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Integrable topological billiards and equivalent dynamical systems

V. V. Vedyushkina (Fokicheva) and A. T. Fomenko

Abstract. We consider several topological integrable billiards and prove that they are Liouville equivalent to many systems of rigid body dynamics. The proof uses the Fomenko–Zieschang theory of invariants of integrable systems. We study billiards bounded by arcs of confocal quadrics and their generalizations, generalized billiards, where the motion occurs on a locally planar surface obtained by gluing several planar domains isometrically along their boundaries, which are arcs of confocal quadrics. We describe two new classes of integrable billiards bounded by arcs of confocal quadrics, namely, non-compact billiards and generalized billiards obtained by gluing planar billiards along non-convex parts of their boundaries. We completely classify non-compact billiards bounded by arcs of confocal quadrics and study their topology using the Fomenko invariants that describe the bifurcations of singular leaves of the additional integral. We study the topology of isoenergy surfaces for some non-convex generalized billiards. It turns out that they possess exotic Liouville foliations: the integral trajectories of the billiard that lie on some singular leaves admit no continuous extension. Such billiards appear to be leafwise equivalent to billiards bounded by arcs of confocal quadrics in the Minkowski metric.

Keywords: integrable system, billiard, Liouville equivalence, Fomenko–Zieschang molecule.

There are many papers devoted to the theory of mathematical billiards, that is, to the problem of the motion of a point mass in a planar domain bounded by a piecewise-smooth curve with absolutely elastic reflection at the boundary. We mention, for example, the book [1] by Kozlov and Treshchev, and Tabachnikov's book [2] containing a survey of classical and modern studies. There is a classical problem on the existence of periodic trajectories and on the integrability of billiards with prescribed boundaries. For example, every acute triangular billiard has a periodic trajectory of three edges, namely, the triangle of smallest perimeter whose vertices are the base points of the altitudes of the original triangle (Fagnano's theorem). A rather popular class of integrable billiards is that of planar billiards bounded by arcs of confocal quadrics.

The integrability of the billiard in the domain bounded by an ellipse was observed by Birkhoff [3]. The integrability of the geodesic flow on an ellipsoid follows from

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the Jacobi–Chasles theorem. When the smallest semi-axis of the ellipsoid tends to zero, the geodesic motion tends to the motion along polygonal arcs lying completely in the image of the ellipsoid, that is, in a planar domain bounded by an ellipse.

The integrability of a billiard persists when we pass to planar domains bounded by arcs of ellipses and hyperbolas of a fixed confocal family provided that the boundary of the domain has no corner points with angles $3\pi/2$. Then all the angles at the corner points are equal to $\pi/2$ since confocal quadrics always intersect at right angles. Kozlov and Treshchev pointed out in [1] that these dynamical systems are completely integrable in the sense of Liouville (that is, they have an additional independent integral Λ). More precisely, their integrability is equivalent to Poncelet's little theorem.

For planar billiards in ellipses, coordinates have been constructed in such a way that the motion is presented as a periodic motion along tori. Such systems, up to Liouville equivalence, were studied in detail by Dragović and Radnović [4], [5] and Fokicheva [6], [7]. In [8], Dragović and Radnović studied the Liouville foliation for the planar billiard in an ellipse as well as the geodesic flows on an ellipsoid in the Minkowski metric. Their answer was given in terms of the Fomenko–Zieschang invariants.

Fokicheva classified all locally planar billiards bounded by arcs of confocal ellipses and hyperbolas (these billiards need not be isometrically embeddable in the plane) as well as those in domains (not necessarily planar) obtained by gluing elementary domains along convex segments of their boundaries. Furthermore, she studied the topology of Liouville foliations on isoenergy surfaces for such billiards and calculated the marked Fomenko–Zieschang molecules (the invariants of Liouville equivalence).

Note that billiards in domains bounded by arcs of confocal parabolas are also integrable. The confocal parabolas may be regarded as a family of confocal ellipses and hyperbolas with one focus at infinity. The topology of Liouville foliations in a domain bounded by arcs of confocal parabolas was studied by Fokicheva [9].

There are many papers dedicated to the study of symmetries and the topology of integrable systems (see, for example, [10]–[12]).

Two smooth integrable systems are said to be Liouville equivalent if there is a diffeomorphism sending the Liouville foliation of one system to that of the other. If the smooth Liouville tori are the closures of non-resonance trajectories on a dense set (this holds in the majority of non-degenerate classical cases of integrability), then the Liouville equivalence of systems means that they have 'the same' closures of the solutions (that is, of the integral trajectories) on the three-dimensional levels of constant energy. In the billiard case, the Liouville tori along with the Liouville foliation and the integral trajectories are piecewise smooth and, therefore, we have not yet studied in detail whether almost all Liouville tori are non-resonant. Nevertheless, the Liouville foliation and Liouville equivalence are well defined here.

The topological type of the Liouville foliation is completely determined by the Fomenko–Zieschang invariant, which is a certain graph with numerical marks (see the theorem of Fomenko and Zieschang in [13] as well as in the book [10] of Bolsinov and Fomenko). Analyzing the many marked molecules that have so far been calculated for various billiards and other integrable systems with two degrees of freedom, Fomenko conjectured that many rather complicated cases of integrability (for example, in rigid body dynamics) can be 'modelled' by the much more visual topological billiards. In particular, this gives an effective way of finding the stable and unstable periodic solutions (trajectories) of integrable systems. This conjecture was confirmed by Fokicheva and Fomenko [14]. Namely, in many integrable cases of rigid body dynamics, a calculation of the Fomenko–Zieschang invariant for several isoenergy surfaces enables one to find a Liouville equivalence between these systems and topological billiards by comparing the marked molecules (see [14]). Thus, to say it more expressively, locally planar topological billiards are 'visual models' of many rather complicated integrable cases in rigid body dynamics. These results are briefly presented and complemented by new results (compared to [14]) in the second section of the present paper.

This paper continues the study of integrable billiards. We find new equivalences between billiards and cases of rigid body dynamics. We construct new classes of integrable billiards bounded by arcs of confocal quadrics, namely, non-compact billiards and generalized billiards obtained by gluing certain planar billiards along non-convex parts of their boundaries (non-convex generalized billiards). We completely classify the non-compact billiards bounded by arcs of confocal quadrics and investigate their topology using the Fomenko invariants that describe the bifurcations of singular leaves of the additional integral. We study the topology of the isoenergy surfaces of some non-convex generalized billiards. It turns out that they possess exotic Liouville foliations: the integral trajectories of the billiard that lie on certain singular leaves admit no continuous extension. We show that such billiards are leafwise equivalent to the billiard in an ellipse in the Minkowski metric as well as to the geodesic flows on an ellipsoid in the Minkowski metric (we mention the papers [15], [16] on billiards and geodesic flows in this metric).

§ 1. Introduction. The Fomenko–Zieschang invariant as a method for the topological (Liouville) classification of integrable billiards in domains bounded by arcs of confocal quadrics

1.1. Hamiltonian systems. Liouville equivalence. The Fomenko–Zieschang invariant.

Definition 1.1. Let $(M_1^4, \omega_1, f_1, g_1)$ and $(M_2^4, \omega_2, f_2, g_2)$ be Liouville integrable systems on symplectic manifolds M_1^4 and M_2^4, with integrals f_1, g_1 and f_2, g_2 respectively. We consider the isoenergy surfaces $Q_1^3 = \{x \in M_1^4 \colon f_1(x) = c_1\}$ and $Q_2^3 = \{x \in M_2^4 \colon f_2(x) = c_2\}$. These integrable Hamiltonian systems are said to be *Liouville equivalent* if there is a leafwise diffeomorphism $Q_1^3 \to Q_2^3$ which, moreover, preserves the orientation of the 3-manifolds Q_1^3 and Q_2^3 and the orientation of all critical circles; see [17].

Let (M^4, ω, f_1, f_2) be a Liouville-integrable non-degenerate Hamiltonian system on a symplectic manifold M^4, with integrals f_1 and f_2. The isoenergy manifold $Q^3 = \{x \in M^4 \colon f_1(x) = c_1\}$ splits into regular two-dimensional level surfaces of f_2, which are tori, cylinders or planes (by Liouville's theorem) and singular leaves. This foliation is called the Liouville foliation. For simplicity we assume for the moment that the manifold Q^3 is compact. Then it can be regarded as obtained by gluing certain regular neighbourhoods of singular leaves to one another along their boundary tori. Consider the base of the resulting Liouville foliation on Q^3. This base is a one-dimensional graph W which is called the Kronrod–Reeb graph of the function $f_2\big|_{Q^3}$. The structure of the foliation in a small neighbourhood of the singular leaf corresponding to a vertex of this graph, is described by a combinatorial object called an atom. The graph all of whose vertices are endowed with the corresponding atoms is called the Fomenko invariant (a rough molecule). The 'atoms' at the vertices of W describe the corresponding bifurcations of Liouville tori. Every edge of W can be endowed with an arrow that indicates its orientation. This is usually done in a global way using the direction of growth of the additional integral (see [10], [17] for details).

We now describe those compact atoms that occur most often in problems of Hamiltonian mechanics.

Definition 1.2. A two-dimensional atom is a pair (P^2, K), where P^2 is a connected orientable or non-orientable compact surface with boundary and K is a connected graph on P^2 such that the following conditions hold.

1) Either K consists of a single point, that is, an isolated vertex of degree zero, or all the vertices of K have degree 4.
2) Every connected component of $P^2 \setminus K$ is homeomorphic to the annulus $S^1 \times (0,1]$. The set of these annuli can be divided into two classes (positive and negative) in such a way that the following condition holds.
3) Every edge of K is adjacent to exactly one positive and exactly one negative annulus.

Here we considere atoms up to natural equivalence: (P^2, K) and (P'^2, K') are equivalent if there is a homeomorphism sending P'^2 to P^2 and K' to K.

We give the examples of two-dimensional atoms most often encountered.

The atom A is homeomorphic to a disc. It is foliated by concentric circles retracting to the singular leaf, the central point. The atom B is a bifurcation of one circle into two, and its singular leaf is a 'figure eight'. The atom C_2 is a bifurcation of two circles into one. These atoms are shown in Fig. 1.

To describe the topology of systems, we need two infinite series of atoms. Since their notation sometimes differs in different papers, we define the atoms of the series B_n and D_n as shown in Fig. 1. Here n is the number of vertices (the complexity) of the atom.

The series of atoms D_n is one of the series of maximally symmetric atoms (see [18]).

Note that the atom B is a particular case of B_n $(n = 1)$, and C_2 is a particular case of the maximally symmetric atoms D_n $(n = 2)$. We shall assume that the atoms B_0 and D_0 are homeomorphic to A, and D_1 is homeomorphic to B.

Besides the two-dimensional atoms described above, we also introduce new atoms which are called atoms with stars. Take an arbitrary atom (P^2, K) and consider its graph K. Along with the atoms considered above, we consider another simple atom: take an annulus for the surface P and its inner circle for the graph K. We

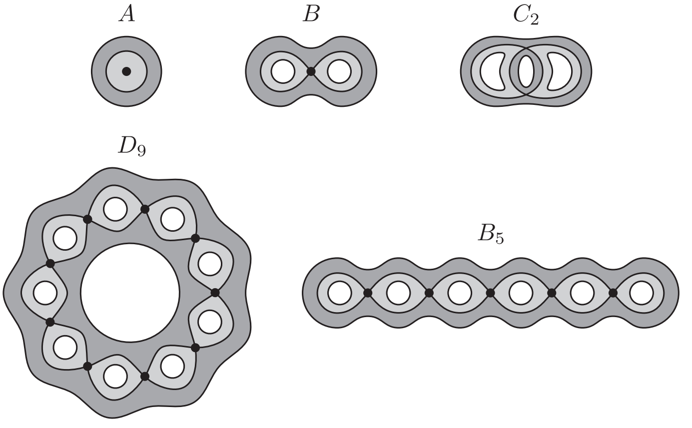


Figure 1. The atoms A, B, C_2 and examples of atoms of the infinite series B_n and D_n, namely, the atoms D_9 and B_5

now construct atoms with stars. On some edges of K, we mark arbitrarily many interior points. They are declared to be new vertices of K and are denoted by stars (see the examples in Fig. 2).

Consider a topologically stable integrable system with a Bott integral f (see [10], [17]) on Q^3, and let L be the connected singular leaf of the Liouville foliation on Q. Let $U(L)$ be a connected invariant three-dimensional neighbourhood of this leaf. Then $U(L)$ is a 3-manifold with the structure of a Liouville foliation. Such manifolds are called 3-atoms. More precisely, two such manifolds are regarded as Liouville equivalent if, first, there is a leafwise diffeomorphism between them and, second, this diffeomorphism preserves the orientation of 3-manifolds and the orientation of the critical circles determined by the Hamiltonian flow.

The equivalence class of a manifold $U(L)$ is called a 3-atom.

Theorem 1.1 (Fomenko; see [10], [17]). *The following assertions hold.*

1) *Every manifold $U(L)$, that is, every 3-atom, carries the structure of a Seifert fibration whose singular fibres are of type $(2, 1)$.*
2) *The base of the Seifert fibration on the 3-atom $U(L)$ possesses the natural structure of a 2-atom.*
3) *The projection $\pi\colon (U(L), L) \to (P^2, K)$ establishes a one-to-one correspondence between 3-atoms and 2-atoms.*
4) *The singular fibres of type $(2, 1)$ in $U(L)$ correspond to the 'stars' on the 2-atom (P^2, K).*

For visual purposes, we construct a map that sends every two-dimensional atom (with or without stars) to a three-dimensional atom. Take an atom (P^2, K) and construct a Morse function f on P^2 whose only critical level coincides with K. This function is unique up to leafwise equivalence. It foliates P^2 by its level lines. It follows from Fomenko's theorem ([10], [17], Theorem 3.1) that the 3-manifold $U(L)$

with the structure of Seifert fibration is uniquely determined (up to leafwise equivalence) by the base P^2 with marked stars (if there are any). Since the non-singular level lines of f on P^2 are circles, their 'images' in the 3-manifold $U(L)$ are tori. In the case when the two-dimensional atom (P^2, K) contains no stars, the singular leaf (the image of K) is the direct product of K and a circle.

Suppose that the atom (P^2, K) contains starred vertices. Then we can construct its double $(\widehat{P^2}, \widehat{K})$ as a branched two-sheeted covering of (P^2, K) whose branch points are exactly the starred vertices. To do this, for example, cut P^2 transversally to K at the starred vertices and glue two copies of the resulting surface along the edges of the slits. Note that f can be extended from P^2 to a Morse function $\widehat{f}$ on the surface $\widehat{P}$. The double $(\widehat{P^2}, \widehat{K})$ is endowed with a natural involution $\tau\colon \widehat{P^2} \to \widehat{P^2}$ that interchanges the two parts of the double, the original surfaces P^2. This involution possesses the following properties:

1) $\tau^2 = id$;
2) τ preserves the function f, that is, $\widehat{f}(\tau(x)) = \widehat{f}(x)$ for all $x \in \widehat{P}$;
3) τ preserves orientation.

To construct a 3-atom, consider the cylinder $\widehat{P} \times [0, 2\pi]$ and glue its bases along the involution τ by identifying the points $(x, 2\pi)$ and $(\tau(x), 0)$. As a result, we obtain an orientable 3-manifold U with boundary. This manifold is referred to as a 3-atom with stars.

For the double of the two-dimensional atom A^* we can take the two-dimensional atom B. Therefore the three-dimensional atom A^* (see Fig. 3) is obtained as a skew product of the two-dimensional atom B and a circle: one must 'twist' B by π. Hence the atom A^* corresponds to a bifurcation of one torus into one (in contrast to the three-dimensional atom B describing a bifurcation of two tori into one). One can similarly obtain the atom A^{**} using the two-dimensional atom C_2 as a double. The atom A^{**} also describes a bifurcation of two tori into one, but there are two critical circles.

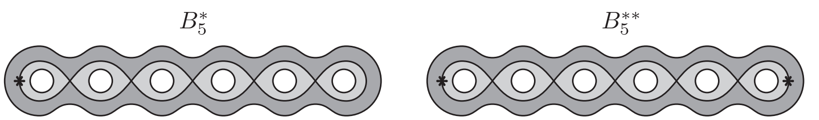


Figure 2. Planar atoms with stars of the series B_n^* and B_n^{**}, namely, the atoms B_5^* and B_5^{**}

We now describe the infinite series B_n^* and B_n^{**}, $n > 0$, of atoms with stars that are used in this paper. The two-dimensional atoms of these series are obtained by adding the starred vertices on the boundary circles of the corresponding graphs K (see Fig. 2). For the doubles of the atoms of the series B_n^* and B_n^{**} we take the atoms B_{2n+1} and D_{2n+2} respectively.

