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Blow-up of solutions of a full non-linear equation of ion-sound waves in a plasma with non-coercive non-linearities

M. O. Korpusov, D. V. Lukyanenko, A. A. Panin, and E. V. Yushkov

Abstract. We consider a series of initial-boundary value problems for the equation of ion-sound waves in a plasma. For each of them we prove the local (in time) solubility and perform an analytical-numerical study of the blow-up of solutions. We use the method of test functions to obtain sufficient conditions for finite-time blow-up and an upper bound for the blow-up time. In concrete numerical examples we improve these bounds numerically using the mesh refinement method. Thus the analytical and numerical parts of the investigation complement each other. The time interval for the numerical modelling is chosen in accordance with the analytically obtained upper bound for the blow-up time. In return, numerical calculations specify the moment and pattern of this blow-up.

Keywords: blow-up of a solution, non-linear initial-boundary value problem, Sobolev-type equations, exponential non-linearity, Richardson extrapolation.

Introduction

This paper is devoted to an analytical-numerical investigation of the equation of ion-sound waves in a plasma in the one-dimensional approximation:

$$(u_{xx} - e^{\varepsilon u})_{tt} + u_{xx} = (|u_x|^q)_t, \qquad x \in (0, l), \quad t > 0.$$

Such equations arise in many problems of mathematical physics, in particular, in the theory of ion-sound waves in a plasma [1]–[3]. However, one usually studies the linearized version of this equation. Our investigation of the full equation of ion-sound waves is motivated by the fact that after linearization of the Boltzmann distribution

$$\exp(\varepsilon u), \qquad \varepsilon > 0,$$

with respect to the small parameter $\varepsilon > 0$, the equation may not account for the essentially non-linear effects such as a finite-time blow-up of solutions. On the other hand, the presence of the non-linear operator

$$u_{xx} - \exp(\varepsilon u)$$

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under the sign of the second time derivative in the full equation makes the situation rather difficult for analytical and numerical investigation.

In this paper we consider three series of boundary conditions: the homogeneous Dirichlet or Neumann conditions and the homogeneous non-local boundary conditions at the endpoints of a finite interval. In all cases we obtain the local (in time) solubility in the classical sense using the method of contraction mapping. We also establish sufficient conditions for the finite-time blow-up of solutions of these initial-boundary value problems. This is done using various versions of the non-linear capacity method (test-function method) of Pokhozhaev and Mitidieri [4].

We also mention other methods for studying blow-up patterns, such as Levine's energy method and its modifications [5]–[8] and the method of self-similar modes based on various comparison criteria and developed by Samarskii, Galaktionov, Kurdyumov and Mikhailov [9] (see also [10]).

However, as a rule, the analytical approach gives only upper bounds for the blow-up time. Therefore numerical diagnostics of the blow-up time (based on ideas of Kalitkin, Al'shin, Al'shina and Koryakin [11], [12], [8]) is of much interest. The main idea is to compare the theoretical order of accuracy with the effective numerical one estimated by Richardson's method. The time layer where these orders become clearly different gives a bound (with accuracy approximately equal to the grid step) for the blow-up time, and localization of blow-up in the space variable is performed according to the distribution of the effective accuracy order on this layer. The analytical and numerical parts of the investigation complement each other. Namely, the time interval for the numerical modelling is chosen in accordance with the analytically obtained upper bound for the blow-up time. In return, numerical calculations specify the moment and pattern of this blow-up (see Examples 2 and 3 in § 5).

This paper continues a series of papers by the authors. It was started in [13], where we considered non-stationary non-linear equations with non-coercive non-linearities. Such equations are difficult because no energy methods can be applied. We considered other model equations of ion-sound waves, in particular, in [14]–[16].

The paper is organized as follows. In § 1 we derive the equation in question from the physical model. In § 2 we give a mathematical statement of the initial-boundary value problems to be studied. § 3 is devoted to proving the local (in time) solubility of these problems, and in § 4 we establish sufficient conditions for blow-up of their solutions.

Finally, in § 5 we describe a method for numerically improving the information about blow-up for any concrete initial data leading to a blow-up regime, and end by stating some conclusions.

§ 1. Derivation of the equation

Consider an electron-ion plasma in a domain $D\subset\mathbb{R}^N$, $N\geqslant 1$. In the approximation of a quasistationary electric field we have the equations

$$\operatorname{div}\mathbf{E}=n_i-n_e,\qquad \operatorname{rot}\mathbf{E}=0, \tag{1}$$

where $\mathbf{E}$ is the electric field strength vector and n_e (resp. n_i) is the density of electrons (resp. ions). Assuming that D is superficially simply connected, we can introduce the potential φ of the electric field. It is related to the vector $\mathbf{E}$ by the formula

$$\mathbf{E} = -\nabla\varphi. \tag{2}$$

Then the density of electrons is adequately described by the Boltzmann distribution

$$n_e = n_0(x)\exp\biggl(\frac{e\varphi}{kT_e}\biggr), \tag{3}$$

where $n_0(x)$ is the equilibrium distribution of the electrons, T_e is their temperature and k is the Boltzmann constant. The concentration n_i of ions satisfies the continuity equation

$$\frac{\partial n_i}{\partial t} = -\operatorname{div}\mathbf{J}, \tag{4}$$

where $\mathbf{J}$ is the current density of the ion component of the plasma. We consider two components of the current,

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2, \tag{5}$$

where $\mathbf{J}_1$ accounts for the time dispersion,

$$\mathbf{J}_1 = \int_0^t \sigma(t,\tau)\mathbf{E}(\tau)\,d\tau, \qquad \sigma(t,\tau) \in \mathbb{C}^2([0,T]\otimes[0,T]), \tag{6}$$

and $\mathbf{J}_2$ accounts for the current induced by 'warm-up' of the plasma. The term $\mathbf{J}_2$ is of the form

$$\mathbf{J}_2 = -\gamma\nabla T_i, \qquad \gamma > 0, \tag{7}$$

where T_i is the temperature of the ions. On the other hand, the temperature T_i of the ions satisfies the equation

$$\varepsilon\frac{\partial T_i}{\partial t} = \Delta T_i + a|\mathbf{E}|^q, \qquad a > 0, \quad q > 0, \tag{8}$$

with a small parameter $\varepsilon > 0$. Therefore, instead of (8), we consider the corresponding stationary equation

$$\Delta T_i + a|\mathbf{E}|^q = 0. \tag{9}$$

We obtain the following differential corollary from the equations (1)–(3):

$$\Delta\varphi - n_0(x)\exp\biggl(\frac{e\varphi}{kT_e}\biggr) = -n_i. \tag{10}$$

The system of equations (4)–(7) and (9) yields the following differential corollary:

$$\frac{\partial n_i}{\partial t} = \int_0^t \sigma(t,\tau)\Delta\varphi(\tau)\,d\tau - \gamma a|\nabla\varphi|^q. \tag{11}$$

The equations (10) and (11) yield the differential corollary

$$\frac{\partial}{\partial t}\biggl(\Delta\varphi - n_0(x)\exp\biggl(\frac{e\varphi}{kT_e}\biggr)\biggr) + \int_0^t \sigma(t,\tau)\Delta\varphi(\tau)\,d\tau = c_0|\nabla\varphi|^q, \tag{12}$$

where $c_0 = \gamma a > 0$. In the important particular case $\sigma(t,\tau) = \sigma_0 > 0$ we arrive at the differential equation

$$\frac{\partial^2}{\partial t^2}\biggl(\Delta\varphi - n_0(x)\exp\biggl(\frac{e\varphi}{kT_e}\biggr)\biggr) + \sigma_0\Delta\varphi(t) = c_0\frac{\partial}{\partial t}|\nabla\varphi|^q. \tag{13}$$

Assuming that $n_0(x) = \text{const}$ and passing to dimensionless quantities, we arrive at the equation

$$\frac{\partial^2}{\partial t^2}(\Delta u - e^{\varepsilon u}) + \Delta u = \frac{\partial}{\partial t}|\nabla u|^q, \qquad x \in (0,l), \quad t > 0, \quad \varepsilon > 0 \text{ is a parameter}, \quad q > 1. \tag{14}$$

In this paper we consider the one-dimensional case and study the initial and boundary conditions for which the solution blows up.

§ 2. Statement of the problem

Thus we consider a series of initial-boundary value problems for the equation

$$(u_{xx} - e^{\varepsilon u})_{tt} + u_{xx} = (|u_x|^q)_t, \qquad x \in (0,l), \quad t > 0, \quad \varepsilon > 0 \text{ is a parameter}, \quad q > 1, \tag{15}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \tag{16}$$

and one of the following boundary conditions: the Dirichlet conditions

$$u(0,t) = u(l,t) = 0, \tag{17}$$

or the Neumann conditions

$$u_x(0,t) = u_x(l,t) = 0, \tag{18}$$

or the non-classical non-local conditions

$$u(0,t) = 0, \qquad u(l,t) = lu_x(0,t). \tag{19}$$

In what follows the problem (15)–(17) is referred to as Problem A, the problem (15), (16), (18) as Problem B, and the problem (15), (16), (19) as Problem C. We assume that the initial data are compatible with the boundary conditions in all these problems, that is, the functions $u_0(x)$, $u_1(x)$ satisfy the corresponding boundary conditions.

§ 3. Local solubility

3.1. Preliminary transformations. We introduce the function

$$f(\eta) = \sum_{n=2}^{\infty}\frac{\varepsilon^n\eta^n}{n!} = e^{\varepsilon\eta} - \varepsilon\eta - 1.$$

Then

$$(e^{\varepsilon u})_{tt} = (f(u) + \varepsilon u + 1)_{tt} = (f(u) + \varepsilon u)_{tt}$$

and the equation (15) can be rewritten in the form

$$(u_{xx} - \varepsilon u - f(u))_{tt} + u_{xx} - \varepsilon u = -\varepsilon u + (|u_x|^q)_t. \tag{20}$$

Our further arguments are applicable to each of the Problems A, B, C unless otherwise stated. However, generally speaking, the values of the constants occurring are different in each case.

We consider the function spaces $C[0,l]$, $C^1[0,l]$ and the space Z^2 of all functions in $C^2[0,l]$ satisfying the chosen boundary conditions (17), (18) or (19). We put

$$\|w\|_{C^1[0,l]} \equiv \|w\|_{C[0,l]} + \|w'\|_{C[0,l]}, \qquad \|w\|_{Z^2} \equiv \|w\|_{C[0,l]} + \|w'\|_{C[0,l]} + \|w''\|_{C[0,l]}.$$

Then Z^2 is a closed subspace of $C^2[0,l]$ and hence a Banach space.

We consider the equation (20) in the space Z^2. More precisely, we introduce the linear operators

$$\mathbb{J}_{20} = \text{the embedding of } Z^2 \text{ in } C[0,l], \qquad \|\mathbb{J}_{20}\| = 1, \tag{21}$$

$$\mathbb{J}_{10} = \text{the embedding of } C^1[0,l] \text{ in } C[0,l], \qquad \|\mathbb{J}_{10}\| = 1, \tag{22}$$

$$\mathbb{J}_{21} = \text{the embedding of } Z^2 \text{ in } C^1[0,l], \qquad \|\mathbb{J}_{21}\| = 1, \tag{23}$$

$$\frac{d}{dx}\colon C^1[0,l] \to C[0,l], \qquad \left\|\frac{d}{dx}\right\| = 1, \tag{24}$$

$$\frac{d^2}{dx^2}\colon Z^2 \to C[0,l], \qquad \left\|\frac{d^2}{dx^2}\right\| = 1, \tag{25}$$

$$\mathbb{L}\colon Z^2 \to C[0,l], \qquad \mathbb{L}w \equiv \frac{d^2}{dx^2}w - \varepsilon\mathbb{J}_{20}w, \tag{26}$$

and the non-linear operators

$$\mathbb{F}\colon C[0,l] \to C[0,l], \qquad \mathbb{F}\colon w(x) \mapsto f(w(x)) \equiv e^{\varepsilon w(x)} - \varepsilon w(x) - 1,$$
$$Q\colon C[0,l] \to C[0,l], \qquad w(x) \mapsto |w(x)|^q.$$

Then the equation (20) takes the form

$$\frac{d^2}{dt^2}\big(\mathbb{L}u(t) - \mathbb{F}(\mathbb{J}_{20}u(t))\big) + \mathbb{L}u(t) = -\varepsilon\mathbb{J}_{20}u(t) + \frac{d}{dt}Q\left(\frac{d}{dx}\mathbb{J}_{21}u(t)\right), \tag{27}$$

where the differentiation with respect to t is understood as strong differentiation in $C[0,l]$.

The operator $\mathbb{L}$ can be inverted. If $\mathbb{L}w = g(x)$, then

$$w = \mathbb{G}g,$$

where $\mathbb{G}\colon C[0,l] \to Z^2$ is the Green operator. Concrete expressions for $\mathbb{G}$ in each of the problems A, B, C will be given below. Applying $\mathbb{G}$ to both sides of (27), we pass to an equivalent equation in the space Z^2:

$$\frac{d^2}{dt^2}\big(u(t) - \mathbb{G}\mathbb{F}(\mathbb{J}_{20}u(t))\big) + u(t) = \mathbb{G}\left(-\varepsilon\mathbb{J}_{20}u(t) + \frac{d}{dt}Q\left(\frac{d}{dx}\mathbb{J}_{21}u(t)\right)\right). \tag{28}$$

For brevity we introduce the notation

$$\mathbb{H} = \mathbb{G} \circ \mathbb{F} \circ \mathbb{J}_{20}, \qquad \mathbb{Q}_1 = Q \circ \frac{d}{dx} \circ \mathbb{J}_{21}.$$

Then the equation (28) takes the form

$$\frac{d^2}{dt^2}\big(u(t) - \mathbb{H}(u(t))\big) + u(t) = \mathbb{G}\biggl(-\varepsilon \mathbb{J}_{20} u(t) + \frac{d}{dt}\mathbb{Q}_1(u(t))\biggr).$$

As a rule, we shall write this in the simplified and more 'classical' form

$$\frac{d^2}{dt^2}(u - \mathbb{H}(u)) + u = \mathbb{G}(-\varepsilon u + (|u_x|^q)_t),$$

which is understood in the sense of (28).

