $t \in [0,T]$, where the $c_j(T)$ are certain constants, $j = 1,\dots,5$.

The following assertion holds (see [27]).

Lemma 2. *For every function* $u(x,t) \in M_\varepsilon(T)$ *we have*

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^1} \mathscr{E}(x-\xi,t-\tau)\mathfrak{M}_{\xi,\tau}u(\xi,\tau)\,d\xi\,d\tau \\ &\quad + \int_{\mathbb{R}^1}\biggl(\mathscr{E}(x-\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi)] + \frac{\partial\mathscr{E}(x-\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)]\biggr)\,d\xi, \end{aligned} \tag{4.8}$$

where

$$u_0(x) := u(x,0), \qquad u_1(x) := \frac{\partial u(x,0)}{\partial t}.$$

Proof. The proof is based on an analogue of Green's second identity for the operator $\mathfrak{M}_{x,t}$; see (4.1). Using the explicit formulae (4.2)–(4.4) for the fundamental solution $\mathscr{E}(x,t)$ and the properties collected in Lemma 1, we deduce from it an analogue of Green's third identity on the interval $[a,b]$ of the variable x. Letting $a \to -\infty$ and $b \to +\infty$ in the result for functions $u(x,t) \in M_\varepsilon(T)$, we obtain (4.8). □

To obtain an analogue of (4.8) for problems on the half-line $[0,+\infty)$ of the variable x, we first define the corresponding class of functions.

Definition 5. Given any $\varepsilon \in (0,1)$ and $T > 0$, we write $M^+_\varepsilon(T)$ for the set of functions $u(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}^{(2)}[0,+\infty))$ satisfying the inequalities

$$\begin{gathered} |\mathfrak{M}_{x,t}u(x,t)| \leqslant c_1(T)\exp(\varepsilon x), \\ |u(x,t)| \leqslant c_2(T)\exp(\varepsilon x), \qquad |u_x(x,t)| \leqslant c_3(T)\exp(\varepsilon x), \\ |u_{xx}(x,0) - u(x,0)| \leqslant c_4(T)\exp(\varepsilon x), \qquad |u_{txx}(x,0) - u_t(x,0)| \leqslant c_5(T)\exp(\varepsilon x) \end{gathered}$$

for all $t \in [0,T]$, where the $c_j(T)$ are certain constants, $j = 1,\dots,5$.

Lemma 3. *For every function* $u(x,t) \in M_\varepsilon^+(T)$ *we have*

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_0^{+\infty} \mathscr{E}_1(x,\xi,t-\tau)\mathfrak{M}_{\xi,\tau}u(\xi,\tau)\,d\xi\,d\tau \\
&\quad + \int_0^{+\infty} \biggl(\mathscr{E}_1(x,\xi,t)[u_{1\xi\xi}(\xi)-u_1(\xi)] + \frac{\partial \mathscr{E}_1(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi)-u_0(\xi)]\biggr)\,d\xi \\
&\quad + 2\int_0^t u(0,\tau)\mathfrak{N}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau + u(0,t)e^{-x}, \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_0^{+\infty} \mathscr{E}_2(x,\xi,t-\tau)\mathfrak{M}_{\xi,\tau}u(\xi,\tau)\,d\xi\,d\tau \\
&\quad + \int_0^{+\infty} \biggl(\mathscr{E}_2(x,\xi,t)[u_{1\xi\xi}(\xi)-u_1(\xi)] + \frac{\partial \mathscr{E}_2(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi)-u_0(\xi)]\biggr)\,d\xi \\
&\quad + 2\int_0^t u_x(0,\tau)\biggl[\frac{\partial^2 \mathscr{E}(x,t-\tau)}{\partial t^2} + \mathscr{E}(x,t-\tau)\biggr]\,d\tau - u_x(0,t)e^{-x}, \qquad (4.10)
\end{aligned}
$$

where

$$
\mathscr{E}_1(x,\xi,t) := \mathscr{E}(x-\xi,t) - \mathscr{E}(x+\xi,t), \qquad \mathscr{E}_2(x,\xi,t) := \mathscr{E}(x-\xi,t) + \mathscr{E}(x+\xi,t),
$$

the operator $\mathfrak{M}_{x,t}$ *is defined in* (4.1) *and the operator* $\mathfrak{N}_{x,t}$ *is defined in* (4.6).

Proof. See [27]. □

It follows from (4.8)–(4.10) that one should study the properties of the potentials occurring in the right-hand sides of (4.8)–(4.10).

§ 5. Properties of the potentials related to the Cauchy problem

We study the properties of the potentials in the right-hand side of (4.8). First consider a potential of the form

$$
V(x,t) = V[\mu](x,t) := \int_{\mathbb{R}^1} \mathscr{E}(x-\xi,t)\mu_0(\xi)\,d\xi.
$$

We put

$$
V_k(x,t) := \int_{-\infty}^{+\infty} \mathscr{E}_k(x-\xi,t)\mu_0(\xi)\,d\xi, \quad \mathscr{E}_k(x,t) := \frac{\partial^k \mathscr{E}(x,t)}{\partial t^k}, \qquad k \in \mathbb{Z}_+.
$$

Lemma 4. *Suppose that* $\mu_0(x) \in \mathbb{C}_b(\mathbb{R}^1)$. *Then*

$$
V(x,t) \in \mathbb{C}^n\bigl([0,+\infty);\mathbb{C}_b^{(2)}(\mathbb{R}^1)\bigr)
$$

for every $n \in \mathbb{N}$ *and we have*

$$
\mathfrak{M}_{x,t}V(x,t) = 0 \quad \text{for all} \quad x \in \mathbb{R}^1, \quad t \geqslant 0,
$$

$$
V(x,0) = 0, \qquad \frac{\partial V}{\partial t}(x,0) = -\int_{\mathbb{R}^1} \frac{e^{-|x-\xi|}}{2}\mu_0(\xi)\,d\xi, \qquad \frac{\partial^2 V}{\partial t^2}(x,0) = 0
$$

for $x \in \mathbb{R}^1$.

Proof. Step 1. It follows from the explicit formula (4.2) for the fundamental solution $\mathscr{E}(x,t)$ that the potential $V[\mu](x,t)$ is infinitely differentiable with respect to $t \in [0,+\infty)$ and one can differentiate with respect to time under the integral sign.

Fix an arbitrary point $(x,t) \in \mathbb{R}^1 \times \mathbb{R}_+$. Choose a sufficiently large $R > 0$ such that $x \in (-R,R)$. Then the expression for $V_k(x,t)$ can be rewritten in the form

$$V_k(x,t) = \left[\int_{-\infty}^{-R} + \int_{R}^{+\infty}\right] \mathscr{E}_k(x-\xi,t)\mu_0(\xi)\,d\xi + \left[\int_{-R}^{x} + \int_{x}^{R}\right] \mathscr{E}_k(x-\xi,t)\mu_0(\xi)\,d\xi.$$

By part 1) of Lemma 1 we have

$$\begin{aligned} \frac{\partial V_k(x,t)}{\partial x} &= \left[\int_{-\infty}^{-R} + \int_{R}^{+\infty}\right] \frac{\partial \mathscr{E}_k(x-\xi,t)}{\partial x}\mu_0(\xi)\,d\xi \\ &\quad + \mathscr{E}_k(0-0,t)\mu_0(x) - \mathscr{E}_k(0+0,t)\mu_0(x) + \left[\int_{-R}^{x} + \int_{x}^{R}\right] \frac{\partial \mathscr{E}_k(x-\xi,t)}{\partial x}\mu_0(\xi)\,d\xi. \end{aligned} \tag{5.1}$$

It follows from the explicit formula (4.2) for the fundamental solution that

$$\mathscr{E}_k(0-0,t) = \mathscr{E}_k(0+0,t), \qquad k \in \mathbb{Z}_+.$$

Therefore (5.1) implies that

$$\frac{\partial V_k(x,t)}{\partial x} = \int_{\mathbb{R}^1} \left\{\frac{\partial \mathscr{E}_k(x-\xi,t)}{\partial x}\right\}\mu_0(\xi)\,d\xi. \tag{5.2}$$

We now differentiate (5.2) with respect to x and obtain

$$\begin{aligned} \frac{\partial^2 V_k(x,t)}{\partial x^2} &= \mu_0(x)\left[\frac{\partial \mathscr{E}_k(0-0,t)}{\partial x} - \frac{\partial \mathscr{E}_k(0+0,t)}{\partial x}\right] \\ &\quad + \int_{\mathbb{R}^1} \left\{\frac{\partial^2 \mathscr{E}_k(x-\xi,t)}{\partial x^2}\right\}\mu_0(\xi)\,d\xi. \end{aligned} \tag{5.3}$$

It follows from the integral representations (4.3) and (4.4) for the fundamental solution that the right-hand sides of (5.2) and (5.3) are infinitely differentiable functions of $t \in [0,+\infty)$. Moreover,

$$\frac{\partial^m}{\partial t^m}\frac{\partial V_k(x,t)}{\partial x},\ \frac{\partial^m}{\partial t^m}\frac{\partial^2 V_k(x,t)}{\partial x^2} \in \mathbb{C}_b(\mathbb{R}^1 \times [0,+\infty))$$

for all $k, m \in \mathbb{Z}_+$.

Step 2. Given any $t \geqslant 0$ and $x \in \mathbb{R}^1$, we apply the operator

$$\mathfrak{D}_{x,t} := \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial t^2} + I\right) - \frac{\partial^2}{\partial t^2} \tag{5.4}$$

to the fundamental solution $\mathscr{E}(x,t)$ defined in (4.2), where I is the identity operator. It follows from (4.2) that one can differentiate it arbitrarily many times with respect to t under the improper integral sign. We easily see that

$$\left(\frac{\partial^2}{\partial t^2} + I\right)\mathscr{E}(x,t) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\mu x}\frac{1}{\mu(\mu^2+1)^{3/2}}\sin\left(\frac{\mu t}{\sqrt{\mu^2+1}}\right)d\mu. \tag{5.5}$$

Note that the right-hand side of (5.5) is twice differentiable in x and one can differentiate under the integral sign. The following formula holds:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial t^2}+I\right)\mathscr{E}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\mu x}\frac{\mu}{(\mu^2+1)^{3/2}}\sin\left(\frac{\mu t}{\sqrt{\mu^2+1}}\right)d\mu. \tag{5.6}$$

Moreover, we have

$$-\frac{\partial^2}{\partial t^2}\mathscr{E}(x,t)=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\mu x}\frac{\mu}{(\mu^2+1)^{3/2}}\sin\left(\frac{\mu t}{\sqrt{\mu^2+1}}\right)d\mu. \tag{5.7}$$

Using (5.6) and (5.7), we arrive at the following equality in view of the definition (5.4) of the operator $\mathfrak{D}_{x,t}$:

$$\mathfrak{D}_{x,t}\mathscr{E}(x,t)=0 \quad \text{for} \quad x\in\mathbb{R}^1, \quad t\geqslant 0. \tag{5.8}$$

Step 3. By the results of Steps 1 and 2 we have a chain of equalities

$$\mathfrak{M}_{x,t}V(x,t)=\mathfrak{D}_{x,t}V(x,t)=\int_{\mathbb{R}^1}\mathfrak{D}_{x,t}\mathscr{E}(x-\xi,t)\mu_0(\xi)\,d\xi=0$$

for all $x\in\mathbb{R}^1$ and $t\geqslant 0$.
Step 4. It follows from Lemma 1 that

$$V(x,0)=0, \qquad \frac{\partial V}{\partial t}(x,0)=-\int_{\mathbb{R}^1}\frac{e^{-|x-\xi|}}{2}\mu_0(\xi)\,d\xi, \qquad \frac{\partial^2 V}{\partial t^2}(x,0)=0. \qquad \square$$

For completeness, we give a proof of the following known result.

Lemma 5. *Suppose that* $\mu_0(x)\in\mathbb{C}_b(\mathbb{R}^1)$. *Then the potential*

$$V_{10}(x)=V_{10}[\mu_0](x)=-\int_{\mathbb{R}^1}\frac{e^{-|x-\xi|}}{2}\mu_0(\xi)\,d\xi$$

belongs to $V_{10}(x)\in\mathbb{C}_b^{(2)}(\mathbb{R}^1)$ *and satisfies the following equation for every* $x\in\mathbb{R}^1$:

$$\frac{d^2V_{10}(x)}{dx^2}-V_{10}(x)=\mu_0(x).$$

Proof. The potential $V_{10}(x)$ can be written in the form

$$V_{10}(x)=-\int_{-\infty}^{x}\frac{e^{-(x-\xi)}}{2}\mu_0(\xi)\,d\xi-\int_{x}^{+\infty}\frac{e^{-(\xi-x)}}{2}\mu_0(\xi)\,d\xi.$$

It follows from the explicit formula for the potential that the function $V_{10}(x)$ is differentiable and its derivative is equal to

$$\begin{aligned}\frac{dV_{10}(x)}{dx}&=-\frac{1}{2}\mu_0(x)+\frac{1}{2}\mu_0(x)+\int_{-\infty}^{x}\frac{e^{-(x-\xi)}}{2}\mu_0(\xi)\,d\xi\\&\qquad-\int_{x}^{+\infty}\frac{e^{-(\xi-x)}}{2}\mu_0(\xi)\,d\xi\in\mathbb{C}_b(\mathbb{R}^1).\end{aligned} \tag{5.9}$$

In its turn, the explicit formula (5.9) for the first derivative of $V_{10}(x)$ implies that this derivative is also differentiable and we have

$$\frac{d^2V_{10}(x)}{dx^2} = \frac{1}{2}\mu_0(x) + \frac{1}{2}\mu_0(x) - \frac{1}{2}\int_{-\infty}^{x} e^{-(x-\xi)}\mu_0(\xi)\,d\xi \\ - \frac{1}{2}\int_{x}^{+\infty} e^{-(x-\xi)}\mu_0(\xi)\,d\xi = \mu_0(x) + V_{10}(x) \in \mathbb{C}_b(\mathbb{R}^1).$$

We obtain from this equality that

$$\frac{d^2V_{10}(x)}{dx^2} - V_{10}(x) = \mu_0(x) \quad \text{for all} \quad x \in \mathbb{R}^1. \qquad \square$$

The following classical result holds.

Lemma 6. *For every function $u_0(x) \in \mathbb{C}_b^{(2)}(\mathbb{R}^1)$ and all $x \in \mathbb{R}^1$ we have*

$$-\int_{\mathbb{R}^1} \frac{e^{-|x-\xi|}}{2}[u_{0\xi\xi}(\xi) - u_0(\xi)]\,d\xi = u_0(x). \tag{5.10}$$

Proof. We have a chain of equalities

$$\int_{\mathbb{R}^1} \frac{e^{-|x-\xi|}}{2}u_{0\xi\xi}(\xi)\,d\xi = \frac{1}{2}\int_{-\infty}^{x} e^{-(x-\xi)}u_{0\xi\xi}(\xi)\,d\xi + \frac{1}{2}\int_{x}^{+\infty} e^{-(\xi-x)}u_{0\xi\xi}(\xi)\,d\xi \\ = \frac{1}{2}u_{0x}(x) - \frac{1}{2}u_{0x}(x) - \frac{1}{2}\int_{-\infty}^{x} e^{-(x-\xi)}u_{0\xi}(\xi)\,d\xi + \frac{1}{2}\int_{x}^{+\infty} e^{-(\xi-x)}u_{0\xi}(\xi)\,d\xi \\ = -\frac{1}{2}u_0(x) - \frac{1}{2}u_0(x) + \frac{1}{2}\int_{-\infty}^{x} e^{-(x-\xi)}u_0(\xi)\,d\xi + \frac{1}{2}\int_{x}^{+\infty} e^{-(\xi-x)}u_0(\xi)\,d\xi \\ = -u_0(x) + \frac{1}{2}\int_{\mathbb{R}^1} e^{-|x-\xi|}u_0(\xi)\,d\xi.$$

This chain of equalities yields (5.10). $\square$

Finally, the following assertion holds.

Lemma 7. *Suppose that $\rho(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b(\mathbb{R}^1))$. Then the volume potential*

$$W(x,t) = W[\rho](x,t) := \int_0^t \int_{\mathbb{R}^1} \mathscr{E}(x-\xi, t-\tau)\rho(\xi,\tau)\,d\xi\,d\tau$$

belongs to $\in \mathbb{C}^{(2)}\big([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1)\big)$ and for all $(x,t) \in \mathbb{R}^1 \times [0,T]$ we have

$$\mathfrak{M}_{x,t}W(x,t) = \rho(x,t), \qquad W(x,0) = \frac{\partial W}{\partial t}(x,0) = 0.$$

Proof. Step 1. Just as at Step 1 of Lemma 4, one can prove that $W(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1))$.