The resulting Fomenko molecule (the graph W) does not describe the topology of the Liouville foliation completely because it does not contain all the information about how the regular neighbourhoods of singular leaves are actually glued. To describe the topology of the foliation, one must choose pairs of so-called admissible

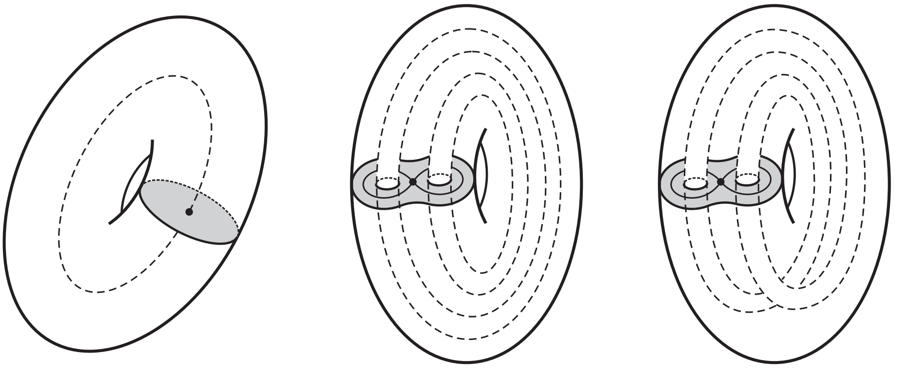


Figure 3. The three-dimensional atoms A, B and A^*

bases on the boundary tori and indicate the transition matrices from one basis to the other. The structure of an atom-bifurcation gives a rule for choosing an admissible basis. See [10], [13], [17], [19] for details. Thus there are two admissible bases at the points of all the edges of the rough molecule W, which is a Liouville torus. For every such pair of bases there is a transition matrix from one to the other. It is also called a gluing matrix. Since only one cycle in an admissible basis is uniquely determined (the complementary cycle can be chosen non-uniquely), the resulting gluing matrix can change when one set of admissible bases is replaced by another. However, the gluing matrix determines certain numerical marks that coincide for all such matrices (see [10], [13], [17], [19]). These marks r, ε and n are invariant under admissible changes of bases on the boundary tori (see Lemmas 4.5 and 4.6 in [10]).

Definition 1.3. The molecule W endowed with the numerical marks r, ε and n_k is called the marked molecule or the Fomenko–Zieschang invariant.

Theorem 1.2 (Fomenko and Zieschang; see [10]). *Two non-degenerate topologically stable integrable Hamiltonian systems on regular isoenergy surfaces* $Q_1^3 = \{x \in M_1^4 \colon f_1(x) = c_1\}$ *and* $Q_2^3 = \{x \in M_2^4 \colon f_2(x) = c_2\}$ *are Liouville equivalent if and only if their marked molecules coincide.*

Thus the calculation of the Fomenko–Zieschang invariant is a visual method for the classification of integrable Hamiltonian systems up to Liouville equivalence.

1.2. Billiards.

1.2.1. *The classical statement of the billiard problem.* Let Ω be a domain in the plane $\mathbb{R}^2$ such that its boundary is piecewise smooth and the angles at the cusps are equal to $\pi/2$. Consider the dynamical system describing the motion of a point mass inside Ω with natural reflection at the boundary $P = \partial\Omega$. This system is called ‘the billiard in the domain’. We assume that at points where P is not smooth (at the cusps with angles $\pi/2$, as said above), the trajectories of the system can be extended by continuity: having hit the corner at a vertex, the point mass bounces off along the same trajectory, without loss of velocity. Thus the phase space of the system is the manifold

$$M^4 := \{(x,v) \colon x \in \Omega,\ v \in T_x\mathbb{R}^2,\ |v| > 0\}/\sim,$$

where the equivalence relation is given by the rule

$$(x_1,v_1) \sim (x_2,v_2) \qquad \Leftrightarrow \qquad x_1 = x_2 \in P, \quad |v_1| = |v_2| \quad \text{and} \quad v_1 - v_2 \perp T_{x_1}P.$$

Here T_xP is the tangent plane to Ω at the point x and $|v|$ is the Euclidean length of the vector v.

1.2.2. *Hamiltonian smoothing.* The billiard system is not smooth in general since, as a rule, we cannot introduce a smooth structure in Cartesian coordinates because of the gluing at some of the boundary points. The definitions above must be modified to take account of the boundary points. The following approach and definitions are due to Fomenko; see also Lazutkin's book [20] and Kudryavtseva's paper [21].

The phase manifold M^4 is piecewise smooth and splits into smooth pieces (whose union will be denoted by $\widetilde{M^4}$), which are glued at the points that project (in the case of a billiard system) to the same points of the boundary of the domain where the billiard is defined. The symplectic structure on M^4 will be introduced only on $\widetilde{M^4}$. We assume that the smooth symplectic structures in adjacent smooth domains match continuously on the interface, that is, their 'right and left' one-sided limits coincide. We say that a piecewise-smooth system on M^4 is integrable (in the piecewise-smooth sense, but these words will be omitted for brevity in what follows) if there exist functionally independent functions f and H which are continuous on M^4 and smooth on $\widetilde{M^4}$ and are in involution on $\widetilde{M^4}$.

Consider a piecewise-smooth isoenergy manifold Q^3 and the connected components of the common level sets of f and H. Suppose that the Hamiltonian flows $\operatorname{sgrad} f$ and $\operatorname{sgrad} H$ are complete. If the connected compact components of the common level sets of f and H can be shown to be homeomorphic to either a piecewise-smooth torus or a piecewise-smooth three-dimensional atom (for finitely many singular values of f), then we say that the piecewise-smooth Liouville theorem holds. In this case we can construct the rough molecule W and define the marks. For a billiard in a compact domain, it is obvious that the Hamiltonian flows are complete.

A piecewise-smooth Liouville foliation for a billiard system differs from a Liouville foliation for a classical integrable Hamiltonian system in the following aspect: as a rule, every common level surface is either a piecewise-smooth atom or a piecewise-smooth torus. However, in what follows we shall use the same notation for atoms and molecules as in the classical case.

Lazutkin [20] showed that if the boundary of the billiard is convex, then the inverse image of a neighbourhood of it under the projection $M^4 \to \Omega$ can be endowed with a smooth structure and a symplectic structure on M^4 in such a way that the functions f and H are smooth and the natural projection $\widetilde{M^4} \to M^4$ is a smooth symplectic map. However, it has not yet been possible to establish the smoothness at the inverse images of cusp boundary points (corners) or points where the boundary is non-convex.

The piecewise-smooth Liouville theorem was proved by Fokicheva [22] for the class of billiards bounded by arcs of confocal ellipses and hyperbolas. We shall use these results in this paper.

1.2.3. *Integrability of billiards in domains bounded by arcs of confocal quadrics.* Let Ω be a plane domain bounded by segments of confocal quadrics.

Theorem 1.3 (Jacobi, Chasles; see [1]). *The tangent lines to a geodesic curve on a quadric in n-dimensional Euclidean space, drawn at all points of this curve, are tangent to this quadric and to $(n-2)$ additional confocal quadrics, which are the same for all points of the geodesic curve.*

Remark 1.1. There are two Russian terms for confocal quadrics in the multidimensional case: 'sofokusnye' and 'konfokal'nye'.

In the planar two-dimensional case, it follows from the Jacobi–Chasles theorem that the tangents at every point of a billiard trajectory inside Ω are also tangent to an ellipse or a hyperbola confocal with the family of quadrics that form the boundary P of Ω.

The functions $|v|$, the modulus of the velocity vector, and Λ, the parameter of the confocal quadric, commute with respect to the standard symplectic structure on the plane. Since they are constant along billiard trajectories, they also commute in the limit at the boundary of the domain. Hence this 'billiard system' possesses two independent (see [1]) integrals:

1) $|v|$, the modulus of the velocity vector,
2) Λ, the parameter of the confocal quadric.

In this paper, by confocal quadrics we mean a family of confocal ellipses and hyperbolas. Billiards in domains bounded by arcs of confocal parabolas are also integrable. The Liouville foliations of such billiards are classified up to Liouville equivalence in [9].

We fix Cartesian coordinates (x, y) on the plane $\mathbb{R}^2$. Consider the family of confocal quadrics

$$(b-\lambda)x^2 + (a-\lambda)y^2 = (a-\lambda)(b-\lambda), \tag{1.1}$$

where $a > b > 0$ are fixed parameters of the family and λ is the parameter of the quadric. When $\lambda < b$ (resp. $b < \lambda < a$) this relation determines a family of confocal ellipses (resp. hyperbolas). When $\lambda = b$, it determines the line Ox, which may be regarded as the union of two degenerate hyperbolas (rays emanating from the foci of the family) and a degenerate ellipse (the interval between the foci). When $\lambda = a$, it determines the line Oy, which may be regarded as a limiting hyperbola. In what follows we regard the line Oy as a hyperbola.

For example, consider the billiard in the ellipse with parameter $\lambda = 0$. The trajectories at various levels of the integral Λ behave as follows (see [1] for details). When $\Lambda = 0$, there are two trajectories corresponding to the circuits around the boundary of the ellipse in opposite directions. When $\Lambda \in (0, b)$, the trajectories are tangent to the ellipse with parameter Λ and also belong to one of two classes: clockwise circuits or anti-clockwise circuits. The value $\Lambda = b$ of the integral corresponds to a singular leaf of the 3-atom B (see [6]): the trajectories lying on this leaf pass successively through the foci of the family (1.1). The critical circle is the trajectory along the major axis of the ellipse. When $\Lambda \in (b, a)$, the tangents to the trajectory are tangent to the hyperbola with parameter Λ. The value $\Lambda = a$ corresponds to the vertical trajectory along the minor axis of the ellipse.

A similar picture can be observed for billiards with corners on the boundary. The trajectory at the level $\Lambda = b$ of the integral consists of intervals of straight lines passing through the foci of the family (1.1). Every convex segment of the billiard's boundary corresponds to a level of the integral such that the trajectories lying on this level pass along this segment (the so-called minimal values of the integral for elliptic convex segments and maximal values of the integral for hyperbolic convex segments in the terminology of [22]).

We mention that no family of confocal ellipses and hyperbolas can bound a classical rectangular billiard table since this family contains only two straight lines (the coordinate axes).

Remark 1.2. In what follows we write 'billiard' instead of 'domain' or 'billiard domain'. It is usually clear from the context whether we sre speaking of two-dimensional domains or billiard dynamics.

Definition 1.4. A *simple elementary compact* (*planar*) *billiard* is a flat compact Riemannian manifold with boundary which can be embedded isometrically in the plane in such a way that the boundary of its image consists of arcs of confocal ellipses and hyperbolas and contains no angles larger than π.

A *composite elementary compact* (*locally planar*) *billiard* is a compact locally flat manifold obtained by gluing together finitely many simple elementary billiards along some boundary arcs of hyperbolas in such a way that, first, the embedded images of these billiards in the plane are locally on different sides of the gluing arc (we omit this requirement when the arc is a straight line segment), second, the angles arising on the boundary of the domain do not exceed π and, third, boundary arcs are glued by means of isometries. Here we do not require the existence of an isometric embedding of the whole composite elementary billiard in the plane.

For brevity, simple and composite elementary billiards will be referred to as *elementary*.

Definition 1.5. An elementary billiard (Ω, U_i) bounded by arcs of quadrics in the confocal family (1.1) is said to be *equivalent* to another elementary billiard (Ω', U_i') bounded by arcs of quadrics in the same family (1.1) if (Ω', U_i') can be obtained from (Ω, U_i) by composing the following transformations.

1) A successive change of boundary segments of the images of some simple elementary billiards U_i under their isometric embeddings in the plane by continuously deforming them in the class of quadrics (1.1) in such a way that the value of the parameter λ for the modified boundary segment never reaches b. Here we require that the values of λ for the quadrics (hyperbolas) containing the images of the common boundary arc of any two simple elementary billiards under their isometric embeddings in the plane that agree on this arc before the deformation, either change simultaneously or remain

equal to each other (so that the isometric embeddings continue to agree on this arc both during and after the deformation). We also require that the values of λ change simultaneously or remain equal to each other for the quadrics (ellipses) that contain the images of elliptic boundary segments (of distinct elementary billiards) having a common vertex.

2) The symmetry with respect to an axis of the family (1.1) in all simple elementary billiards U_i simultaneously.

3) Combining several simple elementary billiards into a single one or splitting an elementary billiard into smaller ones.

Proposition 1.1 (see [22]). *Every elementary billiard is equivalent to a billiard in one of the following three series.*

1) *A finite series consisting of six billiards embeddable in the plane: the billiard A_2 bounded by an ellipse, the billiard A_1 bounded by an elliptic arc and a hyperbolic arc, the billiard A_0 bounded by two elliptic arcs and two hyperbolic arcs, and their upper halves, that is, the domains A'_2, A'_1, A'_0 bounded by the same quadrics as A_2, A_1, A_0 respectively and by the focal line (the subscript is the number of foci of the family* (1.1) *that belong to the billiard).*

2) *An infinite series of ring billiards C_n: the billiard C_2 is bounded by two ellipses, the billiard C_1 is the result of its factorization by the action of $\mathbb{Z}_2$, and the other billiards are their n-sheeted coverings (thus n is the total number of intervals of the focal line that lie in the billiard).*

3) *The series of ribbon billiards consisting of three infinite subseries: B_n, B'_n and B''_n. Their elements are simply connected parts of the billiards C_n with no (B_n), one (B'_n) or two (B''_n) boundary segments lying on the focal line (here n is the total number of intervals of the focal line in the billiard).*

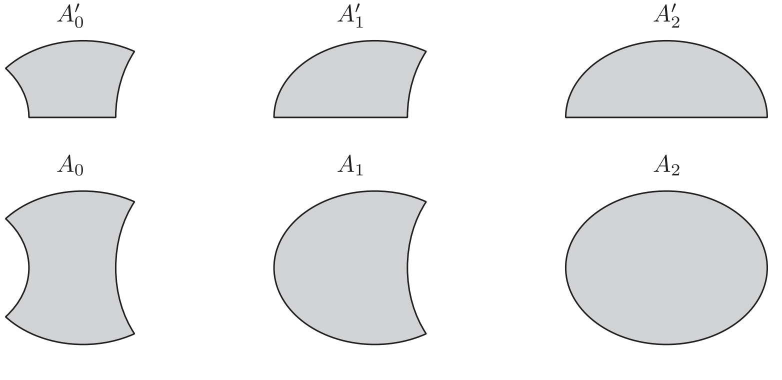


Figure 4. Elementary billiards that form the finite series A

Definition 1.6. Consider a compact billiard in the plane bounded by arcs of confocal quadrics such that the angles at the corner boundary points do not exceed π. Then the boundary of the billiard is either a simple closed curve or the disjoint union of two ellipses. Take the minimal system of arcs of quadrics forming the boundary

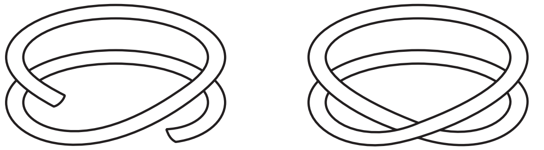


Figure 5. Examples of elementary billiards in the infinite series B and C. The billiards B_4 (left) and C_4 (right) are shown

of the billiard. These arcs are called the compact *segments of the quadrics* bounding the billiard (or the boundary segments of the billiard).

We shall distinguish segments of four types: an ellipse, an arc of a non-degenerate hyperbola confined between two ellipses, an arc of a non-degenerate ellipse confined between two hyperbolas, and an interval of the focal line.

Let Ω_i be a system of compact billiards bounded by arcs of the same family of confocal quadrics. We define the notion of a (compact) generalized billiard. A generalized billiard Δ consists of finitely many elementary billiards Ω_i glued along common convex elliptic segments of their boundaries (and possibly along some hyperbolic segments, which gives rise to so-called conical points). Here we prohibit all gluings giving rise to angles exceeding π on the boundary of the resulting generalized billiard, and angles exceeding 2π at the interior points of the billiard.

We now describe the phase space M^4 of a generalized billiard. Let P_i be the union of all open boundary segments of the billiard Ω_i that are not gluing edges. We define

$$M^4_{\Omega_i} := \{(x,v)\colon x \in \Omega_i,\ v \in T_x\Omega_i,\ |v|>0\}/\sim,$$

where the equivalence relation is given by the rule

$$(x_1,v_1) \sim (x_2,v_2) \qquad \Leftrightarrow \qquad x_1 = x_2 \in P_i, \quad |v_1| = |v_2| \quad \text{and} \quad v_1 - v_2 \perp T_{x_1}P_i.$$

Here T_xP_i is the tangent plane to the billiard Ω_i at the point x and $|v|$ is the Euclidean length of the vector v.

We glue the manifold M^4 from $M^4_{\Omega_i}$. Let Q_{ij} be one of the gluing edges along which the elementary billiards Ω_i and Ω_j are glued (there can be several such edges). In the case when Ω_i and Ω_j are isometrically embedded in the plane in such a way that the images of the gluing segments coincide and are glued by the identity map and the billiards themselves lie on the same side of these segments, the manifolds $M^4_{\Omega_i}$ and $M^4_{\Omega_j}$ are glued by the following rule:

$$(x_1,v_1) \in M^4_{\Omega_i} \sim (x_2,v_2) \in M^4_{\Omega_j} \qquad \Leftrightarrow \qquad x_1 = x_2 \in Q_{ij},$$
$$|v_1| = |v_2| \quad \text{and} \quad v_1 - v_2 \perp T_{x_1}Q_{ij}.$$

We similarly define the rule of gluing $M^4_{\Omega_i}$ and $M^4_{\Omega_j}$ in the general case. This rule is sometimes referred to as the generalized billiard law.