Integrating the resulting equation with respect to t in Z^2, we obtain that

$$\frac{d}{dt}(u-\mathbb{H}(u))-(u_1-\mathbb{H}'_f(u_0)u_1)+\int_0^t u(s)\,ds = -\varepsilon\int_0^t \mathbb{G}u(s)\,ds+\mathbb{G}(|u_x|^q)-\mathbb{G}(|u_{0,x}|^q).$$

Here and in what follows, $u_{0,x}$ stands for the derivative of u_0 with respect to x, and a primed operator with subscript f means the Fréchet derivative of this operator.

Integrating again with respect to t, we obtain that

$$\begin{aligned} u - \mathbb{H}(u) - (u_0 - \mathbb{H}(u_0)) - (u_1 - \mathbb{H}'_f(u_0)u_1)t \\ = -\int_0^t (t-s)(u(s) + \varepsilon(\mathbb{G}u)(s))\,ds + \int_0^t \mathbb{G}(|u_x|^q)(s)\,ds - \mathbb{G}(|u_{0,x}|^q)t. \end{aligned} \tag{29}$$

3.2. Auxiliary assertions.

Lemma 1. *There are ε_1 and C_1 such that for all $\varepsilon \in (0, \varepsilon_1)$ we have*

$$\|\mathbb{G}\| \leqslant \frac{C_1}{\varepsilon}.$$

Proof. Consider the equation

$$v_{xx} - \varepsilon v = g(x) \tag{30}$$

with one of the following boundary conditions: the Dirichlet conditions

$$v(0) = v(l) = 0, \tag{31}$$

or the Neumann conditions

$$v_x(0) = v_x(l) = 0,$$

or the non-classical non-local conditions

$$v(0) = 0, \qquad v(l) = l v_x(0).$$

1. *The Dirichlet problem.* The Green function of the problem (30), (31) is given by the formula

$$G(x,\xi)=\frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\begin{cases}\sinh\sqrt{\varepsilon}\,x\sinh\sqrt{\varepsilon}\,(\xi-l), & 0\leqslant x\leqslant\xi\leqslant l,\\ \sinh\sqrt{\varepsilon}\,\xi\sinh\sqrt{\varepsilon}\,(x-l), & 0\leqslant\xi\leqslant x\leqslant l.\end{cases}$$

Taking this into account, we write the desired function in the explicit form

$$v(x)=\frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\bigg[\sinh\sqrt{\varepsilon}\,(x-l)\int_0^x\sinh\sqrt{\varepsilon}\,\xi g(\xi)\,d\xi + \sinh\sqrt{\varepsilon}\,x\int_x^l\sinh\sqrt{\varepsilon}\,(\xi-l)g(\xi)\,d\xi\bigg].$$

To find the norm of the solution in Z^2, we calculate its first and second derivatives. We have

$$v'(x)=\frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\bigg[\sqrt{\varepsilon}\,\cosh\sqrt{\varepsilon}\,(x-l)\int_0^x\sinh\sqrt{\varepsilon}\,\xi g(\xi)\,d\xi + \sqrt{\varepsilon}\,\cosh\sqrt{\varepsilon}\,x\int_x^l\sinh\sqrt{\varepsilon}\,(\xi-l)g(\xi)\,d\xi\bigg],$$

$$v''(x)=\frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\bigg[\varepsilon\sinh\sqrt{\varepsilon}\,(x-l)\int_0^x\sinh\sqrt{\varepsilon}\,\xi g(\xi)\,d\xi + \varepsilon\sinh\sqrt{\varepsilon}\,x\int_x^l\sinh\sqrt{\varepsilon}\,(\xi-l)g(\xi)\,d\xi\bigg]+g(x).$$

Thus,

$$\begin{aligned}|v(x)|&\leqslant\frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\big[(\sinh\sqrt{\varepsilon}\,l)^2x\|g\|_{C[0,l]}+(\sinh\sqrt{\varepsilon}\,l)^2(l-x)\|g\|_{C[0,l]}\big]\\ &=\frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}(\sinh\sqrt{\varepsilon}\,l)^2l\|g\|_{C[0,l]}=\frac{\sinh\sqrt{\varepsilon}\,l}{\sqrt{\varepsilon}}l\|g\|_{C[0,l]},\\ |v'(x)|&\leqslant\frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\big[\sqrt{\varepsilon}\,(\cosh\sqrt{\varepsilon}\,l)(\sinh\sqrt{\varepsilon}\,l)x\|g\|_{C[0,l]}\\ &\quad+\sqrt{\varepsilon}\,(\cosh\sqrt{\varepsilon}\,l)(\sinh\sqrt{\varepsilon}\,l)(l-x)\|g\|_{C[0,l]}\big]=l\cosh\sqrt{\varepsilon}\,l\|g\|_{C[0,l]},\\ |v''(x)|&\leqslant\frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\big[\varepsilon(\sinh\sqrt{\varepsilon}\,l)^2x\|g\|_{C[0,l]}+(\sinh\sqrt{\varepsilon}\,l)^2(l-x)\|g\|_{C[0,l]}\big]\\ &\quad+\|g\|_{C[0,l]}\leqslant\big(l\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}+1\big)\|g\|_{C[0,l]}.\end{aligned}$$

Estimating the norm of the solution in Z^2, we obtain that

$$\begin{aligned}\|v\|_{Z^2}&\equiv\|v\|_{C[0,l]}+\|v\|_{C^1[0,l]}+\|v\|_{C^2[0,l]}\\ &\leqslant l\bigg(\frac{\sinh\sqrt{\varepsilon}\,l}{\sqrt{\varepsilon}}+\cosh\sqrt{\varepsilon}\,l+\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l\bigg)\|g\|_{C[0,l]}+\|g\|_{C[0,l]}.\end{aligned}$$

Putting

$$\gamma(\varepsilon):=\|\mathbb{G}\|,\qquad\widetilde{\gamma}(\varepsilon):=l\left(\frac{\sinh\sqrt{\varepsilon}\,l}{\sqrt{\varepsilon}}+\cosh\sqrt{\varepsilon}\,l+\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l\right)+1,$$

we obviously have

$$0 < \gamma(\varepsilon) \leqslant \widetilde{\gamma}(\varepsilon) \to l(l+1)+1 \quad \text{as} \quad \varepsilon \to +0.$$

Hence there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\gamma(\varepsilon) < 2(l(l+1)+1).$$

2. *The Neumann problem.* The Green function of this problem is of the form

$$G(x,\xi) = \frac{-1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\begin{cases}\cosh\sqrt{\varepsilon}\,x\cosh\sqrt{\varepsilon}\,(\xi-l), & 0 \leqslant x \leqslant \xi \leqslant l,\\ \cosh\sqrt{\varepsilon}\,\xi\cosh\sqrt{\varepsilon}\,(x-l), & 0 \leqslant \xi \leqslant x \leqslant l.\end{cases}$$

The desired function and its derivatives are given by the formulae

$$\begin{aligned}v(x) &= \frac{-1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\biggl[\cosh\sqrt{\varepsilon}\,(x-l)\int_0^x \cosh\sqrt{\varepsilon}\,\xi g(\xi)\,d\xi\\ &\qquad + \cosh\sqrt{\varepsilon}\,x\int_x^l \cosh\sqrt{\varepsilon}\,(\xi-l)g(\xi)\,d\xi\biggr],\\ v'(x) &= \frac{-1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\biggl[\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,(x-l)\int_0^x \cosh\sqrt{\varepsilon}\,\xi g(\xi)\,d\xi\\ &\qquad + \sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,x\int_x^l \cosh\sqrt{\varepsilon}\,(\xi-l)g(\xi)\,d\xi\biggr],\\ v''(x) &= \frac{-1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\biggl[\varepsilon\cosh\sqrt{\varepsilon}\,(x-l)\int_0^x \cosh\sqrt{\varepsilon}\,\xi g(\xi)\,d\xi\\ &\qquad + \varepsilon\cosh\sqrt{\varepsilon}\,x\int_x^l \cosh\sqrt{\varepsilon}\,(\xi-l)g(\xi)\,d\xi\biggr] + g(x).\end{aligned}$$

The following bounds hold:

$$\begin{aligned}|v(x)| &\leqslant \frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\bigl[(\cosh\sqrt{\varepsilon}\,l)^2(x+(l-x))\|g\|_{C[0,l]}\bigr] = \frac{(\cosh\sqrt{\varepsilon}\,l)^2}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}l\|g\|_{C[0,l]},\\ |v'(x)| &\leqslant \frac{1}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l\cosh\sqrt{\varepsilon}\,l(x+(l-x))\|g\|_{C[0,l]}\\ &= l\cosh\sqrt{\varepsilon}\,l\|g\|_{C[0,l]},\\ |v''(x)| &\leqslant \biggl[\varepsilon\frac{(\cosh\sqrt{\varepsilon}\,l)^2}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}l+1\biggr]\|g\|_{C[0,l]}.\end{aligned}$$

Thus we have

$$\|\mathbb{G}\| =: \gamma(\varepsilon) \leqslant l\biggl(\frac{(\cosh\sqrt{\varepsilon}\,l)^2}{\sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l}(1+\varepsilon) + \cosh\sqrt{\varepsilon}\,l\biggr) + 1 =: \widetilde{\gamma}(\varepsilon).$$

Therefore,

$$\widetilde{\gamma}(\varepsilon) \sim l\biggl(1+\frac{1+\varepsilon}{\varepsilon l}\biggr)+1 = l\biggl(1+\frac{1}{\varepsilon l}+\frac{1}{l}\biggr)+1 = 2+l+\frac{1}{\varepsilon} \quad \text{as} \quad \varepsilon \to +0.$$

Hence there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\gamma(\varepsilon) < 4 + 2l + \frac{2}{\varepsilon},$$

or

$$\varepsilon\gamma(\varepsilon) < (4 + 2l)\varepsilon + 2 < C \quad \text{for some } C > 0.$$

3. *The non-local boundary conditions.* The Green operator is given by the formula

$$\mathbb{G}g = \mathbb{G}_0 g + \mathbb{G}_1 g \equiv \int_0^l G_0(x,\xi)g(\xi)\,d\xi + v_1(x),$$

where

$$G_0(x,\xi) = \frac{-1}{\sqrt{\varepsilon}} \begin{cases} \sinh\sqrt{\varepsilon}\,x \cosh\sqrt{\varepsilon}\,\xi, & 0 \leqslant x \leqslant \xi \leqslant l, \\ \sinh\sqrt{\varepsilon}\,\xi \cosh\sqrt{\varepsilon}\,x, & 0 \leqslant \xi \leqslant x \leqslant l, \end{cases}$$

$$\begin{aligned} v_1(x) &= -\frac{\sinh\sqrt{\varepsilon}\,x}{\sqrt{\varepsilon}} \int_0^l \frac{\sqrt{\varepsilon}\,l \cosh\sqrt{\varepsilon}\,\xi - \cosh\sqrt{\varepsilon}\,l \sinh\sqrt{\varepsilon}\,\xi}{\sinh\sqrt{\varepsilon}\,l - \sqrt{\varepsilon}\,l} g(\xi)\,d\xi \\ &= -\frac{\sinh\sqrt{\varepsilon}\,x}{\sqrt{\varepsilon}} \frac{1}{\sinh\sqrt{\varepsilon}\,l - \sqrt{\varepsilon}\,l} \int_0^l \big(\sqrt{\varepsilon}\,l \cosh\sqrt{\varepsilon}\,\xi - \cosh\sqrt{\varepsilon}\,l \sinh\sqrt{\varepsilon}\,\xi\big) g(\xi)\,d\xi. \end{aligned}$$

Proceeding as in the previous cases, we easily verify that for sufficiently small $\varepsilon > 0$ the term $\mathbb{G}_0 g$ satisfies an ε-independent estimate. Note that

$$0 \leqslant \frac{\sinh\sqrt{\varepsilon}\,x}{\sqrt{\varepsilon}} \leqslant \frac{\sinh\sqrt{\varepsilon}\,l}{\sqrt{\varepsilon}} \to l, \qquad \varepsilon \to +0,$$

$$\sinh t = t + \frac{t^3}{6} + o(t^3) \quad\Longrightarrow\quad \frac{1}{\sinh t - t} = \frac{1}{t^3/6 + o(t^3)} = \frac{6}{t^3}\frac{1}{1 + o(1)} \sim \frac{6}{t^3},$$

whence

$$\frac{1}{\sinh\sqrt{\varepsilon}\,l - \sqrt{\varepsilon}\,l} \sim \frac{6}{l^3\varepsilon^{3/2}}, \qquad \varepsilon \to +0.$$

Estimating the integrand, we have

$$\begin{aligned} |\sqrt{\varepsilon}\,l \cosh\sqrt{\varepsilon}\,\xi - \cosh\sqrt{\varepsilon}\,l \sinh\sqrt{\varepsilon}\,\xi| &\leqslant \sqrt{\varepsilon}\,l \cosh\sqrt{\varepsilon}\,l + \cosh\sqrt{\varepsilon}\,l \sinh\sqrt{\varepsilon}\,l \\ &= \cosh\sqrt{\varepsilon}\,l(\sqrt{\varepsilon}\,l + \sinh\sqrt{\varepsilon}\,l) \sim 2\sqrt{\varepsilon}\,l. \end{aligned}$$

Hence the following bounds hold for all sufficiently small $\varepsilon > 0$:

$$\begin{aligned} |v_1(x)| &\leqslant 2l\frac{12}{l^3\varepsilon^{3/2}} 4\sqrt{\varepsilon}\,l \|g\|_{C[0,l]} = \frac{96}{l\varepsilon}\|g\|_{C[0,l]}, \\ |v_1'(x)| &\leqslant \cosh\sqrt{\varepsilon}\,l \frac{12}{l^3\varepsilon^{3/2}} 4\sqrt{\varepsilon}\,l \|g\|_{C[0,l]} = \frac{C_2}{\varepsilon}\|g\|_{C[0,l]}, \\ |v_1''(x)| &\leqslant \sqrt{\varepsilon}\,\sinh\sqrt{\varepsilon}\,l \frac{12}{l^3\varepsilon^{3/2}} 4\sqrt{\varepsilon}\,l \|g\|_{C[0,l]} = C_3\|g\|_{C[0,l]}. \end{aligned}$$

Thus,

$$\|\mathbb{G}_1\| \leqslant C_3 + \frac{C_4}{\varepsilon} \quad\Longrightarrow\quad \varepsilon\|\mathbb{G}_1\| \leqslant C_3\varepsilon + C_4 < C$$

for some $C > 0$ and all sufficiently small $\varepsilon > 0$. □

Remark 1. It is useful to note that the bound for the norm of the Green operator is uniform with respect to ε in the case of Dirichlet boundary conditions.

