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Periodic solutions of travelling-wave type in circular gene networks

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Abstract. We consider circular chains of unidirectionally coupled ordinary differential equations which are mathematical models of artificial gene networks. We study the problems of the existence and stability of special periodic solutions, the so-called travelling waves, in these chains. We establish that the number of such periodic solutions grows unboundedly as the number of links in the chain grows. However, at most one of these travelling waves can be stable. We give an explicit algorithm for choosing the stable cycle.

Keywords: chain of unidirectionally coupled equations, artificial gene network, travelling wave, asymptotics, stability.

§ 1. Description of mathematical models and methods of study

1.1. Artificial gene networks. Artificial gene oscillators are of interest as simplified models of key biological processes such as the cell-division cycle and circadian rhythms. The simplest gene oscillator, which was suggested in [1] and called a repressilator, consists of at least three elements. Every element unidirectionally inhibits its neighbour (see Fig. 1, where the element A inhibits synthesis of B, the element B inhibits synthesis of C, and the third element C inhibits synthesis of A, completing the chain).

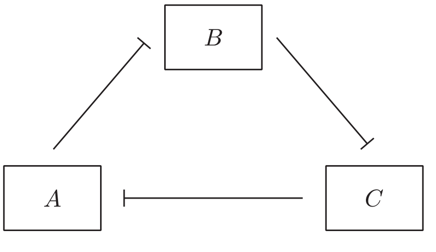

Figure 1

A mathematical model of the system shown in Fig. 1 is of the form

$$\dot{m}_j = -m_j + \frac{\alpha}{1+u_{j-1}^{\gamma}}, \quad \dot{u}_j = \varepsilon(m_j - u_j), \qquad j = 1,2,3, \tag{1.1}$$

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where $u_0 = u_3$. Following [1], we assume that each element of the oscillator consists of mRNA (matrix ribonucleic acid) with concentration m_j and protein with concentration u_j. We further assume that the time evolution of m_j is characterized by the processes of synthesis and degradation. The first process is described by the function $\alpha/(1+u_{j-1}^{\gamma})$, where u_{j-1} is the protein repressor concentration for the jth mRNA, $\gamma = \text{const} > 0$ is the coefficient of cooperation, and $\alpha = \text{const} > 0$ is the transcription rate in the absence of repressors. The second process is described by the linear term $-m_j$.

In the case of proteins with concentration u_j we suppose that their dynamics is given by linear processes of synthesis (the term εm_j in the second equation of (1.1)) and degradation (the term $-\varepsilon u_j$ of the same equation). Here $\varepsilon = \text{const} > 0$ is the ratio of the degradation rates of the protein and the mRNA.

As a rule, the model (1.1) is studied under the biologically natural assumption that ε is small. In this situation, the time change $\varepsilon t \to t$ yields the system

$$\varepsilon \dot{m}_j = -m_j + \frac{\alpha}{1+u_{j-1}^{\gamma}}, \quad \dot{u}_j = m_j - u_j, \qquad j = 1, 2, 3, \tag{1.2}$$

which obeys Tikhonov's well-known reduction principle (see [2]). In accordance with this principle we put $\varepsilon = 0$ in (1.2) and express the variables m_j in terms of u_{j-1}. Thus we arrive at a three-dimensional system

$$\dot{u}_j = -u_j + \frac{\alpha}{1+u_{j-1}^{\gamma}}, \qquad j = 1, 2, 3, \tag{1.3}$$

where $u_0 = u_3$.

The problem of self-oscillations of (1.3) has been studied by many authors (see, for example, [3]–[5]). They considered the cases when a stable cycle results from Andronov–Hopf bifurcations and the case when $\gamma \gg 1$.