A trajectory of the resulting billiard system 'jumps' from one elementary billiard to another when it arrives at a gluing edge, and bounces off by the standard reflection law when it arrives at the boundary of the billiard Δ (see Fig. 6).

We discuss separately the case of a conical point obtained by gluing two corners of two of the distinct elementary billiards Ω comprising the billiard Δ. An easy continuity argument shows that the reflection law will be as follows. A point mass that arrives at the conical point from an elementary billiard Ω reflects back along the same straight line and continues moving on the same elementary billiard Ω (see Fig. 6). Hence a 'jump' of the mass point at an endpoint of the gluing edge is possible only when four elementary billiards are glued together at this cusp vertex. Clearly, the integrability of the system persists under this definition of the phase

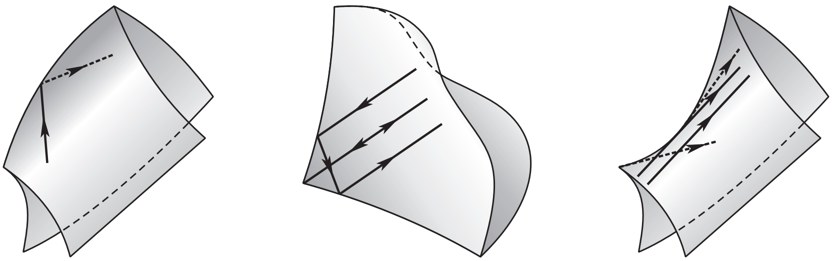


Figure 6. The picture on the left shows the motion in the generalized billiard on a convex gluing edge of two elementary billiards. The middle picture shows how the motion at a conical point is extended by continuity. The picture on the right indicates that if the gluing edge is non-convex, then the trajectories hitting the gluing edge tangentially cannot be extended by continuity: after hitting the non-convex boundary, they can either continue on the same leaf or proceed to the other leaf as limits of two types of trajectories

manifold M^4: we still have an additional integral Λ, the parameter of the confocal quadric to which the billiard trajectory is tangent. This is because the boundaries of all the planar (elementary) billiards Ω_i comprising the generalized billiard Δ and, in particular, all the gluing edges, are formed by arcs of the same family of confocal quadrics.

Definition 1.7. A generalized billiard Δ glued from elementary billiards Ω_i along gluing edges f_{ij} is said to be *equivalent* to another generalized billiard Δ' glued from Ω'_i along gluing edges f'_{ij} if Δ' can be obtained from Δ by replacing the elementary billiards Ω_i with equivalent billiards.

Generalized billiards can conveniently be described using the following notation. We denote generalized billiards without conical points by Δ_α and indicate in brackets the elementary billiards comprising the billiard Δ. When several equivalent billiards are successively glued to each other, we indicate the number of copies, for example, by writing $\Delta_\alpha(kA_0)$. Otherwise we indicate the constituents as separate summands. For example, the notation $\Delta_\alpha(\Omega + kA_0 + \Omega)$ means that two equivalent billiards Ω are glued not to each other but to the billiards A_0. We use a special notation $\Delta_\alpha(kA_0)^2$ for the billiard glued from k copies of A_0 along all elliptic boundaries into a billiard homeomorphic to an annulus.

Generalized billiards with conical points will be denoted by Δ_β. We introduce three types of conical points. Conical points of type x are formed by gluing along a convex elliptic segment l and a horizontal segment m. Conical points of type y are formed by gluing a convex or vertical hyperbolic segment m and a convex elliptic segment l. Conical points of type c or, in other words, central conical points are formed by gluing along a convex or vertical hyperbolic segment m and a horizontal segment l corresponding to the quadric with parameter b.

We introduce the following notation for the gluing rules to indicate precisely the resulting conical point: $\Delta_\beta(\Omega)^2_c$ means that we are gluing two copies of an elementary billiard Ω and obtain a central conical point of type c, $\Delta_\beta(\Omega)^2_y$ means that we are gluing two copies of Ω and obtain a conical point of type y, $\Delta_\beta(\Omega)^2_x$ means that we are gluing two copies of Ω and obtain a critical point of type x. Double subscripts mean that the gluing gives rise to two conical points. For example, $\Delta_\beta(\Omega)^2_{2y}$ is a generalized billiard glued from two copies of Ω in such a way that two conical points of type y arise.

Proposition 1.2 (see [22]). *Every generalized billiard belongs to one of the following four classes.*

1) *The class of generalized billiards glued from elementary billiards of the same type without conical points*: *the five 'simple doubled' billiards of the form $\Delta_\alpha(2\Omega)$, where Ω is equivalent to A'_0, A'_1, A_1, A_2, A'_2, and the four infinite series*

$$\Delta_\alpha(kA_0), \quad k>1, \qquad \Delta_\alpha(kA_0)^2, \quad k>0, \qquad \Delta_\alpha(2B_k), \quad \Delta_\alpha(2C_k).$$

2) *The class of generalized billiards glued from elementary billiards of distinct types without conical points*: *the four billiards $\Delta_\alpha(\Omega_1+\Omega_2)$, where Ω_1 contains the foci of the family and is equivalent to A'_1, A'_2, A_1, A_2, and Ω_2 does not contain the foci and is equivalent to B'_1, B''_2, B_1, C_2 respectively, and the five infinite series $\Delta_\alpha(kA_0+B_0)$, $\Delta_\alpha(kA_0+A'_0)$, $\Delta_\alpha(A'_0+kA_0+B_0)$, $\Delta_\alpha(B_0+kA_0+B_0)$ and $\Delta_\alpha(A'_0+kA_0+A'_0)$, $k>0$.*

3) *The class of generalized billiards glued from elementary billiards of the same type with conical points*: *the thirteen billiards*

$$\Delta_\beta(A'_1)^2_y, \quad \Delta_\beta(A'_1)^2_x, \quad \Delta_\beta(A'_1)^2_c, \quad \Delta_\beta(A'_1)^2_{cxy}, \quad \Delta_\beta((A'_1)^2_c+(A'_1)^2_c),$$
$$\Delta_\beta(A'_0)^2_c, \quad \Delta_\beta(A'_0)^2_y, \quad \Delta_\beta(A'_0)^2_{cy}, \quad \Delta_\beta((A'_0)^2_c+(A'_0)^2_c),$$
$$\Delta_\beta((A'_0)^2_c+2A'_0), \quad \Delta_\beta(A'_2)^2_{2x}, \quad \Delta_\beta(A_1)^2_{2y} \quad \text{and} \quad \Delta_\beta(A_0)^2_{2y}$$

and the nine infinite series

$$\Delta_\beta((A_0)^2_y+2kA_0), \quad \Delta_\beta((A_0)^2_y+2kA_0+(A_0)^2_y), \quad \Delta_\beta(B_k)^2_y,$$
$$\Delta_\beta(B_k)^2_{2y}, \quad \Delta_\beta(B'_k)^2_{yx}, \quad \Delta_\beta(B'_k)^2_y, \quad \Delta_\beta(B'_k)^2_x,$$
$$\Delta_\beta(B''_k)^2_x \quad \text{and} \quad \Delta_\beta(B''_k)^2_{2x}.$$

4) *The class of generalized billiards glued from elementary billiards of distinct types with conical points*: *the billiard*

$$\Delta_\beta((A'_1)^2_c+C_1)$$

and the seven infinite series

$$\Delta_\beta((A_0')^2_c + 2kA_0), \quad \Delta_\beta((A_0')^2_c + 2kA_0 + 2B_0), \quad \Delta_\beta((A_0)^2_y + 2kA_0 + 2B_0),$$
$$\Delta_\beta((A_0')^2_c + 2kA_0 + 2A_0'), \quad \Delta_\beta((A_0)^2_y + 2kA_0 + 2A_0'),$$
$$\Delta_\beta((A_0')^2_c + 2kA_0 + (A_0')^2_c) \quad \textit{and} \quad \Delta_\beta((A_0)^2_y + 2kA_0 + (A_0')^2_c).$$

1.3. Topological classification of integrable billiards.

Theorem 1.4 (see [7]). *The Fomenko–Zieschang invariant (the marked molecule W^* describing the topology of the Liouville foliation on the isoenergy surface Q^3 for an elementary billiard Ω) is of the form $A \xrightarrow{r=0,\ \varepsilon=1} A$ if the billiard Ω contains no segments of the focal line (neither inside the billiard, nor on the boundary), that is, if Ω is equivalent to A_0', A_1', A_2', B_0, B_1' or B_2''. If Ω contains segments of the focal line, then the molecule is of the form shown in the table (see Fig. 7).*

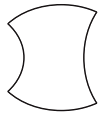
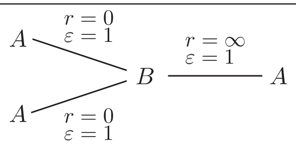
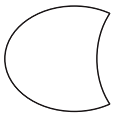
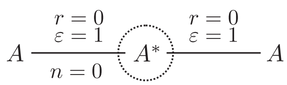
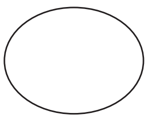
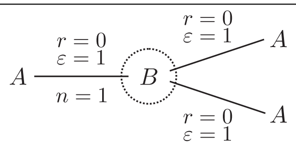
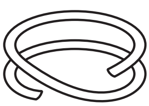
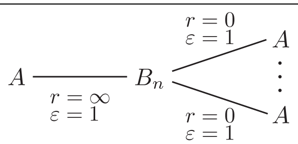
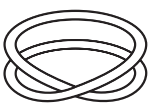
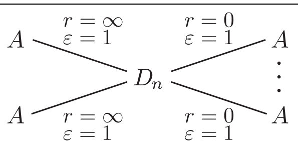
Notation	Billiard	Fomenko–Zieschang invariant
A_0		
A_1		
A_2		
B_n B'_{n+1} B''_{n+2}		
C_n		

Figure 7. The Fomenko–Zieschang invariants (the marked molecules describing the topology of the Liouville foliations on isoenergy surfaces Q^3) for elementary billiards whose interior has non-empty intersection with the focal line

Theorem 1.5 (see [22]). *Let Δ be a generalized billiard consisting of elementary billiards Ω. Suppose that at least one elementary billiard Ω contains a focus of the family* (*inside the billiard or on the boundary*). *Then the Fomenko–Zieschang invariants* (*the marked molecules W^* describing the topology of Liouville foliations on isoenergy surfaces Q^3 for such generalized billiards Δ*) *comprise nine pairwise-inequivalent types. If Δ contains no intervals of the focal line either inside the billiard or as gluing edges, then the Fomenko–Zieschang invariant is of the form $A \xrightarrow{r=0,\ \varepsilon=1} A$ for Δ without conical points, and of the form $A \xrightarrow{r=1/2,\ \varepsilon=1} A$ for Δ with conical points. The invariants for the other billiards are shown in the table* (*see Fig. 8*).

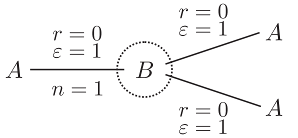
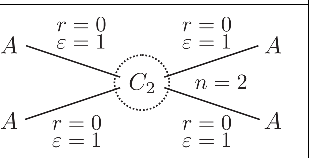
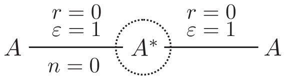
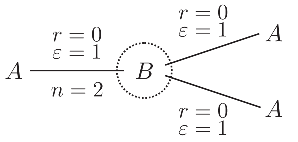
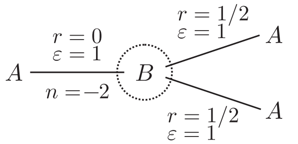
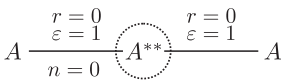
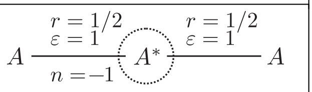
Generalized billiard	Fomenko–Zieschang invariant
$\Delta_\alpha(2A_1)$ $\Delta_\alpha(A_2+C_1)$	
$\Delta_\alpha(2A_2)$	
$\Delta_\alpha(A_1+B_1)$	
$\Delta_\beta(A_1')^2_c$ $\Delta_\beta((A_1')^2_c+C_1)$ $\Delta_\beta(A_1')^2_x$	
$\Delta_\beta(A_2')^2_{2x}$ $\Delta_\beta((A_1')^2_c+(A_1')^2_c)$	
$\Delta_\beta(A_1')^2_{2y}$	
$\Delta_\beta(A_1')^2_{xyc}$	

Figure 8. The Fomenko–Zieschang invariants (the marked molecules describing the topology of the Liouville foliations on isoenergy surfaces Q^3) for generalized billiards containing the foci

Theorem 1.6 (see [22]). *Let Δ be a generalized billiard consisting of elementary billiards Ω. Suppose that none of the billiards Ω contains foci. Then the Fomenko–Zieschang invariant (the marked molecule W^* describing the topology of the Liouville foliation on the isoenergy surface Q^3) takes the following form for such generalized billiards Δ (see Fig. 9 for details).*

1) *The molecule contains one or two lower edges (two edges only when the billiard is homeomorphic to an annulus). On these edges we have $r = \infty$, $\varepsilon = \pm 1$.*
2) *When the billiard is homeomorphic to an annulus, the bifurcation at the level $\Lambda = b$ of the integral is described by the atom D_n, where n is the number of intervals of the focal line that lie inside all the billiards Ω.*
3) *When the billiard is simply connected, the bifurcation at the level $\Lambda = b$ of the integral is described by the atom B_n, where n is the number of intervals of the focal line that lie inside all the elementary billiards Ω, and the atom has stars whose number is equal to the number of conical points of type c or x in the billiard Δ (the conical points of type c and x lie on the axis Ox).*
4) *The upper edges of the molecule are marked by $r = 0$, $\varepsilon = 1$ or $r = 1/2$, $\varepsilon = 1$. The number of fractional marks in the molecule coincides with the number of conical points of type y.*

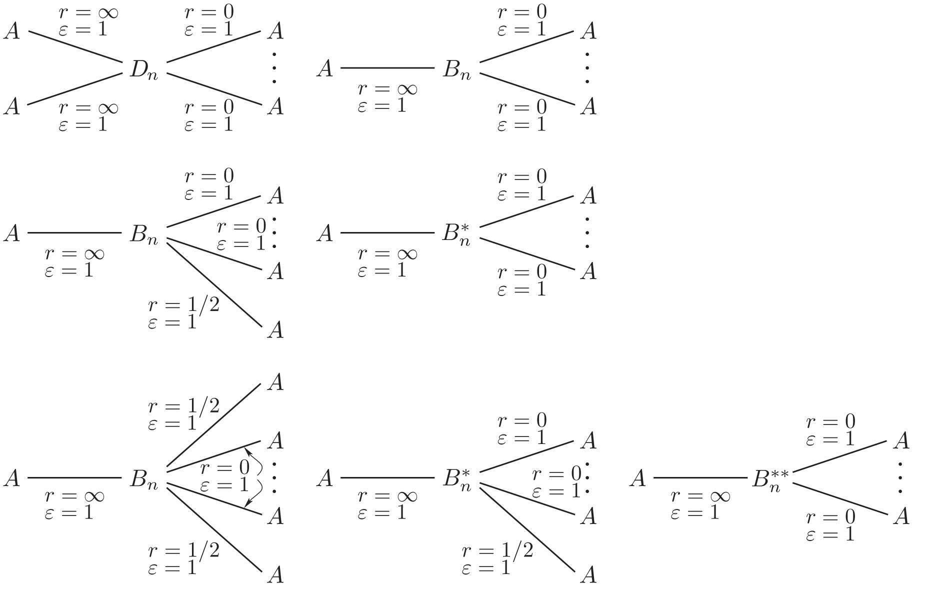


Figure 9. The Fomenko–Zieschang invariants (the marked molecules describing the topology of Liouville foliations on isoenergy surfaces Q^3) for generalized billiards that do not contain foci. The first row (resp. the second and third rows) shows the molecules for billiards without conical points (resp. with one and two conical points)

§ 2. Liouville equivalence of billiards to cases of rigid body dynamics

Having calculated the Fomenko–Zieschang invariants for elementary and generalized billiards, we see that they often coincide with those already calculated in the integrable cases of rigid body dynamics (Euler, Lagrange, Kovalevskaya, Zhukovskii, Goryachev–Chaplygin–Sretenskii, Kovalevskaya–Yehia, Clebsch and Sokolov). This enables us to prove that integrable cases of rigid body dynamics are Liouville equivalent to elementary and generalized billiards. The papers [14], [22] contain a list of Liouville equivalent foliations currently known. They also indicate the domains (on the bifurcation diagrams of the Euler, Lagrange, Kovalevskaya, Zhukovskii and Goryachev–Chaplygin–Sretenskii cases) that correspond to these isoenergy 3-surfaces. For each invariant they describe a billiard that models the behaviour of solutions on these isoenergy surfaces.