Lemma 2. *Let $[0,1] \ni x \mapsto z(x)$ be a continuous function. Then the operator*

$$Q\colon z(x) \mapsto |z(x)|^q, \quad Q\colon C[0,l] \to C[0,l], \quad q > 1,$$

is Fréchet continuously differentiable and

$$(Q'_f(z)h_1)(x) = h_1(x)q|z(x)|^{q-1} \operatorname{sgn} z(x).$$

Proof. Step 1. This operator is differentiable in the sense of Gâteaux:

$$\left(\frac{d}{d\zeta}|z(x)+\zeta h_1(x)|^q\right)\bigg|_{\zeta=0} = \big(h_1(x)q|z(x)+\zeta h_1(x)|^{q-1}\operatorname{sgn}(z(x)+\zeta h_1(x))\big)\big|_{\zeta=0}$$
$$= h_1(x)q|z(x)|^{q-1}\operatorname{sgn} z(x).$$

Step 2. We claim that the Gâteaux derivative is a continuous function of the parameter $z(x) \in C[0,l]$ at every point $z(x) \in C[0,l]$. Fix an arbitrary $z(x) \in C[0,l]$. It suffices to prove that

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, h_2(x) \in C[0,l] \quad \|h_2(x)\|_{C[0,l]} < \delta$$
$$\Longrightarrow \|q|z(x)+h_2(x)|^{q-1}\operatorname{sgn}(z(x)+h_2(x)) - q|z(x)|^{q-1}\operatorname{sgn} z(x)\|_{C[0,l]} < \varepsilon.$$

This is indeed true because if $\|h_2\|_{C[0,l]} \leqslant 1$, then

$$\forall\, x \in [0,l] \quad |z(x)+h_2(x)| \leqslant \|z\|_{C[0,l]} + 1$$

and the function

$$\eta \mapsto |\eta|^{q-1}\operatorname{sgn}\eta$$

is uniformly continuous on the closed interval $[-\|z(x)\|_{C[0,l]} - 1,\ \|z(x)\|_{C[0,l]} + 1]$ (since it is continuous everywhere on $\mathbb{R}$).

Step 3. It follows from the continuity of the Gâteaux derivative that Q is Fréchet differentiable and its Gâteaux and Fréchet derivatives coincide ([17], Theorem 1.3 on p. 14). Then the Fréchet derivative is also continuous. □

In what follows, $R > 0$ is arbitrary and will be specified below. Lemma 2 yields the following assertion.

Lemma 3. *The map*

$$\mathbb{Q}\colon w \mapsto \mathbb{G}(|w_x|^q)$$

is Fréchet continuously differentiable and the following bound holds in the ball $\|w\|_{Z^2} \leqslant 4R$ for $\varepsilon \in (0, \varepsilon_1)$:

$$\|\mathbb{Q}'_f(w)\| \leqslant \frac{C_1}{\varepsilon} q\,(4R)^{q-1}.$$

Proof. In view of (21)–(26), the map $\mathbb{Q}$ is the composite of the following operators:

$$Z^2 \xrightarrow{J_{21}} C^1[0,l] \xrightarrow{d/dx} C[0,l] \xrightarrow{Q} C[0,l] \xrightarrow{\mathbb{G}} Z^2,$$

of which only Q is non-linear and

$$\|J_{21}\| = \left\|\frac{d}{dx}\right\| = 1, \qquad \|\mathbb{G}\| \overset{\text{Lemma 1}}{\leqslant} \frac{C_1}{\varepsilon}, \quad \varepsilon \in (0, \varepsilon_1).$$

The result now follows immediately from Lemma 2. □

Lemma 4. *For* $\varepsilon \in (0,\varepsilon_2)$, $\varepsilon_2 = 1/(2R)$, *and all* $w(x) \in C[0,l]$ *with* $\|w\|_{C[0,l]} \leqslant 4R$ *we have*

$$\|\mathbb{F}'_f(w)\|_{C[0,l]\to C[0,l]} \leqslant 8R\varepsilon^2 =: C_2\varepsilon^2.$$

Proof. Clearly,

$$\big(\mathbb{F}'_f(w)h\big)(x) = \varepsilon(e^{\varepsilon w(x)} - 1)h(x) = \varepsilon e^{\varepsilon w(x)}h(x) - \varepsilon h(x). \tag{32}$$

For every $x \in [0,l]$, Maclaurin's formula with the Lagrange form of the remainder yields the following bound (with $\theta[x] \in [0,1]$):

$$\begin{aligned}|\varepsilon(e^{\varepsilon w(x)} - 1)| &= \left|\varepsilon\left(1 + \varepsilon w(x) + \frac{\varepsilon^2(\theta[x]w(x))^2}{2} - 1\right)\right| \\ &= \varepsilon^2\left|w(x) + \frac{\varepsilon\theta^2[x]w^2(x)}{2}\right| \leqslant \varepsilon^2|4R + 8R^2\varepsilon| \leqslant 8R\varepsilon^2\end{aligned}$$

for $\varepsilon \in (0,\varepsilon_2)$, $\varepsilon_2 = 1/(2R)$. Thus,

$$\|\mathbb{F}'_f(w)\|_{C[0,l]\to C[0,l]} = \sup_{x\in[0,l]} |\varepsilon(e^{\varepsilon w(x)} - 1)| \leqslant 8R\varepsilon^2$$

for $\|w\|_{C[0,l]} \leqslant 4R$, $\varepsilon \in (0,\varepsilon_2)$, $\varepsilon_2 = 1/(2R)$. □

Lemma 5. *The function*

$$\mathbb{F}'_f(w)\colon C[0,l] \to \mathcal{L}(C[0,l], C[0,l])$$

of the parameter w *is Fréchet continuously differentiable and its derivative is equal to*

$$\mathbb{F}''_f(w)(h_1,h_2) = \varepsilon^2 e^{\varepsilon w(x)}h_1(x)h_2(x). \tag{33}$$

Proof. Since the term $\varepsilon h(x)$ in (32) is independent of $w(x)$, it suffices to consider the operator

$$h_1(x) \mapsto \varepsilon e^{\varepsilon w(x)}h_1(x).$$

We have

$$\begin{aligned}e^{\varepsilon w(x)+\varepsilon h_2(x)} - e^{\varepsilon w(x)} - \varepsilon e^{\varepsilon w(x)}h_2(x) &= e^{\varepsilon w(x)}\big(e^{\varepsilon h_2(x)} - 1 - \varepsilon h_2(x)\big) \\ = \frac{e^{\varepsilon w(x)}}{2}\big(\theta_2[x]\varepsilon h_2(x)\big)^2 &= o(\|h_2\|_{C[0,l]}), \qquad \theta_2[x] \in [0,1].\end{aligned}$$

This proves (33). The continuous dependence of this operator on $w(x)$ can easily be proved. □

Lemma 6. *When*

$$\varepsilon \in \big(0, \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)\big), \qquad \varepsilon_3 = \frac{1}{12C_1C_2},$$

the operator $\mathbb{H}$ is twice Fréchet continuously differentiable and we have the following bound for its Fréchet derivative: for all $w \in Z^2$ with $\|w\|_{Z^2} \leqslant 4R$,

$$\|\mathbb{H}'_f(w)\| \leqslant \frac{1}{12}.$$

Proof. This follows from Lemmas 1, 4, 5 and the chain rule for Fréchet derivatives. □

3.3. Main theorems. Thus we arrive at the following equation in Z^2 (see (29)):

$$u(t) = D(u)(t) \equiv (u_0 - \mathbb{H}(u_0)) + (u_1 - \mathbb{H}'_f(u_0)u_1 - \mathbb{G}(|u_{0,x}|^q))t \\ + \mathbb{H}(u) + \int_0^t \mathbb{G}(|u_x|^q)(s)\,ds - \int_0^t (t-s)(u(s) + \varepsilon(\mathbb{G}u)(s))\,ds. \qquad (34)$$

Theorem 1. *There is an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ one can find a $T(\varepsilon)$ such that the equation (34) has the unique solution $u(t) \in C([0, T(\varepsilon)], Z^2)$ on the closed interval $[0, T(\varepsilon)]$.*

Proof. We introduce the following notation:

$$y_0 = u_0 - \mathbb{H}(u_0), \qquad y_1 = u_1 - \mathbb{H}'_f(u_0)u_1 - \mathbb{G}(|u_{0,x}|^q), \qquad y_2(t) = y_0 + y_1 t,$$
$$I_1(u) = \int_0^t \mathbb{G}(|u_x|^q)(s)\,ds, \qquad I_2(u) = \int_0^t (t-s)\big(u(s) + \varepsilon(\mathbb{G}u)(s)\big)\,ds.$$

Clearly, $y_0 \in Z^2$, $y_1 t \in Z^2$ for every t and $y_2(t) \equiv y_0 + y_1 t \in C^\infty([0,T], Z^2)$ for all $T > 0$.

We use the contraction mapping method. Our plan is as follows.

1) Choose $R = \max(1, \|y_0\|_{Z^2})$.

2) Choose T': $\|y_0\|_{Z^2} + T'\|y_1\|_{Z^2} \leqslant 2R$. Then

$$\forall\, T \in (0, T'] \quad \|y_0 + y_1 t\|_{C([0,T],Z^2)} \leqslant 2R.$$

3) Choose a small $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the operator $\mathbb{H}(w)$ is a contraction with contraction constant $\alpha \leqslant 1/12$ in the ball $\|w\|_{Z^2} \leqslant 4R$. (This is possible by Lemma 6.)

4) Choose a small $T(\varepsilon) \leqslant T'$ such that $I_1(u)$ and $I_2(u)$ are contractions in

$$B_{4R}(0) \equiv \big\{u(t) \in C([0, T(\varepsilon)], Z^2) \;\big|\; \forall\, t \in [0, T(\varepsilon)]\ \|u(t)\|_{Z^2} \leqslant 4R\big\}$$

with contraction constant $\alpha \leqslant 1/12$. (The quantity $T(\varepsilon)$ is actually independent of ε for I_2 in each of the Problems A, B, C and for I_1 in Problem A.) A way of performing such a choice will be described below.

Then the following argument shows that the operator $D(\cdot)$ on the right-hand side maps the ball $B_{4R}(0)$ to itself and is a contraction mapping with contraction constant $\alpha \leqslant 1/4$ on this ball.

Indeed, by 3) and 4), for all $u_1(t), u_2(t) \in B_{4R}(0)$ we have

$$\begin{aligned}&\big\|\big(\mathbb{H}(u_1)+I_1(u_1)-I_2(u_1)\big)-\big(\mathbb{H}(u_2)+I_1(u_2)-I_2(u_2)\big)\big\|_{C([0,T],Z^2)}\\&\qquad\leqslant \frac{1}{4}\|u_1-u_2\|_{C([0,T],Z^2)}.\end{aligned}$$

Since $\mathbb{H}(0)=I_1(0)=I_2(0)=0$, the following assertions hold.

1) For every $u(t)\in B_{4R}(0)$ we have

$$\begin{aligned}&\|\mathbb{H}(u)+I_1(u)-I_2(u)\|_{C([0,T],Z^2)}\\&\qquad=\big\|\big(\mathbb{H}(u)+I_1(u)-I_2(u)\big)-\big(\mathbb{H}(0)+I_1(0)-I_2(0)\big)\big\|_{C([0,T],Z^2)}\\&\qquad\leqslant \frac{1}{4}\|u-0\|_{C([0,T],Z^2)}=\frac{1}{4}\|u\|_{C([0,T],Z^2)}\leqslant\frac{1}{4}\cdot 4R=R.\end{aligned}$$

Therefore,

$$\text{for every } u(t)\in B_{4R}(0) \text{ we have } D(u)\in B_R(y_2)\subset B_{3R}(0)\subset B_{4R}(0). \tag{35}$$

2) For all $u_1(t), u_2(t)\in B_{4R}(0)$ we have

$$\|D(u_1)-D(u_2)\|_{C([0,T],Z^2)}\leqslant\frac{1}{4}\|u_1-u_2\|_{C([0,T],Z^2)}.$$

Since $B_R(y_2)\subset B_{4R}(0)$, it follows that for all $u_1,u_2\in B_R(y_2)$ we have

$$\|D(u_1)-D(u_2)\|_{C([0,T],Z^2)}\leqslant\frac{1}{4}\|u_1-u_2\|_{C([0,T],Z^2)}.$$

We now explain how to make the choice of $T(\varepsilon)$ in 4). We separately consider all terms of $D(u)$ not contained in $y_2(t)$: $\mathbb{H}(u)$, $I_1(u)$, $I_2(u)$.

By Lemma 6 the following bound holds for $\varepsilon\in(0,\min(\varepsilon_1,\varepsilon_2,\varepsilon_3))$ and for all $u_1,u_2\in C([0,T],Z^2)$ with $\|u_i\|_{C([0,T],Z^2)}\leqslant 4R$, $i=1,2$, where $T>0$ is arbitrary:

$$\|\mathbb{H}(u_1)-\mathbb{H}(u_2)\|_{C([0,T],Z^2)}\leqslant\frac{1}{12}\|u_1-u_2\|_{C([0,T],Z^2)}.$$

We now consider $I_1(u)$. Recall that $q>1$. By Lemma 3, for all $\varepsilon\in(0,\varepsilon_1)$ and $\|w_k\|_{Z^2}\leqslant 4R$, $k=1,2$, we have

$$\|\mathbb{G}(|w_{1,x}|^q)-\mathbb{G}(|w_{2,x}|^q)\|_{Z^2}\leqslant\frac{C_1}{\varepsilon}q(4R)^{q-1}\|w_1-w_2\|_{Z^2}.$$

Choose $T_1(\varepsilon)=1/(12(C_1/\varepsilon)q(4R)^{q-1})$. Then for every $T\leqslant T_1(\varepsilon)$ we have

$$\|I_1(u_1(t))-I_1(u_2(t))\|_{C([0,T],Z^2)}\leqslant\frac{1}{12}\|u_1(t)-u_2(t)\|_{C([0,T],Z^2)}$$

for all $u_1(t),u_2(t)\in C([0,T],Z^2)$ with $\|u_i(t)\|_{C([0,T],Z^2)}\leqslant 4R$, $i=1,2$.