Step 2. Note that the following equalities hold:

$$\frac{\partial W(x,t)}{\partial t} = \int_0^t \int_{\mathbb{R}^1} \frac{\partial \mathscr{E}(x-\xi,t-\tau)}{\partial t}\rho(\xi,\tau)\,d\xi\,d\tau, \tag{5.11}$$

$$\frac{\partial^2 W(x,t)}{\partial t^2} = -\int_{\mathbb{R}^1} \frac{e^{-|x-\xi|}}{2}\rho(\xi,t)\,d\xi + \int_0^t \int_{\mathbb{R}^1} \frac{\partial^2 \mathscr{E}(x-\xi,t-\tau)}{\partial t^2}\rho(\xi,\tau)\,d\xi\,d\tau. \tag{5.12}$$

It follows from (5.11) and (5.12) that

$$\begin{aligned}
&\mathfrak{M}_{x,t}W(x,t) = \mathfrak{D}_{x,t}W(x,t) = \frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 W(x,t)}{\partial t^2} + W(x,t)\right] - \frac{\partial^2 W(x,t)}{\partial t^2} \\
&= -\left(\frac{\partial^2}{\partial x^2} - I\right)\int_{\mathbb{R}^1} \frac{e^{-|x-\xi|}}{2}\rho(\xi,t)\,d\xi + \int_0^t \int_{\mathbb{R}^1} \mathfrak{D}_{x,t}\mathscr{E}(x-\xi,t-\tau)\rho(\xi,\tau)\,d\xi\,d\tau \\
&= \rho(x,t),
\end{aligned}$$

where we have used the results of Lemmas 4 and 5.
Step 3. The equalities

$$W(x,0) = \frac{\partial W}{\partial t}(x,0) = 0 \quad \text{for} \quad x \in \mathbb{R}^1$$

follow from part 3) of Lemma 1. □

§ 6. Properties of the potentials related to the first boundary-value problem

In this section we study the properties of the potentials in the right-hand side of (4.9). First of all, we study the properties of the potential

$$S(x,t) = S[\nu](x,t) := 2\int_0^t \nu(\tau)\mathfrak{N}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau + \nu(t)e^{-x}, \tag{6.1}$$

where

$$\mathfrak{N}_{x,t}f(x,t) := \frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial t^2} + I\right]f(x,t).$$

Lemma 8. *Suppose that* $\nu(t) \in \mathbb{C}^{(2)}[0,T]$ *and* $\nu(0) = \nu'(0) = 0$. *Then the potential* $S(x,t)$ *belongs to* $\in \mathbb{C}^{(2)}\big([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1_+)\big)$ *and we have*

$$\mathfrak{M}_{x,t}S(x,t) = 0 \quad \textit{for} \quad x \geqslant 0, \quad t \geqslant 0,$$

$$S(0,t) = \nu(t), \quad S(x,0) = \frac{\partial S}{\partial t}(x,0) = 0 \quad \textit{for} \quad t \geqslant 0, \quad x \geqslant 0.$$

Proof. First of all, by part 1) of Lemma 1, the fundamental solution $\mathscr{E}(x,t)$ belongs to $\mathbb{C}^{(n)}\big([0,T];\mathbb{C}_b^{(3)}(\mathbb{R}^1_+)\big)$ for every $n \in \mathbb{Z}_+$ and, therefore,

$$S(x,t) \in \mathbb{C}^{(n)}\big([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1_+)\big).$$

In view of part 3) of Lemma 1 (see (4.5)), one can easily deduce from the explicit representation (6.1) of the potential $S(x,t)$ that

$$S(0,t)=\nu(t).$$

Moreover, since $\nu(0)=\nu'(0)=0$, we have

$$S(x,0)=\nu(0)e^{-x}=0, \qquad \frac{\partial S}{\partial t}(x,0)=2\nu(0)\mathfrak{N}_{x,t}[\mathscr{E}](x,0)+\nu'(0)e^{-x}=0.$$

The following chain of equalities holds:

$$\begin{aligned}\mathfrak{M}_{x,t}[S](x,t)=2\nu(t)\biggl[\frac{\partial^2}{\partial x^2}-I\biggr]\mathfrak{N}_{x,t}\frac{\partial\mathscr{E}}{\partial t}(x,0)\\ +2\int_0^t\nu(\tau)\mathfrak{N}_{x,t}\mathfrak{M}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau+\nu(t)e^{-x}\\ =: I_1(x,t)+I_2(x,t)+I_3(x,t).\end{aligned}$$

Note that

$$\mathfrak{M}_{x,t}\mathscr{E}(x,t)=\mathfrak{D}_{x,t}\mathscr{E}(x,t)=0 \quad \text{for} \quad x\geqslant 0, \quad t\geqslant 0,$$

where the operator $\mathfrak{D}_{x,t}$ is defined in (5.8). Therefore,

$$I_2(x,t)=2\int_0^t\nu(\tau)\mathfrak{N}_{x,t}\mathfrak{D}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau=0, \tag{6.2}$$

where $\vartheta(x,t)\equiv 0$. By part 3) of Lemma 1 we have a chain of equalities

$$I_1(x,t)=2\nu(t)\biggl[\frac{\partial^2}{\partial x^2}-I\biggr]\mathfrak{N}_{x,t}\frac{\partial\mathscr{E}}{\partial t}(x,0)=2\nu(t)\biggl[\frac{\partial^2}{\partial x^2}-I\biggr]\frac{x}{4}e^{-x}=-\nu(t)e^{-x}. \tag{6.3}$$

Thus, we conclude from (6.2) and (6.3) that

$$I_1(x,t)+I_2(x,t)+I_3(x,t)=0. \qquad \square$$

Lemma 9. *Suppose that* $\mu_0(x)\in\mathbb{C}_b[0,+\infty)$. *Then the potential*

$$\widetilde{V}_1(x,t)=\widetilde{V}_1[\mu_0](x,t):=\int_0^{+\infty}\mathscr{E}_1(x,\xi,t)\mu_0(\xi)\,d\xi$$

belongs to $\mathbb{C}_t^{(n)}\bigl([0,+\infty);\mathbb{C}_b^{(2)}[0,+\infty)\bigr)$ *for every* $n\in\mathbb{Z}_+$, *where*

$$\mathscr{E}_1(x,\xi,t)=\mathscr{E}(x-\xi,t)-\mathscr{E}(x+\xi,t),$$

and we have

$$\mathfrak{M}_{x,t}\widetilde{V}_1(x,t)=0 \quad \textit{for} \quad (x,t)\in[0,+\infty)\times[0,+\infty),$$

$$\widetilde{V}_1(0,t)=\frac{\partial\widetilde{V}_1(0,t)}{\partial t}=0 \quad \textit{for} \quad t\geqslant 0,$$

$$\widetilde{V}_1(x,0)=0, \qquad \frac{\partial\widetilde{V}_1}{\partial t}(x,0)=-\int_0^{+\infty}\biggl(\frac{e^{-|x-\xi|}}{2}-\frac{e^{-|x+\xi|}}{2}\biggr)\mu_0(\xi)\,d\xi,$$

$$\frac{\partial^2\widetilde{V}_1}{\partial t^2}(x,0)=0 \quad \textit{for} \quad x\geqslant 0.$$

Proof. Repeat the proof of Lemma 4 using Lemma 1. □

We give the following classical result together with its proof.

Lemma 10. *Suppose that $\mu_0(x) \in \mathbb{C}_b[0,+\infty)$. Then the potential*

$$\widetilde{V}_{10}(x) = -\int_0^{+\infty} \left(\frac{e^{-|x-\xi|}}{2} - \frac{e^{-|x+\xi|}}{2} \right) \mu_0(\xi)\, d\xi$$

belongs to $\mathbb{C}_b^{(2)}[0,+\infty)$ and for every $x \geqslant 0$ we have

$$\frac{d^2 \widetilde{V}_{10}(x)}{dx^2} - \widetilde{V}_{10}(x) = \mu_0(x), \qquad \widetilde{V}_{10}(0) = 0. \tag{6.4}$$

Proof. First of all, note that the expression for $\widetilde{V}_{10}(x)$ can be rewritten in the form

$$\widetilde{V}_{10}(x) = I_{01}(x) + I_{02}(x) := -\int_0^{+\infty} \frac{e^{-|x-\xi|}}{2} \mu_0(\xi)\, d\xi + e^{-x} \int_0^{+\infty} \frac{e^{-\xi}}{2} \mu_0(\xi)\, d\xi.$$

Clearly, $I_{02}(x) \in \mathbb{C}_b^{\infty}[0,+\infty)$ and we have

$$\frac{d^2 I_{02}(x)}{dx^2} - I_{02}(x) = 0 \quad \text{for all} \quad x \geqslant 0. \tag{6.5}$$

Consider the function $I_{01}(x)$. Clearly, $I_{01}(x) \in \mathbb{C}_b[0,+\infty)$. For convenience, we write the expression for $I_{01}(x)$ in the form

$$I_{01}(x) = -\int_0^{x} \frac{e^{-(x-\xi)}}{2} \mu_0(\xi)\, d\xi - \int_x^{+\infty} \frac{e^{-(\xi-x)}}{2} \mu_0(\xi)\, d\xi.$$

We calculate the first derivative:

$$\frac{dI_{01}(x)}{dx} = -\frac{\mu_0(x)}{2} + \frac{\mu_0(x)}{2} + \int_0^{x} \frac{e^{-(x-\xi)}}{2} \mu_0(\xi)\, d\xi - \int_x^{+\infty} \frac{e^{-(\xi-x)}}{2} \mu_0(\xi)\, d\xi.$$

Clearly,

$$\frac{dI_{01}(x)}{dx} \in \mathbb{C}_b[0,+\infty).$$

We calculate the second derivative:

$$\begin{aligned} \frac{d^2 I_{01}(x)}{dx^2} &= \frac{\mu_0(x)}{2} + \frac{\mu_0(x)}{2} - \int_0^{x} \frac{e^{-(x-\xi)}}{2} \mu_0(\xi)\, d\xi - \int_x^{+\infty} \frac{e^{-(\xi-x)}}{2} \mu_0(\xi)\, d\xi \\ &= \mu_0(x) + I_{01}(x) \in \mathbb{C}_b[0,+\infty). \end{aligned} \tag{6.6}$$

The expressions (6.5) and (6.6) yield (6.4) for every $x \geqslant 0$. We also note that the equality $\widetilde{V}_{10}(0) = 0$ is obvious and follows directly from the formula for the potential $\widetilde{V}_{10}(x)$. □

For completeness, we give a proof of the following classical result.

Lemma 11. *For every function* $v_0(x) \in \mathbb{C}_b^{(2)}[0,+\infty)$ *with* $v_0(0)=0$ *one has*

$$-\int_0^{+\infty}\left(\frac{e^{-|x-\xi|}}{2}-\frac{e^{-|x+\xi|}}{2}\right)[v_{0\xi\xi}(\xi)-v_0(\xi)]\,d\xi = v_0(x) \quad \textit{for all} \quad x \geqslant 0. \tag{6.7}$$

Proof. The following chain of equations holds:

$$\begin{aligned}\int_0^{+\infty}\frac{e^{-(x+\xi)}}{2}v_{0\xi\xi}(\xi)\,d\xi &= -v_{0x}(0)\frac{e^{-x}}{2}-\frac{e^{-x}}{2}v_0(0)+\int_0^{+\infty}\frac{e^{-(x+\xi)}}{2}v_0(\xi)\,d\xi\\ &= -v_{0x}(0)\frac{e^{-x}}{2}+\int_0^{+\infty}\frac{e^{-(x+\xi)}}{2}v_0(\xi)\,d\xi.\end{aligned}$$

It follows that

$$\int_0^{+\infty}\frac{e^{-|x+\xi|}}{2}[v_{0\xi\xi}(\xi)-v_0(\xi)]\,d\xi = -v_{0x}(0)\frac{e^{-x}}{2}. \tag{6.8}$$

Moreover, we have a chain of equalities

$$\begin{aligned}&\int_0^{+\infty}\frac{e^{-|x-\xi|}}{2}v_{0\xi\xi}(\xi)\,d\xi = \int_0^{x}\frac{e^{-(x-\xi)}}{2}v_{0\xi\xi}\,d\xi+\int_x^{+\infty}\frac{e^{-(x-\xi)}}{2}v_{0\xi\xi}\,d\xi\\ &\quad = v_{0\xi}(\xi)\frac{e^{-(x-\xi)}}{2}\bigg|_{\xi=0}^{\xi=x}+v_{0\xi}(\xi)\frac{e^{-(\xi-x)}}{2}\bigg|_{\xi=x}^{\xi=+\infty}\\ &\qquad -\int_0^{x}\frac{e^{-(x-\xi)}}{2}v_{0\xi}(\xi)\,d\xi+\int_x^{+\infty}\frac{e^{-(\xi-x)}}{2}v_{0\xi}(\xi)\,d\xi\\ &\quad = \frac{v_{0x}(x)}{2}-\frac{v_{0x}(x)}{2}-v_{0x}(0)\frac{e^{-x}}{2}-v_0(x)+v_0(0)\frac{e^{-x}}{2}+\int_0^{+\infty}\frac{e^{-|x-\xi|}}{2}v_0(\xi)\,d\xi\\ &\quad = -v_0(x)-v_{0x}(0)\frac{e^{-x}}{2}+\int_0^{+\infty}\frac{e^{-|x-\xi|}}{2}v_0(\xi)\,d\xi.\end{aligned}$$

It follows from this chain that

$$\int_0^{+\infty}\frac{e^{-|x-\xi|}}{2}[v_{0\xi\xi}(\xi)-v_0(\xi)]\,d\xi = -v_0(x)-v_{0x}(0)\frac{e^{-x}}{2}. \tag{6.9}$$

The equalities (6.8) and (6.9) yield the desired formula (6.7). □

Lemma 12. *Suppose that* $\rho(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b[0,+\infty))$. *Then the potential*

$$W_1(x,t) = W_1[\rho](x,t) := \int_0^t\int_0^{+\infty}\mathscr{E}_1(x,\xi,t-\tau)\rho(\xi,\tau)\,d\xi\,d\tau$$

belongs to $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b^{(2)}[0,+\infty))$ *and we have*

$$\begin{gathered}\mathfrak{M}_{x,t}W_1(x,t) = \rho(x,t) \quad \textit{for all} \quad (x,t)\in[0,+\infty)\times[0,T],\\ W_1(0,t) = W_1(x,0) = \frac{\partial W_1}{\partial t}(x,0) = 0 \quad \textit{for} \quad x\geqslant 0, \quad t\in[0,T].\end{gathered}$$

Proof. On the whole, the proof of this lemma repeats that of Lemma 7. We only mention that

$$\frac{\partial^2 W_1(x,t)}{\partial t^2} = -\int_0^{+\infty}\biggl(\frac{e^{-|x-\xi|}}{2}-\frac{e^{-(x+\xi)}}{2}\biggr)\rho(\xi,t)\,d\xi + \int_0^t\int_0^{+\infty}\frac{\partial^2\mathscr{E}_1(x,\xi,t-\tau)}{\partial t^2}\rho(\xi,\tau)\,d\xi\,d\tau.$$

This yields the relations

$$\begin{aligned}\mathfrak{M}_{x,t}W_1(x,t)=\mathfrak{D}_{x,t}W_1(x,t)&=\frac{\partial^2}{\partial x^2}\biggl[\frac{\partial^2 W_1(x,t)}{\partial t^2}+W_1(x,t)\biggr]-\frac{\partial^2 W_1(x,t)}{\partial t^2}\\ &=-\biggl(\frac{\partial^2}{\partial x^2}-I\biggr)\int_0^{+\infty}\biggl(\frac{e^{-|x-\xi|}}{2}-\frac{e^{-(x+\xi)}}{2}\biggr)\rho(\xi,t)\,d\xi\\ &\quad+\int_0^t\int_0^{+\infty}\mathfrak{D}_{x,t}\mathscr{E}_1(x,\xi,t-\tau)\rho(\xi,\tau)\,d\xi\,d\tau=\rho(x,t),\end{aligned}$$

where we have used the results of Lemmas 9 and 10.

Finally, using the explicit form of the potential and Step 3 of Lemma 9, we arrive at the equalities

$$W_1(0,t)=W_1(x,0)=\frac{\partial W_1}{\partial t}(x,0)=0 \quad\text{for}\quad x\geqslant 0,\quad t\in[0,T]. \qquad \square$$

§ 7. Properties of the potentials related to the second boundary-value problem

We study the properties of the potentials in the right-hand side of (4.10). First consider the potential

$$P(x,t)=P[\mu](x,t):=2\int_0^t\mu(\tau)\biggl[\frac{\partial^2\mathscr{E}(x,t-\tau)}{\partial t^2}+\mathscr{E}(x,t-\tau)\biggr]\,d\tau-e^{-x}\mu(t). \tag{7.1}$$

Lemma 13. *Suppose that $\mu(t)\in\mathbb{C}^{(2)}[0,T]$ and $\mu(0)=\mu'(0)=0$. Then the potential $P(x,t)$ belongs to $\mathbb{C}^{(2)}\bigl([0,T];\mathbb{C}^{(2)}_b[0,+\infty)\bigr)$ and we have*

$$\mathfrak{M}_{x,t}P(x,t)=0 \quad\textit{for}\quad x\geqslant 0,\quad t\in[0,T],$$
$$\frac{\partial P}{\partial x}(0,t)=\mu(t),\quad P(x,0)=\frac{\partial P}{\partial t}(x,0)=0\quad\textit{for}\quad x\geqslant 0,\quad t\in[0,T].$$

Proof. On the whole, the proof repeats that of Lemma 8. We only mention some key points. We have

$$\begin{aligned}\mathfrak{M}_{x,t}P(x,t)&=2\mu(t)\biggl[\frac{\partial^2}{\partial x^2}-I\biggr]\biggl[\frac{\partial^2}{\partial t^2}+I\biggr]\frac{\partial\mathscr{E}}{\partial t}(x,0)\\ &\quad+2\int_0^t\mu(\tau)\biggl[\frac{\partial^2}{\partial t^2}+I\biggr]\mathfrak{M}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau-\mu(t)e^{-x}\\ &:=P_1(x,t)+P_2(x,t)+P_3(x,t).\end{aligned} \tag{7.2}$$

To calculate the integral $P_1(x,t)$, consider the chain of equalities

$$\left(\frac{\partial^2}{\partial t^2}+I\right)\frac{\partial \mathscr{E}}{\partial t}(x,0)=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\mu x}}{(\mu^2+1)^2}\,d\mu=-\frac{x}{4}e^{-x}-\frac{1}{4}e^{-x},$$

$$\begin{aligned}P_1(x,t)&=2\mu(t)\left[\frac{\partial^2}{\partial x^2}-I\right]\left[\frac{\partial^2}{\partial t^2}+I\right]\frac{\partial \mathscr{E}}{\partial t}(x,0)\\&=2\mu(t)\left[\frac{\partial^2}{\partial x^2}-I\right]\left[-\frac{x}{4}e^{-x}-\frac{1}{4}e^{-x}\right]=\mu(t)e^{-x}.\end{aligned}$$

We now calculate the integral $P_2(x,t)$:

$$P_2(x,t)=2\int_0^t\mu(\tau)\left[\frac{\partial^2}{\partial t^2}+I\right]\mathfrak{D}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau=0. \tag{7.3}$$

Thus, it follows from (7.2) and (7.3) that

$$P_1(x,t)+P_2(x,t)+P_3(x,t)=0\quad\text{for}\quad x\geqslant 0,\quad t\in[0,T].$$

Moreover, we have

$$\frac{\partial P(x,t)}{\partial x}=2\int_0^t\mu(\tau)\mathfrak{N}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau+e^{-x}\mu(t), \tag{7.4}$$

where

$$\mathfrak{N}_{x,t}:=\frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial t^2}+I\right].$$

By the result (4.5) of Lemma 1 we have

$$\mathfrak{N}_{x,t}\mathscr{E}(0,t-\tau)=0\quad\text{for all}\quad 0\leqslant\tau\leqslant t\leqslant T.$$

Hence it follows from (7.4) that

$$\frac{\partial P}{\partial x}(0,t)=\mu(t)\quad\text{for}\quad t\in[0,T].$$

The equality $P(x,0)=0$ is obvious and follows directly from the formula (7.1) for the potential $P(x,t)$. The equality

$$\frac{\partial P}{\partial t}(x,0)=0$$

can be proved in the same way as the equality in Lemma 8. □

Lemma 14. *Suppose that* $\mu_0(x)\in\mathbb{C}_b[0,+\infty)$. *Then the potential*

$$\widetilde{V}_2(x,t)=\widetilde{V}_2[\mu_0](x,t):=\int_0^{+\infty}\mathscr{E}_2(x,\xi,t)\mu_0(\xi)\,d\xi,$$

belongs to $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b^{(2)}[0,+\infty))$, *where*

$$\mathscr{E}_2(x,\xi,t)=\mathscr{E}(x-\xi,t)+\mathscr{E}(x+\xi,t),$$

and we have

$$\mathfrak{M}_{x,t}\widetilde{V}_2(x,t) = 0 \quad for \quad x \geqslant 0, \quad t \geqslant 0, \tag{7.5}$$

$$\frac{\partial \widetilde{V}_2}{\partial x}(0,t) = \frac{\partial}{\partial x}\frac{\partial \widetilde{V}_2}{\partial t}(0,t) = 0, \qquad \widetilde{V}_2(x,0) = 0, \qquad \frac{\partial^2 \widetilde{V}_2}{\partial t^2}(x,0) = 0,$$

$$\frac{\partial \widetilde{V}_2(x,0)}{\partial t} = -\int_0^{+\infty}\left(\frac{e^{-|x-\xi|}}{2} + \frac{e^{-(x+\xi)}}{2}\right)\mu_0(\xi)\,d\xi \quad for \quad x \geqslant 0, \quad t \geqslant 0.$$