Theorem 2.1 (see [14]). *The following cases of rigid body dynamics are modelled by* (*that is, are Liouville equivalent to*) *the generalized billiards indicated.*

1) *The Euler case* (*see* [10]) *is completely modelled by the generalized billiards shown in Fig.* 10, (a), (h), (i). *They correspond to the zones* I, II, III *respectively of energy* H.
2) *The Lagrange case* (*see* [10], [23]) *is modelled by the generalized billiards shown in Fig.* 10, (c) (*energy zone* 5).
3) *The Kovalevskaya case* (*see* [10]) *is modelled by the generalized billiards shown in Fig.* 10, (c) (*energy zone* 5).
4) *The Goryachev–Chaplygin–Sretenskii case* (*see* [10], [23], [24]) *is modelled by the generalized billiards shown in Fig.* 10, (c) (*energy zone* 4, *isoenergy surface* $Q^3 \simeq S^1 \times S^2$) *and Fig.* 10, (g) (*energy zone* 2, *isoenergy surface* $Q^3 \simeq S^3$).
5) *The Zhukovskii case* (*see* [10], [25], [26]) *is modelled by the generalized billiards shown in Fig.* 10, (b) (*energy zone* 11, *isoenergy surface* $Q^3 \simeq RP^3$), *Fig.* 10, (c) (*energy zone* 2, *isoenergy surface* $Q^3 \simeq S^1 \times S^2$), *Fig.* 10, (d) (*energy zone* 8, *isoenergy surface* $Q^3 \simeq S^3$) *and Fig.* 10, (f) (*energy zone* 12, *isoenergy surface* $Q^3 \simeq RP^3$).
6) *The Kovalevskaya–Yehia case* (*see* [27]) *is modelled by the generalized billiards shown in Fig.* 10, (c) (*energy zone* h_{16}, *isoenergy surface* $Q^3 \simeq S^1 \times S^2$) *and Fig.* 10, (e) (*energy zone* h_{18}, *isoenergy surface* $Q^3 \simeq S^3$).
7) *The Clebsch case* (*see* [28]) *is modelled by the generalized billiards shown in Fig.* 10, (e) (*energy zone* 2, *isoenergy surface* $Q^3 \simeq S^3$), *Fig.* 10, (h) (*energy zones* 10, 12, *isoenergy surface* $Q^3 \simeq S^1 \times S^2$) *and Fig.* 10, (i) (*energy zone* 5, *isoenergy surface* $Q^3 \simeq RP^3$).
8) *The Sokolov case* (*see* [29]) *is modelled by the generalized billiards shown in Fig.* 10, (e) (*energy zone* B, *isoenergy surface* $Q^3 \simeq S^3$) *and Fig.* 10, (i) (*energy zone* I, *isoenergy surface* $Q^3 \simeq RP^3$).

A calculation of the Fomenko–Zieschang invariants for the geodesic flow on an ellipsoid (the Jacobi problem) and the Euler case (motion of a rigid body with fixed centre of mass) enabled us to establish their Liouville equivalence and even their

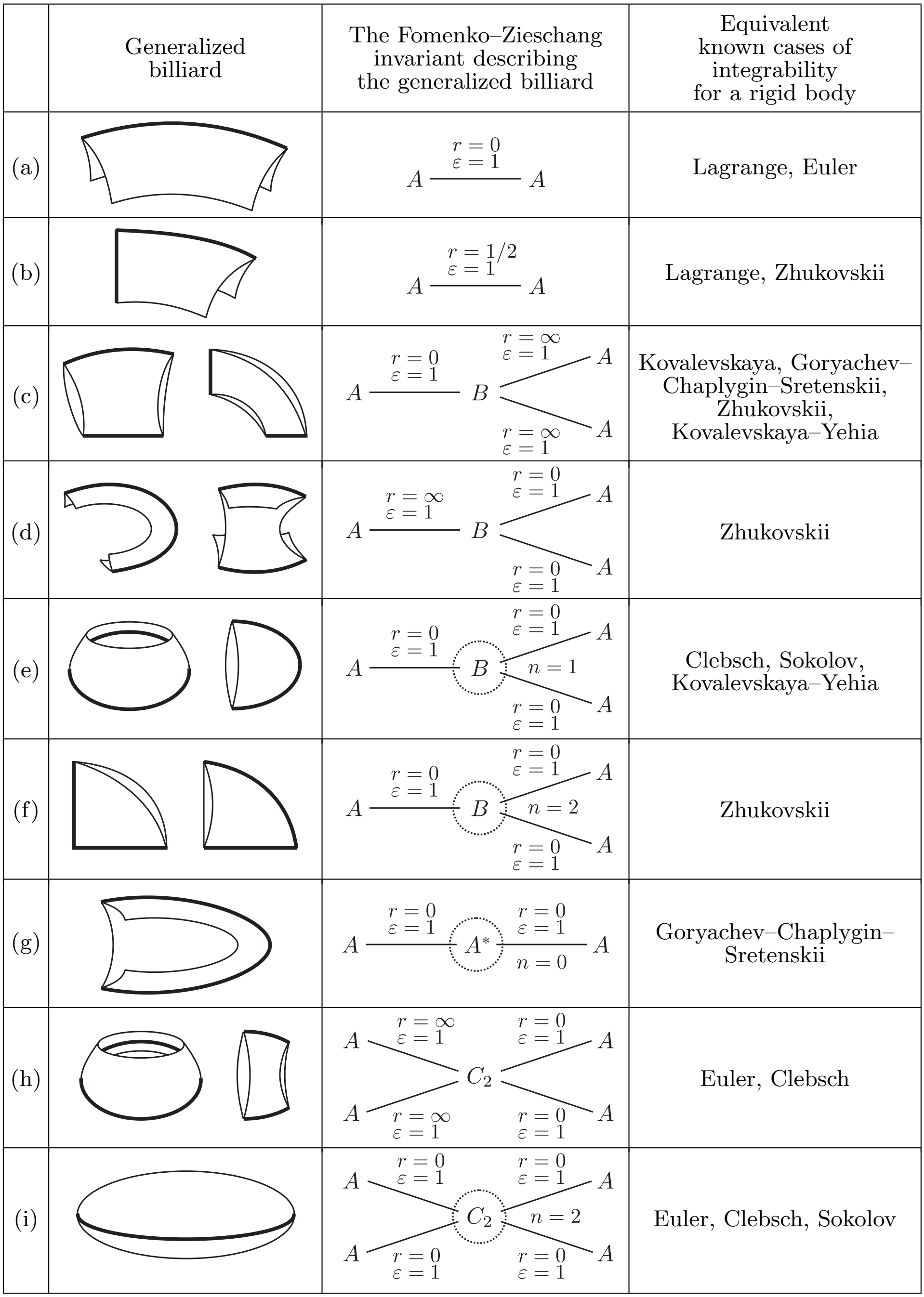

Generalized billiard
The Fomenko–Zieschang invariant describing the generalized billiard
Equivalent known cases of integrability for a rigid body
(a)
A r = 0 ε = 1 A
Lagrange, Euler
(b)
A r = 1/2 ε = 1 A
Lagrange, Zhukovskii
(c)
A r = 0 ε = 1 B
r = ∞ ε = 1 A
r = ∞ ε = 1 A
Kovalevskaya, Goryachev–Chaplygin–Sretenskii, Zhukovskii, Kovalevskaya–Yehia
(d)
A r = ∞ ε = 1 B
r = 0 ε = 1 A
r = 0 ε = 1 A
Zhukovskii
(e)
A r = 0 ε = 1 B n = 1
r = 0 ε = 1 A
r = 0 ε = 1 A
Clebsch, Sokolov, Kovalevskaya–Yehia
(f)
A r = 0 ε = 1 B n = 2
r = 0 ε = 1 A
r = 0 ε = 1 A
Zhukovskii
(g)
A r = 0 ε = 1 A* r = 0 ε = 1 A
n = 0
Goryachev–Chaplygin–Sretenskii
(h)
A r = ∞ ε = 1 C2 r = 0 ε = 1 A
A r = ∞ ε = 1 C2 r = 0 ε = 1 A
Euler, Clebsch
(i)
A r = 0 ε = 1 C2 r = 0 ε = 1 A
A r = 0 ε = 1 C2 r = 0 ε = 1 A
n = 2
Euler, Clebsch, Sokolov

Figure 10

continuous orbital equivalence (a theorem of Bolsinov and Fomenko [30], [31]): the Fomenko–Zieschang molecule in the Euler case with area constant zero is equal to the Fomenko–Zieschang invariant for the Jacobi problem. The same invariant also occurs for generalized billiards, namely, in the case of the billiard $\Delta_\alpha(2A_2)$ glued from two ellipses (see Fig. 10, (i)). Indeed, the billiard glued from two planar ellipses may be regarded as a limit of the geodesic motion on an ellipsoid as the shortest semi-axis of the ellipsoid tends to zero (notice that this is exactly the way in which Birkhoff [3] proved the integrability of classical billiards in ellipses).

In the Euler case, the topology of the foliation of the isoenergy surface Q^3 with area constant zero enables us to give a visual interpretation of the behaviour of periodic solutions. We recall the following familiar experiment. Take an ordinary book (or a piece of wood in the form of a book). Place it in the horizontal plane as shown in Fig. 11 and throw it up causing it to spin around its horizontal axis of symmetry. Then catch the book and look at the position in which it has returned. The result will depend essentially on the original orientation of the book. The book has three perpendicular axes of symmetry. When we throw the book making it spin around the axis of smallest moment of inertia, the book returns to us in the same position as before the throw. The same occurs when the book is thrown while spinning around the axis of largest moment of inertia. But the picture is rather different when we throw the book making it spin around the axis of intermediate moment of inertia. If the binding of the book was originally in your left hand, then, having caught the book, you will find the binding in your right hand.

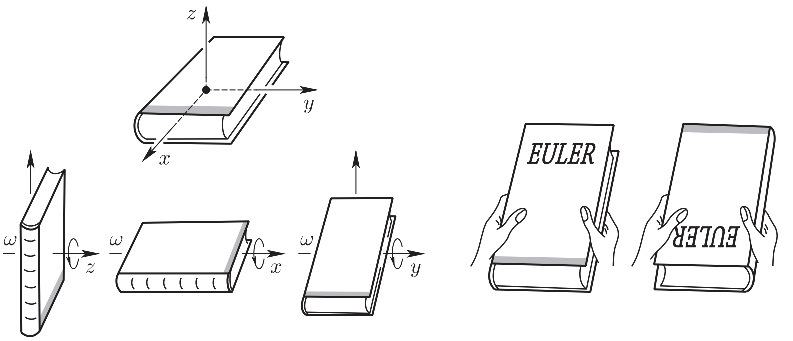


Figure 11

This curious phenomenon is explained as follows. The flight of the book models the Euler case of heavy rigid body dynamics rather well. It suffices to forget about the motion of the centre of mass of the book, that is, consider only its ‘pure rotation’ around the centre of mass. We can also assume that the area constant is equal to zero. Indeed, at each throw we make the book spin around the horizontal axis which is directed parallel to one of the eigendirections of the tensor of inertia. Hence the kinetic momentum vector is proportional to the angular velocity vector. The force of gravity is directed vertically downwards and is therefore orthogonal to the kinetic momentum of the book. Since the area constant is equal to the scalar product of the kinetic momentum and the force of gravity, we see that this constant vanishes in our experiment. Hence we are dealing with the Euler case with area constant zero. The flight of the book can be regarded as a motion along an integral trajectory of the dynamical system of the Euler case on a three-dimensional isoenergy surface. The qualitative description of the motion is determined by the topology of the Liouville foliation. The three motions of the book in the air correspond to the three types of integral trajectories.

The first type is given by stable periodic trajectories of the two 'upper atoms' A of the molecule. Mechanically, this is a rotation of the book around the shortest axis of its ellipsoid of inertia. The motion is stable and the book returns to its original position.

The second type is given by stable periodic trajectories of the two 'lower atoms' A of the molecule. This is a rotation of the book around the longest axis of its ellipsoid of inertia. As we have seen, this motion is also stable.

The third type is given by two hyperbolic periodic trajectories corresponding to the saddle atom C_2. These trajectories pass through the vertices of the atom. The flight of the book is now described by an integral trajectory that starts near the first saddle periodic solution. Theoretically, one could make the book spin in such a way that the corresponding point always moves along the saddle periodic trajectory. But this cannot be done in practice. Small deviations, which always exist, will force the book to move along an integral trajectory which is only initially close to the saddle periodic solution. Then the trajectory quickly leaves this solution and, after some time, begins to approach to the second periodic solution. The integral trajectory actually moves along a planar annulus (on the singular leaf of the 3-atom C_2), 'winding off' its outer boundary and 'winding onto' the inner boundary of the annulus. At the moment when you catch the book, the trajectory has almost reached the second periodic solution. This is precisely the phenomenon of 'the binding turned upside down'. By making the book spin around its intermediate axis of inertia, you force the integral trajectory to move from one saddle vertex of the atom C_2 to the other.

This picture can be modelled more visually using the locally planar generalized billiard $\Delta_\alpha(2A_2)$. Consider a small neighbourhood $B_\varepsilon(x_0)$ of a point x_0 lying on a fixed critical trajectory of the 3-atom C_2. This atom describes the bifurcation of the level curves of the function Λ on an isoenergy surface of the billiard $\Delta_\alpha(2A_2)$, which is glued from two ellipses.

Let $x \in B_\varepsilon(x_0)$ be a point lying in the singular leaf but not on the critical trajectory. Then it belongs to one of the four annuli (the trajectories on two of them come arbitrarily close to the fixed critical circle, and those on the other two 'wind off' this circle and come arbitrarily close to the other critical circle). This behaviour of the trajectories is shown at the upper pictures in Fig. 12.

Now let $x \in B_\varepsilon(x_0)$ be a point outside the singular leaf. Then it lies on an elliptic or hyperbolic torus depending on the type of the quadric touched by its tangents. In both cases (see the lower pictures in Fig. 12) we observe that the trajectory passing through x will soon come close to the other critical circle.

Note that in the case of the book we could not hit the singular leaf exactly because of inaccuracies in the initial data. This gives rise to the reversal of the

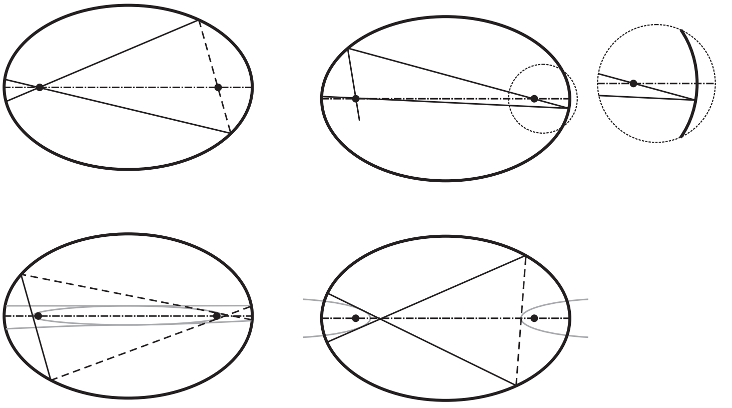


Figure 12. The upper pictures present the trajectories lying on the singular leaf of the atom C_2. This atom describes the bifurcation of the level curves of the function Λ on an isoenergy surface of the billiard $\Delta_\alpha(2A_2)$, which is glued from two ellipses. The lower pictures show the trajectories lying on an elliptic torus (left) and a hyperbolic torus (right). The trajectory is shown by a solid (resp. broken) line when it moves along the upper (resp. lower) copy of the billiard A_2. The foci are shown by the bold dots

binding of the book as it spins around the intermediate axis. In a similar vein, the trajectories of the billiard $\Delta_\alpha(2A_2)$, being initially close to one critical circle, will soon 'wind' in the opposite direction.

Some new invariants of Liouville equivalence have recently been calculated by Fomenko's school. In particular, Nikolaenko [32] completely classified the isoenergy 3-manifolds of the Chaplygin system in the dynamics of a rigid body in a liquid. In [33] he calculated the Fomenko–Zieschang invariants for integrable systems of Goryachev type. Sechkin [34] studied the topology of the dynamics of an ellipsoid of revolution moving along a smooth horizontal plane under the force of gravity. His answer is also in terms of Fomenko–Zieschang molecules. As a result, the authors were able to prove the following assertion.

Proposition 2.1. *The following cases of rigid body dynamics are modelled by* (*or Liouville equivalent to*) *the generalized billiards indicated.*

1) *The Chaplygin case in the dynamics of a rigid body in a liquid* (*see* [32]) *is modelled by the generalized billiards shown in Fig.* 13, (a), (e), (g) *and corresponding to energy zones* (1), (2), (3) *respectively.*
2) *The Goryachev case* (*see* [33]) *is modelled by the generalized billiards shown in Fig.* 13, (a) (*energy zones* (1) *and* (3)), *Fig.* 13, (c) (*energy zone* (2)) *and Fig.* 13, (d) (*energy zone* (4)).
3) *The dynamics of an ellipsoid of revolution moving along a smooth horizontal plane under the force of gravity* (*see* [34]) *is modelled by the generalized billiards shown in Fig.* 13, (a), (b), (f).