Finally, we consider $I_2(u)$. We have

$$u+\varepsilon\mathbb{G}(u)=(I+\varepsilon\mathbb{G}\mathbb{J}_{20})u,\quad \|I+\varepsilon\mathbb{G}\mathbb{J}_{20}\|\leqslant 1+\varepsilon\frac{C_1}{\varepsilon}=1+C_1,\qquad \varepsilon\in(0,\varepsilon_1).$$

Then, for all $T > 0$,

$$\left\| \int_0^t (t-s)(u(s) + \varepsilon(\mathbb{G}u)(s))\, ds \right\|_{C([0,T],Z^2)} \leqslant T^2 \|u + \varepsilon \mathbb{G}u\|_{C([0,T],Z^2)}.$$

Thus for every $T \leqslant T_2 = \sqrt{1/(12(1+C_1))}$ we have

$$\|I_2(u_1(t)) - I_2(u_2(t))\|_{C([0,T],Z^2)} \leqslant \frac{1}{12} \|u_1(t) - u_2(t)\|_{C([0,T],Z^2)}.$$

As a result, we put $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, fix an arbitrary $\varepsilon \in (0, \varepsilon_0)$ and take

$$T(\varepsilon) = \min(T', T_1(\varepsilon), T_2).$$

By what was said above, the operator $D(u)$ on the right-hand side of (34) maps the ball $B_{4R}(0)$ to itself in every Banach space $C([0, \widetilde{T}], Z^2)$ with $\widetilde{T} \leqslant T(\varepsilon)$ and is a contraction map on this ball. Hence, in every Banach space $C([0, \widetilde{T}], Z^2)$ with $\widetilde{T} \leqslant T(\varepsilon)$, the integral equation (34) has a unique solution among all functions of norm not exceeding $4R$.

We can now prove the uniqueness of solutions of (34) in $C([0, T(\varepsilon)], Z^2)$ (not only in the ball $B_{4R}(0)$). Indeed, by (35), every solution of (34) in $C([0, T(\varepsilon)], Z^2)$ that lies in $B_{4R}(0)$ is actually contained in $B_{3R}(0)$. On the other hand, assume that there is a solution $\widetilde{u}(t) \in C([0, T(\varepsilon)], Z^2)$ different from $u(t)$. Then it does not lie in the ball $B_{4R}(0)$ (since the uniqueness of solutions in this ball has already been established). Then the solution $\widetilde{u}(t)$ leaves $B_{4R}(0)$ for some $t = t_2 < T(\varepsilon)$ and leaves $B_{3R}(0)$ for some $t = t_1$ and we have the strict inequality $t_1 < t_2$ because $\widetilde{u}(t)$ is continuous. Hence the restriction of $\widetilde{u}(t)$ to $[0, t_2]$ is a solution in $C([0, t_2], Z^2)$ lying in $B_{4R}(0)$, but not in $B_{3R}(0)$. The resulting contradiction proves the uniqueness of solutions. □

We shall use the following theorem.

Theorem 2 (see [18], Theorem 12.3.3 on p. 651). *Let F be a continuously differentiable map from an open ball $U = B_r(x_0)$ in a Banach space X to a Banach space Y. Suppose that $\Lambda := F'_f(x_0)$ is a one-to-one map of X onto Y. Then F is a one-to-one map of some neighbourhood V of x_0 onto some neighbourhood W of $F(x_0)$. Moreover, the map $G := F^{-1} \colon W \to V$ is continuously differentiable and we have*

$$G'_f(y) = \bigl(F'_f(F^{-1}(y))\bigr)^{-1}, \qquad y \in W.$$

Then the following theorem can be proved.

Theorem 3. *Let $u(t) \in C([0, T(\varepsilon)], Z^2)$ be the solution whose existence and uniqueness were established in Theorem 1. Then $u(t)$ possesses the following smoothness property*: $u(t) \in C^2([0, T_0(\varepsilon)], Z^2)$, $T_0(\varepsilon) \leqslant T(\varepsilon)$.

Proof. Step 1. Consider again the equation (34) and rewrite it in the form

$$u(t) - \mathbb{H}(u) = v(t), \tag{36}$$

where

$$v(t) = (u_0 - \mathbb{H}(u_0)) + \big(u_1 - \mathbb{H}'_f(u_0)u_1 - \mathbb{G}(|u_{0,x}|^q)\big)t + I_1(u) - I_2(u),$$
$$I_1(u) = \int_0^t \mathbb{G}(|u_x|^q)(s)\,ds, \qquad I_2(u) = \int_0^t (t-s)\big(u(s) + \varepsilon(\mathbb{G}u)(s)\big)\,ds.$$

We know that $u(t) \in C([0,T(\varepsilon)],Z^2)$. Hence $u(t)+\varepsilon(\mathbb{G}u)(t) \in C([0,T(\varepsilon)],Z^2)$ and it follows from the general properties of integrals with variable upper limits (depending on a parameter or not) that

$$I_1(u)(t) \in C^1([0,T(\varepsilon)],Z^2), \qquad I_2(u)(t) \in C^1([0,T(\varepsilon)],Z^2).$$

Since we obviously have

$$(u_0 - \mathbb{H}(u_0)) + \big(u_1 - \mathbb{H}'_f(u_0)u_1 - \mathbb{G}(|u_{0,x}|^q)\big)t \in C^2([0,T(\varepsilon)],Z^2),$$

it follows that $v(t) \in C^1([0,T(\varepsilon)],Z^2)$.

Step 2. Using Theorem 2, we establish that the operator

$$w \mapsto w - \mathbb{H}(w) =: \mathbb{K}(w)$$

has a continuously differentiable inverse in some neighbourhood Ω of y_0. Indeed, since $\|u(t)\|_{Z^2} \in B_{4R}(0)$ for all $t \in [0,T(\varepsilon)]$, it follows from Lemma 6 that

$$\|\mathbb{H}'_f(u(t))\| \leqslant \frac{1}{12}, \qquad t \in [0,T(\varepsilon)], \tag{37}$$

and $\mathbb{H}'_f(w)$ is continuous. Hence for all $t \in [0,T(\varepsilon)]$ the operator

$$\mathbb{K}'_f(u(t)) \equiv I - \mathbb{H}'_f(u(t))$$

is invertible. In particular, the hypotheses of Theorem 2 hold at y_0. Since $u(t)$ is continuous, there is a $T_0(\varepsilon) \leqslant T(\varepsilon)$ such that $u(t) \in \Omega$ for $t \in [0,T_0(\varepsilon)]$. Therefore,

$$u(t) = \mathbb{K}^{-1}(v(t)) \in C^1([0,T_0(\varepsilon)],Z^2).$$

But then $I_2(u)(t) \in C^2([0,T_0(\varepsilon)];Z^2)$ and Lemma 3 yields that $I_1(u)(t) \in C^2([0,T_0(\varepsilon)];Z^2)$, whence

$$v(t) \in C^2([0,T(\varepsilon)],Z^2).$$

Step 3. Differentiating (36), we obtain that

$$\big(I - \mathbb{H}'_f(u)\big)u'(t) = v'(t).$$

It was shown in Step 2 that the operator $I - \mathbb{H}'_f(u(t))$ (depending on the parameter $u(t)$) has an inverse for every $t \in [0,T_0(\varepsilon)]$:

$$u'(t) = \big(I - \mathbb{H}'_f(u(t))\big)^{-1} v'(t).$$

By (37), the operator on the right-hand side can be represented by a Neumann series. Therefore it is continuously differentiable with respect to $t \in [0,T_0(\varepsilon)]$. Thus $u(t) \in C^2([0,T_0(\varepsilon)],Z^2)$. $\square$

§ 4. Blow-up of solutions

4.1. The method of test functions for Problems B and C. Blow-up of solutions of Problem B can be proved using a very simple version of the method of test functions. Namely, integrating the equation (15) with respect to x from 0 to l and using the boundary conditions (18), we obtain that

$$-\frac{d^2}{dt^2}\int_0^l e^{\varepsilon u(x,t)}\,dx = \frac{d}{dt}\int_0^l |u_x(x,t)|^q\,dx,$$

or

$$\frac{d^2}{dt^2}\int_0^l e^{\varepsilon u(x,t)}\,dx + \frac{d}{dt}\int_0^l |u_x(x,t)|^q\,dx = 0.$$

Integrate with respect to t:

$$\frac{d}{dt}\int_0^l e^{\varepsilon u(x,t)}\,dx - \biggl(\frac{d}{dt}\int_0^l e^{\varepsilon u(x,t)}\,dx\biggr)\bigg|_{t=0} + \int_0^l |u_x(x,t)|^q\,dx - \int_0^l |u_x(x,0)|^q\,dx = 0,$$

or

$$\frac{d}{dt}\int_0^l e^{\varepsilon u(x,t)}\,dx - \varepsilon\int_0^l e^{\varepsilon u_0(x)}u_1(x)\,dx + \int_0^l |u_x(x,t)|^q\,dx - \int_0^l |u_{0,x}(x)|^q\,dx = 0.$$

Integrating again with respect to t, we obtain

$$\begin{aligned}\int_0^l e^{\varepsilon u(x,t)}\,dx - \int_0^l e^{\varepsilon u_0(x)}\,dx - \varepsilon t\int_0^l e^{\varepsilon u_0(x)}u_1(x)\,dx - t\int_0^l |u_{0,x}(x)|^q\,dx \\ + \int_0^t ds\int_0^l |u_x(x,s)|^q\,dx = 0.\end{aligned}$$

We group the terms as follows:

$$\begin{aligned}&\int_0^l e^{\varepsilon u(x,t)}\,dx + \int_0^t ds\int_0^l |u_x(x,s)|^q\,dx \\ &\qquad = \int_0^l e^{\varepsilon u_0(x)}\,dx + t\biggl(\varepsilon\int_0^l e^{\varepsilon u_0(x)}u_1(x)\,dx + \int_0^l |u_{0,x}(x)|^q\,dx\biggr).\end{aligned}$$

Since the terms on the left-hand side are non-negative, it follows that

$$\int_0^l e^{\varepsilon u_0(x)}\,dx + t\biggl(\varepsilon\int_0^l e^{\varepsilon u_0(x)}u_1(x)\,dx + \int_0^l |u_{0,x}(x)|^q\,dx\biggr) \geqslant 0.$$

Therefore, if the term in brackets is negative, then Problem B cannot be globally (in time) soluble and we have the following upper bound for the blow-up time:

$$T_{\mathrm{bl}} = -\frac{\int_0^l e^{\varepsilon u_0(x)}\,dx}{\varepsilon\int_0^l e^{\varepsilon u_0(x)}u_1(x)\,dx + \int_0^l |u_{0,x}(x)|^q\,dx}. \tag{38}$$

The problem with non-local conditions (19) can be studied in a similar way replacing the test function 1 by the test function $l-x$. In view of the boundary conditions (19) we have

$$\int_0^l (l-x)u_{xx}(x,t)\,dx = (l-x)u_x(x,t)\big|_{x=0}^{x=l} + \int_0^l u_x(x,t)\,dx$$
$$= -lu_x(0,t) + u(l,t) - u(0,t) = 0.$$

Therefore, multiplying (15) by $l-x$ and integrating over $[0,l]$, we obtain that

$$-\frac{d^2}{dt^2}\int_0^l (l-x)e^{\varepsilon u(x,t)}\,dx = \frac{d}{dt}\int_0^l (l-x)|u_x(x,t)|^q\,dx,$$

which can be rewritten (in the same way as above) in the form

$$\int_0^l (l-x)e^{\varepsilon u(x,t)}\,dx + \int_0^t ds \int_0^l (l-x)|u_x(x,s)|^q\,dx$$
$$= \int_0^l (l-x)e^{\varepsilon u_0(x)}\,dx + t\biggl(\varepsilon\int_0^l (l-x)e^{\varepsilon u_0(x)}u_1(x)\,dx + \int_0^l (l-x)|u_{0,x}(x)|^q\,dx\biggr).$$

Under the condition

$$\varepsilon\int_0^l (l-x)e^{\varepsilon u_0(x)}u_1(x)\,dx + \int_0^l (l-x)|u_{0,x}(x)|^q\,dx < 0$$

we see that solutions of Problem C cannot exist for all $t > 0$ and the following upper bound for the blow-up time holds:

$$T_{\mathrm{bl}} = -\frac{\int_0^l (l-x)e^{\varepsilon u_0(x)}\,dx}{\varepsilon\int_0^l (l-x)e^{\varepsilon u_0(x)}u_1(x)\,dx + \int_0^l (l-x)|u_{0,x}(x)|^q\,dx}.$$

4.2. The method of test functions (a universal version). In this subsection we use the non-linear capacity method of Pokhozhaev and Mitidieri [4] to prove that for any boundary conditions there are initial data (16) for which the problem is not globally (in time) soluble. Since the method works in the multi-dimensional case (not only the one-dimensional case considered in this paper), we shall write Δu instead of u_{xx} and regard x as a multi-dimensional variable. In view of these changes we consider in this subsection the general equation (14) instead of (15):

$$\frac{\partial^2}{\partial t^2}(\Delta u - e^{\varepsilon u}) + \Delta u = \frac{\partial}{\partial t}|\nabla u|^q. \tag{39}$$

We introduce a test function

$$\varphi(x,t) = \varphi_1(x)\varphi_2(t),$$

where

$$\varphi_1(x) \in C_0^2(\Omega), \qquad \varphi_1(x) \geqslant 0, \tag{40}$$

and $\varphi_2(t)$ is of the form

$$\varphi_2(t) = \left(1 - \frac{t}{T}\right)^{\lambda}. \tag{41}$$

The parameters $T > 0$ and $\lambda > 2$ will be specified below.