Proof. Step 1. First of all, by the integral representation (4.2) we have

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial t^2} + I\right]\frac{\partial}{\partial x}\mathscr{E}_2(x,\xi,t)\bigg|_{x=0} \\
&\quad = \frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial t^2} + I\right]\mathscr{E}(x-\xi,t)\bigg|_{x=0} + \frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial t^2} + I\right]\mathscr{E}(x+\xi,t)\bigg|_{x=0} \\
&\quad = -\frac{i}{2\pi}\int_{-\infty}^{+\infty}[e^{i\mu\xi} + e^{-i\mu\xi}]\frac{1}{(\mu^2+1)^{3/2}}\sin\left(\frac{\mu t}{\sqrt{\mu^2+1}}\right)d\mu \\
&\quad = -\frac{i}{\pi}\int_{-\infty}^{+\infty}\cos(\mu\xi)\frac{1}{(\mu^2+1)^{3/2}}\sin\left(\frac{\mu t}{\sqrt{\mu^2+1}}\right)d\mu = 0
\end{aligned} \tag{7.6}$$

for all $t \geqslant 0$ and $\xi \in \mathbb{R}^1_+$ since the integrand is an odd function of μ. Moreover,

$$\mathscr{E}_2(x,\xi,0) = \mathscr{E}(x-\xi,0) + \mathscr{E}(x+\xi,0) = 0 \quad \Longrightarrow \quad \frac{\partial \mathscr{E}_2}{\partial x}(0,\xi,0) = 0 \tag{7.7}$$

and for $\xi \geqslant x$ we have

$$\begin{aligned}
\frac{\partial \mathscr{E}_2}{\partial t}(x,\xi,0) &= \frac{\partial \mathscr{E}}{\partial t}(x-\xi,0) + \frac{\partial \mathscr{E}}{\partial t}(x+\xi,0) \\
&= -\frac{e^{-|x-\xi|}}{2} - \frac{e^{-(x+\xi)}}{2} = -\frac{e^{-(\xi-x)}}{2} - \frac{e^{-(x+\xi)}}{2}.
\end{aligned} \tag{7.8}$$

Thus, in view of (7.8), we have

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{E}_2}{\partial x}(0,\xi,0) = -\frac{1}{2}e^{-\xi} + \frac{1}{2}e^{-\xi} = 0. \tag{7.9}$$

By (7.6), (7.7) and (7.9), the function

$$Y(t) := \frac{\partial \mathscr{E}_2}{\partial x}(0,\xi,t)$$

satisfies the Cauchy conditions

$$\frac{d^2Y(t)}{dt^2} + Y(t) = 0, \qquad Y(0) = 0, \qquad \frac{dY}{dt}(0) = 0, \qquad t \geqslant 0.$$

In the classical sense, it follows that $Y(t) = 0$ for all $t \geqslant 0$. Therefore,

$$\frac{\partial \mathscr{E}_2}{\partial x}(0,\xi,t) = 0 \quad \text{for all} \quad \xi \geqslant 0, \quad t \geqslant 0. \tag{7.10}$$

Moreover, it follows that

$$\frac{\partial^2 \mathscr{E}_2}{\partial x\,\partial t}(0,\xi,t)=0 \quad \text{for all} \quad \xi\geqslant 0, \quad t\geqslant 0. \tag{7.11}$$

Step 2. Write $\widetilde{V}_2(x,t)$ and $\widetilde{V}_{2t}(x,t)$ in the form

$$\widetilde{V}_2(x,t)=\int_0^x \mathscr{E}_2(x,\xi,t)\mu_0(\xi)\,d\xi+\int_x^{+\infty}\mathscr{E}_2(x,\xi,t)\mu_0(\xi)\,d\xi,$$
$$\frac{\partial \widetilde{V}_2(x,t)}{\partial t}=\int_0^x \frac{\partial \mathscr{E}_2(x,\xi,t)}{\partial t}\mu_0(\xi)\,d\xi+\int_x^{+\infty}\frac{\partial \mathscr{E}_2(x,\xi,t)}{\partial t}\mu_0(\xi)\,d\xi.$$

These equalities yield the following expressions for the derivatives:

$$\frac{\partial \widetilde{V}_2(x,t)}{\partial x}=\mu_0(x)[\mathscr{E}_2(x,x,t)-\mathscr{E}_2(x,x,t)]+\int_0^{+\infty}\left\{\frac{\partial \mathscr{E}_2(x,\xi,t)}{\partial x}\right\}\mu_0(\xi)\,d\xi$$
$$=\int_0^{+\infty}\left\{\frac{\partial \mathscr{E}_2(x,\xi,t)}{\partial x}\right\}\mu_0(\xi)\,d\xi,$$
$$\frac{\partial^2 \widetilde{V}_2(x,t)}{\partial x\,\partial t}=\int_0^{+\infty}\left\{\frac{\partial^2 \mathscr{E}_2(x,\xi,t)}{\partial x\,\partial t}\right\}\mu_0(\xi)\,d\xi.$$

In view of (7.10) and (7.11), we arrive at the conclusion that

$$\frac{\partial \widetilde{V}_2}{\partial x}(0,t)=0 \quad \text{for} \quad t\geqslant 0, \qquad \frac{\partial^2 \widetilde{V}_2}{\partial x\,\partial t}(0,t)=0 \quad \text{for} \quad t\geqslant 0.$$

Step 3. The equalities

$$\widetilde{V}_2(x,0)=0,$$
$$\frac{\partial \widetilde{V}_2}{\partial t}(x,0)=-\int_0^{+\infty}\left(\frac{e^{-|x-\xi|}}{2}+\frac{e^{-(x+\xi)}}{2}\right)\mu_0(\xi)\,d\xi, \qquad \frac{\partial^2 \widetilde{V}_2}{\partial t^2}(x,0)=0$$

for $x\geqslant 0$ and $t\geqslant 0$ follow immediately from the results of Lemma 1.
Step 4. The equality (7.5) can be proved in exactly the same way as the corresponding equality in Lemma 9. □

Lemma 15. *Suppose that $\mu_0(x)\in\mathbb{C}_b[0,+\infty)$. Then the potential*

$$\widetilde{V}_{20}(x)=\widetilde{V}_{20}[\mu_0](x)=-\int_0^{+\infty}\left(\frac{e^{-|x-\xi|}}{2}+\frac{e^{-|x+\xi|}}{2}\right)\mu_0(\xi)\,d\xi$$

belongs to $\mathbb{C}_b^{(2)}[0,+\infty)$ and for every $x\geqslant 0$ we have

$$\frac{d^2\widetilde{V}_{20}(x)}{dx^2}-\widetilde{V}_{20}(x)=\mu_0(x), \qquad \frac{dV_{20}}{dx}(0)=0.$$

Proof. A direct verification yields all these equalities as in the proof of Lemma 10. □

Lemma 16. *For every function* $v_0(x) \in \mathbb{C}_b^{(2)}[0,+\infty)$ *with* $v_{0x}(0) = 0$ *we have*

$$-\int_0^{+\infty} \left(\frac{e^{-|x-\xi|}}{2} + \frac{e^{-|x+\xi|}}{2} \right) [v_{0\xi\xi}(\xi) - v_0(\xi)]\, d\xi = v_0(x)$$

for all $x \geqslant 0$.

Proof. Just repeat the proof of Lemma 11. □

Lemma 17. *Suppose that* $\rho(x,t) \in \mathbb{C}([0,T]; \mathbb{C}_b[0,+\infty))$. *Then the potential*

$$W_2(x,t) = W_2[\rho](x,t) := \int_0^t \int_0^{+\infty} \mathscr{E}_2(x,\xi,t-\tau)\rho(\xi,\tau)\, d\xi\, d\tau$$

belongs to $\mathbb{C}^{(2)}([0,T]; \mathbb{C}_b^{(2)}[0,+\infty))$ *and we have*

$$\mathfrak{M}_{x,t} W_2(x,t) = \rho(x,t) \quad \textit{for all} \quad (x,t) \in [0,+\infty) \times [0,T],$$
$$W_2(x,0) = \frac{\partial W_2}{\partial t}(x,0) = \frac{\partial W_2}{\partial x}(0,t) = 0 \quad \textit{for} \quad x \geqslant 0, \quad t \in [0,T].$$

Proof. This can be proved in the same way as Lemma 12. □

§ 8. Solubility of the Cauchy problems for the equations (3.1) and (3.2)

Theorem 1. *The classical solutions of the Cauchy problems for the equations* (3.1) *and* (3.2) *in the sense of Definition* 1 *in the class*

$$u(x,t) \in \mathbb{C}^{(2)}\bigl([0,T]; \mathbb{C}_b^{(2)}(\mathbb{R}^1)\bigr), \qquad u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}(\mathbb{R}^1)$$

are equivalent to the following integral equations respectively:

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^1} \mathscr{E}(x-\xi,t-\tau)\, \frac{\partial^2 u^2(\xi,\tau)}{\partial \tau^2}\, d\xi\, d\tau \\ &\quad + \int_{\mathbb{R}^1} \biggl(\mathscr{E}(x-\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi)] + \frac{\partial \mathscr{E}(x-\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)] \biggr)\, d\xi, \quad (8.1) \\ u(x,t) &= -\int_0^t \int_{\mathbb{R}^1} \mathscr{E}(x-\xi,t-\tau)\, \frac{\partial(|u|^q u)(\xi,\tau)}{\partial \tau}\, d\xi\, d\tau \\ &\quad + \int_{\mathbb{R}^1} \biggl(\mathscr{E}(x-\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi)] + \frac{\partial \mathscr{E}(x-\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)] \biggr)\, d\xi. \quad (8.2) \end{aligned}$$

Proof. We prove the theorem for the integral equation (8.1) since the proof for (8.2) is exactly the same.

By Lemma 2, every classical solution of the Cauchy problem for (3.1) in the sense of Definition 1 satisfies the integral equation (8.1). Consider the function

$$\begin{aligned} U(x,t) &:= \int_0^t \int_{\mathbb{R}^1} \mathscr{E}(x-\xi,t-\tau)\, \frac{\partial^2 u^2(\xi,\tau)}{\partial \tau^2}\, d\xi\, d\tau \\ &\quad + \int_{\mathbb{R}^1} \biggl(\mathscr{E}(x-\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi)] + \frac{\partial \mathscr{E}(x-\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)] \biggr)\, d\xi. \end{aligned}$$

Suppose that $u(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1))$. Then, by Lemmas 4–7, the function $U(x,t)$ belongs to $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1))$ and we have

$$\begin{aligned}
\mathfrak{M}_{x,t}U(x,t) &= \frac{\partial^2 u^2(x,t)}{\partial t^2}, \qquad (x,t) \in \mathbb{R}^1 \times [0,T], \\
U(x,0) &= \int_{\mathbb{R}^1} \frac{\partial \mathscr{E}(x-\xi,0)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)]\,d\xi \\
&= -\int_{\mathbb{R}^1} \frac{e^{-|x-\xi|}}{2}[u_{0\xi\xi}(\xi) - u_0(\xi)]\,d\xi = u_0(x), \\
\frac{\partial U(x,0)}{\partial t} &= \int_{\mathbb{R}^1} \frac{\partial \mathscr{E}(x-\xi,0)}{\partial t}[u_{1\xi\xi}(\xi) - u_1(\xi)]\,d\xi \\
&= -\int_{\mathbb{R}^1} \frac{e^{-|x-\xi|}}{2}[u_{1\xi\xi}(\xi) - u_1(\xi)]\,d\xi = u_1(x), \qquad x \in \mathbb{R}^1.
\end{aligned}$$

Hence every solution $u(x,t)$ of the integral equation (8.1) in this class is a classical solution of (3.1) in the sense of Definition 1. □

Our next task is to prove the local-in-time solubility of the integral equation (8.1) in the Banach space $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1))$ with the norm

$$\|v\|_T := \sup_{(x,t)\in\mathbb{R}^1\times[0,T]} \sum_{j=0}^{2} \left|\frac{\partial^j v(x,t)}{\partial t^j}\right|.$$

Theorem 2. *For every $T>0$ one can find sufficiently small numbers $R_2>0$ and $R_3>0$ such that under the conditions*

$$\begin{aligned}
u_0(x) &\in \mathbb{C}_b^2(\mathbb{R}^1), \qquad \|u_0\|_{\mathbb{C}_b^2(\mathbb{R}^1)} \leqslant R_2, \\
u_1(x) &\in \mathbb{C}_b^2(\mathbb{R}^1), \qquad \|u_1\|_{\mathbb{C}_b^2(\mathbb{R}^1)} \leqslant R_3
\end{aligned} \tag{8.3}$$

the solution of (8.1) in the class $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1)) \cap B_R$ for sufficiently small $R>0$ exists and is unique, where

$$B_R = \bigl\{v \in \mathbb{C}^{(2)}\bigl([0,T];\mathbb{C}_b(\mathbb{R}^1)\bigr) \colon \|v\|_T \leqslant R\bigr\}. \tag{8.4}$$

Proof. Rewrite the integral equation (8.1) in the form

$$u(x,t) = A(u)(x,t), \tag{8.5}$$

where

$$A(u)(x,t) := f(x,t) + W(\rho)(x,t), \qquad \rho(\xi,\tau) := \frac{\partial^2 u^2(\xi,\tau)}{\partial \tau^2}, \tag{8.6}$$

$$f(x,t) := \int_{\mathbb{R}^1} \biggl(\mathscr{E}(x-\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi)] + \frac{\partial \mathscr{E}(x-\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)]\biggr)\,d\xi, \tag{8.7}$$

$$W(\rho)(x,t) := \int_0^t \int_{\mathbb{R}^1} \mathscr{E}(x-\xi,t-\tau)\rho(\xi,\tau)\,d\xi\,d\tau.$$

The function $f(x,t)$ belongs to $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1))$ by Lemma 4. For the volume potential, we have Lemma 7 and the equalities (5.11) and (5.12). They are of the form

$$\frac{\partial W(x,t)}{\partial t} = \int_0^t \int_{\mathbb{R}^1} \frac{\partial \mathscr{E}(x-\xi,t-\tau)}{\partial t}\rho(\xi,\tau)\,d\xi\,d\tau,$$
$$\frac{\partial^2 W(x,t)}{\partial t^2} = -\int_{\mathbb{R}^1} \frac{e^{-|x-\xi|}}{2}\rho(\xi,t)\,d\xi + \int_0^t \int_{\mathbb{R}^1} \frac{\partial^2 \mathscr{E}(x-\xi,t-\tau)}{\partial t^2}\rho(\xi,\tau)\,d\xi\,d\tau.$$

The following bounds hold in view of the estimate (4.7) with $\varepsilon \in (0,1)$ in Lemma 1:

$$\begin{aligned}
&|W(\rho_1)(x,t) - W(\rho_2)(x,t)| \\
&\qquad \leqslant c_1(\varepsilon)e^{a_1(\varepsilon)T}\int_0^t\int_{\mathbb{R}^1}\exp(-\varepsilon|x-\xi|)|\rho_1(\xi,\tau)-\rho_2(\xi,\tau)|\,d\xi\,d\tau, \qquad (8.8)\\
&\left|\frac{\partial W(\rho_1)(x,t)}{\partial t} - \frac{\partial W(\rho_2)(x,t)}{\partial t}\right| \\
&\qquad \leqslant c_2(\varepsilon)e^{a_1(\varepsilon)T}\int_0^t\int_{\mathbb{R}^1}\exp(-\varepsilon|x-\xi|)|\rho_1(\xi,\tau)-\rho_2(\xi,\tau)|\,d\xi\,d\tau,\\
&\left|\frac{\partial^2 W(\rho_1)(x,t)}{\partial t^2} - \frac{\partial^2 W(\rho_2)(x,t)}{\partial t^2}\right| \\
&\qquad \leqslant \frac{1}{2}\int_{\mathbb{R}^1}\exp(-|x-\xi|)|\rho_1(\xi,t)-\rho_2(\xi,t)|\,d\xi \\
&\qquad\qquad + c_3(\varepsilon)e^{a_1(\varepsilon)T}\int_0^t\int_{\mathbb{R}^1}\exp(-\varepsilon|x-\xi|)|\rho_1(\xi,\tau)-\rho_2(\xi,\tau)|\,d\xi\,d\tau.
\end{aligned}$$

Suppose that $u_1(\xi,\tau)$, $u_2(\xi,\tau) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1))$ and

$$\rho_1(\xi,\tau) = \frac{\partial^2 u_1^2(\xi,\tau)}{\partial\tau^2}, \qquad \rho_2(\xi,\tau) = \frac{\partial^2 u_2^2(\xi,\tau)}{\partial\tau^2}.$$

We have a chain of inequalities

$$\begin{aligned}
|\rho_1(\xi,\tau)-\rho_2(\xi,\tau)| &\leqslant 2\left|\left(\frac{\partial u_1}{\partial\tau}\right)^2 - \left(\frac{\partial u_2}{\partial\tau}\right)^2\right| + 2\left|u_1\frac{\partial^2 u_1}{\partial\tau^2} - u_2\frac{\partial^2 u_2}{\partial\tau^2}\right| \\
&\leqslant 4\max\left\{\left|\frac{\partial u_1}{\partial\tau}\right|,\left|\frac{\partial u_2}{\partial\tau}\right|\right\}\left|\frac{\partial u_1}{\partial\tau}-\frac{\partial u_2}{\partial\tau}\right| \\
&\qquad + 2\left|\frac{\partial^2 u_1}{\partial\tau^2}\right||u_1-u_2| + 2|u_2|\left|\frac{\partial^2 u_1}{\partial\tau^2}-\frac{\partial^2 u_2}{\partial\tau^2}\right| \\
&\leqslant 8\max\{\|u_1\|_T,\|u_2\|_T\}\,\|u_1-u_2\|_T. \qquad (8.9)
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^1}\exp(-\varepsilon|y|)\,dy = \frac{2}{\varepsilon}.$$

Thus, it follows from (8.8) and (8.9) that

$$\|W(\rho_1)(x,t) - W(\rho_2)(x,t)\|_T \leqslant \bigl(TK_1(\varepsilon)e^{a_1(\varepsilon)T} + 8\bigr)\max\{\|u_1\|_T,\|u_2\|_T\}\|u_1-u_2\|_T, \qquad (8.10)$$

where

$$K_1 := 16\frac{c_1(\varepsilon)+c_2(\varepsilon)+c_3(\varepsilon)}{\varepsilon}.$$

Putting $u_1 = u$ and $u_2 = 0$ in (8.10), we obtain the following bound for $\rho = \partial^2 u^2/\partial\tau^2$:

$$\|W(\rho)\|_T \leqslant (8 + TK_1 e^{a_1 T})\|u\|_T^2. \tag{8.11}$$

Consider the closed ball

$$B_R = \bigl\{u(x,t) \in \mathbb{C}^{(2)}\bigl([0,T];\mathbb{C}_b(\mathbb{R}^1)\bigr) \colon \|u\|_T \leqslant R\bigr\}.$$

Put

$$R_1 := \|f(x,t)\|_T,$$

where the function $f(x,t)$ is defined in (8.7). It follows from the definition (8.6) of the operator A and the bound (8.11) that

$$\|A(u)\|_T \leqslant \|f\|_T + (TK_1 e^{a_1 T} + 8)\|u\|_T^2. \tag{8.12}$$

We claim that for every $T > 0$ the operator A acts from B_R to B_R provided that $0 < R_1 < R$ and $R > 0$ is sufficiently small. Indeed, it follows from (8.12) for $u \in B_R$ that

$$\|A(u)\|_T \leqslant R_1 + (TK_1 e^{a_1 T} + 8)R^2.$$

The inequality

$$R_1 + (TK_1 e^{a_1 T} + 8)R^2 \leqslant R \quad \text{if} \quad R_1 < R$$

clearly holds for every fixed $T > 0$ when $R > 0$ is sufficiently small. Suppose that $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}(\mathbb{R}^1)$ and the following inequalities hold:

$$\|u_0\|_{\mathbb{C}_b^{(2)}(\mathbb{R}^1)} \leqslant R_2, \qquad \|u_1\|_{\mathbb{C}_b^{(2)}(\mathbb{R}^1)} \leqslant R_3. \tag{8.13}$$

Then we deduce the following chain of inequalities from the explicit formula (8.7) for $f(x,t)$:

$$\begin{aligned}\|f\|_T &\leqslant K_2(\varepsilon)e^{a_1(\varepsilon)T}\int_{\mathbb{R}^1} e^{-\varepsilon|x-\xi|}|u_{0\xi\xi}(\xi) - u_0(\xi)|\,d\xi \\ &\qquad + K_2(\varepsilon)e^{a_1(\varepsilon)T}\int_{\mathbb{R}^1} e^{-\varepsilon|x-\xi|}|u_{1\xi\xi}(\xi) - u_1(\xi)|\,d\xi \\ &\leqslant \biggl(\frac{4K_2(\varepsilon)}{\varepsilon}R_2 + \frac{4K_3(\varepsilon)}{\varepsilon}R_3\biggr)e^{a_1(\varepsilon)T} =: R_1.\end{aligned}$$

Therefore, for small $R_2 > 0$ and $R_3 > 0$, the number $R_1 > 0$ can be made arbitrarily small. This proves the following assertion.