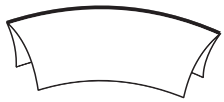
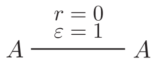
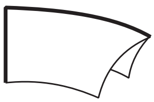
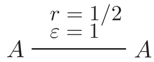
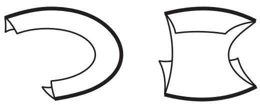
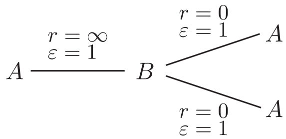
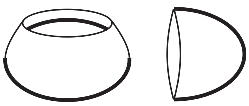
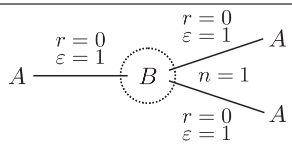
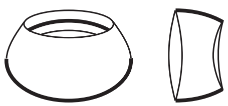
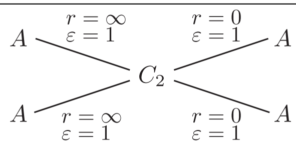
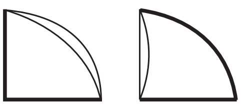
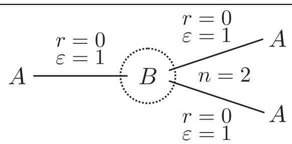
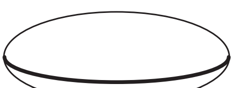
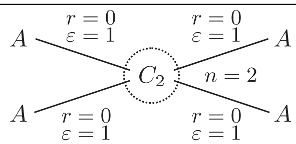
	Generalized billiard	The Fomenko–Zieschang invariant describing the generalized billiard	The new known cases of Liouville equivalence
(a)			Goryachev, Chaplygin, ellipsoid of revolution on a smooth surface
(b)			ellipsoid of revolution on a smooth surface
(c)			Goryachev
(d)			Goryachev
(e)			Chaplygin
(f)			ellipsoid of revolution on a smooth surface
(g)			Chaplygin

Figure 13

§ 3. A new class of integrable billiards: non-compact locally planar billiards bounded by confocal quadrics

3.1. Classification of billiards.

3.1.1. *Elementary non-compact billiards.*

Definition 3.1. A simple non-compact elementary (planar) billiard Θ is a connected flat two-dimensional Riemannian manifold (with boundary) having an isometric embedding in the plane such that its boundary $\partial\Theta$ consists of arcs of confocal ellipses and hyperbolas in the family (1.1) and the angles on the boundary of Θ are equal to $\pi/2$.

Definition 3.2. A simple non-compact elementary billiard is said to be non-singular if none of its boundary parts lies on the quadric with parameter b, that is, on the focal line. All other simple non-compact elementary billiards are said to be singular.

Definition 3.3. A simple non-compact elementary billiard Θ_1 is said to be equivalent to another simple non-compact elementary billiard Θ_2 if Θ_2 can be obtained from Θ_1 by a composition of symmetries with respect to the axes of the family (1.1) (the lines Ox and Oy) and successive deformations of boundary arcs such that, first, the deformation is performed in the class of arcs of quadrics in the family (1.1) and, second, the parameter of the quadric where the boundary is modified never takes the value b. In other words, the modified boundary part never lies on the line Ox during the deformation.

Remark 3.1. In particular, the equivalence relation postulates that a non-singular billiard cannot be equivalent to a singular one.

Proposition 3.1. *Every simple non-singular non-compact elementary billiard is equivalent to one of the following seven billiards: the billiards $A_0^{2\infty}$ and A_1^{∞} which are bounded by branches of hyperbolas, the billiard C_2^{∞} which is bounded by an ellipse, and the billiards A_0^{∞}, B_0^{∞}, B_1^{∞} and B_2^{∞} which are bounded by two hyperbolic arcs and one elliptic arc (see Fig. 14).*

Every simple singular elementary billiard is equivalent to one of the following six billiards: the billiard $A_0^{'\infty}$ which is bounded by the focal line and two hyperbolic arcs, the billiards $B_1^{'\infty}$, $B_2^{'\infty}$ and $B_2^{''\infty}$ which are bounded by an elliptic arc and arcs of hyperbolas, possibly degenerate, the billiard $A_1^{'\infty}$ which is bounded by the focal line and one hyperbolic arc, and the billiard $A_2^{'\infty}$ in the upper half-plane (see Fig. 14).

Proof. Let Θ be a simple non-compact billiard. We denote its boundary by $\partial\Theta$.

Suppose that the union of arcs $\partial\Theta$ contains only arcs of non-degenerate hyperbolas. Then these arcs are whole branches of hyperbolas because they are not confined by any arcs of ellipses. We obtain two possible cases: either $\partial\Theta$ consists of one branch of a hyperbola and hence Θ is equivalent to A_1^{∞}, or it consists of two branches (possibly of different hyperbolas) and hence Θ is equivalent to $A_0^{2\infty}$.

Suppose that the union of arcs $\partial\Theta$ contains only arcs of non-degenerate ellipses. Then, clearly, $\partial\Theta$ consists of whole ellipses (any other elliptic arc is a curvilinear segment whose endpoints must lie on hyperbolic boundary arcs). If there are at least two ellipses, then Θ is either compact or disconnected. Hence $\partial\Theta$ consists of only one ellipse, and the corresponding Θ is equivalent to C_2^{∞}.

Suppose that the union of arcs $\partial\Theta$ contains both elliptic and hyperbolic arcs. Let e be an elliptic arc (possibly degenerate) in $\partial\Theta$. The endpoints of e lie on some hyperbolic arcs h_1 and h_2 (possibly degenerate). We claim that the successive union of the arcs h_1, e, h_2 exhausts the whole boundary $\partial\Theta$.

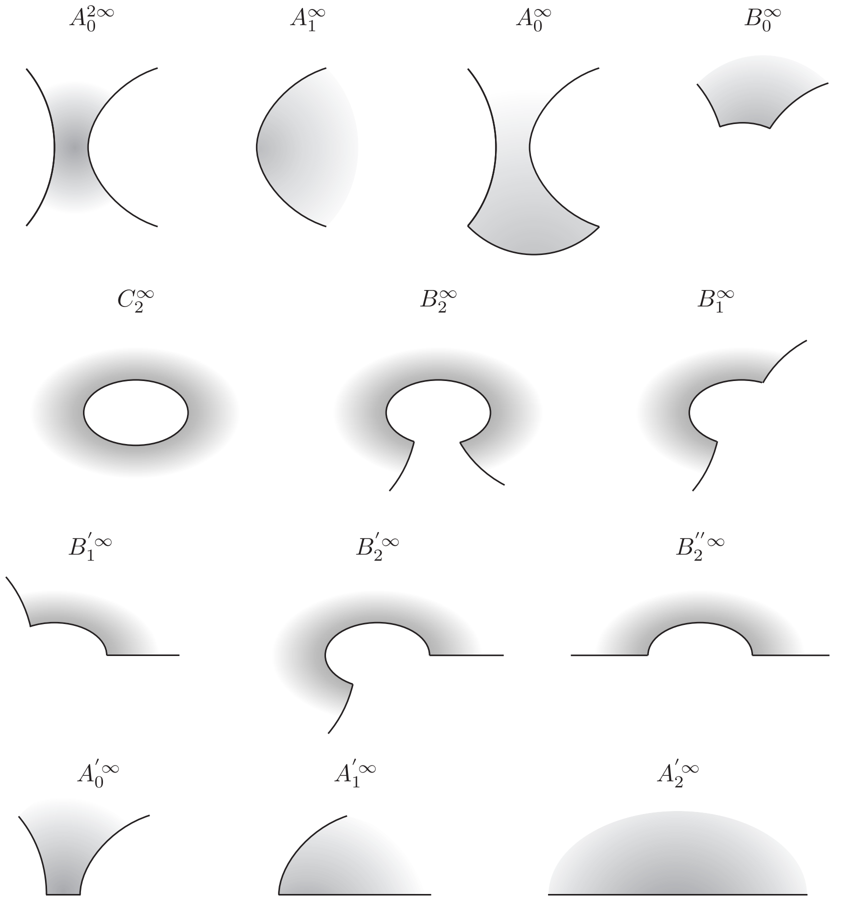


Figure 14. Simple non-compact elementary billiards

Assume that this is not the case. Then $\partial\Theta$ contains a segment q of a quadric distinct from h_1, e, h_2. Note that q is neither an ellipse nor a branch of a hyperbola since otherwise it would have points in common with h_1, h_2 or with e respectively. Suppose that q is an elliptic arc. If its endpoints lie on the hyperbolic arcs h_1, h_2, then the union of the arcs h_1, e, h_2, q bounds a quadrilateral, whence it follows that Θ is either compact or disconnected. If at least one endpoint of the elliptic segment q does not lie on the hyperbolic arcs h_1 and h_2, then there is a hyperbolic boundary segment q' having points in common with q. In this case, the angle formed by q and q' at their intersection point will be greater than π since Θ lies locally between the elliptic segments e and q. Suppose that q is a hyperbolic segment. Then there is an elliptic segment $\widetilde{q}$ one of whose endpoints coincides with an endpoint of the hyperbolic segment q. But the existence of $\widetilde{q}$ has already been disproved.

Since the successive union of the arcs h_1, e, h_2 exhausts the whole $\partial\Theta$, Θ can be described case by case for all possible arcs e, h_1, h_2 (see Fig. 14). $\square$

Remark 3.2. It should be noted that simple non-compact elementary billiards can be obtained, just like composite compact elementary billiards, by gluing infinitely many simple elementary compact billiards.

Definition 3.4. Let Θ be a simple non-compact elementary billiard. Consider the minimal system of arcs of quadrics constituting the boundary of the image of Θ under its isometric embedding in the plane. We call these arcs *segments of quadrics* bounding the billiard (or its boundary segments). Compact boundary segments are ellipses, elliptic arcs confined between hyperbolas, or hyperbolic arcs confined between ellipses. Non-compact boundary segments are branches of hyperbolas, half-open intervals on branches of non-degenerate hyperbolas, rays on the line Ox with the origin on non-degenerate hyperbolas, or the line Ox itself.

Definition 3.5. A composite elementary non-compact billiard Θ is a two-dimensional locally flat non-compact manifold (with boundary) that cannot be embedded isometrically in the plane but can be obtained from compact elementary billiards and simple non-compact elementary billiards by gluing them isometrically along common boundary segments. Here we assume that the images of simple elementary billiards under isometric embeddings in the plane locally lie on different sides of the common gluing segment. This enables us to endow a neighbourhood of this segment with a flat metric which is compatible with the flat metrics on the simple elementary billiards to be glued.

The set of elementary billiards to be glued is required to be finite or countable.

Definition 3.6. A composite non-compact elementary billiard Θ_1 is said to be equivalent to another composite non-compact elementary billiard Θ_2 if Θ_2 can be obtained from Θ_1 by a composition of the following transformations.

1) Symmetries with respect to the axes of the family (1.1) (the lines Ox and Oy) in all simple billiards simultaneously.
2) Replacing a simple elementary billiard by an equivalent billiard in the sense of Definitions 1.6 and 3.3.
3) Combining several simple elementary billiards into a single one or partitioning an elementary billiard into smaller ones.

Proposition 3.2. *Every composite elementary non-compact billiard belongs to one of the following five series.*

1) $(B_0')_\infty$ (*resp.* $(B_0)_\infty$) *is the billiard obtained by gluing infinitely many compact elementary billiards of type* B, *where the compact hyperbolic boundary segment is rectilinear* (*resp. curvilinear*).
2) B_∞ *is the billiard obtained by gluing infinitely many compact elementary billiards of type* B *and having no hyperbolic boundary segments.*
3) B_n^∞, $B_n'^\infty$ *and* $B_n''^\infty$, *as well as the billiards* C_1^∞ *and* C_n^∞, $n > 2$, *are analogues of simple non-compact elementary billiards.*
4) $(B_0')_\infty^\infty$ (*resp.* $(B_0)_\infty^\infty$) *is the billiard obtained by gluing infinitely many simple elementary non-compact billiards of type* B^∞, *where the non-compact hyperbolic boundary segment is rectilinear* (*resp. curvilinear*).
5) B_∞^∞ *is the billiard obtained by gluing infinitely many non-compact elementary billiards of type* B^∞.

Proof. Let Θ be a composite non-compact elementary billiard obtained by gluing members of a certain set of billiards U_i. We claim that the billiards U_i can only be compact billiards of series B and C (B_n, B'_n, B''_n and C_n) as well as non-compact billiards of the series B_n^∞ (B_0^∞, B_1^∞, B_2^∞, $B_1^{'\infty}$, $B_2^{'\infty}$ and $B_2^{''\infty}$). Indeed, compact billiards of series A, when glued with finitely many other compact billiards, still give compact billiards of series A. When the set of admissible gluing operations is countable, the resulting billiard will be in the class of non-compact billiards of series A. After admissible gluing of non-compact billiards of series A along boundary segments with other elementary billiards, the result still belongs to the same class (we recall that the result of gluing must be a manifold with boundary). The billiards U_i cannot be equivalent to the billiards C_2^∞ since these can be glued only with the billiard A_2 and the result will again be a manifold without boundary.

We now describe all possible results of gluing compact and non-compact ribbon billiards of series B and compact ring billiards of series C_n. Note that compact and non-compact ribbon billiards cannot be glued together (there are no common boundary segments).

For the result of gluing compact billiards of series B to be non-compact, the number of gluings must be infinite. Note that such billiards can be glued along hyperbolic as well as elliptic segments. In the case of successive elliptic gluings, if the number of billiards is finite, then all billiards to be glued are equivalent to one another and to the result of the gluing. But if we glue infinitely many ribbon billiards along elliptic segments, then the result of the gluing will be a non-compact billiard (for example, the billiard B_1^∞ can be obtained by gluing a countable set of compact billiards B_1). When infinitely many billiards are glued along only hyperbolic segments, we obtain a billiard belonging to one of the first two series.

If at least one of U_i is equivalent to C_n, then so are all the others. The requirement of non-compactness of Θ implies that the set of admissible gluings must be infinite. As a result, we obtain the billiards C_n^∞. The remaining billiards of the third series can be obtained in a similar way, by gluing infinitely many billiards along elliptic segments or finitely many billiards along hyperbolic segments.

Admitting infinitely many gluings along both elliptic and hyperbolic segments, we obtain billiards belonging to the last two series.

Note that non-compact ribbon billiards can be partitioned into an infinite union of compact billiards. Hence, replacing such non-compact ribbon billiards by a union of compact billiards, we arrive at the previous case. □

Definition 3.7. Unless otherwise stated, simple and composite non-compact elementary billiards as well as compact elementary billiards will be referred to as elementary billiards and denoted by Θ.

We note that the notion of a boundary segment (see Definition 3.4) is now defined only for simple non-compact elementary billiards, that is, for elementary billiards that can be embedded isometrically in the plane. We now extend this notion to composite elementary billiards.

Definition 3.8. Let Θ be a non-compact elementary billiard glued from some billiards U_i. Suppose that two simple elementary billiards U_i and U_j are glued along a common boundary segment l. Then among the new boundary segments of the result there will be boundary segments of U_i and U_j that are disjoint from l, as well as the new boundary segments obtained by gluing the segments intersecting l under the restriction of the gluing along l to these segments. We define the *boundary segments of the composite elementary billiard* as the boundary segments resulting from gluing all the constituent elementary billiards to each other in accordance with this rule.

It is convenient to use the following classification of boundary segments.

Definition 3.9. A boundary segment of an elementary billiard is said to be *hyperbolic* (resp. *vertical hyperbolic*, *elliptic*) if it is obtained by gluing hyperbolic (resp. vertical, elliptic) boundary segments of simple elementary billiards. A boundary segment is said to be *degenerate or horizontal* if it is obtained by gluing boundary segments (degenerate or horizontal) that lie on the focal line. A boundary segment of an elementary billiard Ω is said to be *convex* (resp. *non-strictly convex*) if every point of this segment has a neighbourhood in Ω which is isometric to a strictly convex (resp. non-strictly convex) subset of the plane.

3.1.2. *Generalized non-compact billiards.*

Definition 3.10. Let l_1 and l_2 be convex elliptic or horizontal boundary segments of elementary billiards Θ_1 and Θ_2 (compact or non-compact) such that the images of these segments under the local isometric embeddings of Θ_1 and Θ_2 (that is, under the embeddings of the corresponding simple elementary billiards) in the plane coincide and lie in the quadric with parameter $\lambda_{l_1} = \lambda_{l_2}$ of the family (1.1). We define the *gluing* of the billiards Θ_1 and Θ_2 along the elliptic segments l_1 and l_2 (whose images after gluing will be referred to as the *gluing edge*) as the gluing along l_1 and l_2 by means of the homeomorphism between l_1 and l_2 which is compatible with the given local isometric embeddings of Θ_1 and Θ_2 (that is, with the embeddings of the corresponding simple elementary billiards) in the plane. The endpoints of gluing edges are called *gluing vertices*.