For the reader's convenience we present some elementary relations that will be used below. These include the following expressions for the derivatives of $\varphi_2(t)$ and their values at the endpoints:

$$\begin{aligned}
&\varphi_2(t) = \left(1 - \frac{t}{T}\right)^{\lambda}, && \varphi_2(0) = 1, && \varphi_2(T) = 0,\\
&\varphi_2'(t) = -\frac{\lambda}{T}\left(1 - \frac{t}{T}\right)^{\lambda-1}, && \varphi_2'(0) = -\frac{\lambda}{T}, && \varphi_2'(T) = 0,\\
&\varphi_2''(t) = \frac{\lambda(\lambda-1)}{T^2}\left(1 - \frac{t}{T}\right)^{\lambda-2}, && \varphi_2''(0) = \frac{\lambda(\lambda-1)}{T^2}, && \varphi_2''(T) = 0,
\end{aligned} \tag{42}$$

and integration-by-parts formulae involving $\varphi_2(t)$ (these follow from (42)):

$$\int_0^T dt\, v'(t)\varphi_2(t) = -v(0) + \frac{\lambda}{T}\int_0^T dt\, v(t)\left(1 - \frac{t}{T}\right)^{\lambda-1}, \tag{43}$$

$$\int_0^T dt\, v''(t)\varphi_2(t) = -v'(0) - \frac{\lambda}{T}v(0) + \frac{\lambda(\lambda-1)}{T^2}\int_0^T dt\, v(t)\left(1 - \frac{t}{T}\right)^{\lambda-2}. \tag{44}$$

Multiplying both sides of (39) by the test function $\varphi(x,t)$, we have

$$\left(\frac{\partial^2}{\partial t^2}(\Delta u) + \Delta u\right)\varphi(x,t) = \left(\frac{\partial}{\partial t}|\nabla u|^q + \frac{\partial^2}{\partial t^2}e^{\varepsilon u}\right)\varphi(x,t).$$

Integrating this relation over $[0,T]\times\Omega$, we obtain that

$$\begin{aligned}
&\int_0^T dt\,\varphi_2(t)\frac{d^2}{dt^2}\int_\Omega dx\,\varphi_1(x)\Delta u + \int_0^T dt\,\varphi_2(t)\int_\Omega dx\,\varphi_1(x)\Delta u\\
&\quad= \int_0^T dt\,\varphi_2(t)\,\frac{d}{dt}\int_\Omega dx\,\varphi_1(x)|\nabla u|^q + \int_0^T dt\,\varphi_2(t)\frac{d^2}{dt^2}\int_\Omega dx\,\varphi_1(x)e^{\varepsilon u(x,t)}.
\end{aligned}$$

Integrating by parts and using (43), (44), we arrive at

$$\begin{aligned}
&-\int_\Omega dx\,\varphi_1(x)\Delta u_1(x) - \frac{\lambda}{T}\int_\Omega dx\,\varphi_1(x)\Delta u_0(x)\\
&\qquad+ \frac{\lambda(\lambda-1)}{T^2}\int_0^T dt\left(1 - \frac{t}{T}\right)^{\lambda-2}\int_\Omega dx\,\varphi_1(x)\Delta u(x,t)\\
&\qquad+ \int_0^T dt\left(1 - \frac{t}{T}\right)^{\lambda}\int_\Omega dx\,\varphi_1(x)\Delta u(x,t)
\end{aligned}$$

$$= -\int_\Omega dx\,\varphi_1(x)|\nabla u_0(x)|^q + \frac{\lambda}{T}\int_0^T dt\left(1-\frac{t}{T}\right)^{\lambda-1}\int_\Omega dx\,\varphi_1(x)|\nabla u(x,t)|^q$$
$$-\varepsilon\int_\Omega dx\,\varphi_1(x)e^{\varepsilon u_0(x)}u_1(x) - \frac{\lambda}{T}\int_\Omega dx\,\varphi_1(x)e^{\varepsilon u_0(x)}$$
$$+\frac{\lambda(\lambda-1)}{T^2}\int_0^T dt\left(1-\frac{t}{T}\right)^{\lambda-2}\int_\Omega dx\,\varphi_1(x)e^{\varepsilon u(x,t)}.$$

Collect all the u-independent terms on one side and use Green's formula:

$$-\int_\Omega dx\,\Delta u_1\varphi_1 - \frac{\lambda}{T}\int_\Omega dx\,\Delta u_0\varphi_1 + \int_\Omega dx\,\varphi_1|\nabla u_0|^q + \varepsilon\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0}u_1$$
$$+\frac{\lambda}{T}\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0} - \frac{\lambda(\lambda-1)}{T^2}\int_0^T dt\left(1-\frac{t}{T}\right)^{\lambda-2}\int_\Omega dx\,(\nabla\varphi_1,\nabla u)$$
$$-\int_0^T dt\left(1-\frac{t}{T}\right)^{\lambda}\int_\Omega dx\,(\nabla\varphi_1,\nabla u)$$
$$= \frac{\lambda}{T}\int_0^T dt\left(1-\frac{t}{T}\right)^{\lambda-1}\int_\Omega \varphi_1|\nabla u|^q\,dx$$
$$+\frac{\lambda(\lambda-1)}{T^2}\int_0^T dt\left(1-\frac{t}{T}\right)^{\lambda-2}\int_\Omega dx\,\varphi_1 e^{\varepsilon u}.$$

We now pass from the repeated integral to a double interval over $[0,T]\times\Omega$ and take into account that $\varphi_1(x)$ is different from 0 only on the set $\Omega_1 \Subset \Omega$:

$$-\int_\Omega dx\,\Delta u_1\varphi_1 - \frac{\lambda}{T}\int_\Omega dx\,\Delta u_0\varphi_1 + \int_\Omega dx\,\varphi_1|\nabla u_0|^q + \varepsilon\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0}u_1$$
$$+\frac{\lambda}{T}\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0} - \frac{\lambda(\lambda-1)}{T^2}\iint_{[0,T]\times\Omega_1} dt\,dx\left(1-\frac{t}{T}\right)^{\lambda-2}(\nabla\varphi_1,\nabla u)$$
$$-\iint_{[0,T]\times\Omega_1} dt\,dx\left(1-\frac{t}{T}\right)^{\lambda}(\nabla\varphi_1,\nabla u)$$
$$= \frac{\lambda}{T}\iint_{[0,T]\times\Omega_1} dt\,dx\left(1-\frac{t}{T}\right)^{\lambda-1}|\nabla u|^q\varphi_1$$
$$+\frac{\lambda(\lambda-1)}{T^2}\iint_{[0,T]\times\Omega_1} dt\,dx\left(1-\frac{t}{T}\right)^{\lambda-2}\varphi_1 e^{\varepsilon u(x,t)}. \tag{45}$$

We use 'Young's inequality with ε'

$$ab \leqslant \varepsilon^q\frac{a^q}{q} + \frac{1}{\varepsilon^{q'}}\frac{b^{q'}}{q'}, \qquad \frac{1}{q}+\frac{1}{q'} = 1$$

to estimate the terms on the left-hand side above. With q as in (15), we have

$$\begin{aligned}\left|(\nabla\varphi_1,\nabla u)\left(1-\frac{t}{T}\right)^{\lambda-2}\right| &\leqslant |\nabla\varphi_1|\cdot|\nabla u|\left(1-\frac{t}{T}\right)^{\lambda-2}\\ &= |\nabla\varphi_1|\cdot|\nabla u|\left(1-\frac{t}{T}\right)^{(\lambda-1)/q}\left(1-\frac{t}{T}\right)^{\lambda-2-(\lambda-1)/q}\varphi_1^{1/q}\varphi_1^{-1/q}\\ &\leqslant \frac{\varepsilon_1^q}{q}|\nabla u|^q\left(1-\frac{t}{T}\right)^{\lambda-1}\varphi_1+\frac{1}{\varepsilon_1^{q'}}\frac{1}{q'}|\nabla\varphi_1|^{q'}\left(1-\frac{t}{T}\right)^{(\lambda-2-(\lambda-1)/q)q'}\varphi_1^{-q'/q},\end{aligned} \tag{46}$$

$$\begin{aligned}\left|(\nabla\varphi_1,\nabla u)\left(1-\frac{t}{T}\right)^{\lambda}\right| &\leqslant |\nabla\varphi_1|\cdot|\nabla u|\left(1-\frac{t}{T}\right)^{\lambda}\\ &= |\nabla\varphi_1|\cdot|\nabla u|\left(1-\frac{t}{T}\right)^{(\lambda-1)/q}\left(1-\frac{t}{T}\right)^{\lambda-(\lambda-1)/q}\varphi_1^{1/q}\varphi_1^{-1/q}\\ &\leqslant \frac{\varepsilon_2^q}{q}|\nabla u|^q\left(1-\frac{t}{T}\right)^{\lambda-1}\varphi_1+\frac{1}{\varepsilon_2^{q'}}\frac{1}{q'}|\nabla\varphi_1|^{q'}\left(1-\frac{t}{T}\right)^{(\lambda-(\lambda-1)/q)q'}\varphi_1^{-q'/q}.\end{aligned} \tag{47}$$

These inequalities hold throughout the set where $\varphi_1(x) > 0$. However, a calculation in [4] shows that they can be extended by continuity to the closure of this set, that is, to the support of φ_1. We fix the set Ω_1 to be equal to this support. Then, in view of (46) and (47), we obtain from (45) that

$$\begin{aligned}&-\int_\Omega dx\,\Delta u_1\varphi_1-\frac{\lambda}{T}\int_\Omega dx\,\Delta u_0\varphi_1+\int_\Omega dx\,\varphi_1|\nabla u_0|^q+\varepsilon\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0}u_1\\ &\qquad+\frac{\lambda}{T}\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0}+\frac{\varepsilon_1^q}{q}\frac{\lambda(\lambda-1)}{T^2}\iint_{[0,T]\times\Omega_1}dt\,dx\,|\nabla u|^q\left(1-\frac{t}{T}\right)^{\lambda-1}\varphi_1\\ &\qquad+\frac{1}{\varepsilon_1^{q'}}\frac{1}{q'}\frac{\lambda(\lambda-1)}{T^2}\iint_{[0,T]\times\Omega_1}dt\,dx\,|\nabla\varphi_1|^{q'}\left(1-\frac{t}{T}\right)^{(\lambda-2-(\lambda-1)/q)q'}\varphi_1^{-q'/q}\\ &\qquad+\frac{\varepsilon_2^q}{q}\iint_{[0,T]\times\Omega_1}dt\,dx\,|\nabla u|^q\left(1-\frac{t}{T}\right)^{\lambda-1}\varphi_1\\ &\qquad+\frac{1}{\varepsilon_2^{q'}}\frac{1}{q'}\iint_{[0,T]\times\Omega_1}dt\,dx\,|\nabla\varphi_1|^{q'}\left(1-\frac{t}{T}\right)^{(\lambda-(\lambda-1)/q)q'}\varphi_1^{-q'/q}\\ &\quad\geqslant\frac{\lambda}{T}\iint_{[0,T]\times\Omega_1}dt\,dx\left(1-\frac{t}{T}\right)^{\lambda-1}|\nabla u|^q\varphi_1\\ &\qquad+\frac{\lambda(\lambda-1)}{T^2}\iint_{[0,T]\times\Omega_1}dt\,dx\left(1-\frac{t}{T}\right)^{\lambda-2}\varphi_1 e^{\varepsilon u}.\end{aligned}$$

Collecting the homogeneous terms in the last inequality, we have

$$
\begin{aligned}
&-\int_\Omega dx\,\Delta u_1\varphi_1-\frac{\lambda}{T}\int_\Omega dx\,\Delta u_0\varphi_1+\int_\Omega dx\,\varphi_1|\nabla u_0|^q\\
&\qquad+\varepsilon\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0}u_1+\frac{\lambda}{T}\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0}\\
&\qquad+\frac{1}{\varepsilon_1^{q'}}\,\frac{1}{q'}\,\frac{\lambda(\lambda-1)}{T^2}\iint_{[0,T]\times\Omega_1}dt\,dx\,|\nabla\varphi_1|^{q'}\biggl(1-\frac{t}{T}\biggr)^{(\lambda-2-(\lambda-1)/q)q'}\varphi_1^{-q'/q}\\
&\qquad+\frac{1}{\varepsilon_2^{q'}}\,\frac{1}{q'}\iint_{[0,T]\times\Omega_1}dt\,dx\,|\nabla\varphi_1|^{q'}\biggl(1-\frac{t}{T}\biggr)^{(\lambda-(\lambda-1)/q)q'}\varphi_1^{-q'/q}\\
&\quad\geqslant\biggl(\frac{\lambda}{T}-\frac{\varepsilon_1^q}{q}\,\frac{\lambda(\lambda-1)}{T^2}-\frac{\varepsilon_2^q}{q}\biggr)\iint_{[0,T]\times\Omega_1}dt\,dx\,\biggl(1-\frac{t}{T}\biggr)^{\lambda-1}|\nabla u|^q\varphi_1\\
&\qquad+\frac{\lambda(\lambda-1)}{T^2}\iint_{[0,T]\times\Omega_1}dt\,dx\,\biggl(1-\frac{t}{T}\biggr)^{\lambda-2}\varphi_1 e^{\varepsilon u}. \qquad (48)
\end{aligned}
$$

Note that the right-hand side is positive for sufficiently small ε_1, ε_2 and $\varphi_1 \not\equiv 0$. All the terms on the left-hand side depend only on the initial data and test functions, but not on the solution. We separate those terms on the left-hand side that depend only on the initial data:

$$
\int_\Omega dx\,\biggl(-\frac{\lambda}{T}\Delta u_0+|\nabla u_0|^q+\frac{\lambda}{T}e^{\varepsilon u_0}\biggr)\varphi_1+\int_\Omega dx\,(-\Delta u_1+\varepsilon u_1 e^{\varepsilon u_0})\varphi_1
$$

and, using Green's formula, rewrite them as

$$
\int_\Omega dx\,\biggl(-\frac{\lambda}{T}\Delta u_0+|\nabla u_0|^q+\frac{\lambda}{T}e^{\varepsilon u_0}\biggr)\varphi_1+\int_\Omega dx\,(-\Delta \varphi_1+\varepsilon \varphi_1 e^{\varepsilon u_0})u_1.
$$

Thus, putting

$$
u_1(x)=-r\bigl(-\Delta\varphi_1(x)+\varepsilon\varphi_1(x)e^{\varepsilon u_0(x)}\bigr) \qquad (49)
$$

and choosing r sufficiently large, we obtain that the left-hand side of (48) is negative or equal to zero and, therefore, (48) does not hold. It follows that the solution with the chosen initial data cannot exist on the whole interval $[0, T]$.