A number of systems more general than (1.3) were suggested for describing artificial gene networks. For example, four classes of such systems (not necessary circular) were introduced in [6], [7]. Restricting ourselves to the simplest circular systems, we have the following typical representatives of these classes:

$$\dot{u}_j = -u_j + \frac{\alpha}{1+\delta_1 u_{j-1}^{\gamma_1} + \delta_2 u_{j-2}^{\gamma_2} + \dots + \delta_s u_{j-s}^{\gamma_s}}, \qquad j = 1, 2, \dots, m; \tag{1.4}$$

$$\dot{u}_j = -u_j + \frac{\alpha}{(1+\delta_1 u_{j-1}^{\gamma_1})(1+\delta_2 u_{j-2}^{\gamma_2}) \cdots (1+\delta_s u_{j-s}^{\gamma_s})}, \qquad j = 1, 2, \dots, m; \tag{1.5}$$

$$\dot{u}_j = -u_j + \frac{\alpha}{1+u_{j-1}^{\gamma_1} u_{j-2}^{\gamma_2} \cdots u_{j-s}^{\gamma_s}}, \qquad j = 1, 2, \dots, m; \tag{1.6}$$

$$\dot{u}_j = -u_j + \sum_{k=1}^{s} \frac{\alpha_k}{1+\delta_k u_{j-k}^{\gamma_k}}, \qquad j = 1, 2, \dots, m. \tag{1.7}$$

In all the formulae (1.4)–(1.7) we assume that $m > s+1$, $u_{-k} = u_{m-k}$ for $k = 0, 1, \dots, s-1$, $\alpha = \text{const} > 0$, $\alpha_k, \delta_k, \gamma_k = \text{const} > 0$, $k = 1, \dots, s$.

By *travelling waves* of the systems (1.4)–(1.7) we mean special periodic solutions of the form

$$u_j = u(t + (j-1)\Delta), \qquad j = 1, 2, \dots, m, \quad \Delta = \text{const} > 0. \tag{1.8}$$

We shall study the problems of the existence and stability of such solutions for the systems (1.4), (1.5) under the additional assumptions

$$\gamma_k = \frac{\gamma_k^0}{\varepsilon}, \qquad \gamma_k^0, \delta_k = \mathrm{const} > 0, \qquad \alpha = \mathrm{const} > 1, \qquad 0 < \varepsilon \ll 1. \tag{1.9}$$

We mention that the travelling waves (1.8) are the most natural attractors of circular chains of unidirectionally coupled equations. However, these chains admit more complicated oscillations. For example, it was shown by numerical analysis in [8] that a ring of three unidirectionally coupled generators possesses a chaotic self-induced oscillatory regime. A series of papers (see, for example, [9]–[11]) was dedicated to the transition to chaos in unidirectionally coupled chains of oscillators as the number of links increases in an appropriate way.

1.2. General scheme of investigation. Our methods of analyzing travelling waves are based on the further development of methods suggested in [12]–[14]. Since they are rather universal, it makes sense to explain them in maximal generality, that is, for an arbitrary circular system

$$\dot{x}_j = f(x_j, x_{j-1}, \dots, x_{j-s}), \qquad j = 1, 2, \dots, m. \tag{1.10}$$

Here $m > s+1$, $x_j = x_j(t) \in \mathbb{R}^n$, $x_{-k} = x_{m-k}$ for $k = 0, 1, \dots, s-1$, and the vector-valued function $f(x, y_1, \dots, y_s)$ with values in $\mathbb{R}^n$ is infinitely differentiable with respect to all variables $x \in \mathbb{R}^n$, $y_k \in \mathbb{R}^n$, $k = 1, \dots, s$.

To find cycles of the form (1.8) for the system (1.10), we consider an auxiliary equation with delay:

$$\dot{x} = f\big(x, x(t-\Delta), x(t-2\Delta), \dots, x(t-s\Delta)\big), \tag{1.11}$$

where $x = x(t) \in \mathbb{R}^n$. We assume that equation (1.11) has a non-trivial (that is, non-constant) periodic solution $x = x_*(t, \Delta)$ of period $T_* = T_*(\Delta) > 0$ for all values of Δ in some interval $(\Delta_1, \Delta_2) \subset (0, +\infty)$. Then the following assertion holds.