Definition 3.11. Let m_1 and m_2 be convex hyperbolic or horizontal boundary segments of elementary billiards Θ_1 and Θ_2 (compact or non-compact) such that the images of these segments under the local isometric embeddings of Θ_1 and Θ_2 (that is, under the embeddings of the corresponding simple elementary billiards) in the plane coincide and lie in the quadric with parameter $\lambda_{m_1} = \lambda_{m_2}$ of the family (1.1). We define the *gluing* of the billiards Θ_1 and Θ_2 along the hyperbolic or horizontal segments m_1 and m_2 (whose images after gluing will be referred to as the *gluing edge*) as the gluing along m_1 and m_2 by means of the homeomorphism between m_1 and m_2 which is compatible with the given local isometric embeddings of Θ_1 and Θ_2 (that is, with the embeddings of the corresponding simple elementary billiards) in the plane. The endpoints of gluing edges are called *gluing vertices*.

Recall that since elementary billiards are manifolds with a flat smooth Riemannian metric, the manifold resulting from their gluing will also be locally flat although, in general, it has a piecewise-smooth Riemannian metric.

Definition 3.12. A *generalized* (*locally planar*) *non-compact billiard* Δ *without conical points* is a two-dimensional orientable non-compact manifold (with boundary) endowed with a piecewise-smooth Riemannian metric and resulting from gluing (as defined above) certain compact and non-compact elementary billiards along elliptic segments (Definition 3.10). Note that then each gluing vertex is incident to one gluing edge and two free edges (such gluing vertices are called *boundary gluing vertices*).

A *generalized* (*locally planar*) *non-compact billiard* Δ *with conical points* is a two-dimensional orientable non-compact manifold (with boundary) endowed with a piecewise-smooth Riemannian metric and resulting from gluing (as defined above) certain elementary billiards along segments (see Definitions 3.10 and 3.11) provided that the following conditions hold. First, we require that each gluing vertex be incident to either one gluing edge and two free edges (such gluing vertices are called *boundary gluing vertices*), two gluing edges and no free edges (such gluing vertices are called *conical points*), or four gluing edges and no free edges (such gluing vertices are called *interior gluing vertices*). Interior gluing vertices are incident to two hyperbolic or horizontal gluing edges m_{i_1}, m_{i_2} and two elliptic or horizontal gluing edges l_{j_1}, l_{j_2}. We denote the connected component of the union of all hyperbolic (or horizontal) gluing edges by $\bigcup_i m_i$, $i \in \{1, \ldots, n\}$, where the edges m_i are joined successively. Second, we require that at least one of the gluing edges m_1 and m_n be incident to a conical point. Third, we require that the number of conical points in a generalized billiard Δ with conical points be greater than zero.

We point out that for every such billiard Δ we fix a set of elementary billiards Θ_i and a set of gluing edges f_{ij} between them such that the result of gluing these billiards along these edges is Δ. The boundary segments of the billiards Θ_i that are not gluing edges are called *free edges*, and their union for a fixed billiard Δ is called the *free boundary*. We use the notation Δ_α for billiards glued without conical points, and Δ_β for those with conical points.

Definition 3.13. A generalized billiard Δ is said to be *equivalent* to another generalized billiard Δ' if Δ' can be obtained from Δ by replacing the constituent elementary billiards with equivalent billiards in such a way that the set of elementary billiards constituting Δ is mapped bijectively onto the set of elementary billiards constituting Δ' and the gluing edges between them coincide identically.

Proposition 3.3. *Every generalized non-compact billiard* Δ *is equivalent to a billiard in one of the following four classes.*

1) *Generalized billiards glued from elementary billiards that are equivalent to one another and containing no conical points*:
   - $\Delta_\alpha(2B_\infty)$;
   - $\Delta_\alpha(2(B_0)_\infty)$ *and* $\Delta_\alpha(2(B_0')_\infty)$;
   - $\Delta_\alpha\left(\sum_{i=1}^{\infty} A_0\right)$ *and* $\Delta_\alpha\left(\sum_{-\infty}^{+\infty} A_0\right)$.
2) *Generalized billiards glued from elementary billiards that are equivalent to one another and having conical points*:
   - $\Delta_\beta\left(\sum_{i=1}^{\infty} 2A_0\right)_y$;
   - $\Delta_\beta(2A_0^\infty)_y$;

- $\Delta_\beta(2(B_1)_\infty)_y$;
- $\Delta_\beta(2(B_0')_\infty)_x$;
- $\Delta_\beta(2A_1'^\infty)_c$;
- $\Delta_\beta(2A_0'^\infty)_c$.

3) *Generalized billiards glued from elementary billiards that belong to more than one equivalence class and containing no conical points*:
   - $\Delta_\alpha\bigl(\sum_{i=1}^\infty A_0 + A_0'\bigr)$;
   - $\Delta_\alpha\bigl(\sum_{i=1}^\infty A_0 + B_0\bigr)$;
   - $\Delta_\alpha\bigl(A_0^\infty + \sum_{i=1}^k A_0\bigr)$, $k \leqslant \infty$;
   - $\Delta_\alpha\bigl(A_0^\infty + \sum_{i=1}^k A_0 + B_0\bigr)$, $k < \infty$;
   - $\Delta_\alpha\bigl(A_0^\infty + \sum_{i=1}^k A_0 + A_0'\bigr)$, $k < \infty$;
   - $\Delta_\alpha\bigl(A_0^\infty + \sum_{i=1}^k A_0 + A_0^\infty\bigr)$, $k < \infty$.
4) *Generalized billiards glued from elementary billiards that belong to more than one equivalence class and having conical points*:
   - $\Delta_\beta\bigl((2A_0')_c + \sum_{i=1}^k 2A_0 + 2A_0^\infty\bigr)$, $k < \infty$;
   - $\Delta_\beta\bigl((2A_0')_c + \sum_{i=1}^\infty 2A_0\bigr)$;
   - $\Delta_\beta\bigl((2A_0)_y + \sum_{i=1}^k 2A_0 + 2A_0^\infty\bigr)$, $k < \infty$.

Proof. We describe the class of billiards from which generalized non-compact billiards can be glued. The billiards A_1^∞ and $A_0^{2\infty}$ must be excluded from this class since they have only hyperbolic boundary segments. We also exclude the billiard A_2^∞ since it can be glued only to itself and the resulting billiard has no boundary. The billiards C_n^∞, B_∞^∞, B_k^∞ have no convex elliptic segments and, therefore, also cannot belong to the desired class of billiards. The billiards C_n, A_1, A_1', A_2, A_2', B_k', B_k'', $k \geqslant 0$, B_n, $n > 0$, are also outside this class since admissible gluing along elliptic segments between them gives only compact billiards, and gluing with other elementary billiards is impossible (there are no common segments).

Thus the class of elementary billiards that constitute generalized non-compact billiards consists of

1) the compact elementary billiards A_0, A_0' and B_0,
2) the simple non-compact elementary billiards A_0^∞, $A_1'^\infty$ and $A_0'^\infty$,
3) the non-compact elementary ribbon billiards $(B_0')_\infty$, $(B_0)_\infty$ and B_∞.

For a generalized billiard Δ to be non-compact, it must contain either a non-compact elementary billiard, or infinitely many compact elementary billiards. Hence the class of generalized non-compact billiards glued from elementary billiards of the same equivalence class and having no conical points consists of the simple doubled billiards $\Delta_\alpha(2B_\infty)$, $\Delta_\alpha(2(B_0)_\infty)$, $\Delta_\alpha(2(B_0')_\infty)$ and the billiards $\Delta_\alpha\bigl(\sum_{i=1}^\infty A_0\bigr)$ and $\Delta_\alpha\bigl(\sum_{-\infty}^{+\infty} A_0\bigr)$, which are glued from infinitely many compact elementary billiards A_0.

Note that, by the classification of elementary billiards, two boundary segments with a common point uniquely determine the equivalence class of an elementary billiard. Hence a conical point can be obtained only by gluing two copies of the same billiard. This gives rise to the following description of billiards glued from elementary billiards in the same equivalence class and having conical points. After gluing a pair of non-compact elementary billiards, nothing can be added to the result and, therefore, we obtain the billiards $\Delta_\beta(2A_0^\infty)_y$, $\Delta_\beta(2(B_0)_\infty)_y$, $\Delta_\beta(2(B_0')_\infty)_x$, $\Delta_\beta(2A_1'^\infty)_c$, $\Delta_\beta(2A_0'^\infty)_c$. After gluing the compact billiard A_0, we obtain the billiard $\Delta_\beta\bigl(\sum_{i=1}^\infty 2A_0\bigr)_y$ with one conical point.

Note that both classes contain no billiards glued from A_0' and B_0. These billiards have only one convex elliptic gluing segment and one cannot glue infinitely many such billiards.

We now observe that if a generalized non-compact billiard is glued from elementary billiards of different equivalence classes, then its constituent elementary billiards can only be the compact elementary billiards A_0, A_0', B_0 and the non-compact elementary billiard A_0^∞. We exclude non-compact ribbon billiards as well as the billiards $A_0'^\infty$ and $A_1'^\infty$ since each of them can be glued only with an equivalent billiard and nothing can be added to the result.

The elementary billiards A_0, A_0', B_0 and A_0^∞ can be glued to each other only along convex elliptic segments. Note that only A_0 has two convex elliptic segments, which enables one to repeat the process of gluing infinitely many times and gives rise to the compact 'pieces' $\sum_{i=1}^k A_0$ or non-compact 'tails' $\sum_{i=1}^\infty A_0$ made from A_0. For a generalized billiard to be non-compact, it must contain either a non-compact 'tail' made from A_0, or the non-compact billiard A_0^∞. It follows that the billiards glued from elementary billiards that belong to more than one equivalence class and having no conical points, are either of the form

$$\Delta_\alpha\biggl(\sum_{i=1}^\infty A_0 + A_0'\biggr) \quad\text{and}\quad \Delta_\alpha\biggl(\sum_{i=1}^\infty A_0 + B_0\biggr)$$

(a non-compact 'tail' glued to a compact billiard), or of the form

$$\Delta_\alpha\biggl(A_0^\infty + \sum_{i=1}^k A_0\biggr), \qquad k \leqslant \infty$$

(the successive gluing of a non-compact billiard and either a 'piece' or 'tail' made from the billiards A_0), or of the form

$$\Delta_\alpha\biggl(A_0^\infty + \sum_{i=1}^k A_0 + A_0'\biggr), \quad \Delta_\alpha\biggl(A_0^\infty + \sum_{i=1}^k A_0 + B_0\biggr), \qquad k < \infty$$

(the successive gluing of a non-compact billiard, which may be empty, to a 'piece' made from A_0 and to a compact billiard), or of the form

$$\Delta_\alpha\biggl(A_0^\infty + \sum_{i=1}^k A_0 + A_0^\infty\biggr), \qquad k < \infty$$

(the gluing of two non-compact billiards to a 'piece' made from A_0).

When a composite billiard with conical points is glued from billiards of distinct equivalence classes A_0, A_0', B_0 and A_0^∞, the conical points may result only from gluing two billiards A_0' (which gives a conical point of type c) or two billiards A_0

(which gives a conical point of type y). It follows that all such billiards are glued from a pair of billiards A_0' or A_0 (which form a conical point), a 'piece' (possibly empty) made from the billiards A_0, and either the non-compact billiard A_0^∞ or an infinite 'tail' made from the billiards A_0. Thus all billiards glued from elementary billiards of distinct equivalence classes and having conical points, belong to one of the following three series of billiards:

$$\Delta_\beta\biggl((2A_0')_c + \sum_{i=1}^{\infty} 2A_0\biggr), \quad \Delta_\beta\biggl((2A_0')_c + \sum_{i=1}^{k} 2A_0 + 2A_0^\infty\biggr),$$
$$\Delta_\beta\biggl((2A_0)_y + \sum_{i=1}^{k} 2A_0 + 2A_0^\infty\biggr), \qquad k < \infty. \qquad \square$$

3.2. Non-compact bifurcation-atoms. To describe the topology of non-compact isoenergy surfaces Q^3, one must define non-compact bifurcation-atoms describing the bifurcations of tori, cylinders and planes. A theory of non-compact atoms has not yet been constructed. Therefore we restrict ourselves to certain examples that enable us to describe the Liouville foliation for some non-compact integrable billiards.

Consider a three-dimensional non-compact isoenergy surface Q^3 foliated by the level surfaces of the integral f. Two phenomena may arise from the non-compactness of Q^3. First, the leaves of f may be non-compact and then one must introduce new, non-compact atoms describing the bifurcations of non-compact leaves. Second, the function $f\colon Q^3 \to \mathbb{R}$ may happen to take infinite values. Then the neighbourhoods of the inverse images of the infinite values are described by so-called 'empty atoms', which are homeomorphic to direct products of cylinders (C_∞) or planes (P_∞) with the half-open interval $[0,1)$.

Remark 3.3. We introduce 'empty atoms' for two purposes. On the one hand, this enables us to describe Q^3 by a graph, avoiding those edges that are half-open intervals with only one endpoint incident to a vertex of the graph. On the other hand, this fixes the topology of the leaves of Q^3 on such edges: in contrast to the compact case when all leaves are tori, they can be either cylinders or planes in the non-compact case. Our notation makes it unnecessary to indicate the type of leaf (a cylinder or a plane) on the edges.

We describe atoms that are non-compact bifurcations of the leaves of f into one another.

We first describe some two-dimensional non-compact atoms which will be used in what follows. Consider the atom B. Its singular leaf is a figure eight. Take a point on one 'eyelet' of this figure eight and consider an interval passing through this point transversally to the singular leaf of B. This interval meets each non-singular leaf (a circle) at exactly one point. Removing the interval, we obtain a non-compact atom. Denote it by B' (see Fig. 15). It describes a bifurcation of an interval into an interval and a circle. Removing two intervals that intersect the singular leaf of B on different sides of the singular point, we obtain the atom B'' shown in Fig. 15. This atom describes a bifurcation of two intervals into two intervals.

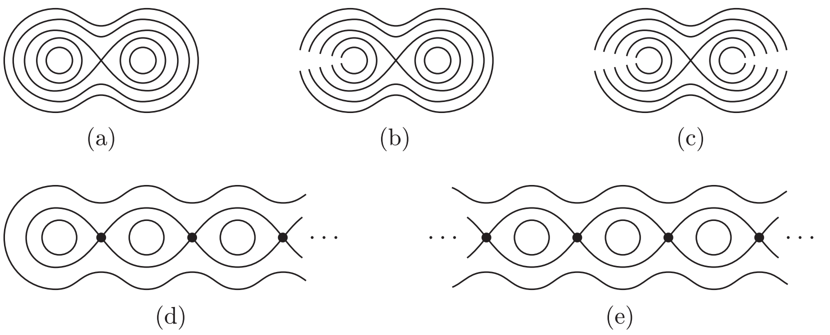


Figure 15. The upper row shows (a) the two-dimensional compact atom B, (b) the non-compact atom B' obtained from B by removing a point of the singular leaf and a neighbourhood of it on the neighbouring circles, and (c) the non-compact atom B'' obtained from B by removing a point of the singular leaf, a neighbourhood of it on the neighbouring circles, and the symmetric points

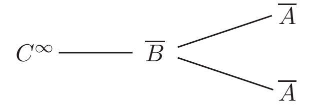
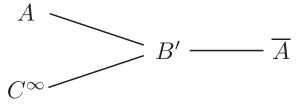
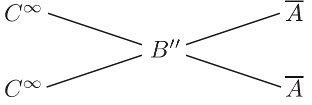
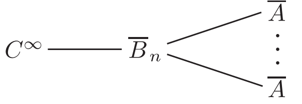
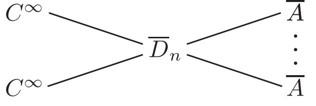
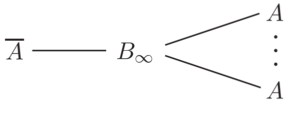
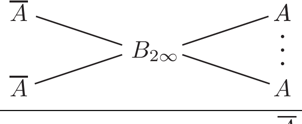
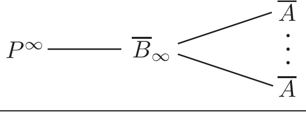
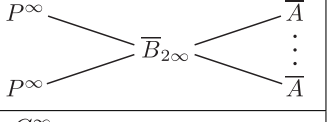
We similarly construct non-compact atoms of the series B'_n and B''_n. These non-compact atoms describe bifurcations of intervals and circles whose singular leaves contain finitely many singular points. But we shall also need atoms with infinitely many singular points, namely, analogues of atoms of the series B_n, to be denoted by B_∞ and $B_{2\infty}$. In other words, these are the limits of atoms of the series B_n as $n \to \infty$. These atoms describe bifurcations of infinitely many circles into one interval (the atom B_∞) or two intervals (the atom $B_{2\infty}$).

Non-compact 3-atoms are defined similarly to compact 3-atoms by taking the direct or skew product of a two-dimensional (now non-compact) atom P and the circle S^1 (the skew product is taken when the atom has stars). The atoms $(B'_n)^*$ and B^*_∞ with stars are natural analogues of the atom B^*_n. The doubles of these atoms are denoted by B''_{2n} and $B_{2\infty}$. Thus we first add a 'prime' which makes the 2-atom non-compact and then obtain a non-compact 3-atom as a skew product of the non-compact 2-atom and the circle (the 'star' operation, now with brackets).