§ 5. Numerical diagnostics of the blow-up of solutions

In this section we discuss in detail the methods for finding numerical diagnostics of the fact of blow-up and specifying its localization in space and time. We recall that the *a priori* results obtained analytically in § 4 give a bound for the blow-up time but fail to give a detailed description of the process of blow-up. The numerical approach, which uses this analytical information, can help in specifying the blow-up pattern and the moment of blow-up.

Remark 2. We use the notation $u_{\mathrm{init}_0}(x)$, $u_{\mathrm{init}_1}(x)$ instead of $u_0(x)$, $u_1(x)$ because such subscripts will be used to define mesh values of functions.

We start by reducing the original equation (15) to a system of first order in time. This will enable us to use the effective numerical methods described below. Thus the problem[1]

$$
\begin{cases}
\dfrac{\partial^2}{\partial t^2}(u_{xx} - e^{\varepsilon u}) + u_{xx} = \dfrac{\partial}{\partial t}(u_x)^2, \qquad x \in (0,l), \quad t \in (0, T_{\mathrm{bl}}], \\
u(x,0) = u_{\mathrm{init}_0}(x), \quad u_t(x,0) = u_{\mathrm{init}_1}(x), \\
u_x(0,t) = u_x(l,t) = 0
\end{cases}
\tag{50}
$$

takes the form

$$
\begin{cases}
\dfrac{\partial}{\partial t}(u_{xx} - e^{\varepsilon u}) = v, \qquad x \in (0,l), \quad t \in (0, T_{\mathrm{bl}}], \\
\dfrac{\partial}{\partial t}(v - (u_x)^2) + u_{xx} = 0, \\
u(x,0) = u_{\mathrm{init}_0}(x), \\
v(x,0) = u_{\mathrm{init}_1}(x)_{xx} - \varepsilon u_{\mathrm{init}_1}(x) e^{\varepsilon u_{\mathrm{init}_0}(x)}.
\end{cases}
\tag{51}
$$

Here we fix $q = 2$ for convenience of the numerical modelling. Our approach can also be realized for other values of $q > 1$, but then the problem (51) and subsequent formulae change somewhat.

The stiff method of lines and the Rosenbrock scheme with a complex coefficient. The numerical modelling of (51) involves using the stiff method of lines (SMOL) [19], [20] in order to reduce the original system of partial differential equations to an implicit system of ordinary differential equations (unfortunately not resolved with respect to the derivatives), which can be solved effectively by the one-stage Rosenbrock scheme with a complex coefficient CROS1 [21].

We first introduce a uniform grid X_N only in the spatial variable x with step $h = (l-0)/N$ containing $N+1$ nodes (and, accordingly, N intervals): $X_N = \{x_n,\ 0 \leqslant n \leqslant N\colon x_n = 0 + nh\}$. Then, after a finite-difference approximation of the spatial derivatives in (51) within the second order of accuracy, we obtain the following system of ordinary differential equations for $N-1$ unknown functions $u_n \equiv u_n(t) \equiv u(x_n,t)$ ($n = 1, \dots, N-1$, u_0 and u_N are found from the relations induced by the boundary conditions) and $N-1$ auxiliary functions $v_n \equiv v_n(t) \equiv v(x_n,t)$ ($n = 1, \dots, N-1$, v_0 and v_N do not occur in the system):

$$
\begin{cases}
\dfrac{du_{n-1}}{dt} - (2 + \varepsilon h^2 e^{\varepsilon u_n})\dfrac{du_n}{dt} + \dfrac{du_{n+1}}{dt} = h^2 v_n, \\
\dfrac{dv_n}{dt} + \dfrac{u_{n+1} - u_{n-1}}{2h^2}\dfrac{du_{n-1}}{dt} - \dfrac{u_{n+1} - u_{n-1}}{2h^2}\dfrac{du_{n+1}}{dt} = -\dfrac{u_{n+1} - 2u_n + u_{n-1}}{h^2}, \\
u_n(0) = u_{\mathrm{init}_0}(x_n), \\
v_n(0) = \dfrac{u_{\mathrm{init}_1}(x_{n+1}) - 2u_{\mathrm{init}_1}(x_n) + u_{\mathrm{init}_1}(x_{n-1})}{h^2} - \varepsilon u_{\mathrm{init}_1}(x_n) e^{\varepsilon u_{\mathrm{init}_0}(x_n)}, \\
u_0 = \dfrac{4}{3}u_1 - \dfrac{1}{3}u_2, \\
u_N = \dfrac{4}{3}u_{N-1} - \dfrac{1}{3}u_{N-2}.
\end{cases}
\tag{52}
$$

[1]We pose the problem of finding the solution for all times not exceeding T_{bl}, although we know that the solution does not exist at the last moment and may even blow up earlier. This is because we want to detect the blow-up of the solution numerically and, therefore, we need to have a *numerical* solution for all these times.

This system may be rewritten in the vector form

$$\begin{cases} \mathbf{M}(\mathbf{u})\,\dfrac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}), \\ \mathbf{u}(0) = \mathbf{u}_{\text{init}}, \end{cases} \tag{53}$$

where $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_{N-1}, v_1, v_2, v_3, \ldots, v_{N-1})^T$, $\mathbf{f} = (f_1, f_2, f_3, \ldots, f_{2N-2})^T$ and $\mathbf{u}_{\text{init}} = (u_1(0), u_2(0), u_3(0), \ldots, u_{N-1}(0), v_1(0), v_2(0), v_3(0), \ldots, v_{N-1}(0))^T$.

The vector-valued function $\mathbf{f}$ has the following structure:

$$f_n = \begin{cases} h^2 v_n & \text{if } n = 1, \ldots, N-1, \\ -\dfrac{1}{h^2}\left(u_2 - 2u_1 + \left(\dfrac{4}{3}u_1 - \dfrac{1}{3}u_2\right)\right) & \text{if } n = N, \\ -\dfrac{1}{h^2}(u_{n-N+2} - 2u_{n+N+1} + u_{n-N}) & \text{if } n = N+1, \ldots, 2N-3, \\ -\dfrac{1}{h^2}\left(\left(\dfrac{4}{3}u_{N-1} - \dfrac{1}{3}u_{N-2}\right) - 2u_{N-1} + u_{N-2}\right) & \text{if } n = 2N-2 \end{cases}$$

and the matrix-valued function $\mathbf{M}$ has the following entries:

$$M_{n,n} = \begin{cases} \dfrac{4}{3} - (2 + \varepsilon h^2 e^{\varepsilon u_1}) & \text{if } n = 1, \\ -(2 + \varepsilon h^2 e^{\varepsilon u_n}) & \text{if } n = 2, \ldots, N-2, \\ \dfrac{4}{3} - (2 + \varepsilon h^2 e^{\varepsilon u_{N-1}}) & \text{if } n = N-1, \\ 1 & \text{if } n = N, \ldots, 2N-2, \end{cases}$$

$$M_{n,n-1} = \begin{cases} 1 & \text{if } n = 2, \ldots, N-2, \\ -\dfrac{1}{3} + 1 & \text{if } n = N-1, \end{cases}$$

$$M_{n,n+1} = \begin{cases} -\dfrac{1}{3} + 1 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \ldots, N-2, \end{cases}$$

$$M_{n,n-N} = \begin{cases} \dfrac{1}{2h^2}(u_{n-N+2} - u_{n-N}) & \text{if } n = N+1, \ldots, 2N-3, \\ \dfrac{1}{2h^2}\left(\dfrac{4}{3}u_{n-N+1} - \dfrac{4}{3}u_{n-N}\right)\left(1 + \dfrac{1}{3}\right) & \text{if } n = 2N-2, \end{cases}$$

$$M_{n,n-N+1} = \begin{cases} \dfrac{1}{2h^2}\left(\dfrac{4}{3}u_{n-N+2} - \dfrac{4}{3}u_{n-N+1}\right)\left(\dfrac{4}{3}\right) & \text{if } n = N, \\ \dfrac{1}{2h^2}\left(\dfrac{4}{3}u_{n-N+1} - \dfrac{4}{3}u_{n-N}\right)\left(-\dfrac{4}{3}\right) & \text{if } n = 2N-2, \end{cases}$$

$$M_{n,n-N+2} = \begin{cases} \dfrac{1}{2h^2}\left(\dfrac{4}{3}u_{n-N+2} - \dfrac{4}{3}u_{n-N+1}\right)\left(-\dfrac{1}{3} - 1\right) & \text{if } n = N, \\ -\dfrac{1}{2h^2}(u_{n-N+2} - u_{n-N}) & \text{if } n = N+1, \ldots, 2N-3. \end{cases}$$

The other entries of $\mathbf{M}$ are equal to zero.

The numerical modelling of (53) will be done using the Rosenbrock scheme with a complex coefficient (CROS1). This is the best choice for solving such problems since this scheme has a high order of accuracy ($O(\tau^1)$) and is monotone and stable (L_2), see [22].

Remark 3. It is important to note that the Rosenbrock scheme has order of accuracy $O(\tau^2)$ in the case of a constant matrix $\mathbf{M}$ (see [11], [12]). An implicit system of ordinary differential equations with such a matrix can be obtained by introducing two auxiliary variables in (51): $w = u_{xx} - e^{\varepsilon u}$ and $g = v - (u_x)^2$. Then we obtain an implicit system of ordinary differential equations (similar to (52)) with a constant (although degenerate) matrix twice the size of the original matrix. This increases the computation time. Which of these approaches is more effective from the computational point of view is still an open question. Our main purpose is to demonstrate the numerical diagnostics of the blow-up of solutions in space and/or time. Thus the question raised in this remark will not be studied in this paper.

To implement this scheme, we introduce a uniform grid (one can also use a quasi-uniform grid without changing the algorithm) T_M in t having $M+1$ nodes (that is, M intervals): $T_M = \{t_m,\ 0 \leqslant m \leqslant M\colon 0 = t_0 < t_1 < t_2 < \cdots < t_{M-1} < t_M = T_{\rm bl}\}$.

We can now use the scheme CROS1 to solve the system (53):

$$\mathbf{u}(t_{m+1}) = \mathbf{u}(t_m) + (t_{m+1} - t_m)\operatorname{Re}\mathbf{w},$$

where $\mathbf{w}$ is a solution of the system

$$\left[\mathbf{M}(\mathbf{u}(t_m)) - \frac{1+i}{2}(t_{m+1} - t_m)\mathbf{f_u}(\mathbf{u}(t_m))\right]\mathbf{w} = \mathbf{f}(\mathbf{u}(t_m)). \tag{54}$$

Here $\mathbf{f_u}$ is the Jacobian, whose structure for the system under consideration is as follows:

$$(f_u)_{n,n+N-1} = h^2 \quad \text{if } \ n = 1, \dots, N-1,$$

$$(f_u)_{n,n-N} = \begin{cases} -\dfrac{1}{h^2} & \text{if } \ n = N+1, \dots, 2N-3, \\ -\dfrac{1}{h^2} + \dfrac{1}{3h^2} & \text{if } \ n = 2N-2, \end{cases}$$

$$(f_u)_{n,n-N+1} = \begin{cases} \dfrac{2}{h^2} - \dfrac{4}{3h^2} & \text{if } \ n = N, \\ \dfrac{2}{h^2} & \text{if } \ n = N+1, \dots, 2N-3, \\ \dfrac{2}{h^2} - \dfrac{4}{3h^2} & \text{if } \ n = 2N-2, \end{cases}$$

$$(f_u)_{n,n-N+2} = \begin{cases} -\dfrac{1}{h^2} + \dfrac{1}{3h^2} & \text{if } \ n = N, \\ -\dfrac{1}{h^2} & \text{if } \ n = N+1, \dots, 2N-3. \end{cases}$$

The other entries of $\mathbf{f_u}$ are equal to zero for the equation under consideration.

Thus the matrix of the system (54) consists of four blocks of size $(N-1)\times(N-1)$ (the structure of this matrix is shown in Fig. 1). This enables us to use an algorithm for solving systems of linear algebraic equations (SLAE) that finds the solution of (54) in $O(N)$ operations.

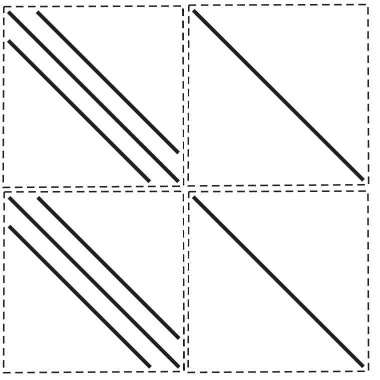

Figure 1. The structure of the matrix of SLAE (54)

Calculations on refined grids. In numerical calculations, it is important to obtain not only an approximate numerical result, but also a bound for its accuracy. The method of calculating the *a posteriori* asymptotically exact bound for the error (see [11]) enables us to do this. It can also help to detect the fact of blow-up of an exact solution [12]. The main formulae and assertions of this subsection were first given in [23], [11], [12].

We have approximated all spatial derivatives in (51) with accuracy $O(h^2)$, and the numerical modelling of (53) uses the scheme CROS1 of accuracy $O(\tau^1)$. Hence the resulting method of solving (51) has accuracy $O(\tau^1+h^2)$.