Lemma 1.1. *Suppose that the equation*

$$T_*(\Delta) = \frac{m\Delta}{p} \tag{1.12}$$

has a root $\Delta = \Delta_{(p)} \in (\Delta_1, \Delta_2)$ for some positive integer p. Then, for every such root, the system (1.10) *possesses a cycle* (*travelling wave*)

$$C_p \colon x_j = x_{(p)}(t + (j-1)\Delta_{(p)}), \qquad j = 1, 2, \dots, m, \tag{1.13}$$

of period $T_{(p)} = m\Delta_{(p)}/p$, where $x_{(p)}(t) = x_*(t, \Delta)|_{\Delta = \Delta_{(p)}}$.

Proof. Since all the functions

$$x_j(t) = x_{(p)}(t + (j-1)\Delta_{(p)}), \qquad j = 1, 2, \dots, m, \tag{1.14}$$

are solutions of the same equation (1.11) for $\Delta = \Delta_{(p)}$, we have

$$\begin{gathered} \dot{x}_j(t) = f\big(x_j(t), x_j(t-\Delta_{(p)}), x_j(t-2\Delta_{(p)}), \dots, x_j(t-s\Delta_{(p)})\big), \\ j = 1, 2, \dots, m. \end{gathered} \tag{1.15}$$

In (1.15), we take into account that the following relations hold by (1.12), (1.14):

$$x_j(t-k\Delta_{(p)}) = x_{j-k}(t), \qquad j=k+1,\dots,m, \quad k=1,\dots,s,$$
$$x_j(t-k\Delta_{(p)}) = x_j(t+(m-k)\Delta_{(p)}) = x_{m-k+j}(t), \qquad j=1,\dots,k, \quad k=1,\dots,s.$$

We thus obtain that the functions (1.14) satisfy the system (1.10). □

The problem of the stability of the cycle (1.13) reduces to studying the disposition of the multipliers of the linear system

$$\dot h_j = \sum_{k=0}^{s} A_{k,p}(t+(j-1)\Delta_{(p)})h_{j-k}, \qquad j=1,2,\dots,m, \tag{1.16}$$

where $h_j = h_j(t) \in \mathbb{R}^n$, $h_{-k} = h_{m-k}$, $k=0,1,\dots,s-1$, and the matrices $A_{k,p}(t)$ are given by the equations

$$A_{0,p}(t) = \frac{\partial f}{\partial x}\bigl(x_{(p)}(t), x_{(p)}(t-\Delta_{(p)}),\dots,x_{(p)}(t-s\Delta_{(p)})\bigr),$$
$$A_{k,p}(t) = \frac{\partial f}{\partial y_k}\bigl(x_{(p)}(t), x_{(p)}(t-\Delta_{(p)}),\dots,x_{(p)}(t-s\Delta_{(p)})\bigr), \qquad k=1,\dots,s.$$

Along with (1.16) we shall use the following auxiliary linear equation with delay:

$$\dot h = \sum_{k=0}^{s} \varkappa^k A_{k,p}(t) h(t-k\Delta_{(p)}), \tag{1.17}$$

where $h(t) \in \mathbb{C}^n$, $\varkappa$ is an arbitrary complex parameter. More precisely, we are interested in the multipliers $\nu_l(\varkappa)$, $l=1,2,\dots$, indexed in such a way that their absolute values decrease.

To clarify the meaning of the term 'multiplier' for the delay equation (1.17), we consider the Banach space $E = C([-s\Delta_{(p)},0];\mathbb{C}^n)$ of all vector-valued functions $h^0(t) = (h_1^0(t),\dots,h_n^0(t))$ that are continuous for $-s\Delta_{(p)} \leqslant t \leqslant 0$, with the norm

$$\|h^0\|_E = \max_{1\leqslant k\leqslant n} \max_{-s\Delta_{(p)}\leqslant t\leqslant 0} |h_k^0(t)|.$$

By the *monodromy operator* of equation (1.17) we mean a bounded linear operator $\mathscr{V}\colon E \to E$ acting on an arbitrary function $h^0(t) \in E$ by the rule

$$\mathscr{V} h^0 = h\biggl(t + \frac{m\Delta_{(p)}}{p}\biggr), \qquad -s\Delta_{(p)} \leqslant t \leqslant 0, \tag{1.18}$$

where $h(t)$ is a solution of (1.17) on the time interval $0 \leqslant t \leqslant m\Delta_{(p)}/p$ with initial function $h^0(t)$, $-s\Delta_{(p)} \leqslant t \leqslant 0$. Note that the spectrum of this operator is certainly discrete since some power of it is compact (if $m/p \geqslant s$, then $\mathscr{V}$ itself is compact). As in the case of ordinary differential equations, we call the eigenvalues of the operator (1.18) *multipliers of the equation* (1.17).