Non-compact 3-atoms can also be obtained as direct products of a two-dimensional (compact or non-compact) atom and the line $\mathbb{R}$. We denote such non-compact atoms by $\overline{P}$, where P is the corresponding compact or non-compact 2-atom.

3.3. Topological classification of non-compact billiards. A theory of invariants of Liouville equivalence for non-compact isoenergy surfaces has not yet been constructed. Therefore we restrict ourselves to rough Liouville equivalence (without marks).

Theorem 3.1. *The Liouville foliations and the isoenergy surfaces* $Q^3 := \{(x,v) \in M^4 \colon x \in \Theta,\ |v| = 1\}$ *of the non-compact billiards* Θ *are roughly classified by the Fomenko molecules shown in the table below.*

<table>
<tr><th>Non-compact billiard</th><th>The molecule describing the topology of Liouville foliation for the non-compact billiard</th></tr>
<tr><td>B_0^{∞}, $B_1'^{\infty}$, $B_2''^{\infty}$, $A_1'^{\infty}$, $A_2'^{\infty}$</td><td>$\overline{A}$ —— C^{∞}</td></tr>
<tr><td>A_1^{∞}, B_1^{∞}</td><td>
</td></tr>
<tr><td>A_0^{∞}, $\Delta_\beta(2A_0^{\infty})_y$, $\Delta_\alpha(A_0^{\infty}+B_0)$,
$\Delta_n(A_0^{\infty}+A_0')$</td><td>
</td></tr>
<tr><td>$A_0^{2\infty}$</td><td>
</td></tr>
<tr><td>B_n^{∞}, $B_n'^{\infty}$, $B_n''^{\infty}$</td><td>
</td></tr>
<tr><td>C_n^{∞}</td><td>
</td></tr>
<tr><td>$(B_0')_\infty$, $(B_0)_\infty$, $\Delta_\alpha(2(B_0')_\infty)$, $\Delta_\alpha(2(B_0')_\infty)$,
$\Delta_\alpha(2(B_0)_\infty)$, $\Delta_\alpha(\sum_{i=1}^{\infty} A_0)$, $\Delta_\beta(\sum_{i=1}^{\infty} 2A_0)_y$,
$\Delta_\beta(2(B_1)_\infty)_y$, $\Delta_\alpha(\sum_{i=1}^{\infty} A_0 + A_0')$,
$\Delta_\alpha(\sum_{i=1}^{\infty} A_0 + B_0)$</td><td>
</td></tr>
<tr><td>B_∞, $\Delta_\alpha(2B_\infty)$, $\Delta_\alpha(\sum_{-\infty}^{\infty} A_0)$</td><td>
</td></tr>
<tr><td>$(B_0')_\infty^\infty$, $(B_0')_\infty^\infty$</td><td>
</td></tr>
<tr><td>B_∞^∞</td><td>
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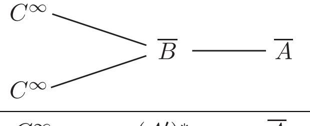
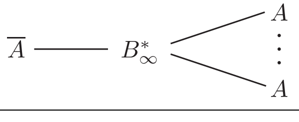
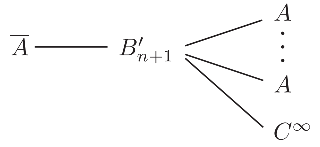
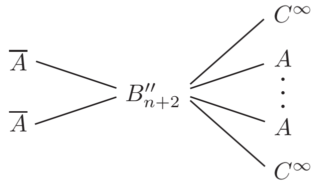
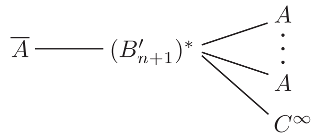
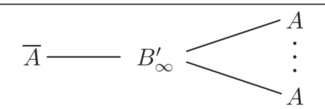
<tr><td>$\Delta_\beta(2A_1'^{\infty})_c$</td><td>
</td></tr>
<tr><td>$\Delta_\beta(2A_0'^{\infty})_c$</td><td>$C^{\infty}$ —— $(A')^*$ —— $\overline{A}$</td></tr>
<tr><td>$\Delta_\beta(2(B_0')_\infty)_x$, $\Delta_\beta(2(A_0')_c + \sum_{i=1}^{\infty} 2A_0)$</td><td>
</td></tr>
</table>

Non-compact billiard	The molecule describing the topology of Liouville foliation of the non-compact billiard
$\Delta_\alpha(A_0^\infty + \sum_{i=1}^n A_0)$, $\Delta_\alpha(A_0^\infty + \sum_{i=1}^n A_0 + B_0)$, $\Delta_\alpha(A_0^\infty + \sum_{i=1}^n A_0 + A_0')$, $n < \infty$, $\Delta_\beta((2A_0)_y + \sum_{i=1}^{n-1} 2A_0 + 2A_0^\infty)$, $1 \leqslant n < \infty$	
$\Delta_\alpha(A_0^\infty + \sum_{i=1}^n A_0 + A_0^\infty)$, $n < \infty$	
$\Delta_\beta((2A_0')_c + \sum_{i=1}^{n-1} 2A_0 + 2A_0^\infty)$, $n < \infty$	
$\Delta_\alpha(A_0^\infty + \sum_{i=1}^\infty A_0)$	

Lemma 3.1. *Let Ω be a compact billiard. Then the inverse image of every boundary segment of Ω on the leaves-tori of an isoenergy surface is a union of circles that are non-contractible cycles.*

The proof of this lemma follows from the proofs of theorems on the topological classification of compact integrable billiards (see [22], Theorems 4.1 and 4.2).

Proof. It follows from the lemma that if we can represent a non-compact billiard as a limit of compact billiards equivalent to one another (that is, we simply let one or two boundary segments of a given billiard tend to infinity), then the non-singular leaves-tori either remain tori (if their projection to the domain of the compact billiard contains no segments tending to infinity), become cylinders (if their projection to the domain of the compact billiard contains exactly one segment tending to infinity), or become planes (if their projection to the domain of the compact billiard contains three consecutive segments tending to infinity).

First step. *The billiards B_0^∞, $B_1'^\infty$, $B_2''^\infty$, $A_0'^\infty$, $A_1'^\infty$ and $A_2'^\infty$.* Each of these non-compact billiards can be obtained as a limit of compact billiards equivalent to one another when a convex elliptic boundary segment is located at infinity. The molecule describing the topology of the Liouville foliation of the compact billiard is of the form $A - A$. It follows from the lemma that, as we pass to the limit, all tori become cylinders (since they will be cut along the cycles corresponding to the elliptic segment at infinity). Hence one of the atoms A becomes the atom C^∞ (the empty limit of cylinders) because its singular circle represented the motion along the convex elliptic segment. The other atom A becomes the atom $\overline{A}$: its singular line corresponds to the motion along either a convex hyperbolic boundary arc, or the vertical line Oy.

Second step. *Focusless billiards (analogues of compact billiards glued from the ribbon billiards of series B and ring billiards of series C): B_n^{∞}, $B_n^{'\infty}$, $B_n^{''\infty}$ and C_n^{∞}.* The proof for these billiards is similar to that in the previous step. Note that the inverse image of the convex elliptic segment in the isoenergy surface of the compact billiard contains the singular leaf of the two-dimensional atom B_n (or D_n for the billiard C_n). Thus, passage to the limit is equivalent to cutting the compact 3-atom transversally to the singular circle. This gives rise to a non-compact 3-atom obtained as the product of the same base and the line.

Third step. *The billiards glued from A_0^{∞}, $A_0^{2\infty}$ and finitely many billiards A_0, A_0' and B_0.* Suppose that the billiard Θ has no conical points and can be obtained as a limit of a compact billiard Ω of the form

$$\Delta_\alpha\biggl(P+\sum_{i=1}^{k}A_0+A_0\biggr)$$

or

$$\Delta_\alpha\biggl(A_0+\sum_{i=1}^{k}A_0+A_0\biggr),$$

where the billiard P is either empty, equivalent to A_0', or equal to B_0, and $k\geqslant 0$ is a certain number. The limit is taken as the free convex segment of the extreme billiards A_0 (the first and last to be glued) tends to infinity.

Note that the 3-atom describing the bifurcation at the saddle level of the integral Λ on the isoenergy surface of the compact billiard Ω, is now equal to B_n, where n is the number of billiards A_0 constituting the billiard Ω. The circles of the saddle leaf are the motions along the focal line in each billiard A_0. If we consider the union of the arcs of some hyperbola that lie in the billiards constituting Ω and endow this union with unit velocity vectors, then we obtain a section of the 3-atom, and this section will be the corresponding 2-atom B_n. Fig. 16 shows the partition of this 2-atom into parts whose points lie in the corresponding elementary billiards constituting Ω.

As we pass to the limit, the hyperbolic arcs lying in the extreme billiards A_0 become non-compact. This gives rise to a discontinuity in the 2-atom B_n, which therefore becomes either the atom B_n' (one discontinuity, the billiard of the form $\Delta_\alpha(P+\sum_{i=1}^{k}A_0+A_0)$), or the atom B_n'' (two discontinuities, the billiard of the form $\Delta_\alpha(A_0+\sum_{i=1}^{k}A_0+A_0)$).

A similar argument can be performed for the billiard $A_0^{2\infty}$. Here the discontinuities occur in the 2-atom B.

Suppose that Θ is equivalent to $\Delta_\beta(2A_0^{\infty})_y$. We regard this billiard as a limit of the billiards $\Delta_\beta(A_0)_y^2$. In contrast to the cases above, we cannot now construct a section transversal to the critical circle of the 3-atom B that describes the bifurcation at the saddle level of the integral Λ on the isoenergy surface of the billiard $\Delta_\beta(A_0)_y^2$. The obstacle is the additional gluing of convex elliptic segments that

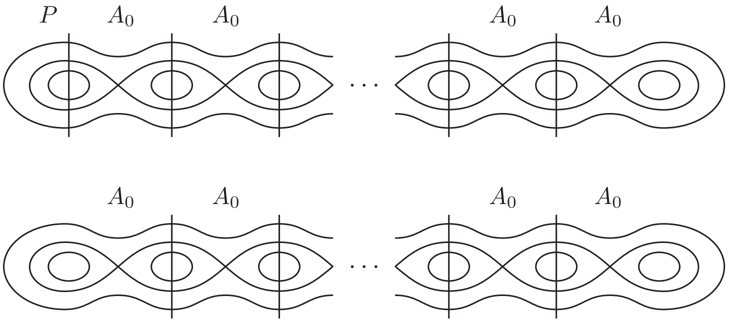


Figure 16. The partition of the section of the 3-atom describing the bifurcation at the saddle level of the integral Λ on the isoenergy surface of the compact billiard Ω, into parts whose points lie in the corresponding elementary billiards that constitute Ω. The upper and lower pictures show the partition for the billiards $\Delta_\alpha\bigl(P+\sum_{i=1}^k A_0+A_0\bigr)$ and $\Delta_\alpha\bigl(A_0+\sum_{i=1}^k A_0+A_0\bigr)$ respectively

forms the conical point. However, we can construct this section in such a way that it coincides with the section described above (the equipped arcs of hyperbolas) on those parts of the billiards A_0 that lie on one side of the focal line and are not glued into the conical point. As we pass to the limit, the convex elliptic segments of these parts tend to infinity. This gives rise to discontinuities in those two-dimensional atoms B that are homeomorphic to the equipped parts of hyperbolas. A similar construction can also be performed in the case when Θ is equivalent to the billiard

$$\Delta_\beta\biggl((2A_0)_y+\sum_{i=1}^n 2A_0+2A_0^\infty\biggr).$$

Suppose that Θ is equivalent to the billiard $\Delta_\beta(2A_0'^\infty)_c$. We regard this billiard as a limit of the billiards $\Delta_\beta(A_0')_c^2$. Then the 3-atom describing the bifurcation at the saddle level of the integral Λ on the isoenergy surface of this billiard is the atom A^*. The section transversal to the critical circle is the 2-atom B. It can also be obtained as equipped arcs of hyperbolas filling the billiards A_0'. When we pass to the limit, both 'eyelets' of these atoms tear. The resulting non-compact 2-atom still has the structure of a skew product with the circle. Thus we obtain the 3-atom $(A')^*$. A similar construction can be performed for the billiard

$$\Delta_\beta\biggl(\bigl(2A_0'^\infty\bigr)_c+\sum_{i=1}^n 2A_0+2A_0^\infty\biggr).$$

Fourth step. *The billiards A_1^∞ and $\Delta_\beta(2A_1'^\infty)_c$.* We regard the billiard A_1^∞ as a limit of the billiards A_1. Since this billiard contains the foci of the family (1.1), it would be non-trivial to construct a fibration of the 3-atom A^* into two-dimensional atoms B transversal to the critical circle. However, this section can be constructed near an elliptic boundary segment using the arcs of confocal ellipses equipped with velocity vectors. We fix the ellipse with parameter $b/2$. The arcs of ellipses belong to the domain of the billiard A_1 when $\lambda \in [t, b/2]$, where t is the parameter of the elliptic boundary segment of A_1. These arcs, equipped with their unit velocity vectors, are homeomorphic to two two-dimensional atoms B (two because there are two directions of the velocity vectors: left and right) when $\lambda \neq t$ and one atom B when $\lambda = t$. The union of these arcs (equipped with the velocity vectors) in the isoenergy manifold Q^3 is homeomorphic to the direct product of the 2-atom B and the closed interval $[t, b/2]$. As we pass to the limit, t tends to $-\infty$ and, therefore, this union of arcs tears into two products of the 'left' and 'right' atoms B (according to the direction of their velocity vectors) and the interval $(-\infty, b/2]$. These intervals are glued with the remaining compact part of the atom A^* (the product of a closed interval and the atom B) and form the non-compact 3-atom $\overline{B}$. A similar argument can be performed for the billiard $\Delta_\beta(2A_1^{'\infty})_c$, which can be regarded as a limit of the billiards $\Delta_\beta(A_1')_c^2$.

Fifth step. *Billiards with infinitely many intervals of the focal line.* Suppose that Θ contains no conical points and can be obtained as a limit of compact billiards Ω_k of the form $\Delta_\alpha\bigl(\sum_{i=1}^{k} A_0 + P\bigr)$ as $k \to \infty$. Here the billiard P is either empty, or equivalent to A_0' or B_0, or equal to A_0^∞. For finite k, the atom describing the bifurcation at the saddle level of the integral Λ will be homeomorphic either to the compact atom B_k (if the billiard Ω_k is compact), or to the non-compact atom B_k'. As k tends to infinity, the number of critical circles of these atoms grows and, as a result, the limit is either the non-compact 3-atom B_∞ (if the domain of Θ contains no domains of A_0^∞ as a subdomain), or the 3-atom B_∞'. If Θ can be represented as a limit of the billiards Ω_k of the form $\Delta_\alpha\bigl(\sum_{i=-k}^{k} A_0\bigr)$ as $k \to \infty$, then the limit of the 3-atoms B_k (which describe the bifurcation at the saddle level of the integral Λ for the billiards Ω_k) will be the non-compact atom $B_{2\infty}$.

Suppose that Θ is equivalent to the billiard $\Delta_\beta\bigl(\sum_{i=1}^{\infty} 2A_0\bigr)_y$. Then it can be represented as a limit of the billiards Ω_k of the form $\Delta_\beta\bigl(\sum_{i=1}^{k} 2A_0\bigr)_y$ as $k \to \infty$. Then the limits of the compact two- and three-dimensional saddle atoms B_k (both in sections transversal to the critical circle and in the whole 3-atoms) will be equal to the non-compact 2- and 3-atoms B_∞. A similar argument can be performed for the billiard $\Delta_\beta\bigl((2A_0')_c + \sum_{i=1}^{\infty} 2A_0\bigr)$, which can be represented as a limit of the billiards Ω_k of the form $\Delta_\beta\bigl((2A_0')_c + \sum_{i=1}^{k} 2A_0\bigr)$ as $k \to \infty$.

Suppose that Θ is equivalent to $(B_0')_\infty$ or $(B_0)_\infty$. Then Θ can be cut along arcs of confocal hyperbolas into infinitely many billiards B_1 and B_0', namely, $\sum_{i=1}^{\infty} B_1 + B_0'$ or $\sum_{i=1}^{\infty} B_1$. In this case we can regard the billiards $(B_0')_\infty$ and $(B_0)_\infty$ as limits as $k \to \infty$ of the compact billiards Ω_k of the form $\sum_{i=1}^{k} B_1 + B_0'$ and $\sum_{i=1}^{k} B_1$, respectively. For finite k, the atom describing the bifurcation at the saddle level of the integral Λ will be homeomorphic to the compact 3-atom B_k, which tends to the 3-atom B_∞ as k tends to infinity. This argument can also be performed for the billiards $\Delta_\beta((2B_0)_\infty)_y$ and $\Delta_\beta((2B_0')_\infty)_x$, which can be represented as limits of the compact billiards Ω_k of the form $\sum_{i=1}^{k} 2B_1 + (B_0)_y^2$ and $\sum_{i=1}^{k} B_1 + (B_0')_x^2$ (respectively) as $k \to \infty$. When Θ is equivalent to B_∞, we can cut it along arcs of confocal hyperbolas into infinitely many billiards B_1 and thus represent it as a limit of the billiards Ω_k of the form $\sum_{i=-k}^{k} B_1$ as $k \to \infty$. Then the 3-atom describing the bifurcation at the saddle level of the integral Λ will be homeomorphic to the non-compact atom $B_{2\infty}$.