We begin by introducing the base grid $X_N\times T_M$: $\{x_n,t_m\}$, $0\leqslant n\leqslant N$, $0\leqslant m\leqslant M$. Then we successively refine the grids, starting with the base, and compute the solutions $u(x,t)$ on the resulting grids. Since the theoretical order of accuracy in time (resp. space) is equal to 1 (resp. 2), we successively refine the mesh in time (resp. space) by an integer factor r_t (resp. r_x) in such a way that $r_t^1=r_x^2$ (see [23] for details). For computations, it is most convenient to choose $r_t=4$ and $r_x=2$. Then each subsequent grid $X_{r_x^{s-1}N}\times T_{r_t^{s-1}M}$ (s is the number of the grid) has nodes coinciding with the nodes (x_n,t_m) of the base grid. At these nodes (x,t) we can obtain an *a posteriori* asymptotically exact bound for the error (see [23], [12]),

$$\Delta^{(r_x^sN,r_t^sM)}(x,t)=\frac{u^{(r_x^sN,r_t^sM)}(x,t)-u^{(r_x^{s-1}N,r_t^{s-1}M)}(x,t)}{{r_t}^1-1}+o(\tau^1+h^2),$$

and estimate the effective order of accuracy ([23], [12]),

$$p_s^{\mathrm{eff}}(x,t)=\log_{r_t}\frac{u^{(r_x^{s-1}N,r_t^{s-1}M)}(x,t)-u^{(r_x^{s-2}N,r_t^{s-2}M)}(x,t)}{u^{(r_x^sN,r_t^sM)}(x,t)-u^{(r_x^{s-1}N,r_t^{s-1}M)}(x,t)}.$$

At those points (x,t) where the solution of the original problem has continuous first temporal and second spatial derivatives, we have the convergence

$$p_s^{\mathrm{eff}}(x,t)\xrightarrow[s\to\infty]{}p^{\mathrm{theor}}=1 \tag{55}$$

and the corresponding error bound is asymptotically exact as $s \to \infty$ (or, equivalently, $N, M \to \infty$). Absence of the convergence (55) indicates that the exact solution loses smoothness. In particular, in the case of a power-type 'singularity' $u(x,t) \sim (t^* - t)^{-\beta}$ for every $t > t^*$, the effective order of accuracy $p^{\text{eff}}(x,t) \xrightarrow[N,M\to\infty]{} -\beta$. This enables us to find the exponent β. If $p^{\text{eff}}(x,t) \xrightarrow[N,M\to\infty]{} -\infty$ for every $t > t^*$, we can assert that the solution increases exponentially, that is, $u(x,t) = \infty$. If $p^{\text{eff}}(x,t) \xrightarrow[N,M\to\infty]{} 0$ for every $t > t^*$, then the solution grows logarithmically in a neighbourhood of the 'singularity': $u(x,t) \sim \ln(t^* - t)$. The blow-up time t^* of the solution can be found within accuracy equal to the grid step.

If the solution loses smoothness simultaneously on the whole domain of the spatial variable, then the deviation of $p^{\text{eff}}(x,t)$ from convergence to 1 occurs simultaneously at all points of the grid $\{x_n\}$ beginning with the first temporal layer $t \geqslant t^*$ (see Examples 1 and 2). If the solution blows up at a single point x^*, then the method described enables us to follow in time the process of blow up of the solution at the other points (see Example 3). This diagnostics of the blow-up process becomes possible because the scheme CROS never overfills, even when the solution of the problem tends to infinity ([11], [12]).

Example 1. We first consider an example with the input data

$$u_{\text{init}_0}(x) \equiv 0, \qquad u_{\text{init}_1}(x) \equiv -1, \qquad l = \pi, \qquad \varepsilon = 1.$$

Then the solution of (50) can be written down explicitly:

$$u(x,t) = \ln(1-t).$$

Clearly, the blow-up time of this solution is $t^* = 1$ and the solution grows logarithmically in a neighbourhood of this point.

We apply our numerical algorithm to detect the time and location of the blow-up and compare the result with the theoretical answer. To solve the problem (50) numerically, we take the following set of parameters: $T_{\text{bl}} = 1$ (calculated from the formula (38); we assume that $q = 2$ here and in what follows), $N = 50$, $M = 50$, $r_x = 2$, $r_t = 4$, $S = 6$ (the number of successive grids used in this computation, including the original grid).

Having obtained the approximate numerical solution on various grids, we can verify the convergence of the effective order of accuracy to the theoretical one for every temporal layer using the formula

$$p_s^{\text{eff}}(t_m) = \log_{r_t} \frac{\sqrt{\sum_{n=1}^{N} \left(u^{(r_x^{s-1}N, r_t^{s-1}M)}(x_n,t_m) - u^{(r_x^{s-2}N, r_t^{s-2}M)}(x_n,t_m)\right)^2}}{\sqrt{\sum_{n=1}^{N} \left(u^{(r_x^{s}N, r_t^{s}M)}(x_n,t_m) - u^{(r_x^{s-1}N, r_t^{s-1}M)}(x_n,t_m)\right)^2}}, \tag{56}$$

where s is the number of the grid. We stress once again that the notation (x_n, t_m) is used for the nodes coinciding with the corresponding nodes of the base grid. After calculations on S nested grids, the effective order of accuracy p^{eff} converges to $p^{\text{theor}} = 1$ (see Fig. 2) for every temporal layer t_m except for the layer corresponding to t_{50}: $p^{\text{eff}}(t_{50}) \to 0$, and this means that the solution blows up at time $t^* \in (t_{49}, t_{50}] \equiv (0.98, 1]$ and grows logarithmically at t^*: $u(x,t) \sim \ln(1-t)$.

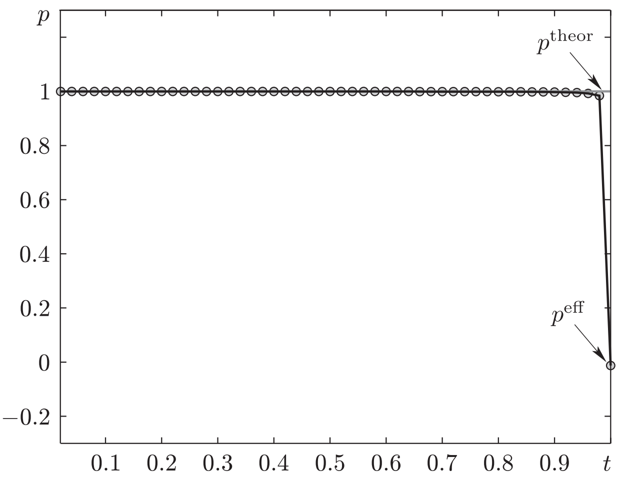

Figure 2. Example 1. The effective order of accuracy for each temporal layer. Blow-up of the solution is detected at time $t^* \in (t_{49}, t_{50}] \equiv (0.98, 1]$. Here and in what follows the zero temporal layer is not shown

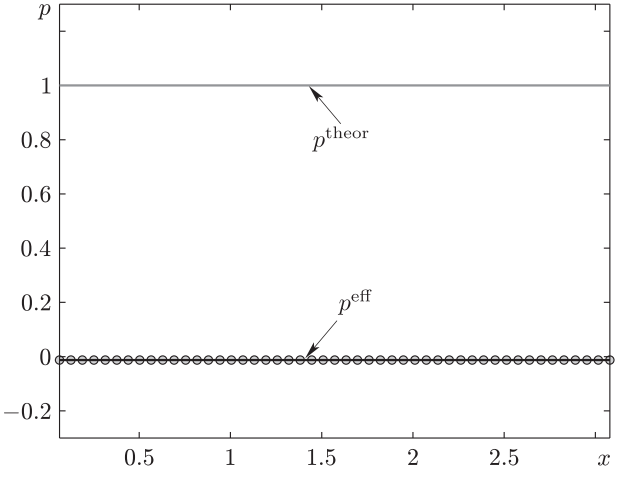

Figure 3. Example 1. The effective order of accuracy for each spatial point of the temporal layer corresponding to the moment $t_{50} = 1$, at which the fact of blow-up has been detected. The blow-up of the solution is detected at all spatial points of this temporal layer

For each spatial point of the chosen temporal layer corresponding to the moment t_m, we can also estimate the effective order of accuracy by the formula

$$p_s^{\text{eff}}(x_n, t_m) = \log_{r_t} \frac{u^{(r_x^{s-1}N, r_t^{s-1}M)}(x_n, t_m) - u^{(r_x^{s-2}N, r_t^{s-2}M)}(x_n, t_m)}{u^{(r_x^{s}N, r_t^{s}M)}(x_n, t_m) - u^{(r_x^{s-1}N, r_t^{s-1}M)}(x_n, t_m)}. \tag{57}$$

For example, this formula can be used for the temporal layer corresponding to the moment t_{50} (at which the blow-up of the solution has been detected) in order to specify whether the solution blows up on the whole temporal layer or only at separate points in the spatial variable. We see in Fig. 3 that the solution blows up at all spatial points of this temporal layer.

Example 2. We now consider the following example:

$$u_{\mathrm{init}_0}(x) \equiv 0, \qquad u_{\mathrm{init}_1}(x) \equiv -[x(\pi - x)]^2 \sin x, \qquad l = \pi, \qquad \varepsilon = 10^{-1}.$$

In this case, the exact solution of (50) cannot be obtained analytically. We again use the algorithm described above for the numerical diagnostics of the blow-up of the solution in time and space. The upper bound T_{bl} for the blow-up time t^* can be calculated numerically by the formula (38): $T_{\mathrm{bl}} \approx 3.7$. To solve the problem (50) numerically, we use the following set of parameters: $T_{\mathrm{bl}} = 3.7$, $N = 50$, $M = 50$, $r_x = 2$, $r_t = 4$, $S = 5$ (the number of grids used in the calculation, including the original grid).

As in Example 1, we can estimate the effective order of accuracy using the formula (56). After calculations on S nested grids, the effective order of accuracy p^{eff} converges to $p^{\mathrm{theor}} = 1$ (see Fig. 4) for every temporal layer before the layer corresponding to t_7: $p^{\mathrm{eff}}(t_7) \to 0$. This means that the solution blows up at a moment $t^* \in (t_6, t_7] \equiv (0.44, 0.52]$ and grows logarithmically at t^*: $u(x,t) \sim \ln(t^* - t)$.

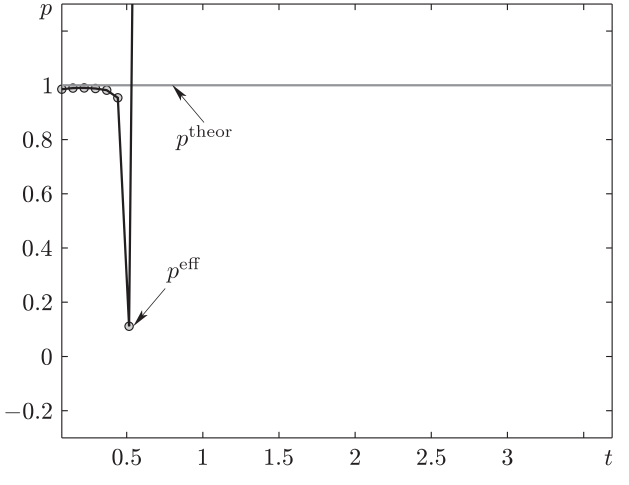

Figure 4. Example 2. The effective order of accuracy for each temporal layer. Blow-up of the solution is detected at $t^* \in (t_6, t_7] \equiv (0.44, 0.52]$

Thus, the analytically obtained blow-up time turns out to be considerably overvalued in this example. This demonstrates the utility of numerical diagnostics. On the other hand, the analytical estimate is also necessary. It gives us a rough localization of the blow-up time, so that we know the time interval for launching the numerical analysis.

The formula (57) for the time layer corresponding to the moment t_7, at which the blow-up of the solution had been detected, can also be used to specify whether the solution blows up on the whole temporal layer or only at separate points in the spatial variable. We see in Fig. 5 that the solution blows up at all spatial points of this temporal layer.

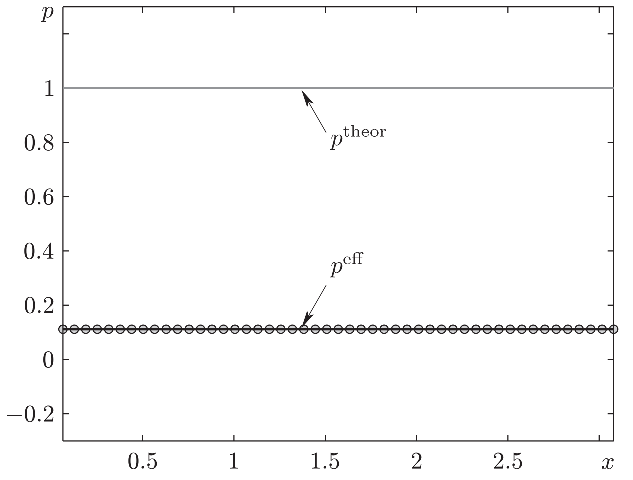

Figure 5. Example 2. The effective order of accuracy for each spatial point of the first temporal layer where the blow-up had been detected

Example 3. We now consider one of the most interesting examples:

$$u_{\text{init}_0}(x) \equiv 0, \qquad u_{\text{init}_1}(x) \equiv -[x(\pi - x)]^2 \sin x, \qquad l = \pi, \qquad \varepsilon = 10^{10}.$$

It is similar to Example 2 except for the value of ε, which is now much larger. The exact solution of (50) cannot be found analytically. The upper bound (38) for the blow-up time t^* is given by $T_{\text{bl}} \approx 3.7 \cdot 10^{-11}$. To solve the problem (50) numerically, we use the following set of parameters: $T_{\text{bl}} = 3.7 \cdot 10^{-11}$, $N = 50$, $M = 50$, $r_x = 2$, $r_t = 4$, $S = 6$ (the number of grids used for the calculations, including the original grid).

We can also estimate the effective order of accuracy for each temporal layer using the formula (56) (see Fig. 6). We see that blow-up occurs near the midpoint $t^* \in [t_{22}, t_{23}) \equiv (1.6 \cdot 10^{-11}, 1.7 \cdot 10^{-11}]$ of the modelled time interval and $p^{\text{eff}}(t_{24}) \to 0$. This means that the solution grows logarithmically at t^*: $u(x,t) \sim \ln(t^* - t)$.

Using the formula (57) for various temporal layers, we can specify (see Fig. 7) that the blow-up first occurs at the midpoint of the interval $(0, \pi)$ and then spreads to all other points of the domain of the spatial variable.