Lemma 18. *For every $T > 0$ one can find small numbers $R_2 > 0$ and $R_3 > 0$ such that, for all $u_0(x)$, $u_1(x) \in \mathbb{C}_b^{(2)}(\mathbb{R}^1)$ in the balls (8.13), the operator A defined in (8.6) acts from the ball B_R to the ball B_R for some small $R > 0$.*

We now study whether A is a contraction on B_R. Indeed, by (8.10) we have the following bound for all $u_1, u_2 \in B_R$:

$$\begin{aligned}\|A(u_1)(x,t) - A(u_2)\|_T &\leqslant \bigl(TK_1(\varepsilon)e^{a_1(\varepsilon)T} + 8\bigr)\max\{\|u_1\|_T, \|u_2\|_T\}\|u_1 - u_2\|_T \\ &\leqslant \bigl(TK_1(\varepsilon)e^{a_1(\varepsilon)T} + 8\bigr)R\|u_1 - u_2\|_T.\end{aligned}$$

For every $T > 0$ and any sufficiently small $R > 0$ with

$$\bigl(TK_1(\varepsilon)e^{a_1(\varepsilon)T} + 8\bigr)R \leqslant \frac{1}{2},$$

the operator A is clearly a contraction on B_R. □

We now study the question of the solubility of the integral equation (8.2) in the Banach space $\mathbb{C}([0,T]; \mathbb{C}_b((1+x^2)^{\alpha/2}; \mathbb{R}^1))$.

Theorem 3. *For all $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2}; \mathbb{R}^1)$ with*

$$\gamma \geqslant 1 + \alpha, \qquad \alpha \geqslant \frac{1}{q}, \qquad q > 0$$

one can find a $T_0 = T_0(u_0, u_1) > 0$ such that for every $T \in (0, T_0)$ there is a unique solution $u(x,t) \in \mathbb{C}^{(1)}([0,T]; \mathbb{C}_b((1+x^2)^{\alpha/2}; \mathbb{R}^1))$ of the integral equation (8.2). Moreover, either $T_0 = +\infty$ or $T_0 < +\infty$, and the following limit property holds in this latter case:

$$\lim_{T \uparrow T_0} \|u\|_{\alpha,T} = +\infty, \qquad \|u\|_{\alpha,T} := \sup_{(x,t)\in\mathbb{R}^1\times[0,T]} (1+x^2)^{\alpha/2}|u(x,t)|.$$

Proof. Integration by parts yields the equality

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^1} \mathscr{E}(x-\xi, t-\tau)\frac{\partial(|u|^q u)(\xi,\tau)}{\partial\tau}\,d\xi\,d\tau &= \int_{\mathbb{R}^1} \mathscr{E}(x-\xi, 0)(|u|^q u)(\xi, t)\,d\xi \\ &\quad - \int_{\mathbb{R}^1} \mathscr{E}(x-\xi, t)(|u_0|^q u_0)(\xi)\,d\xi - \int_0^t \int_{\mathbb{R}^1} \frac{\partial \mathscr{E}(x-\xi, t-\tau)}{\partial\tau}(|u|^q u)(\xi,\tau)\,d\xi\,d\tau \\ &= -\int_{\mathbb{R}^1} \mathscr{E}(x-\xi, t)(|u_0|^q u_0)(\xi)\,d\xi - \int_0^t \int_{\mathbb{R}^1} \frac{\partial \mathscr{E}(x-\xi, t-\tau)}{\partial\tau}(|u|^q u)(\xi,\tau)\,d\xi\,d\tau\end{aligned}$$

for all $u(x,t) \in \mathbb{C}^{(1)}([0,T]; \mathbb{C}_b(\mathbb{R}^1))$, where we have used the equality $\mathscr{E}(x,0) = 0$ (see part 3 of Lemma 1). Rewrite the integral equation (8.2) in the form

$$u(x,t) = B(u)(x,t), \tag{8.14}$$

where

$$B(u)(x,t) = f_1(x,t) + W_1(\rho)(x,t), \qquad \rho(\xi,\tau) := (|u|^q u)(\xi,\tau). \tag{8.15}$$

Here the volume potential $W(\rho)(x,t)$ is deifned by the equality

$$W_1(\rho)(x,t) := \int_0^t \int_{\mathbb{R}^1} \frac{\partial \mathscr{E}(x-\xi, t-\tau)}{\partial\tau}\rho(\xi,\tau)\,d\xi\,d\tau,$$

and the function $f_1(x,t)$ by the equality

$$f_1(x,t) := \int_{\mathbb{R}^1} \biggl(\mathscr{E}(x-\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi) - (|u_0|^q u_0)(\xi)] + \frac{\partial \mathscr{E}(x-\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)] \biggr)\, d\xi.$$

The function $f_1(x,t)$ belongs to $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1))$ by Lemma 4. We claim that actually

$$f_1(x,t) \in \mathbb{C}^{(2)}\bigl([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};\mathbb{R}^1)\bigr).$$

Indeed, if $u_1(\xi,\tau)$, $u_2(\xi,\tau) \in \mathbb{C}([0,T];\mathbb{C}_b((1+x^2)^{\gamma/2};\mathbb{R}^1))$ with $\gamma \geqslant \alpha+1$, then the following chain of estimates holds by Lemma 1 and Lemma 21 (see § 14):

$$\begin{aligned}
(1+x^2)^{\alpha/2}&\biggl|\frac{\partial^k f(x,t)}{\partial t^k}\biggr| \\
&\leqslant (1+x^2)^{\alpha/2} \int_{\mathbb{R}^1} \biggl(\biggl|\frac{\partial^k \mathscr{E}(x-\xi,t)}{\partial t^k}\biggr| \bigl|u_{1\xi\xi}(\xi) - u_1(\xi) - (|u_0|^q u_0)(\xi)\bigr| \\
&\qquad + \biggl|\frac{\partial^{k+1} \mathscr{E}(x-\xi,t)}{\partial t^{k+1}}\biggr| |u_{0\xi\xi}(\xi) - u_0(\xi)| \biggr)\, d\xi \\
&\leqslant a_0(\varepsilon) e^{a_1(\varepsilon)T} \|u_{1xx}(x) - u_1(x) - (|u_0|^q u_0)(x))\|_{\gamma,T} \\
&\qquad\qquad \times \sup_{x\in\mathbb{R}^1} \int_{\mathbb{R}^1} \frac{(1+x^2)^{\alpha/2}}{(1+\xi^2)^{\gamma/2}} \exp(-\varepsilon|x-\xi|)\, d\xi \\
&\qquad + a_0(\varepsilon) e^{a_1(\varepsilon)T} \|u_{0xx}(x) - u_0(x)\|_{\gamma,T} \sup_{x\in\mathbb{R}^1} \int_{\mathbb{R}^1} \frac{(1+x^2)^{\alpha/2}}{(1+\xi^2)^{\gamma/2}} \exp(-\varepsilon|x-\xi|)\, d\xi \\
&\leqslant c(\varepsilon,\gamma,\alpha,T,k)\bigl(\|u_{1xx}(x) - u_1(x) - (|u_0|^q u_0)(x)\|_{\gamma,T} + \|u_{0xx}(x) - u_0(x)\|_{\gamma,T}\bigr)
\end{aligned}$$

for all $(x,t) \in \mathbb{R}^1 \times [0,T]$ with $k = 0,1,2$ provided that $\gamma \geqslant \alpha+1$ and $\varepsilon \in (0,1)$.

Put

$$\rho_1(\xi,\tau) = (|u_1|^q u_1)(\xi,\tau), \qquad \rho_2(\xi,\tau) = (|u_2|^q u_2)(\xi,\tau).$$

Since $q > 0$, we have

$$|\rho_1 - \rho_2| \leqslant (q+1)\max\{|u_1|^q, |u_2|^q\}|u_1 - u_2|.$$

The following chain of estimates holds by the bound (4.7) with $\varepsilon \in (0,1)$ in Lemma 1:

$$\begin{aligned}
|W_1(\rho_1)(x,t) &- W_2(\rho_2)(x,t)| \\
&\leqslant c_2(\varepsilon) e^{a_1(\varepsilon)T} \int_0^t \int_{\mathbb{R}^1} \exp(-\varepsilon|x-\xi|) |\rho_1(\xi,\tau) - \rho_2(\xi,\tau)|\, d\xi\, d\tau \\
&\leqslant c_2(\varepsilon) e^{a_1(\varepsilon)T}(q+1) \int_0^t \int_{\mathbb{R}^1} \exp(-\varepsilon|x-\xi|) \max\{|u_1|^q(\xi,\tau), |u_2|^q(\xi,\tau)\} \\
&\qquad\qquad \times |u_1(\xi,\tau) - u_2(\xi,\tau)|\, d\xi\, d\tau.
\end{aligned}$$

It follows that

$$(1+x^2)^{\alpha/2}|W_1(\rho_1)(x,t)-W_2(\rho_2)(x,t)| \\ \leqslant J_1 c_2(\varepsilon) e^{a_1(\varepsilon)T}(q+1)T\max\{\|u_1\|^q_{\alpha,T}, \|u_2\|^q_{\alpha,T}\}\|u_1-u_2\|_{\alpha,T} \tag{8.16}$$

for all functions $u_1(x,t)$ and $u_2(x,t)$ such that

$$\|u_1\|_{\alpha,T}, \|u_2\|_{\alpha,T} < +\infty,$$

where

$$J_1 := \sup_{x\in\mathbb{R}^1}(1+x^2)^{\alpha/2}\int_{\mathbb{R}^1}\frac{\exp(-\varepsilon|x-\xi|)}{(1+\xi^2)^{(q+1)\alpha/2}}\,d\xi.$$

By Lemma 21 we have $0<J_1<+\infty$ provided that $\alpha\geqslant 1/q$.

Thus, if $\alpha\geqslant 1/q$, then it follows from (8.16) that

$$\|W_1(\rho_1)-W_1(\rho_2)\|_{\alpha,T}\leqslant K_4(\varepsilon,q,\alpha)Te^{a_1(\varepsilon)T}\max\{\|u_1\|^q_{\alpha,T}, \|u_2\|^q_{\alpha,T}\}\|u_1-u_2\|_{\alpha,T}. \tag{8.17}$$

We now put $u_1=u$ and $u_2=0$ in (8.17) and obtain

$$\|W_1(\rho)\|_{\alpha,T}\leqslant K_4(\varepsilon,q,\alpha)Te^{a_1(\varepsilon)T}\|u\|^{q+1}_{\alpha,T}, \qquad \rho=|u|^q u. \tag{8.18}$$

Suppose that

$$u_0(x),\ u_1(x)\in\mathbb{C}^{(2)}_b\bigl((1+x^2)^{\gamma/2};\mathbb{R}^1\bigr).$$

In particular, it follows that $(|u_0|^q u_0)(x)\in\mathbb{C}_b((1+x^2)^{\gamma/2};\mathbb{R}^1)$. Suppose that $R\geqslant 2\|f\|_{\alpha,T}$. Consider the ball

$$B_R := \bigl\{\|u\|_{\alpha,T}\leqslant R\colon u(x,t)\in\mathbb{C}\bigl([0,T];\mathbb{C}_b\bigl((1+x^2)^{\alpha/2};\mathbb{R}^1\bigr)\bigr)\bigr\}.$$

The following inequality holds for all $u(x,t)\in B_R$ in view of (8.18):

$$\|B(u)\|_{\alpha,T}\leqslant\|f_1\|_{\alpha,T}+K_4(\varepsilon,q,\alpha)Te^{a_1(\varepsilon)T}\|u\|^{q+1}_{\alpha,T} \\ \leqslant\frac{R}{2}+K_4(\varepsilon,q,\alpha)Te^{a_1(\varepsilon)T}R^{q+1}.$$

For every $R\geqslant 2\|f\|_{\alpha,T}$ one can clearly find a small number $T>0$ such that

$$K_4(\varepsilon,q,\alpha)Te^{a_1(\varepsilon)T}R^{q+1}\leqslant\frac{R}{2}.$$

Thus, for every $R>0$ there is a small $T>0$ such that the operator B defined in (8.15) acts from B_R to B_R. We claim that it is a contraction on B_R.

Indeed, the bound (8.17) yields that

$$\|B(u_1)-B(u_2)\|_{\alpha,T}\leqslant K_4(\varepsilon,q,\alpha)Te^{a_1(\varepsilon)T}\max\{\|u_1\|^q_{\alpha,T}, \|u_2\|^q_{\alpha,T}\}\|u_1-u_2\|_{\alpha,T},$$

whence it follows that for every $R\geqslant 2\|f\|_{\alpha,T}$ one can find a small $T>0$ such that

$$K_4(\varepsilon,q,\alpha)Te^{a_1(\varepsilon)T}R^q\leqslant\frac{1}{2}$$

and, therefore, $B(u)$ is a contraction on B_R. This proves that for all initial data

$$u_0(x),\, u_1(x) \in \mathbb{C}_b^{(2)}\bigl((1+x^2)^{\gamma/2};\mathbb{R}^1\bigr), \qquad \gamma \geqslant \alpha+1, \quad \alpha \geqslant \frac{1}{q},$$

one can find a small $T>0$ such that the integral equation (8.14) has a unique solution $u(x,t) \in \mathbb{C}([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};\mathbb{R}^1))$.

Note that the right-hand side of (8.14) belongs to $\mathbb{C}^{(1)}([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};\mathbb{R}^1))$. Hence the function $u(x,t)$ in the left-hand side also belongs to this class. Indeed, on the one hand, $f_1(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};\mathbb{R}^1))$. On the other hand, by Lemma 1 we have

$$\frac{\partial W_1(\rho)(x,t)}{\partial t} = -\frac{1}{2}\int_{\mathbb{R}^1} \exp(-|x-\xi|)\rho(\xi,t)\,d\xi - \int_0^t \int_{\mathbb{R}^1} \frac{\partial^2 \mathscr{E}(x-\xi,t-\tau)}{\partial \tau^2}\rho(\xi,\tau)\,d\xi\,d\tau,$$

where $\rho(\xi,\tau) = (|u|^q u)(\xi,\tau)$. The following inequality holds for all $t\in[0,T]$ in view of (4.7):

$$\begin{aligned}(1+x^2)^{\alpha/2}\biggl|\frac{\partial W_1(\rho)(x,t)}{\partial t}\biggr| &\leqslant \frac{1}{2}\|u\|_{\alpha,T}^{q+1}\int_{\mathbb{R}^1}\frac{(1+x^2)^{\alpha/2}}{(1+\xi^2)^{(q+1)\alpha/2}}\exp(-|x-\xi|)\,d\xi \\ &\quad + a_0(\varepsilon)e^{a_1(\varepsilon)T}T\|u\|_{\alpha,T}^{q+1}\int_{\mathbb{R}^1}\frac{(1+x^2)^{\alpha/2}}{(1+\xi^2)^{(q+1)\alpha/2}}\exp(-\varepsilon|x-\xi|)\,d\xi \\ &\leqslant C(\varepsilon,T,q,\alpha)\|u\|_{\alpha,T}^{q+1},\end{aligned}$$

where we have used the result of Lemma 21. It follows from this bound and the integral equation (8.14) that $u(x,t) \in \mathbb{C}^{(1)}([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};\mathbb{R}^1))$.

Using the standard algorithm of extension in time for solutions of integral equations with Volterra operators (as described, for example, in [20]), we arrive at the conclusion of the theorem. □

Theorem 4. *For every $T>0$ one can find sufficiently small numbers $R_2>0$ and $R_3>0$ such that if the conditions (8.3) hold, then there is a classical solution of the Cauchy problem for (3.1) in the sense of Definition 1 lying in the ball B_R defined in (8.4).*

Proof. Since $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}(\mathbb{R}^1)$, we conclude from Lemma 4 that the function $f(x,t)$ defined in (8.7) belongs to $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1))$. Moreover, we have $u(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1))$ by Theorem 2. Thus $\rho(x,t) = \partial^2 u^2(x,t)/\partial t^2 \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1))$. Then, by Lemma 7, we have

$$W(\rho)(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1)).$$

Thus the right-hand side of the equation (8.5) belongs to $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b^{(2)}(\mathbb{R}^1))$ and, therefore, so does the left-hand side. □

Theorem 5. *For any $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2};\mathbb{R}^1)$ with*

$$\gamma \geqslant \alpha+1, \qquad \alpha \geqslant \frac{1}{q}, \qquad q>0,$$

one can find a $T_0 = T_0(u_0, u_1) > 0$ *such that for every* $T \in (0, T_0)$ *there is a unique solution*

$$u(x,t) \in \mathbb{C}^{(2)}\big([0,T]; \mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2}; \mathbb{R}^1)\big)$$

of the Cauchy problem for (3.2) *in the sense of Definition* 1*. Moreover, either* $T_0 = +\infty$ *or* $T_0 < +\infty$, *and the following limit property holds in the latter case*:

$$\lim_{T \uparrow T_0} \|u\|_{\alpha,T} = +\infty,$$
$$\|u\|_{\alpha,T} := \sup_{(x,t) \in \mathbb{R}^1 \times [0,T]} (1+x^2)^{\alpha/2} |u(x,t)|.$$

Proof. First of all, as in the proof of Theorem 4, one can prove that the solution $u(x,t)$ of the integral equation (8.14) belongs to $\mathbb{C}^{(2)}([0,T]; \mathbb{C}_b^{(2)}(\mathbb{R}^1))$ for every $T \in (0, T_0)$. We claim that it actually belongs to $\mathbb{C}^{(2)}([0,T]; \mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2}; \mathbb{R}^1))$ with $\alpha \geqslant 1/q$ for every $T \in (0, T_0)$.