Suppose that Θ is equivalent to one of the billiards $\Delta_\alpha(2(B_0')_\infty)$, $\Delta_\alpha(2(B_0)_\infty)$ or $\Delta_\alpha(2B_\infty)$, that is, Θ can be partitioned into an infinite union of the doubled billiards $\Delta_\alpha(2B_1)$ and $\Delta_\alpha(2B_0')$. Note that taking the 'doubled' billiard in the case of a ribbon billiard of series B does not change the topology of the Liouville foliation. Therefore the molecules for the billiards $\Delta_\alpha(2(B_0')_\infty)$, $\Delta_\alpha(2(B_0)_\infty)$ and $\Delta_\alpha(2B_\infty)$ coincide with those for the billiards $(B_0')_\infty$, $(B_0)_\infty$ and B_∞.

Suppose that Θ is equivalent to one of the billiards $(B_0')_\infty^\infty$, $(B_0)_\infty^\infty$ or B_∞^∞. Then it can be obtained as a limit of the billiard $\widetilde{\Theta}$ which is equivalent to $(B_0')_\infty$, $(B_0)_\infty$ or B_∞ respectively. We now describe the foliation of the 3-atom corresponding to the bifurcation at the saddle level of the integral Λ on the isoenergy surface Q^3 for the billiard $\widetilde{\Theta}$. Suppose that the elliptic boundary segments of $\widetilde{\Theta}$ lie on the ellipses with parameters $\lambda_e > \lambda_E$. Fix a parameter $\lambda \in (\lambda_E, \lambda_e)$. The (infinite) union of those arcs of the ellipse with parameter λ that lie in $\widetilde{\Theta}$, when equipped with velocity vectors, is homeomorphic to a pair of planar atoms B_∞ (or $B_{2\infty}$ in the case when $\widetilde{\Theta}$ is equivalent to B_∞). The infinite union of the arcs of the ellipse with the boundary parameter (that is, either λ_e or λ_E) is homeomorphic to one planar atom B_∞ ($B_{2\infty}$ in the case when $\widetilde{\Theta}$ is equivalent to B_∞). Thus the whole 3-atom is the direct product of a circle and the corresponding planar saddle atom. We pass to the limit as $E \to -\infty$. Then the billiard $\widetilde{\Theta}$ tends to Θ, and the circle (which was earlier multiplied by the 2-atom) tends to a line. Hence the 3-atom describing the bifurcation at the saddle level of the integral Λ on the isoenergy surface Q^3 will be homeomorphic to the atom $\overline{B}_\infty$ for the billiards $(B_0')_\infty^\infty$ and $(B_0)_\infty^\infty$, and to the atom $\overline{B}_{2\infty}$ for the billiard B_∞^∞. $\square$

Remark 3.4. The equivalence relation on the set of non-compact billiards enables us to pass to certain billiards having only rectilinear boundaries: the billiard $A_1^{'\infty}$ in a non-compact angle and the billiards $A_2^{'\infty}$, A_1^∞ in a half-plane. Note that by the choice of integral we obtain two different Liouville foliations, which are described by different Fomenko molecules.

§ 4. Non-convex generalized billiards glued from elementary ones along non-convex segments

Definitions 3.10, 3.11 of gluing of elementary billiards prohibit gluing along non-convex segments. To explain this, suppose that two elementary billiards C_2 are glued along a non-convex elliptic segment lying on the ellipse with parameter λ_0. Denote the resulting billiard by $\Delta_\gamma(2C_2)$.

The intervals that form any given trajectory (as a polygonal arc) lie on straight lines that are tangent to some quadric (an ellipse or a hyperbola). This quadric is said to be integral (an integral ellipse or an integral hyperbola).

The trajectories lying at the levels $\Lambda < \lambda_0$ of the integral do not intersect the gluing segment: moving towards the gluing segment, they reach the integral ellipse,

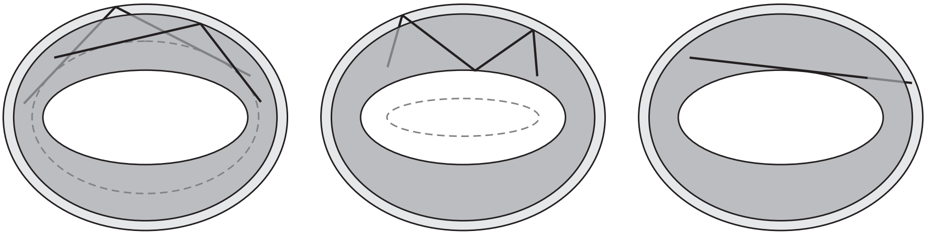


Figure 17. Trajectories of the billiard $\Delta_\gamma(2C_2)$ at the integral levels $\lambda < \lambda_0$ (left) and $\lambda > \lambda_0$ (middle). Trajectories at the level $\Lambda = \lambda_0$ (right) cannot be defined as their simultaneous continuous limit. The dashed lines indicate integral ellipses

touch it and move away staying always on the same copy of C_2. The trajectories lying at the levels with $b > \lambda > \lambda_0$ pass from one copy of C_2 to the other whenever they reach the gluing segment. The generalized billiard law gives no answer about the behaviour of the trajectories at the level $\Lambda = \lambda_0$. At this level of the integral, the interior gluing edge lies on the integral ellipse. Having reached the integral ellipse, the trajectory must stay on the same copy of C_2 (being a continuous limit of trajectories lying at levels with $\lambda < \lambda_0$) and, on the other hand, pass to the other copy (being a continuous limit of trajectories lying at levels with $b > \lambda > \lambda_0$).

Although the trajectories at the level $\Lambda = \lambda_0$ of the integral are undefined, all other non-singular leaves for $\Lambda \neq \lambda_0$ are tori and all singular leaves can be described by three-dimensional atoms. Hence Q^3 possesses the structure of a Liouville foliation. A circle lying in the inverse image of the points of the non-convex gluing segment on the singular leaf $\Lambda = \lambda_0$ is referred to as a singular circle. It turns out that the topology of a neighbourhood of the leaf $\Lambda = \lambda_0$ is nonetheless described by the three-dimensional atom B.

In this section we describe the topology of the Liouville foliation for generalized billiards with non-convex gluing in the example of the billiards $\Delta_\gamma(2B_1)$ and $\Delta_\gamma(2C_2)$ that are glued from two copies of B_1 and C_2 respectively.

Proposition 4.1. *Let $\Delta_\gamma(2B_1)$ be the non-convex generalized billiard glued from two copies of the domain B_1 along a non-convex boundary segment lying on the ellipse with parameter λ_0. Then the topology of the Liouville foliation on the three-dimensional isoenergy surface Q^3 of $\Delta_\gamma(2B_1)$ is described by the 3-atom B.*

Let $\Delta_\gamma(2C_2)$ be the non-convex generalized billiard glued from two copies of the domain C_2 along a non-convex boundary segment lying on the ellipse with parameter λ_0. Then the topology of the Liouville foliation on the three-dimensional isoenergy surface Q^3 of $\Delta_\gamma(2C_2)$ is described by the disjoint union of two 3-atoms B.

Proof. There is no loss of generality in assuming that the convex elliptic boundary segments of the elementary billiards B_1 and C_2 whose gluing yields the non-convex generalized billiards $\Delta_\gamma(2B_1)$ and $\Delta_\gamma(2C_2)$, lie on the ellipse with parameter 0.

The level sets of the function Λ for B_1 and C_2 consist of two-dimensional tori when $\Lambda < b$ (see [22]): one torus for B_1 and two tori (which differ by the direction of motion, clockwise or anticlockwise) for C_2. The generalized billiard law introduced on the gluing edge in the billiards $\Delta_\gamma(2B_1)$ and $\Delta_\gamma(2C_2)$ does not influence the tori when $\Lambda < \lambda_0$. Therefore we have two tori (each on its own copy of B_1) for $\Delta_\gamma(2B_1)$ and four tori for $\Delta_\gamma(2C_2)$. When $\Lambda > \lambda_0$, we must cut the tori corresponding to one copy of the domain of the elementary billiard along certain cycles (see § 3.1) in the inverse image of the gluing segment and then glue them by the generalized billiard law with slit tori that are Liouville leaves in the other copy of the domain of the elementary billiard. This again results in a torus.

We now suppose that $\Lambda = \lambda_0$. Then the inverse image of the gluing segment on the tori is one cycle. The singular leaf of the atom B arises when we glue two tori (each torus corresponds to its own copy of the elementary billiard) along this cycle.

Fig. 18 shows a section of the piecewise-smooth 3-atom B which is a three-dimensional neighbourhood of the singular leaf of the integral. Here we do not consider integral trajectories but look only at pairs (point, vector). This approach yields a Liouville foliation which in this case coincides with the foliation on the ordinary 3-atom B. Its singular leaf is the direct product of a figure eight and a circle (see Fig. 18). The behaviour of integral trajectories on the Liouville tori close to the singular leaf can be found in the standard way. Nevertheless, the limit of these trajectories as we approach the singular leaf is not well defined (see Fig. 17 and Fig. 6). This is because a trajectory of the mass point, having touched the non-convex gluing edge, 'does not know' which of the two leaves to choose for its further motion. Therefore the critical circle on the singular leaf of the 3-atom B is not an integral trajectory and is not a continuous limit of close trajectories. □

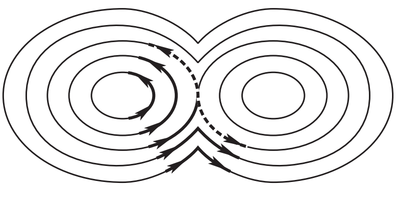


Figure 18. A section of the 3-atom B. It shows the projections of the trajectories of the billiards $\Delta_\gamma(2B_1)$ and $\Delta_\gamma(2C_2)$ near the integral level that contains the non-convex gluing edge. The undefined trajectory is shown by the dashed line

Remark 4.1. In other words, although the bifurcation at the level $\Lambda = \lambda_0$ is described by the 3-atom B (that is, the change of level sets of Λ can be explained using B), the trajectories on the singular leaf of this 3-atom admit no continuous extension. The singular leaf of the classical atom B consists of two annuli, and the trajectories on them infinitely 'wind off' one boundary of the annulus and infinitely 'wind onto' the other. On the same 3-atom, trajectories attain the boundary of the annulus for a finite time, and then the motion is undefined: there are equal reasons for a trajectory to stay on the same annulus or proceed to the other. Because of this non-uniqueness, the motion along the singular circle (the non-convex gluing edge) cannot be represented as an integral trajectory.

We have already mentioned that the Liouville foliation is well defined for non-convex generalized billiards. Therefore, as above, we can study its topology using the Fomenko–Zieschang invariant. However, this invariant does not now give complete information, for example, about the stability or instability of closed integral trajectories since they are now undefined in the case when the saddle atom corresponds to the non-convex gluing edge of the billiard. At the same time, the behaviour of the trajectories on the other atoms is standard.

Proposition 4.2. *The topology of the Liouville foliation on the isoenergy surface Q^3 of the non-convex generalized billiards $\Delta_\gamma(2B_1)$ and $\Delta_\gamma(2C_2)$ is described by the Fomenko–Zieschang invariants shown in Fig. 19.*

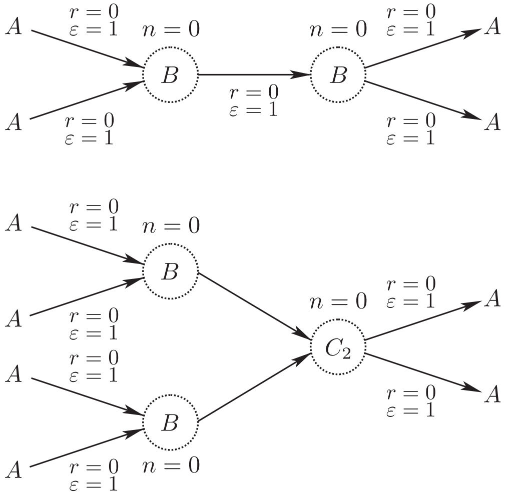


Figure 19. The Fomenko–Zieschang molecules describing the topology of the Liouville foliation on the isoenergy surface Q^3 for the non-convex generalized billiards $\Delta_\gamma(2B_1)$ (top) and $\Delta_\gamma(2C_2)$ (bottom)

Proof. We first prove the assertion for $\Delta_\gamma(2B_1)$. Choose pairs of cycles λ as shown in Fig. 20. Here the dashed lines indicate arcs of the integral quadric that determines the Liouville torus. To choose a cycle on the torus, we take a curve in its projection to the billiard domain and equip it with velocity vectors in such a way that the marked pairs point-vector form a non-trivial cycle with required properties on the torus (see [10] for the cycle choice rules). The cycles λ on the boundary tori of the minimax atoms A must be contracted to a point as the torus approaches the critical circle of the atom A. In the left part of Fig. 20, the cycles whose projections lie on the arcs of hyperbolas vanish as the integral quadric (dashed) tends to the convex boundary of the domain. We observe the same behaviour in the right part of Fig. 20 for the cycle whose projection lies on the gluing ellipse between the boundary hyperbola and the integral hyperbola. The other cycles λ are chosen on the boundary tori of the saddle atoms B. When the tori approach the singular leaf of the 3-atom B, these cycles must tend to its singular circle. We first consider the atom B that describes the bifurcation of the Liouville foliation at the level $\Lambda = \lambda_0$ of the integral. Its singular circle is the gluing arc equipped with the tangent velocity vectors. The left and middle pictures in Fig. 20 show that the curves lying on ellipses and equipped with the velocity vectors tend to the desired cycles as the tori approach the singular leaf of B. We now consider the atom B that describes the bifurcation at the level $\Lambda = b$ of the integral. Its singular circle is the motion along intervals of the focal line. The middle and right pictures in Fig. 20 show that the curves lying on hyperbolas and equipped with the velocity vectors tend to the desired cycles as the tori approach the singular leaf of B.

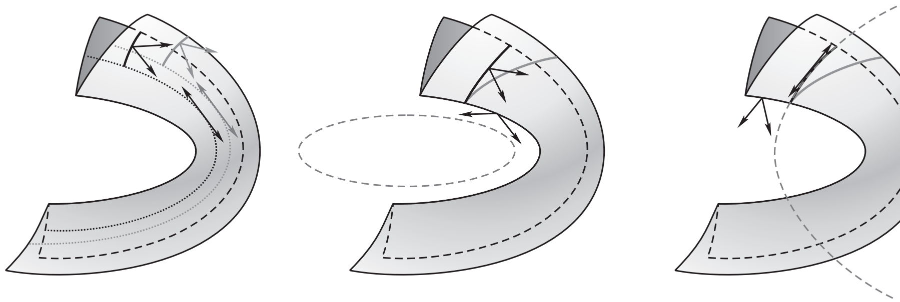


Figure 20. Pairs of cycles λ on the projection of Liouville tori for the billiard $\Delta_\gamma(2B_1)$

Note that on each torus, the cycles λ intersect at one point and, therefore, form a basis on this torus. Then on each torus, the pairs of these cycles may be regarded as a basis (λ, μ) chosen by the rule for the minimax atom A. Note that the cycles complementary to the cycles λ on the tori corresponding to the saddle atoms are constrained by the existence of a global section, and this again enables us to regard them as the cycles μ. The orientation of the cycles λ on the boundary tori of the saddle atoms B, as well as the orientation of the cycles μ on the boundary tori of the minimax atoms A, is given by the flow of the vector field, that is, it coincides with the direction of the velocity vectors in the equipment of the curves in the projections of these tori to the domain of the billiard. We obtain that the gluing matrices on all edges are of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The marks are uniquely calculated from the gluing matrices.

The proof for the billiard $\Delta_\gamma(2C_2)$ is perfectly analogous. □

Corollary 4.1. *The billiard $\Delta_\gamma(2B_1)$ is Liouville equivalent to the billiard in an ellipse in the Minkowski metric.*

Proof. The topology of the Liouville foliation for the billiards in ellipses and the geodesic flows on ellipsoids has been studied by Dragović and Radnović [8]. Note that the molecule in their paper has the mark $\varepsilon = -1$ between the saddle atoms B. However, it follows from § 4.5.2 in [10] that reversal of the orientation on Q^3 implies that the sign of this mark ε is switched while all other marks are preserved. Hence the Liouville foliations given by this molecule coincide. □

Remark 4.2. One of the three Fomenko–Zieschang molecules classifying the Liouville foliation of the geodesic flow in the Minkowski metric (studied in [8]) coincides with the Fomenko–Zieschang molecule describing the foliation on the isoenergy surface Q^3 for the billiard $\Delta_\gamma(2C_2)$. The other two invariants coincide, after reversing the orientation of Q^3, with the molecule for the billiard in an ellipse in the ordinary metric.

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