Example 4. The following Examples 4a and 4b are of a different kind. Here we give a practical application of the results of § 4.2. Without giving a specific bound for the blow-up time, we shall show that for a large class of boundary conditions (containing all the conditions (17)–(19)) one can choose the initial data (16) in such

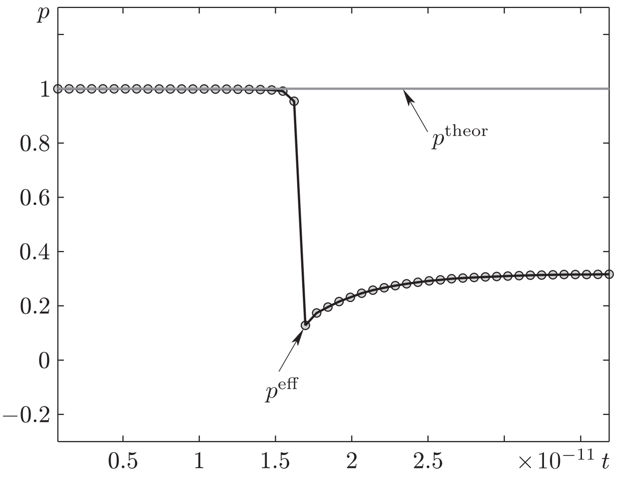

Figure 6. Example 3. The effective order of accuracy for each temporal layer. Blow-up of the solution is detected at $t^* \in (t_{22}, t_{23}] \equiv (1.6\cdot 10^{-11},\ 1.7\cdot 10^{-11}]$

a way that the solution blows up no later than the prescribed moment T of time. To do this, we proceed as follows.

1) Fix a test function $\varphi_1(x)$ (see (40)) satisfying the boundary conditions and choose the parameter $\lambda > 2$ in (41).

2) For the same q as in (15) and for the chosen T and λ, fix the values of ε_1, ε_2 in (48) in such a way that the coefficient

$$\frac{\lambda}{T} - \frac{\varepsilon_1^q}{q}\frac{\lambda(\lambda-1)}{T^2} - \frac{\varepsilon_2^q}{q}$$

is positive.

3) Choose r in (49) so large that the left-hand side of (48) is equal to 0 for the chosen T (this can always be achieved since only the first and fourth terms on the left-hand side of (48) depend on $u_1(x)$).

We have chosen the following values of the parameters:

$$l=\pi, \qquad q=2, \qquad \varepsilon=0.2, \qquad \lambda=4, \qquad T=1, \qquad \varepsilon_1=\varepsilon_2=\frac{1}{2}, \qquad u_0(x)\equiv 0.$$

Then

$$\frac{\lambda}{T} - \frac{\varepsilon_1^q}{q}\frac{\lambda(\lambda-1)}{T^2} - \frac{\varepsilon_2^q}{q} = 4 - \frac{3}{2} - \frac{1}{8} > 0.$$

This guarantees that the right-hand side of (48) is positive. For $\varphi_1(x)$ we take the function

$$\varphi_1(x) = \begin{cases} 0, & x \in [0,a] \cup [b,\pi], \\ \sin^3 \pi \dfrac{x-a}{b-a}, & x \in (a,b), \end{cases}$$

$a = 0.1\pi$, $b = 0.9\pi$. Note that this function satisfies the conditions (18) and (17) and is twice continuously differentiable on $[0,\pi]$.

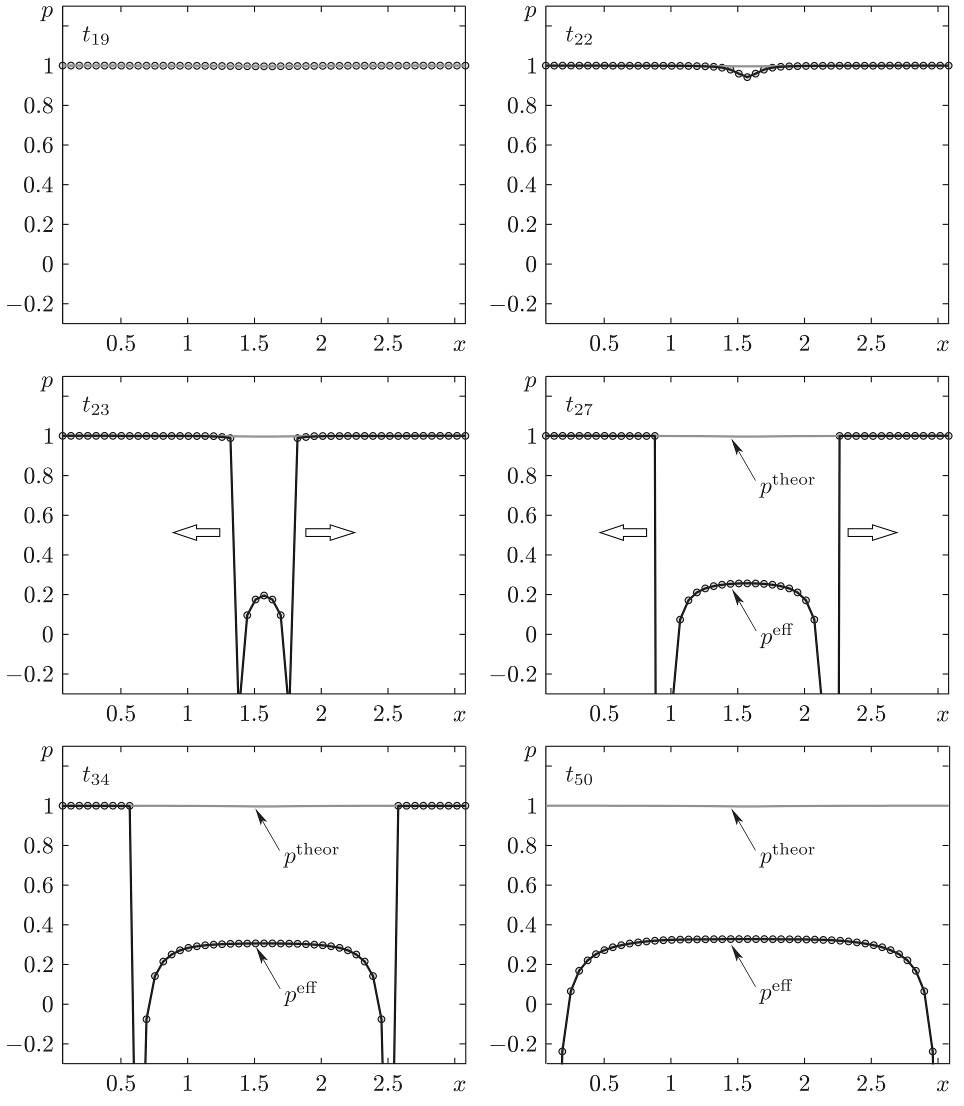

Figure 7. Example 3. The effective orders of accuracy for all spatial points of several temporal layers (t_{19}, t_{22}, t_{23}, t_{27}, t_{34} and t_{50}). The blow-up of the solution is first detected at the midpoint of the interval $(0,\pi)$. Then it spreads to all other points of this interval

To choose the parameter r in (49), we require the left-hand side of (48) be equal to zero:

$$
\begin{aligned}
&-\int_\Omega dx\,\Delta u_1\varphi_1 - \frac{\lambda}{T}\int_\Omega dx\,\Delta u_0\varphi_1 + \int_\Omega dx\,\varphi_1|\nabla u_0|^q \\
&\quad + \varepsilon\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0}u_1 + \frac{\lambda}{T}\int_\Omega dx\,\varphi_1 e^{\varepsilon u_0}
\end{aligned}
$$

$$+ \frac{1}{\varepsilon_1^{q'}} \frac{1}{q'} \frac{\lambda(\lambda-1)}{T^2} \iint_{[0,T]\times\Omega_1} dt\,dx\,|\nabla\varphi_1|^{q'} \left(1-\frac{t}{T}\right)^{(\lambda-2-(\lambda-1)/q)q'} \varphi_1^{-q'/q}$$
$$+ \frac{1}{\varepsilon_2^{q'}} \frac{1}{q'} \iint_{[0,T]\times\Omega_1} dt\,dx\,|\nabla\varphi_1|^{q'} \left(1-\frac{t}{T}\right)^{(\lambda-(\lambda-1)/q)q'} \varphi_1^{-q'/q} = 0.$$

In view of the choice of the parameters above, we arrive at

$$r = \frac{4\int_0^\pi \varphi_1(x)\,dx + \frac{37}{3}\int_0^\pi \frac{(\varphi_1')^2}{\varphi_1}\,dx}{\int_0^\pi (-\varphi_1'' + \varepsilon\varphi_1)^2\,dx} \approx 5.3.$$

We now choose the Neumann boundary conditions (18) in Example 4a and the Dirichlet conditions (17) in Example 4b. The bound (38) for the blow-up time is also available in the first case, and we state it for purposes of comparison.

Example 4a (choosing the Neumann boundary conditions (18)). In accordance with what was said above, we put

$$u_{\mathrm{init}_0}(x) \equiv 0, \qquad u_{\mathrm{init}_1}(x) \equiv -r(-\varphi_1''(x) + \varepsilon\varphi_1(x)), \qquad l = \pi, \qquad \varepsilon = 0.2.$$

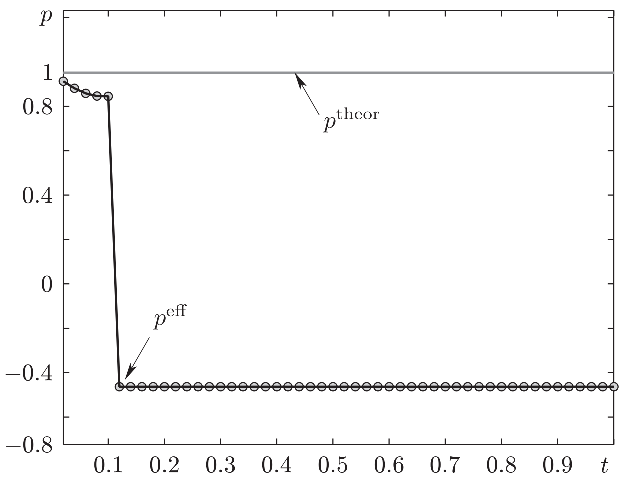

Figure 8. Example 4a. The effective order of accuracy for each temporal layer. Blow-up of the solution is detected at time $t^* \in (t_4, t_5] \equiv (0.08, 0.10]$

To solve the problem (50), we take the following set of parameters: $T = 1$, $N = 50$, $M = 50$, $r_x = 2$, $r_t = 4$, $S = 6$ (the number of grids used in the calculation, including the original grid). By estimating the effective order of accuracy for each temporal layer by the formula (56) (see Fig. 8), we can see that the solution blows up at a moment $t^* \in (t_5, t_6] \equiv (0.10, 0.12]$. Unfortunately, we were not able to obtain the exact asymptotic behaviour of $p^{\mathrm{eff}}(t_5)$ and, therefore, we cannot be certain about the pattern of blow-up of $u(x,t)$.

Again using (57) for various temporal layers, we can specify that the blow-up of the solution on the corresponding temporal layer occurs at all spatial points in x.

Note that the bound (38) for the blow-up time is equal to $T_{\rm bl} = 13.83$.

Example 4b (choosing the Dirichlet boundary conditions (17)). In accordance with what was said above, we put

$$u_{\mathrm{init}_0}(x) \equiv 0, \qquad u_{\mathrm{init}_1}(x) \equiv -r(-\varphi_1''(x) + \varepsilon\varphi_1(x)), \qquad l = \pi, \qquad \varepsilon = 0.2.$$

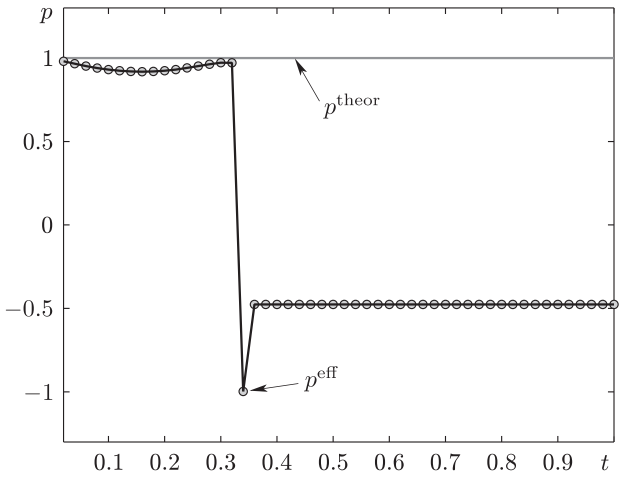

Figure 9. Example 4b. The effective order of accuracy for each temporal layer. Blow-up of the solution is detected at $t^* \in (t_{16}, t_{17}] \equiv (0.32, 0.34]$

To solve the problem (50) numerically, we take the following set of parameters: $T = 1,\ N = 50,\ M = 50,\ r_x = 2,\ r_t = 4,\ S = 6$ (the number of grids used in the calculation, including the original grid). Estimating the effective order of accuracy for each temporal layer using the formula (56) (see Fig. 9), we can see that blow-up of the solution occurs at a moment $t^* \in (t_{16}, t_{17}] \equiv (0.32, 0.34]$ and $p^{\rm eff}(t_{17}) \to -1$. This means that the solution has power-type growth $u(x,t) \sim (t^* - t)^{-1}$ at the point t^*.

Conclusion

In this paper we have considered the equation of ion-sound waves in a plasma in the one-dimensional approximation. We have performed an analytic-numerical investigation of a series of initial-boundary value problems for this equation. We have established the local solubility, found sufficient conditions for finite-time blow-up of solutions and obtained an upper bound for the blow-up time. In the example of the problems considered, we have demonstrated that the time and pattern of blow-up can be specified numerically using estimates of Richardson's effective order of accuracy. It is important to mention that the analytic and numerical parts of this approach are interrelated: the numerical method enables one to improve the results obtained analytically (for any concrete data) and uses these results as a rough estimate for the blow-up time.

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Maxim O. Korpusov
Faculty of Physics,
Moscow State University
E-mail: korpusov@gmail.com, korpusov@rsci.ru

Dmitry V. Lukyanenko
Faculty of Physics,
Moscow State University
E-mail: lukyanenko@physics.msu.ru

Alexander A. Panin
Faculty of Physics,
Moscow State University
S. M. Nikol'skii Mathematical Institute,
Russian University of Peoples' Friendship,
Moscow
E-mail: a-panin@yandex.ru

Egor V. Yushkov
Faculty of Physics,
Moscow State University
Space Research Institute of RAS, Moscow
E-mail: yushkov.msu@mail.ru