Indeed, note that the equation (8.14) for functions $u(x,t) \in \mathbb{C}^{(1)}([0,T]; \mathbb{C}_b(\mathbb{R}^1))$ can be rewritten in the form

$$u(x,t) = U(x,t), \tag{8.19}$$

where

$$U(x,t) = W(\rho)(x,t) + f(x,t),$$
$$W(x,t) := W(\rho)(x,t) = \int_0^t \int_{\mathbb{R}^1} \mathscr{E}(x-\xi, t-\tau) \rho(\xi,\tau)\, d\xi\, d\tau, \qquad \rho(\xi,\tau) := -\frac{\partial u^2(\xi,\tau)}{\partial \tau}, \tag{8.20}$$

$$f(x,t) := \int_{\mathbb{R}^1} \bigg(\mathscr{E}(x-\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi)] + \frac{\partial \mathscr{E}(x-\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)] \bigg)\, d\xi. \tag{8.21}$$

We have the following representations for the derivatives of $W(x,t)$:

$$\frac{\partial W(x,t)}{\partial x} = \int_0^t \int_{\mathbb{R}^1} \bigg\{ \frac{\partial \mathscr{E}(x-\xi,t-\tau)}{\partial x} \bigg\} \rho(\xi,\tau)\, d\xi\, d\tau, \tag{8.22}$$

$$\begin{aligned} \frac{\partial^2 W(x,t)}{\partial x^2} &= \int_0^t \int_{\mathbb{R}^1} \bigg\{ \frac{\partial^2 \mathscr{E}(x-\xi,t-\tau)}{\partial x^2} \bigg\} \rho(\xi,\tau)\, d\xi\, d\tau \\ &\quad + \int_0^t \bigg(\frac{\partial \mathscr{E}}{\partial x}(0-0,t-\tau) - \frac{\partial \mathscr{E}}{\partial x}(0+0,t-\tau) \bigg) \rho(x,\tau)\, d\tau, \end{aligned} \tag{8.23}$$

$$\frac{\partial W(x,t)}{\partial t} = \int_0^t \int_{\mathbb{R}^1} \frac{\partial \mathscr{E}(x-\xi,t-\tau)}{\partial t} \rho(\xi,\tau)\, d\xi\, d\tau,$$

$$\frac{\partial^2 W(x,t)}{\partial x\, \partial t} = \int_0^t \int_{\mathbb{R}^1} \bigg\{ \frac{\partial^2 \mathscr{E}(x-\xi,t-\tau)}{\partial x\, \partial t} \bigg\} \rho(\xi,\tau)\, d\xi\, d\tau,$$

$$\begin{aligned} \frac{\partial^3 W(x,t)}{\partial x^2\, \partial t} &= \int_0^t \int_{\mathbb{R}^1} \bigg\{ \frac{\partial^3 \mathscr{E}(x-\xi,t-\tau)}{\partial x^2\, \partial t} \bigg\} \rho(\xi,\tau)\, d\xi\, d\tau \\ &\quad + \int_0^t \bigg(\frac{\partial^2 \mathscr{E}}{\partial x\, \partial t}(0-0,t-\tau) - \frac{\partial^2 \mathscr{E}}{\partial x\, \partial t}(0+0,t-\tau) \bigg) \rho(x,\tau)\, d\tau, \end{aligned}$$

$$\frac{\partial^2 W(x,t)}{\partial t^2} = -\frac{1}{2}\int_{\mathbb{R}^1} \exp(-|x-\xi|)\rho(\xi,t)\,d\xi + \int_0^t \int_{\mathbb{R}^1} \frac{\partial^2 \mathscr{E}(x-\xi,t-\tau)}{\partial t^2}\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\frac{\partial^3 W(x,t)}{\partial t^2\,\partial x} = -\frac{1}{2}\int_{\mathbb{R}^1} \left\{\frac{d\exp(-|x-\xi|)}{dx}\right\}\rho(\xi,t)\,d\xi + \int_0^t \int_{\mathbb{R}^1} \left\{\frac{\partial^3 \mathscr{E}(x-\xi,t-\tau)}{\partial t^2\,\partial x}\right\}\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\frac{\partial^4 W(x,t)}{\partial t^2\,\partial x^2} = \rho(x,t) - \frac{1}{2}\int_{\mathbb{R}^1} \exp(-|x-\xi|)\rho(\xi,t)\,d\xi + \int_0^t \int_{\mathbb{R}^1} \left\{\frac{\partial^4 \mathscr{E}(x-\xi,t-\tau)}{\partial t^2\,\partial x^2}\right\}\rho(\xi,\tau)\,d\xi\,d\tau + \int_0^t \left(\frac{\partial^3 \mathscr{E}}{\partial t^2\,\partial x}(0-0,t-\tau) - \frac{\partial^3 \mathscr{E}}{\partial t^2\,\partial x}(0+0,t-\tau)\right)\rho(x,\tau)\,d\tau.$$

By the results of Lemma 1 for the integral

$$W_{k,l}(x,t) := \int_0^t \int_{\mathbb{R}^1} \left\{\frac{\partial^{k+l} \mathscr{E}(x-\xi,t-\tau)}{\partial x^k\,\partial t^l}\right\}\rho(\xi,\tau)\,d\xi\,d\tau, \quad \rho(\xi,\tau) = -\frac{\partial(|u|^q u)(\xi,\tau)}{\partial \tau} \tag{8.24}$$

we have the bound

$$\begin{aligned}(1+x^2)^{\alpha/2}|W_{k,l}(x,t)| &\leqslant (q+1)a_0(\varepsilon,k)e^{a_1(\varepsilon)T} \times \int_0^t \int_{\mathbb{R}^1} (1+x^2)^{\alpha/2}\exp(-\varepsilon|x-\xi|)|u(\xi,\tau)|^q \left|\frac{\partial u(\xi,\tau)}{\partial \tau}\right| d\xi\,d\tau \\ &\leqslant (q+1)a_0(\varepsilon,k)e^{a_1(\varepsilon)T}\|u\|^q_{\alpha,T}\left\|\frac{\partial u}{\partial t}\right\|_{\alpha,T} \times \sup_{x\in\mathbb{R}^1}\int_{\mathbb{R}^1} \frac{(1+x^2)^{\alpha/2}}{(1+\xi^2)^{(q+1)\alpha/2}}\exp(-\varepsilon|x-\xi|)\,d\xi \\ &\leqslant C(\varepsilon,q,k,\alpha,T)\|u\|^q_{\alpha,T}\left\|\frac{\partial u}{\partial t}\right\|_{\alpha,T},\end{aligned} \tag{8.25}$$

where we used Lemma 21.

Consider the integral

$$U_k(x,t) := \int_0^t \left(\frac{\partial^{k+1}\mathscr{E}(0-0,t-\tau)}{\partial t^2\,\partial x} - \frac{\partial^{k+1}\mathscr{E}(0+0,t-\tau)}{\partial t^2\,\partial x}\right)\rho(x,\tau)\,d\tau. \tag{8.26}$$

Note that, by the results of Lemma 1, we have

$$\frac{\partial^{k+1}\mathscr{E}(0-0,t-\tau)}{\partial t^2\,\partial x}, \frac{\partial^{k+1}\mathscr{E}(0+0,t-\tau)}{\partial t^2\,\partial x} \in \mathbb{C}[0,+\infty).$$

This yields the following bound for the integral $U_k(x,t)$:

$$(1+x^2)^{\alpha/2}|U_k(x,t)| \leqslant C(k,T)\frac{1}{(1+x^2)^{\alpha q/2}}\|u\|^q_{\alpha,T}\left\|\frac{\partial u}{\partial t}\right\|_{\alpha,T} \leqslant C(k,T)\|u\|^q_{\alpha,T}\left\|\frac{\partial u}{\partial t}\right\|_{\alpha,T}.$$

Finally, we have the following estimates:

$$\begin{aligned}(1+x^2)^{\alpha/2}&\left|\int_{\mathbb{R}^1}\exp(-|x-\xi|)\rho(\xi,t)\,d\xi\right|\\ &\leqslant (q+1)\|u\|^q_{\alpha,T}\left\|\frac{\partial u}{\partial t}\right\|_{\alpha,T}\sup_{x\in\mathbb{R}^1}\int_{\mathbb{R}^1}\frac{(1+x^2)^{\alpha/2}}{(1+\xi^2)^{(q+1)\alpha/2}}\exp(-|x-\xi|)\,d\xi\\ &\leqslant C(q,\alpha,T)\|u\|^q_{\alpha,T}\left\|\frac{\partial u}{\partial t}\right\|_{\alpha,T},\end{aligned}$$

$$(1+x^2)^{\alpha/2}|\rho(x,t)| \leqslant (q+1)\frac{1}{(1+x^2)^{\alpha q/2}}\|u\|^q_{\alpha,T}\left\|\frac{\partial u}{\partial t}\right\|_{\alpha,T} \leqslant C(q,\alpha,T)\|u\|^q_{\alpha,T}\left\|\frac{\partial u}{\partial t}\right\|_{\alpha,T}. \tag{8.27}$$

One can similarly prove that the function $f(x,t)$ defined in (8.21) belongs to $\mathbb{C}^{(2)}_b([0,T];\mathbb{C}^{(2)}_b((1+x^2)^{\alpha/2};\mathbb{R}^1))$ if $u_0(x)$, $u_1(x)\in\mathbb{C}^{(2)}_b((1+x^2)^{\gamma/2};\mathbb{R}^1)$ with $\gamma\geqslant\alpha+1$, $\alpha\geqslant 1/q$.

Thus we conclude from (8.20)–(8.24), (8.26) in view of the bounds (8.25)–(8.27) that the right-hand side of the integral equation (8.19) belongs to $\mathbb{C}^{(2)}([0,T];\mathbb{C}^{(2)}_b((1+x^2)^{\alpha/2};\mathbb{R}^1))$ for every $T\in(0,T_0)$ provided that $\gamma\geqslant\alpha+1$, $\alpha\geqslant 1/q$. □

§ 9. Solubility of the first boundary-value problems for (3.1) and (3.2)

Theorem 6. *Suppose that $\nu(0)=\nu'(0)=u_0(0)=u_1(0)=0$. Then the classical solutions of the first boundary-value problem for the equations (3.1) and (3.2) in the sense of Definition 2 in the class*

$$u(x,t)\in\mathbb{C}^{(2)}([0,T];\mathbb{C}^{(2)}_b[0,+\infty)),$$
$$u_0(x),u_1(x)\in\mathbb{C}^{(2)}_b[0,+\infty),\qquad \nu(t)\in\mathbb{C}^{(2)}[0,T]$$

are equivalent to the following integral equations respectively:

$$\begin{aligned}u(x,t)&=\int_0^t\int_0^{+\infty}\mathscr{E}_1(x,\xi,t-\tau)\frac{\partial^2u^2(\xi,\tau)}{\partial\tau^2}\,d\xi\,d\tau\\ &\quad+\int_0^{+\infty}\biggl(\mathscr{E}_1(x,\xi,t)[u_{1\xi\xi}(\xi)-u_1(\xi)]+\frac{\partial\mathscr{E}_1(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi)-u_0(\xi)]\biggr)\,d\xi\\ &\quad+2\int_0^t\nu(\tau)\mathfrak{N}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau+\nu(t)e^{-x},\end{aligned} \tag{9.1}$$

$$u(x,t) = -\int_0^t \int_0^{+\infty} \mathscr{E}_1(x,\xi,t-\tau)\frac{\partial(|u|^q u)(\xi,\tau)}{\partial \tau}\,d\xi\,d\tau$$
$$+ \int_0^{+\infty}\left(\mathscr{E}_1(x,\xi,t)[u_{1\xi\xi}(\xi)-u_1(\xi)] + \frac{\partial \mathscr{E}_1(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi)-u_0(\xi)]\right)d\xi$$
$$+ 2\int_0^t \nu(\tau)\mathfrak{N}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau + \nu(t)e^{-x}, \tag{9.2}$$

where

$$\mathscr{E}_1(x,\xi,t) := \mathscr{E}(x-\xi,t) - \mathscr{E}(x+\xi,t), \qquad \mathfrak{N}_{x,t} := \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial t^2} + I\right).$$

Proof. This follows directly from the equality (4.9) in Lemma 3 and Lemmas 8–12. □

Theorem 7. *For every $T > 0$ one can find small numbers $R > 0$, $R_2 > 0$, $R_3 > 0$, $R_4 > 0$ such that under the conditions*

$$u_0(x) \in \mathbb{C}_b^{(2)}[0,+\infty), \qquad \|u_0\|_{\mathbb{C}_b^2[0,+\infty)} \leqslant R_2, \tag{9.3}$$
$$u_1(x) \in \mathbb{C}_b^{(2)}[0,+\infty), \qquad \|u_1\|_{\mathbb{C}_b^2[0,+\infty)} \leqslant R_3,$$
$$\nu(t) \in \mathbb{C}^{(2)}[0,T], \qquad \|\nu\|_{\mathbb{C}^2[0,T]} \leqslant R_4 \tag{9.4}$$

the equation (9.1) has a unique solution in $\mathbb{C}^{(2)}([0,T];\mathbb{C}_b(\mathbb{R}^1_+)) \cap B_R$, where

$$B_R = \left\{v(x,t) \in \mathbb{C}^{(2)}\big([0,T];\mathbb{C}_b[0,+\infty)\big) \colon \|v\|_T^+ \leqslant R\right\}, \tag{9.5}$$
$$\|v\|_T^+ := \sup_{(x,t)\in[0,+\infty)\times[0,T]} \sum_{j=0}^2 \left|\frac{\partial^j v(x,t)}{\partial t^j}\right|.$$

Proof. We rewrite the integral equation (9.1) in the form

$$u(x,t) = A_1(u)(x,t),$$

where

$$A_1(u)(x,t) := f_1(x,t) + f_2(x,t) + W_1(\rho)(x,t), \qquad \rho(\xi,\tau) := \frac{\partial^2 u^2(\xi,\tau)}{\partial \tau^2},$$
$$f_1(x,t) := \int_0^{+\infty}\left(\mathscr{E}_1(x,\xi,t)[u_{1\xi\xi}(\xi)-u_1(\xi)] + \frac{\partial \mathscr{E}_1(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi)-u_0(\xi)]\right)d\xi,$$
$$f_2(x,t) := 2\int_0^t \nu(\tau)\mathfrak{N}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau + \nu(t)e^{-x},$$
$$W_1(\rho)(x,t) := \int_0^t \int_0^{+\infty} \mathscr{E}_1(x,\xi,t-\tau)\rho(\xi,\tau)\,d\xi\,d\tau$$

and

$$\mathscr{E}_1(x,\xi,t) := \mathscr{E}(x-\xi,t) - \mathscr{E}(x+\xi,t), \qquad \mathfrak{N}_{x,t} := \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial t^2} + I\right).$$

We have $f_1(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b[0,+\infty))$ by Lemma 9 and $f_2(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}_b[0,+\infty))$ by Lemma 8. The function $W_1(x,t) := W_1(\rho)(x,t)$ satisfies the following equalities in view of Lemma 1:

$$\frac{\partial W_1(x,t)}{\partial t} := \int_0^t \int_0^{+\infty} \frac{\partial \mathscr{E}_1(x,\xi,t-\tau)}{\partial t}\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\frac{\partial^2 W_1(x,t)}{\partial t^2} := -\frac{1}{2}\int_0^{+\infty}(e^{-|x-\xi|} - e^{-|x+\xi|})\rho(\xi,t)\,d\xi + \int_0^t \int_0^{+\infty} \frac{\partial \mathscr{E}_1(x,\xi,t-\tau)}{\partial t}\rho(\xi,\tau)\,d\xi\,d\tau.$$

The rest of the proof repeats the proof of Theorem 2 using Lemma 1. □

We have the following assertion about the integral equation (9.2).

Theorem 8. *For any functions $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2};[0,+\infty))$ with*

$$\gamma \geqslant \alpha + 1, \qquad \alpha \geqslant \frac{1}{q}, \qquad q > 0$$

and $\nu(t) \in \mathbb{C}_b^{(1)}[0,+\infty)$ one can find a maximal $T_0 = T_0(u_0,u_1,\nu) > 0$ such that for every $T \in (0,T_0)$ there is a unique solution $u(x,t) \in \mathbb{C}^{(1)}([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};[0,+\infty)))$ of the integral equation (9.2). Moreover, either $T_0 = +\infty$ or $T_0 < +\infty$, and the following limit property holds in the latter case:

$$\lim_{T \uparrow T_0} \|u\|^+_{\alpha,T} = +\infty,$$

where

$$\|v\|^+_{\alpha,T} := \sup_{(x,t)\in[0,+\infty)\times[0,T]} (1+x^2)^{\alpha/2}|v(x,t)|.$$

Proof. The integral equation (9.2) for functions $u(x,t) \in \mathbb{C}^{(1)}([0,T];\mathbb{C}_b[0,+\infty))$ can be rewritten in the form

$$u(x,t) = B_1(u)(x,t), \qquad B_1(u)(x,t) = f_1(x,t) + f_2(x,t) + W_1(\rho)(x,t),$$

where

$$f_1(x,t) := \int_0^{+\infty}\biggl(\mathscr{E}_1(x,\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi) - (|u_0|^q u_0)(\xi)] + \frac{\partial \mathscr{E}_1(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)]\biggr)\,d\xi,$$

$$f_2(x,t) := 2\int_0^t \nu(\tau)\mathfrak{N}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau + \nu(t)e^{-x},$$

$$W_1(\rho)(x,t) := \int_0^t \int_0^{+\infty} \frac{\partial \mathscr{E}_1(x,\xi,t-\tau)}{\partial \tau}\rho(\xi,\tau)\,d\xi\,d\tau, \qquad \rho(\xi,\tau) := (|u|^q u)(\xi,\tau),$$

$$\mathscr{E}_1(x,\xi,t) := \mathscr{E}(x-\xi,t) - \mathscr{E}(x+\xi,t), \qquad \mathfrak{N}_{x,t} := \frac{\partial}{\partial x}\biggl(\frac{\partial^2}{\partial t^2} + I\biggr).$$

The rest of the proof repeats the corresponding parts of the proof of Theorem 3. It is necessary to use the results of Lemmas 1 and 22. □

Theorems 6 and 7 yield the following assertion.

Theorem 9. *Suppose that $\nu(0)=\nu'(0)=u_0(0)=u_1(0)=0$. Then for every $T>0$ one can find small numbers $R>0$, $R_2>0$, $R_3>0$, $R_4>0$ such that under the conditions (9.3), (9.4) there is a unique classical solution $u(x,t)\in\mathbb{C}^{(2)}([0,T];\mathbb{C}_b^{(2)}[0,+\infty))$ of the first boundary-value problem for (3.1) in the sense of Definition 2. This solution lies in the ball B_R defined in (9.5).*

Proof. Use the smoothness properties of the potentials in § 5. □

Theorems 6 and 8 yield the following assertion.

Theorem 10. *Suppose that $\nu(0)=\nu'(0)=u_0(0)=u_1(0)=0$. Then for all $u_0(x)$, $u_1(x)\in\mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2};[0,+\infty))$ with*

$$\gamma\geqslant\alpha+1,\qquad \alpha\geqslant\frac{1}{q},\qquad q>0$$

and $\nu(t)\in\mathbb{C}_b^{(2)}[0,+\infty)$ one can find a maximal $T_0=T_0(u_0,u_1,\nu)>0$ such that for every $T\in(0,T_0)$ there is a unique solution $u(x,t)\in\mathbb{C}^{(2)}([0,T];\mathbb{C}_b((1+x^2)^{\alpha/2};[0,+\infty)))$ of the first boundary-value problem for (3.2) in the sense of Definition 2. Moreover, either $T_0=+\infty$ or $T_0<+\infty$, and the following limit property holds in the latter case:

$$\lim_{T\uparrow T_0}\|u\|^+_{\alpha,T}=+\infty.$$

Proof. On the whole, the proof repeats that of Theorem 5. Consider the integral equation (9.2). Put

$$u(x,t)=U_1(x,t),\qquad U_1(x,t)=W_1(\rho)(x,t)+f_1(x,t)+f_2(x,t),$$

$$f_1(x,t):=\int_0^{+\infty}\biggl(\mathscr{E}_1(x,\xi,t)[u_{1\xi\xi}(\xi)-u_1(\xi)]+\frac{\partial\mathscr{E}_1(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi)-u_0(\xi)]\biggr)\,d\xi,$$

$$f_2(x,t):=2\int_0^t\nu(\tau)\mathfrak{N}_{x,t}\mathscr{E}(x,t-\tau)\,d\tau+\nu(t)e^{-x},$$

$$W_1(x,t):=W_1(\rho)(x,t)=\int_0^t\int_0^{+\infty}\mathscr{E}_1(x,\xi,t-\tau)\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\rho(\xi,\tau):=-\frac{\partial(|u|^qu)(\xi,\tau)}{\partial\tau}.$$

We have the following formulae for the derivatives of $W_1(x,t)$:

$$\frac{\partial W_1(x,t)}{\partial x}=\int_0^t\int_0^{+\infty}\biggl\{\frac{\partial\mathscr{E}_1(x-\xi,t-\tau)}{\partial x}\biggr\}\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\frac{\partial^2 W_1(x,t)}{\partial x^2}=\int_0^t\int_0^{+\infty}\biggl\{\frac{\partial^2\mathscr{E}_1(x-\xi,t-\tau)}{\partial x^2}\biggr\}\rho(\xi,\tau)\,d\xi\,d\tau$$
$$+\int_0^t\biggl(\frac{\partial\mathscr{E}_1}{\partial x}(0-0,t-\tau)-\frac{\partial\mathscr{E}_1}{\partial x}(0+0,t-\tau)\biggr)\rho(x,\tau)\,d\tau,$$

$$\frac{\partial W_1(x,t)}{\partial t}=\int_0^t\int_0^{+\infty}\frac{\partial\mathscr{E}_1(x-\xi,t-\tau)}{\partial t}\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\frac{\partial^2 W_1(x,t)}{\partial x\,\partial t} = \int_0^t \int_0^{+\infty} \biggl\{\frac{\partial^2 \mathscr{E}_1(x-\xi,t-\tau)}{\partial x\,\partial t}\biggr\}\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\frac{\partial^3 W_1(x,t)}{\partial x^2\,\partial t} = \int_0^t \int_0^{+\infty} \biggl\{\frac{\partial^3 \mathscr{E}_1(x-\xi,t-\tau)}{\partial x^2\,\partial t}\biggr\}\rho(\xi,\tau)\,d\xi\,d\tau + \int_0^t \biggl(\frac{\partial^2 \mathscr{E}_1}{\partial x\,\partial t}(0-0,t-\tau) - \frac{\partial^2 \mathscr{E}_1}{\partial x\,\partial t}(0+0,t-\tau)\biggr)\rho(x,\tau)\,d\tau,$$

$$\frac{\partial^2 W_1(x,t)}{\partial t^2} = -\frac{1}{2}\int_0^{+\infty} (e^{-|x-\xi|} - e^{-|x+\xi|})\rho(\xi,t)\,d\xi + \int_0^t \int_0^{+\infty} \frac{\partial^2 \mathscr{E}_1(x-\xi,t-\tau)}{\partial t^2}\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\frac{\partial^3 W_1(x,t)}{\partial t^2\,\partial x} = -\frac{1}{2}\int_0^{+\infty} \biggl\{\frac{d}{dx}(e^{-|x-\xi|} - e^{-|x+\xi|})\biggr\}\rho(\xi,t)\,d\xi + \int_0^t \int_0^{+\infty} \biggl\{\frac{\partial^3 \mathscr{E}_1(x-\xi,t-\tau)}{\partial t^2\,\partial x}\biggr\}\rho(\xi,\tau)\,d\xi\,d\tau,$$

$$\frac{\partial^4 W_1(x,t)}{\partial t^2\,\partial x^2} = \rho(x,t) - \frac{1}{2}\int_0^{+\infty} (e^{-|x-\xi|} - e^{-|x+\xi|})\rho(\xi,t)\,d\xi + \int_0^t \int_0^{+\infty} \biggl\{\frac{\partial^4 \mathscr{E}_1(x-\xi,t-\tau)}{\partial t^2\,\partial x^2}\biggr\}\rho(\xi,\tau)\,d\xi\,d\tau + \int_0^t \biggl(\frac{\partial^3 \mathscr{E}_1}{\partial t^2\,\partial x}(0-0,t-\tau) - \frac{\partial^3 \mathscr{E}_1}{\partial t^2\,\partial x}(0+0,t-\tau)\biggr)\rho(x,\tau)\,d\tau.$$

Then we use the results of Lemmas 1 and 22. Analogous formulae hold for the functions $f_1(x,t)$ and $f_2(x,t)$. $\square$

§ 10. Solubility of the second boundary-value problems for (3.1) and (3.2)

We have the following theorem whose proof repeats that of Theorem 6.

Theorem 11. *Suppose that $\mu(0) = \mu'(0) = u_{0x}(0) = u_{1x}(0) = 0$. Then the classical solutions of the second boundary-value problem for the equations (3.1) and (3.2) in the sense of Definition 3 in the class*

$$u(x,t) \in \mathbb{C}^{(2)}\bigl([0,T]; \mathbb{C}_b^{(2)}[0,+\infty)\bigr),$$
$$u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}[0,+\infty), \qquad \mu(t) \in \mathbb{C}^{(2)}[0,T]$$

are equivalent to the following integral equations respectively:

$$u(x,t) = \int_0^t \int_0^{+\infty} \mathscr{E}_2(x,\xi,t-\tau)\frac{\partial^2 u^2(\xi,\tau)}{\partial \tau^2}\,d\xi\,d\tau + \int_0^{+\infty} \biggl(\mathscr{E}_2(x,\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi)] + \frac{\partial \mathscr{E}_2(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)]\biggr)\,d\xi + 2\int_0^t \mu(\tau)\biggl[\frac{\partial^2 \mathscr{E}(x,t-\tau)}{\partial t^2} + \mathscr{E}(x,t-\tau)\biggr]\,d\tau - \mu(t)e^{-x}, \qquad (10.1)$$

$$
\begin{aligned}
u(x,t) = &-\int_0^t \int_0^{+\infty} \mathscr{E}_2(x,\xi,t-\tau) \frac{\partial(|u|^q u)(\xi,\tau)}{\partial \tau}\, d\xi\, d\tau \\
&+ \int_0^{+\infty} \left(\mathscr{E}_2(x,\xi,t)[u_{1\xi\xi}(\xi) - u_1(\xi)] + \frac{\partial \mathscr{E}_2(x,\xi,t)}{\partial t}[u_{0\xi\xi}(\xi) - u_0(\xi)] \right) d\xi \\
&+ 2\int_0^t \mu(\tau) \left[\frac{\partial^2 \mathscr{E}(x,t-\tau)}{\partial t^2} + \mathscr{E}(x,t-\tau) \right] d\tau - \mu(t) e^{-x}. \qquad (10.2)
\end{aligned}
$$

Studying the integral equations (10.1) and (10.2), we obtain the following assertions.

Theorem 12. *Suppose that $\mu(0) = \mu'(0) = u_{0x}(0) = u_{1x}(0) = 0$. Then for every $T > 0$ one can find small numbers $R > 0$, $R_2 > 0$, $R_3 > 0$, $R_4 > 0$ such that under the conditions*

$$
\begin{aligned}
u_0(x) &\in \mathbb{C}_b^{(2)}[0,+\infty), \qquad \|u_0\|_{\mathbb{C}_b^{(2)}[0,+\infty)} \leqslant R_2, \\
u_1(x) &\in \mathbb{C}_b^{(2)}[0,+\infty), \qquad \|u_1\|_{\mathbb{C}_b^{(2)}[0,+\infty)} \leqslant R_3, \\
\mu(t) &\in \mathbb{C}^{(2)}[0,T], \qquad \|\mu\|_{\mathbb{C}^{(2)}[0,T]} \leqslant R_4
\end{aligned}
$$

the second boundary-value problem for (3.1) has a unique solution in the sense of Definition 3 in the class $\mathbb{C}^{(2)}([0,T]; \mathbb{C}_b^{(2)}(\mathbb{R}^1_+)) \cap B_R$, where

$$
B_R = \left\{ v(x,t) \in \mathbb{C}^{(2)}\big([0,T]; \mathbb{C}_b^{(2)}[0,+\infty)\big) \colon \|v\|_T^+ \leqslant R \right\},
$$

$$
\|v\|_T^+ := \sup_{(x,t)\in[0,+\infty)\times[0,T]} \sum_{j=0}^{2} \left| \frac{\partial^j v(x,t)}{\partial t^j} \right|.
$$

Proof. Just repeat the proof of Theorems 7 and 15. □

Theorem 13. *Suppose that $\mu(0) = \mu'(0) = u_{0x}(0) = u_{1x}(0) = 0$. For any functions $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2}; [0,+\infty))$ with*

$$
\gamma \geqslant \alpha + 1, \qquad \alpha \geqslant \frac{1}{q}, \qquad q > 0
$$

and $\mu(t) \in \mathbb{C}_b^{(2)}[0,+\infty)$ one can find a maximal $T_0 = T_0(u_0, u_1, \mu) > 0$ such that for every $T \in (0, T_0)$ there is a unique classical solution

$$
u(x,t) \in \mathbb{C}^{(2)}\big([0,T]; \mathbb{C}_b((1+x^2)^{\alpha/2}; [0,+\infty))\big)
$$

of the second boundary-value problem for (3.2) in the sense of Definition 3. Moreover, either $T_0 = +\infty$ or $T_0 < +\infty$, and the following limit property holds in the latter case:

$$
\lim_{T \uparrow T_0} \|u\|_{\alpha,T}^+ = +\infty,
$$

where

$$
\|v\|_{\alpha,T}^+ := \sup_{(x,t)\in[0,+\infty)\times[0,T]} (1+x^2)^{\alpha/2} |v(x,t)|.
$$

§ 11. *A priori* estimates for solutions of the Cauchy problem and the second boundary-value problem for (3.1)

We first obtain *a priori* estimates for classical solutions of the second boundary-value problems for the equation (3.1) in the sense of Definition 3. Let $u(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}^{(2)}[0,+\infty))$ be such a solution. We rewrite (3.1) in the following equivalent form:

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2}\biggl(u(x,t)+\frac{1}{2}\biggr)+\frac{\partial^2}{\partial x^2}\biggl(u(x,t)+\frac{1}{2}\biggr)=\frac{\partial^2}{\partial t^2}\biggl(u(x,t)+\frac{1}{2}\biggr)^2. \tag{11.1}$$

Take a test function of the form

$$\phi(x,t)=\phi_1(x)\phi_2(t), \qquad \phi_2(t)=\biggl(1-\frac{t}{T}\biggr)^2, \tag{11.2}$$

$$\phi_1(x)=\begin{cases}1 & \text{if } x\in[0,1],\\ 0 & \text{if } x\geqslant 2,\end{cases} \qquad 0\leqslant\phi_1(x)\leqslant 1, \tag{11.3}$$

where the function $\phi_1(x)$ belongs to $\mathbb{C}^{(2)}[0,+\infty)$ and is monotone non-increasing. Integrating by parts we have the following formulae:

$$\begin{aligned}\int_0^{+\infty}\frac{\partial^2}{\partial x^2}\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_1(x)\,dx&=\frac{\partial}{\partial x}\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_1(x)\biggr|_{x=0}^{x=+\infty}\\&\quad-\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_{1x}(x)\biggr|_{x=0}^{x=+\infty}+\int_0^{+\infty}\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_{1xx}(x)\,dx\\&=-u_x(0,t)+\int_0^{+\infty}\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_{1xx}(x)\,dx,\end{aligned} \tag{11.4}$$

$$\begin{aligned}&\int_0^{T}\frac{\partial^2}{\partial t^2}\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_2(t)\,dt\\&\quad=\frac{\partial}{\partial t}\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_2(t)\biggr|_{t=0}^{t=T}-\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_2'(t)\biggr|_{t=0}^{t=T}\\&\qquad+\int_0^T\biggl(u(x,t)+\frac{1}{2}\biggr)\phi_2''(t)\,dt=-u'(x,0)-\frac{2}{T}\biggl(u(x,0)+\frac{1}{2}\biggr)\\&\qquad+\frac{2}{T^2}\int_0^T\biggl(u(x,t)+\frac{1}{2}\biggr)\,dt.\end{aligned} \tag{11.5}$$

Here we have used the equalities

$$\phi_1(0)=1, \qquad \phi_{1x}(0)=0, \qquad \phi_2(0)=1, \qquad \phi_2'(0)=-\frac{2}{T}, \qquad \phi_2''(t)=\frac{2}{T^2},$$
$$\phi_2(T)=\phi_2'(T)=0,$$

which follow from the definitions (11.2) and (11.3).

It follows from (11.4) and (11.5) that

$$\int_0^T\int_0^{+\infty}\frac{\partial^2}{\partial x^2}\left(u(x,t)+\frac{1}{2}\right)\phi_1(x)\phi_2(t)\,dx\,dt$$
$$=-\int_0^T\mu(t)\left(1-\frac{t}{T}\right)^2dt+\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)\left(1-\frac{t}{T}\right)^2\phi_{1xx}(x)\,dx\,dt, \tag{11.6}$$

$$\int_0^T\int_0^{+\infty}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial t^2}\left(u(x,t)+\frac{1}{2}\right)\phi_1(x)\phi_2(t)\,dx\,dt$$
$$=-\int_0^T\frac{\partial^3 u}{\partial t^2\,\partial x}(0,t)\phi_2(t)\,dt+\int_0^T\int_0^{+\infty}\frac{\partial^2}{\partial t^2}\left(u(x,t)+\frac{1}{2}\right)\phi_{1xx}(x)\phi_2(t)\,dx\,dt$$
$$=-\int_0^T\frac{\partial^3 u}{\partial t^2\,\partial x}(0,t)\phi_2(t)\,dt-\int_0^{+\infty}\frac{\partial u}{\partial t}(x,0)\phi_{1xx}(x)\,dx$$
$$-\frac{2}{T}\int_0^{+\infty}\left(u(x,0)+\frac{1}{2}\right)\phi_{1xx}(x)\,dx+\frac{2}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)\phi_{1xx}(x)\,dx\,dt$$
$$=-\int_0^T\mu''(t)\left(1-\frac{t}{T}\right)^2dt-\int_0^{+\infty}u_1(x)\phi_{1xx}(x)\,dx$$
$$-\frac{2}{T}\int_0^{+\infty}\left(u_0(x)+\frac{1}{2}\right)\phi_{1xx}(x)\,dx+\frac{2}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)\phi_{1xx}(x)\,dx\,dt,$$
$$\int_0^T\int_0^{+\infty}\frac{\partial^2}{\partial t^2}\left(u(x,t)+\frac{1}{2}\right)^2\phi_1(x)\phi_2(t)\,dx\,dt$$
$$=-2\int_0^{+\infty}u_1(x)\left(u_0(x)+\frac{1}{2}\right)\phi_1(x)\,dx-\frac{2}{T}\int_0^{+\infty}\left(u_0(x)+\frac{1}{2}\right)^2\phi_1(x)\,dx$$
$$+\frac{2}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)^2\phi_1(x)\,dx\,dt. \tag{11.7}$$

Multiplying both sides of (11.1) by the test function $\phi_1(x)\phi_2(t)$ and integrating by parts in view of (11.6) and (11.7), we obtain the following equality:

$$\frac{2}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)\phi_{1xx}(x)\,dx\,dt$$
$$+\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)\left(1-\frac{t}{T}\right)^2\phi_{1xx}(x)\,dx\,dt$$
$$-\int_0^{+\infty}\left[u_1(x)+\frac{2}{T}\left(u_0(x)+\frac{1}{2}\right)\right]\phi_{1xx}(x)\,dx-\int_0^T[\mu''(t)+\mu(t)]\left(1-\frac{t}{T}\right)^2dt$$
$$+2\int_0^{+\infty}u_1(x)\left(u_0(x)+\frac{1}{2}\right)\phi_1(x)\,dx+\frac{2}{T}\int_0^{+\infty}\left(u_0(x)+\frac{1}{2}\right)^2\phi_1(x)\,dx$$
$$=\frac{2}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)^2\phi_1(x)\,dx\,dt. \tag{11.8}$$

The following estimates hold:

$$\begin{aligned}
&\frac{2}{T^2}\left|\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)\phi_{1xx}(x)\,dx\,dt\right| \\
&\quad\leqslant \frac{2\lambda}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)^2\phi_1(x)\,dx\,dt+\frac{1}{2T^2\lambda}\int_0^T\int_0^{+\infty}\frac{|\phi_{1xx}(x)|^2}{\phi_1(x)}\,dx\,dt \\
&\quad= \frac{2\lambda}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)^2\phi_1(x)\,dx\,dt+\frac{1}{2T\lambda}\int_0^{+\infty}\frac{|\phi_{1xx}(x)|^2}{\phi_1(x)}\,dx, \qquad (11.9)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)\left(1-\frac{t}{T}\right)^2\phi_{1xx}(x)\,dx\,dt\right| \\
&\quad\leqslant \frac{2\lambda}{T^2}\int_0^T\int_0^{+\infty}\phi_1(x)\left(u(x,t)+\frac{1}{2}\right)^2dx\,dt+\frac{T^2}{8\lambda}\int_0^T\int_0^{+\infty}\frac{|\phi_{1xx}|^2}{\phi_1(x)}\left(1-\frac{t}{T}\right)^4dx\,dt \\
&\quad= \frac{2\lambda}{T^2}\int_0^T\int_0^{+\infty}\phi_1(x)\left(u(x,t)+\frac{1}{2}\right)^2dx\,dt+\frac{T^3}{40\lambda}\int_0^{+\infty}\frac{|\phi_{1xx}(x)|^2}{\phi_1(x)}\,dx. \qquad (11.10)
\end{aligned}$$

Using (11.9) and (11.10), we obtain from (11.8) that

$$\begin{aligned}
&\left(\frac{1}{2T\lambda}+\frac{T^3}{40\lambda}\right)\int_0^{+\infty}\frac{|\phi_{1xx}(x)|^2}{\phi_1(x)}\,dx \\
&\qquad+\int_0^{+\infty}\left[u_1(x)+\frac{2}{T}\left(u_0(x)+\frac{1}{2}\right)\right]\phi_{1xx}(x)\,dx-\int_0^T[\mu''(t)+\mu(t)]\left(1-\frac{t}{T}\right)^2dt \\
&\qquad+2\int_0^{+\infty}u_1(x)\left(u_0(x)+\frac{1}{2}\right)\phi_1(x)\,dx+\frac{2}{T}\int_0^{+\infty}\left(u_0(x)+\frac{1}{2}\right)^2\phi_1(x)\,dx \\
&\quad\geqslant \frac{2(1-2\lambda)}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)^2\phi_1(x)\,dx\,dt. \qquad (11.11)
\end{aligned}$$

We take the test function $\phi_1(x)$ to be of the form

$$\phi_1(x)=\phi_0\left(\frac{x}{R}\right), \qquad \phi_0(s)=\begin{cases}1 & \text{if } s\in[0,1],\\ 0 & \text{if } s\geqslant 2\end{cases}$$

for $R\geqslant 1$, where the function $\phi_0(s)$ belongs to $\mathbb{C}_0^{(2)}[0,+\infty)$ and is monotone non-increasing. Note that there is such a function $\phi_0(s)$ with the following properties:

$$\int_0^{+\infty}\frac{|\phi_{1xx}(x)|^2}{\phi_1(x)}\,dx=\frac{c_0}{R^3}, \qquad c_0:=\int_0^2\frac{|\phi_{0ss}(s)|^2}{\phi_0(s)}\,ds<+\infty. \qquad (11.12)$$

We now make an important assumption. Suppose that

$$u_0(x)+\frac{1}{2}\in L^2(0,+\infty), \qquad u_1(x)\in L^2(0,+\infty).$$

Then the following inequalities hold:

$$\begin{aligned}&\left|\int_0^{+\infty}\left[u_1(x)+\frac{2}{T}\left(u_0(x)+\frac{1}{2}\right)\right]\phi_{1xx}(x)\,dx\right|\\&\qquad\leqslant\left(\int_0^{+\infty}\left[u_1(x)+\frac{2}{T}\left(u_0(x)+\frac{1}{2}\right)\right]^2dx\right)^{1/2}\left(\int_0^{+\infty}|\phi_{1xx}|^2\,dx\right)^{1/2}=\frac{c_1}{R^{3/2}},\\&\quad c_1=\left(\int_0^{+\infty}\left[u_1(x)+\frac{2}{T}\left(u_0(x)+\frac{1}{2}\right)\right]^2dx\right)^{1/2}\left(\int_0^{2}|\phi_{0ss}|^2\,ds\right)^{1/2}<+\infty,\end{aligned}\tag{11.13}$$

and we also obtain the following limit properties:

$$\begin{aligned}\int_0^{+\infty}u_1(x)\left(u_0(x)+\frac{1}{2}\right)\phi_1(x)\,dx&\to\int_0^{+\infty}u_1(x)\left(u_0(x)+\frac{1}{2}\right)dx,\\\int_0^{+\infty}\left(u_0(x)+\frac{1}{2}\right)^2\phi_1(x)\,dx&\to\int_0^{+\infty}\left(u_0(x)+\frac{1}{2}\right)^2dx\end{aligned}\tag{11.14}$$

as $R\to+\infty$. Put $R=N\in\mathbb{N}$. Using the Beppo Levi theorem, we deduce the following *a priori* estimate from the inequality (11.11) in view of the bounds and limiting properties (11.12)–(11.14):

$$\begin{aligned}&-\int_0^T[\mu''(t)+\mu(t)]\left(1-\frac{t}{T}\right)^2dt\\&\qquad+2\int_0^{+\infty}u_1(x)\left(u_0(x)+\frac{1}{2}\right)dx+\frac{2}{T}\int_0^{+\infty}\left(u_0(x)+\frac{1}{2}\right)^2dx\\&\quad\geqslant\frac{2(1-2\lambda)}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)^2dx\,dt.\end{aligned}$$

This estimate holds for all $\lambda>0$. Letting $\lambda\to 0+0$, we arrive at the desired *a priori* estimate

$$\begin{aligned}&-\int_0^T[\mu''(t)+\mu(t)]\left(1-\frac{t}{T}\right)^2dt\\&\qquad+2\int_0^{+\infty}u_1(x)\left(u_0(x)+\frac{1}{2}\right)dx+\frac{2}{T}\int_0^{+\infty}\left(u_0(x)+\frac{1}{2}\right)^2dx\\&\quad\geqslant\frac{2}{T^2}\int_0^T\int_0^{+\infty}\left(u(x,t)+\frac{1}{2}\right)^2dx\,dt.\end{aligned}$$

Thus, the following assertion holds.

Theorem 14. *Suppose that $u(x,t)\in\mathbb{C}^{(2)}([0,T];\mathbb{C}^{(2)}[0,+\infty))$ is a classical solution of the second boundary-value problem for (3.1) in the sense of Definition 3 with initial conditions*

$$u_0(x)+\frac{1}{2}\in L^2(0,+\infty),\qquad u_1(x)\in L^2(0,+\infty).$$

Then the following assertions hold.

1) *If* $u_0(x) = -1/2$ *and*

$$\int_0^T \big(\mu''(t) + \mu(t)\big)\left(1 - \frac{t}{T}\right)^2 dt > 0 \quad \textit{for small} \quad T > 0,$$

then the classical solution $u(x,t)$ *is absent even locally in time.*

2) *If* $u_0(x) = -1/2$ *and we have* $\mu(t) = 0$ *and* $u_1(x) \neq 0$, *then then the classical solution* $u(x,t)$ *is absent even locally in time.*

3) *If*

$$\int_0^T \big(\mu''(t) + \mu(t)\big)\left(1 - \frac{t}{T}\right)^2 dt \to A \quad \textit{as} \quad T \to +\infty$$

and

$$A > 2\int_0^{+\infty} u_1(x)\left(u_0(x) + \frac{1}{2}\right) dx,$$

then the solution $u(x,t)$ *does not exist globally in time*: $T < +\infty$.

The following theorem about classical solutions of the Cauchy problem for (3.1) in the sense of Definition 1 can be proved in a similar way.

Theorem 15. *Suppose that* $u(x,t) \in \mathbb{C}^{(2)}([0,T];\mathbb{C}^{(2)}(\mathbb{R}^1))$ *is a classical solution of the Cauchy problem for* (3.1) *in the sense of Definition* 1 *with initial conditions*

$$u_0(x) + \frac{1}{2} \in L^2(-\infty,+\infty), \qquad u_1(x) \in L^2(-\infty,+\infty).$$

Then we have an a priori estimate

$$2\int_{-\infty}^{+\infty} u_1(x)\left(u_0(x) + \frac{1}{2}\right) dx + \frac{2}{T}\int_{-\infty}^{+\infty}\left(u_0(x) + \frac{1}{2}\right)^2 dx \geqslant \frac{2}{T^2}\int_0^T\int_{-\infty}^{+\infty}\left(u(x,t) + \frac{1}{2}\right)^2 dx\,dt,$$

which yields the following results.

1) *If* $u_0(x) = -1/2$ *and* $u_1(x) \neq 0$, *then the classical solution* $u(x,t)$ *is absent even locally in time.*

2) *If* $u_0(x) = -1/2$ *and* $u_1(x) = 0$, *then the only classical solution is* $u(x,t) = -1/2$.

3) *If*

$$\int_{-\infty}^{+\infty} u_1(x)\left(u_0(x) + \frac{1}{2}\right) dx < 0,$$

then the classical solution $u(x,t)$ *does not exist globally in time and we have the following bound*:

$$T \leqslant -\frac{\int_{-\infty}^{+\infty}(u_0(x) + 1/2)^2\,dx}{\int_{-\infty}^{+\infty} u_1(x)(u_0(x) + 1/2)\,dx} < +\infty.$$

§ 12. Blow up of solutions of the Cauchy problem for (3.2)

Suppose that $u_0(x), u_1(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2}; \mathbb{R}^1)$ with

$$\gamma \geqslant \alpha + 1, \qquad \alpha > \max\left\{1, \frac{1}{q}\right\}, \qquad q > 0. \tag{12.1}$$

Then by Theorem 5, one can find a maximal $T_0 = T_0(u_0, u_1) > 0$ such that for every $T \in (0, T_0)$ there is a unique solution $u(x,t) \in \mathbb{C}^{(2)}([0,T]; \mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2}; \mathbb{R}^1))$ of the Cauchy problem for (3.2) in the sense of Definition 1. In particular, this means that $u(x,t) \in \mathbb{C}^{(2)}([0,T]; H^2(\mathbb{R}^1))$ for every $T \in (0, T_0)$. In this section we obtain sufficient conditions for the inequality $T_0 < +\infty$.

Consider the equation

$$u_{1xx}(x) - u_1(x) = -(|u_0|^q u_0)(x) + f(x), \qquad x \in \mathbb{R}^1. \tag{12.2}$$

Lemma 19. *Suppose that* $u_0(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2}; \mathbb{R}^1)$, $f(x) \in \mathbb{C}_b((1+x^2)^{\mu/2}; \mathbb{R}^1)$ *with*

$$\mu \geqslant \gamma + 1, \qquad \gamma \geqslant \frac{1}{q}, \qquad q > 0.$$

Then the solution $u_1(x)$ *of* (12.2) *exists, is unique, belongs to* $\mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2}; \mathbb{R}^1)$ *and is given by the explicit formula*

$$u_1(x) = \int_{\mathbb{R}^1} \frac{\exp(-|x-\xi|)}{2} \bigl((|u_0|^q u_0)(\xi) - f(\xi)\bigr)\, d\xi. \tag{12.3}$$

Proof. The uniqueness of solution of the linear (in $u_1(x)$) equation (12.2) is clear. Just as in the proof of Lemma 5, one can prove in these classes that the right-hand side of (12.3) is a classical solution of class $\mathbb{C}_b^{(2)}(\mathbb{R}^1)$ of equation (12.2). We claim that this solution actually belongs to $\mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2}; \mathbb{R}^1)$.

Indeed, the following equalities hold:

$$u_{1x}(x) = \int_{\mathbb{R}^1} \left\{\frac{d}{dx} \frac{\exp(-|x-\xi|)}{2}\right\} \bigl((|u_0|^q u_0)(\xi) - f(\xi)\bigr)\, d\xi, \tag{12.4}$$

$$u_{1xx}(x) = -(|u_0|^q u_0)(x) + f(x) + \int_{\mathbb{R}^1} \frac{\exp(-|x-\xi|)}{2} \bigl((|u_0|^q u_0)(\xi) - f(\xi)\bigr)\, d\xi, \tag{12.5}$$

where, for $x \neq \xi$,

$$\frac{d}{dx} \frac{\exp(-|x-\xi|)}{2} = \begin{cases} -\exp(-|x-\xi|) & \text{if } x > \xi, \\ \exp(-|x-\xi|) & \text{if } x < \xi. \end{cases} \tag{12.6}$$

It follows from (12.3) that

$$\sup_{x\in\mathbb{R}^1} (1+x^2)^{\gamma/2} |u_1(x)| \leqslant J_1 \sup_{x\in\mathbb{R}^1} (1+x^2)^{(q+1)\gamma/2} |u_0(x)|^{q+1} + J_2 \sup_{x\in\mathbb{R}^1} (1+x^2)^{\mu/2} |f(x)|,$$

where

$$J_1 = \sup_{x\in\mathbb{R}^1} \int_{\mathbb{R}^1} \frac{\exp(-|x-\xi|)}{2} \frac{(1+x^2)^{\gamma/2}}{(1+\xi^2)^{(q+1)\gamma/2}}\, d\xi,$$
$$J_2 = \sup_{x\in\mathbb{R}^1} \int_{\mathbb{R}^1} \frac{\exp(-|x-\xi|)}{2} \frac{(1+x^2)^{\gamma/2}}{(1+\xi^2)^{\mu/2}}\, d\xi.$$

By the results of Lemma 21 we have

$$J_1,\ J_2 < +\infty \quad \text{provided that} \quad \gamma \geqslant \frac{1}{q}, \quad \mu \geqslant \gamma + 1, \quad q > 0.$$

In a similar way, we deduce the following bound from (12.4) in view of (12.6):

$$\sup_{x\in\mathbb{R}^1} (1+x^2)^{\gamma/2}|u_{1x}(x)| \leqslant J_1 \sup_{x\in\mathbb{R}^1} (1+x^2)^{(q+1)\gamma/2}|u_0(x)|^{q+1} + J_2 \sup_{x\in\mathbb{R}^1} (1+x^2)^{\mu/2}|f(x)|.$$

Finally, it follows from (12.5) that

$$\sup_{x\in\mathbb{R}^1} (1+x^2)^{\gamma/2}|u_{1xx}(x)| \leqslant (1+J_1) \sup_{x\in\mathbb{R}^1} (1+x^2)^{\gamma(q+1)/2}|u_0(x)|^{q+1} + (1+J_2) \sup_{x\in\mathbb{R}^1} (1+x^2)^{\mu/2}|f(x)|.$$

Thus the lemma is proved. □

Integrating the equation (3.2) over time and using the initial conditions and (12.2), we obtain the equation

$$\frac{\partial}{\partial t}(u_{xx} - u) + \int_0^t u_{xx}(x,\tau)\, d\tau + |u|^q u = f(x). \tag{12.7}$$

We now use a method applied in [6]. Consider the quadratic functionals

$$\Phi(t) := \frac{1}{2}\|u_x\|_2^2 + \frac{1}{2}\|u\|_2^2, \qquad J(t) := \|u'_x\|_2^2 + \|u'\|_2^2, \tag{12.8}$$

where

$$\|v\|_p := \left(\int_{\mathbb{R}^1} |v(x)|^p\, dx\right)^{1/p}, \qquad p \geqslant 1. \tag{12.9}$$

Since $u(x,t) \in \mathbb{C}^{(2)}([0,T]; \mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2}; \mathbb{R}^1))$ for all $T \in (0,T_0)$ under the conditions (12.1), we have $\Phi(t) \in \mathbb{C}^{(2)}[0,T]$ and $J(t) \in \mathbb{C}^{(1)}[0,T]$ for all $T \in (0,T_0)$.

Lemma 20. *Let $\Phi(t)$ and $J(t)$ be the functionals defined in (12.8), where $u(x,t) \in \mathbb{C}^{(1)}([0,T_0); H^1(\mathbb{R}^1))$. Then*

$$(\Phi'(t))^2 \leqslant 2\Phi(t)J(t) \quad \textit{for all} \quad t \in [0,T], \qquad 0 < T < T_0. \tag{12.10}$$

Proof. The following equality holds:

$$\Phi'(t) = \int_{\mathbb{R}^1} u_x(x,t)u'_x(x,t)\, dx + \int_{\mathbb{R}^1} u(x,t)u'(x,t)\, dx \quad \text{for} \quad t \in [0,T_0). \tag{12.11}$$

Using Hölder's inequality, we obtain a chain of inequalities

$$\begin{aligned}(\Phi'(t))^2 &\leqslant (\|u_x\|_2\|u'_x\|_2 + \|u\|_2\|u'\|_2)^2\\ &\leqslant \|u_x\|_2^2\|u'_x\|_2^2 + 2\|u_x\|_2\|u'_x\|_2\|u\|_2\|u'\|_2 + \|u\|_2^2\|u'\|_2^2\\ &\leqslant \|u_x\|_2^2\|u'_x\|_2^2 + \|u\|_2^2\|u'\|_2^2 + \|u_x\|_2^2\|u'\|_2^2 + \|u'_x\|_2^2\|u\|_2^2\\ &= (\|u_x\|_2^2 + \|u\|_2^2)(\|u'_x\|_2^2 + \|u'\|_2^2) = \frac{1}{2}\Phi(t)J(t). \qquad \square\end{aligned}$$

Let $u(x,t) \in \mathbb{C}^{(2)}([0,T_0);\mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2};\mathbb{R}^1))$ be the classical solution of the Cauchy problem for (3.2). It exists under the conditions (12.1) by Theorem 5.

We multiply both sides of (12.7) by $u(x,t)$ and integrate with respect to $x \in \mathbb{R}^1$. Integrating by parts and using the notation (12.8) and (12.9), we obtain that

$$\frac{d\Phi(t)}{dt} + \int_0^t \int_{\mathbb{R}^1} u_x(x,t)u_x(x,\tau)\,dx\,d\tau + \int_{\mathbb{R}^1} f(x)u(x,t)\,dx = \|u\|_{q+2}^{q+2}. \tag{12.12}$$

We now multiply both sides of (12.7) by $u'(x,t)$ and integrate with respect to $x \in \mathbb{R}^1$. Integrating by parts and using the notation (12.8) and (12.9), we obtain that

$$J(t) + \frac{d}{dt}\int_{\mathbb{R}^1} f(x)u(x,t)\,dx + \int_0^t \int_{\mathbb{R}^1} u_x(x,\tau)u'_x(x,t)\,dx\,d\tau = \frac{1}{q+2}\,\frac{d}{dt}\|u\|_{q+2}^{q+2}. \tag{12.13}$$

Substituting the expression for $\|u\|_{q+2}^{q+2}$ from (12.12) into (12.13), we have

$$\begin{aligned}J(t) = \frac{1}{q+2}\frac{d^2\Phi(t)}{dt^2} + \frac{1}{q+2}\|u_x\|_2^2 - \frac{q+1}{q+2}\int_0^t \int_{\mathbb{R}^1} u'_x(x,t)u_x(x,\tau)\,dx\,d\tau\\ - \frac{q+1}{q+2}\int_{\mathbb{R}^1} f(x)u'(x,t)\,dx.\end{aligned} \tag{12.14}$$

The following estimates hold:

$$\frac{1}{q+2}\|u_x\|_2^2 \leqslant \frac{2}{q+2}\Phi(t), \tag{12.15}$$

$$\begin{aligned}-\frac{q+1}{q+2}\int_0^t \int_{\mathbb{R}^1} u_x(x,\tau)u'_x(x,t)\,dx\,d\tau &\leqslant \frac{q+1}{q+2}\int_0^t \|u_x\|_2(\tau)\|u'_x\|_2(t)\,d\tau\\ &\leqslant \frac{q+1}{q+2}\biggl(\int_0^t \|u_x\|_2^2(\tau)\,d\tau\biggr)^{1/2}\biggl(\int_0^t \|u'_x\|_2^2(t)\,d\tau\biggr)^{1/2}\\ &\leqslant \frac{q+1}{q+2}t^{1/2}J^{1/2}(t)\biggl(2\int_0^t \Phi(\tau)\,d\tau\biggr)^{1/2} \leqslant \delta J(t) + \frac{1}{4\delta}\biggl(\frac{q+1}{q+2}\biggr)^2 2T\int_0^t \Phi(\tau)\,d\tau\\ &= \delta J(t) + T\frac{1}{2\delta}\biggl(\frac{q+1}{q+2}\biggr)^2\int_0^t \Phi(\tau)\,d\tau,\end{aligned}$$

$$-\frac{q+1}{q+2}\int_{\mathbb{R}^1} f(x)u'(x,t)\,dx \leqslant \frac{q+1}{q+2}\|u'\|_2\|f\|_2 \leqslant \delta J(t) + \frac{1}{4\delta}\biggl(\frac{q+1}{q+2}\biggr)^2\|f\|_2^2. \tag{12.16}$$

Thus it follows from (12.14) in view of (12.15) and (12.16) that

$$(1-2\delta)J(t) \leqslant \frac{1}{q+2}\Phi''(t) + \frac{2}{q+2}\Phi(t) + T\frac{1}{2\delta}\left(\frac{q+1}{q+2}\right)^2 \int_0^t \Phi(\tau)\,d\tau + \frac{1}{4\delta}\left(\frac{q+1}{q+2}\right)^2 \|f\|_2^2 \tag{12.17}$$

for $\delta \in (0,1/2)$. The following estimate is obtained from (12.10) and (12.17) for $\delta \in (0,1/2)$:

$$\frac{q+2}{2}(1-2\delta)(\Phi'(t))^2 \leqslant \Phi(t)\Phi''(t) + 2\Phi^2(t) + T\frac{1}{2\delta}\frac{(q+1)^2}{q+2}\Phi(t)\int_0^t \Phi(\tau)\,d\tau + \frac{1}{4\delta}\frac{(q+1)^2}{q+2}\|f\|_2^2\Phi(t). \tag{12.18}$$

We rewrite the integro-differential inequality (12.18) in the following general form:

$$\Phi(t)\Phi''(t) - \alpha(\Phi'(t))^2 + \beta\Phi^2(t) + \gamma_1\Phi(t) + \gamma_2 T\Phi(t)\int_0^t \Phi(s)\,ds \geqslant 0 \tag{12.19}$$

for $t \in [0,T]$, where

$$\alpha = \frac{q+2}{2}(1-2\delta), \qquad \beta = 2, \qquad \gamma_1 = \frac{1}{4\delta}\frac{(q+1)^2}{q+2}\|f\|_2^2, \qquad \gamma_2 = \frac{1}{2\delta}\frac{(q+1)^2}{q+2}.$$

We require that the following relation holds:

$$\alpha > 1 \quad \Longrightarrow \quad \delta \in \left(0, \frac{1}{2}\frac{q}{q+2}\right).$$

Then we want to use the result of Theorem 17 in § 13.

The coefficients of the integro-differential inequality (12.19) contain a parameter δ, which should be chosen in an optimal way. We choose it so as to make the coefficient

$$\frac{2\gamma_1}{2\alpha-1}$$

minimal. The following equality holds:

$$\frac{2\gamma_1}{2\alpha-1} = \left(\frac{q+1}{q+2}\right)^2 \frac{\|f\|_2^2}{2}\frac{1}{\delta(1-2\delta)}.$$

Note that the minimum value of the function

$$g(x) = \frac{1}{x(1-2x)}$$

is attained at $x = 1/4$. We also have

$$\frac{1}{4} < \frac{q}{2(q+2)} \quad \text{for} \quad q > 2.$$

Therefore we choose $\delta > 0$ in the following way:

$$\delta = \begin{cases} \dfrac{1}{4} & \text{if } q > 2, \\ \dfrac{q}{2(q+2)} - \delta_1 & \text{if } 0 < q \leqslant 2, \end{cases}$$

where $\delta_1 \in (0, q/(2(q+2)))$ is arbitrarily small.

Thus,

$$\Phi(t) = \frac{1}{2}\int_{\mathbb{R}^1} [u^2(x,t) + u_x^2(x,t)]\,dx,$$

$$\Phi_0 := \Phi(0) = \frac{1}{2}\int_{\mathbb{R}^1} [u_0^2(x) + u_{0x}^2(x)]\,dx, \tag{12.20}$$

$$\Phi_1 := \Phi'(0) = \int_{\mathbb{R}^1} [u_0(x)u_1(x) + u_{0x}(x)u_{1x}(x)]\,dx. \tag{12.21}$$

Multiplying both sides of (12.2) by $u_0(x)$ and integrating by parts, we obtain the equality

$$\Phi_1 = \int_{\mathbb{R}^1} [u_1(x)u_0(x) + u_{1x}(x)u_{0x}(x)]\,dx = \int_{\mathbb{R}^1} |u_0(x)|^{q+2}\,dx - \int_{\mathbb{R}^1} f(x)u_0(x)\,dx > 0$$

provided that

$$\int_{\mathbb{R}^1} |u_0(x)|^{q+2}\,dx > \int_{\mathbb{R}^1} f(x)u_0(x)\,dx.$$

We also require that $\Phi_0 > 0$. Putting $x = T_1^2$ and using the notation (12.20) and (12.21), we can rewrite the equation (13.14) (see Theorem 17) in the form

$$ax^2 + bx + c = 0, \tag{12.22}$$

$$a = \frac{\gamma_2}{\alpha - 1}\Phi_0^2, \qquad b = \frac{2\gamma_1}{2\alpha - 1}\Phi_0 + \frac{\beta}{\alpha - 1}\Phi_0^2 - \Phi_1^2, \qquad c = \frac{1}{(\alpha - 1)^2}\Phi_0^2.$$

We need to prove that the quadratic equation (12.22) has a positive solution. To do this, we replace the function $u_0(x)$ by $Ru_0(x)$, where $R > 0$. For sufficiently large $R > 0$ we have

$$\Phi_1 \sim R^{q+2}, \qquad \Phi_0 \sim R^2.$$

Therefore,

$$\mathscr{D} = b^2 - 4ac > 0 \quad \text{and} \quad b < 0$$

for sufficiently large $R > 0$. Hence the quadratic equation (12.22) has a positive solution. Thus all the hypotheses of Theorem 17 hold.

This proves the following theorem.

Theorem 16. *Let* $u_0(x) \in \mathbb{C}_b^{(2)}((1 + x^2)^{\gamma/2}; \mathbb{R}^1)$ *and* $f(x) \in \mathbb{C}_b^{(2)}((1 + x^2)^{\mu/2}; \mathbb{R}^1)$, *where*

$$\mu \geqslant \gamma + 1, \qquad \gamma \geqslant \alpha + 1, \qquad \alpha > \max\left\{1, \frac{1}{q}\right\}, \qquad q > 0,$$

and let $u_1(x) \in \mathbb{C}_b^{(2)}((1+x^2)^{\gamma/2};\mathbb{R}^1)$ *be a solution of the equation* (12.2). *If* $u_0(x)$ *is so large that, in particular, we have*

$$\Phi_0 = \frac{1}{2}\int_{\mathbb{R}^1}[u_0^2(x) + u_{0x}^2(x)]\,dx > 0,$$

$$\Phi_1 = \int_{\mathbb{R}^1}|u_0(x)|^{q+2}\,dx - \int_{\mathbb{R}^1} f(x)u_0(x)\,dx > 0,$$

then the solution $u(x,t) \in \mathbb{C}^{(2)}([0,T_0);\mathbb{C}_b^{(2)}((1+x^2)^{\alpha/2};\mathbb{R}^1))$ *of the Cauchy problem for* (3.2) *in the sense of Definition* 1 *does not exist globally in time. Hence the time* $T_0 = T_0(u_0,f) > 0$ *in Theorem* 5 *is finite and, therefore,*

$$\lim_{T\uparrow T_0}\|u\|_{\alpha,T} = +\infty.$$

Moreover, we have the upper bound $T_0 \leqslant T_1$, *where* $T_1 > 0$ *is a solution of* (13.14).

§ 13. Appendix. Solution of the ordinary integro-differential inequality (12.19)

In this appendix we obtain a lower bound for the functional $\Phi(t) \in \mathbb{C}^{(2)}[0,T]$ satisfying the integro-differential inequality (12.19) with $\alpha > 1$. Suppose that

$$\Phi'(0) > 0. \tag{13.1}$$

Then one can find a $t_1 \in (0,T)$ such that

$$\Phi'(t) \geqslant 0 \quad \text{for all} \quad t \in [0,t_1].$$

The following chain of relations holds:

$$\int_0^t \Phi(s)\,ds = s\Phi(s)\Big|_{s=0}^{s=t} - \int_0^t s\Phi'(s)\,ds \leqslant T\Phi(t) \quad \text{for all} \quad t\in[0,t_1]. \tag{13.2}$$

In view of the inequality (13.2), we can pass from the integro-differential inequality (12.19) to the differential inequality

$$\Phi\Phi'' - \alpha(\Phi')^2 + [\beta+\gamma_2 T^2]\Phi^2 + \gamma_1\Phi \geqslant 0 \quad \text{for} \quad t\in[0,t_1]. \tag{13.3}$$

Dividing both sides of (13.3) by $\Phi^{1+\alpha}(t)$, we obtain the inequality

$$\frac{\Phi''}{\Phi^\alpha} - \alpha\frac{(\Phi')^2}{\Phi^{1+\alpha}} + [\beta+\gamma_2T^2]\Phi^{1-\alpha} + \gamma_1\Phi^{-\alpha} \geqslant 0 \quad \text{for all} \quad t\in[0,t_1]. \tag{13.4}$$

We introduce a new function

$$Z(t) := \Phi^{1-\alpha}(t), \qquad \alpha > 1. \tag{13.5}$$

Then (13.4) yields the inequality

$$Z''(t) \leqslant (\alpha-1)(\beta+\gamma_2T^2)Z(t) + (\alpha-1)\gamma_1 Z^{\alpha/(\alpha-1)}(t) \quad \text{for} \quad t\in[0,t_1]. \tag{13.6}$$

Note that

$$Z'(t) = (1-\alpha)\Phi^{-\alpha}(t)\Phi'(t) \leqslant 0 \quad \text{for} \quad t \in [0, t_1]. \tag{13.7}$$

Multiplying both sides of (13.6) by $Z'(t)$, we obtain the inequality

$$Z'(t)Z''(t) \geqslant (\alpha-1)(\beta+\gamma_2 T^2)Z(t)Z'(t) + (\alpha-1)\gamma_1 Z^{\alpha/(\alpha-1)}(t)Z'(t)$$

for $t \in [0, t_1]$, which can be rewritten in the form

$$\frac{1}{2}\frac{d}{dt}(Z')^2 \geqslant \frac{(\alpha-1)(\beta+\gamma_2 T^2)}{2}\frac{d}{dt}Z^2(t) + \frac{(\alpha-1)^2\gamma_1}{2\alpha-1}\frac{d}{dt}Z^{(2\alpha-1)/(\alpha-1)}(t)$$

for $t \in [0, t_1]$. Integrating this inequality with respect to time, we obtain that

$$(Z'(t))^2 \geqslant A + (\alpha-1)(\beta+\gamma_2 T^2)Z^2(t) + \frac{2(\alpha-1)^2\gamma_1}{2\alpha-1}Z^{(2\alpha-1)/(\alpha-1)}(t) \tag{13.8}$$

for $t \in [0, t_1]$, where

$$A := (Z'(0))^2 - (\alpha-1)(\beta+\gamma_2 T^2)Z^2(0) - \frac{2(\alpha-1)^2\gamma_1}{2\alpha-1}Z^{(2\alpha-1)/(\alpha-1)}(0). \tag{13.9}$$

We now need to require that $A > 0$. In view of (13.5), (13.7) and (13.9), this requirement is equivalent to the inequality

$$(\Phi'(0))^2 > \frac{\beta+\gamma_2 T^2}{\alpha-1}\Phi^2(0) + \frac{2\gamma_1}{2\alpha-1}\Phi(0). \tag{13.10}$$

Suppose that (13.10) holds and, therefore, $A > 0$. It follows from (13.8) that

$$(Z'(t))^2 > A > 0 \quad \text{for} \quad t \in [0, t_1]. \tag{13.11}$$

Using this inequality and (13.7), we obtain the following chain of relations for $t \in [0, t_1]$:

$$|Z'(t)| \geqslant A^{1/2} \quad \Longrightarrow \quad Z'(t) \leqslant -A^{1/2} < 0$$
$$\Longrightarrow \quad (1-\alpha)\Phi^{-\alpha}(t)\Phi'(t) \leqslant -A^{1/2} < 0 \quad \Longrightarrow \quad \Phi'(t) \geqslant \frac{A^{1/2}}{\alpha-1}\Phi^{\alpha}(t). \tag{13.12}$$

Suppose that

$$\Phi(0) > 0. \tag{13.13}$$

Since $\Phi'(t) \geqslant 0$ for $t \in [0, t_1]$, we have

$$\Phi(t) \geqslant \Phi(0) > 0 \quad \text{for} \quad t \in [0, t_1].$$

Using this and (13.12), we arrive at the following inequalities:

$$\Phi'(t) \geqslant \frac{A^{1/2}}{\alpha-1}\Phi^{\alpha}(t) \geqslant \frac{A^{1/2}}{\alpha-1}\Phi^{\alpha}(0) > 0.$$

Hence, in particular, $\Phi'(t_1) > 0$. Therefore, repeating the whole argument from the very beginning, we obtain that

$$\Phi'(t) \geqslant \frac{A^{1/2}}{\alpha - 1}\Phi^{\alpha}(0) > 0 \quad \text{for} \quad t \in [0, T].$$

We conclude from this and (13.7) that

$$Z'(t) < 0 \quad \text{for} \quad t \in [0, T].$$

Therefore, starting with the inequality (13.11), which holds for $t \in [0, T]$, we successively arrive at the following relations:

$$Z(t) \leqslant Z(0) - A^{1/2}t \\ \quad \Longrightarrow \quad \Phi^{1-\alpha}(t) \leqslant \Phi^{1-\alpha}(0) - A^{1/2}t \quad \Longrightarrow \quad \Phi(t) \geqslant \frac{1}{[\Phi^{1-\alpha}(0) - A^{1/2}t]^{1/(\alpha-1)}},$$

which are obtained under the conditions (13.1), (13.10) and (13.13). We now require that

$$A^{1/2}T = \Phi^{1-\alpha}(0).$$

This equality can be rewritten in the form

$$(\Phi'(0))^2 = \frac{1}{T^2(\alpha-1)^2}(\Phi(0))^2 + \frac{\beta + \gamma_2 T^2}{\alpha - 1}(\Phi(0))^2 + \frac{2\gamma_1}{2\alpha - 1}\Phi(0).$$

Generally speaking, this equation may have four roots. We are interested only in the minimal positive root $T = T_1 > 0$. We have thus proved the following theorem.

Theorem 17. *Let $\Phi(t) \in \mathbb{C}^{(2)}[0, T_0)$ be a function satisfying the differential inequality (12.19) with*

$$\Phi(0) > 0, \qquad \Phi'(0) > 0, \qquad \alpha > 1,$$

and let the initial conditions $\Phi(0)$ and $\Phi'(0)$ be such that there is a minimal positive root T_1 of the equation

$$(\Phi'(0))^2 = \frac{1}{T_1^2(\alpha-1)^2}(\Phi(0))^2 + \frac{\beta + \gamma_2 T_1^2}{\alpha - 1}(\Phi(0))^2 + \frac{2\gamma_1}{2\alpha - 1}\Phi(0). \tag{13.14}$$

Then $\Phi(t)$ satisfies the inequality

$$\Phi(t) \geqslant \frac{1}{[\Phi^{1-\alpha}(0) - A^{1/2}t]^{1/(\alpha-1)}},$$

for all $t \in [0, T_0)$ and $T_0 \leqslant T_1 < +\infty$, where

$$A := (\alpha-1)^2\Phi^{-2\alpha}(0)\left[(\Phi'(0))^2 - \frac{\beta + \gamma_2 T_1^2}{\alpha - 1}(\Phi(0))^2 - \frac{2\gamma_1}{2\alpha - 1}\Phi(0)\right] > 0.$$

§ 14. Estimates of integrals

Consider the following integral for $\varepsilon \in (0,1)$:

$$I(x)=\int_{-\infty}^{+\infty}\frac{e^{-\varepsilon|x-y|}}{(1+y^2)^{\beta/2}}\,dy, \qquad \beta\geqslant\alpha+1, \quad \alpha>0.$$

We have a chain of equalities

$$\begin{aligned}I(x)&=\int_{-\infty}^{+\infty}\frac{e^{-\varepsilon|x-y|}}{(1+y^2)^{\beta/2}}\,dy=\int_{-\infty}^{x}\frac{e^{-\varepsilon(x-y)}}{(1+y^2)^{\beta/2}}\,dy+\int_{x}^{+\infty}\frac{e^{-\varepsilon(y-x)}}{(1+y^2)^{\beta/2}}\,dy\\&=e^{-\varepsilon x}\int_{-\infty}^{x}\frac{e^{\varepsilon y}}{(1+y^2)^{\beta/2}}\,dy+e^{\varepsilon x}\int_{x}^{+\infty}\frac{e^{-\varepsilon y}}{(1+y^2)^{\beta/2}}\,dy:=I_1(x)+I_2(x).\end{aligned} \tag{14.1}$$

Consider the case when $x\geqslant 1$. We have the following bound for I_2:

$$I_2(x)\leqslant\frac{e^{\varepsilon x}}{(1+x^2)^{\beta/2}}\int_{x}^{+\infty}e^{-\varepsilon y}\,dy=\frac{1}{\varepsilon}\,\frac{1}{(1+x^2)^{\beta/2}}. \tag{14.2}$$

Represent $I_1(x)$ in the form

$$I_1(x)=e^{-\varepsilon x}\int_{-\infty}^{x/2}\frac{e^{\varepsilon y}}{(1+y^2)^{\beta/2}}\,dy+e^{-\varepsilon x}\int_{x/2}^{x}\frac{e^{\varepsilon y}}{(1+y^2)^{\beta/2}}\,dy:=I_{11}(x)+I_{12}(x).$$

We have the following bound for $I_{11}(x)$:

$$I_{11}(x)\leqslant e^{-\varepsilon x/2}\int_{-\infty}^{+\infty}\frac{1}{(1+y^2)^{\beta/2}}\,dy=M_1(\beta)e^{-\varepsilon x/2},$$

and $I_{12}(x)$ satisfies the bound:

$$I_{12}(x)\leqslant\int_{x/2}^{x}\frac{1}{(1+y^2)^{\beta/2}}\,dy\leqslant\int_{x/2}^{x}\frac{1}{y^{\beta}}\,dy=\frac{2^{\beta-1}-1}{\beta-1}\,\frac{1}{x^{\beta-1}}. \tag{14.3}$$

Thus we obtain from (14.1) in view of the bounds (14.2) and (14.3) that

$$I(x)\leqslant\frac{M_2(\beta,\varepsilon)}{x^{\beta-1}} \quad \text{for} \quad x\geqslant 1. \tag{14.4}$$

The case $x\leqslant -1$ can be considered in a similar way, and we arrive at the same bound,

$$I(x)\leqslant\frac{M_2(\beta,\varepsilon)}{(-x)^{\beta-1}} \quad \text{for} \quad x\leqslant -1.$$

Finally, the following inequality clearly holds for $x\in[-1,1]$:

$$I(x)\leqslant M_3(\beta,\varepsilon) \quad \text{for} \quad x\in[-1,1]. \tag{14.5}$$

Hence it follows from (14.4), (14.5) that one can find a constant $M_4=M_4(\beta,\varepsilon)$ such that

$$I(x)\leqslant\frac{M_4(\beta,\varepsilon)}{(1+|x|)^{\beta-1}} \quad \text{for} \quad x\in\mathbb{R}^1. \tag{14.6}$$

We now consider the expression

$$G(x) := (1+x^2)^{\alpha/2} \int_{-\infty}^{+\infty} \frac{e^{-\varepsilon|x-y|}}{(1+y^2)^{\beta/2}}\,dy. \tag{14.7}$$

By (14.6) we have a bound

$$G(x) \leqslant \frac{M_5(\beta,\varepsilon)}{(1+|x|)^{\beta-\alpha-1}} \quad \text{for} \quad x \in \mathbb{R}^1. \tag{14.8}$$

In view of (14.7) and (14.8), this proves the following lemma.

Lemma 21. *Suppose that*

$$\beta \geqslant \alpha+1, \qquad \alpha > 0. \tag{14.9}$$

Then the function $G(x)$ defined in (14.7) is bounded. Moreover, (14.8) holds.

Consider the following two integrals for $x \geqslant 0$ and for $\varepsilon \in (0,1)$:

$$G_1(x) := (1+x^2)^{\alpha/2} \int_{0}^{+\infty} \frac{e^{-\varepsilon|x-y|}}{(1+y^2)^{\beta/2}}\,dy, \tag{14.10}$$
$$G_2(x) := (1+x^2)^{\alpha/2} \int_{0}^{+\infty} \frac{e^{-\varepsilon|x+y|}}{(1+y^2)^{\beta/2}}\,dy.$$

The following assertion is easily proved.

Lemma 22. *Suppose that the inequalities (14.9) hold. Then the function $G_1(x)$ defined in (14.10) is bounded for $x \geqslant 0$. Moreover, it satisfies a bound of the form (14.8). The function $G_2(x)$ is bounded for $x \geqslant 0$ for any $\alpha \geqslant 0$ and $\beta \geqslant 0$.*

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