The problem of the connection between the multipliers of the systems (1.16) and (1.17) was solved (in the case when $s=1$) in [12] by the method of tuning with respect to the parameter $\varkappa$. We shall see that this method is also applicable when $s>1$, but first we explain its essence.

Choose any multiplier $\nu(\varkappa)$ in the countable set of multipliers of (1.17) and consider the corresponding equation $(\nu(\varkappa))^p=\varkappa^m$. It will be shown that for every root $\varkappa=\varkappa_0\neq 0$ of this equation there is a multiplier of (1.16) given by the equality $\nu=\nu(\varkappa_0)$. The converse is also true: one can 'tune' to a multiplier ν of the cycle (1.13) by selecting an appropriate parameter $\varkappa$. In other words, every multiplier ν of this cycle admits a representation $\nu=\nu(\varkappa_0)$, where $\nu(\varkappa)$ is one of the multipliers of the auxiliary system (1.17) and $\varkappa_0$ is a root of the corresponding equation $(\nu(\varkappa))^p=\varkappa^m$.

It was mentioned in [13], [14] that the tuning method is analogous to the process of tuning a radio receiver to a chosen station. Indeed, the set of all non-zero roots of the equations in question may be regarded as a set of broadcast 'frequencies', and the act of 'tuning to a station' corresponds to a concrete choice of one of these roots, which results in getting the corresponding multiplier $\nu=\nu(\varkappa_0)$ of the system (1.16).

To make the tuning method rigorous, we consider the family of equations

$$[\nu_l(\varkappa)]^p=\varkappa^m,\qquad l\in\mathbb{N}. \tag{1.19}$$

Lemma 1.2. *For every multiplier ν of the system (1.16) one can find a positive integer l_0 such that*

$$\nu=\nu_{l_0}(\varkappa_0), \tag{1.20}$$

where $\varkappa_0$ is one of the roots of the equation (1.19) for $l=l_0$. Conversely, if the equation (1.19) has a root $\varkappa=\varkappa_0$ for some $l=l_0$, then the system (1.16) has a multiplier of the form (1.20).

Proof. Fix any multiplier $\nu=\rho_0\exp(i\varphi_0)$ of the system (1.16), where $\rho_0>0$, $0\leqslant\varphi_0<2\pi$, and assume that it is simple. Then there is a unique (up to a factor) Lyapunov–Floquet solution, that is, a solution of the form

$$\begin{gathered} h_j=\exp(\mu_0 t)h_{*,j}(t),\qquad h_{*,j}(t)\in\mathbb{C}^n,\quad h_{*,j}\biggl(t+\frac{m\Delta_{(p)}}{p}\biggr)\equiv h_{*,j}(t),\\ j=1,2,\dots,m,\qquad \mu_0=\frac{p}{m\Delta_{(p)}}(\ln\rho_0+i\varphi_0). \end{gathered} \tag{1.21}$$

We note that the system (1.16) is invariant under changes of the form

$$t-\Delta_{(p)}\to t,\qquad h_{j-1}\to h_j,\quad j=1,2,\dots,m, \tag{1.22}$$

and the solution (1.21), because of its uniqueness, is transformed by such a change into a solution

$$h_j=\lambda\exp(\mu_0(t+\Delta_{(p)}))h_{*,j}(t),\qquad j=1,2,\dots,m,$$

where $\lambda\neq 0$ is a complex constant. Thus we have

$$\Lambda h_*(t+\Delta_{(p)})=\lambda\, h_*(t), \tag{1.23}$$

where $h_*(t) =: (h_{*,1}(t), \dots, h_{*,m}(t))$, and the entries of the nm-dimensional square matrix

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & I \\ I & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 \end{pmatrix}$$

are the zero and identity matrices of order n.

It follows from (1.23) that

$$h_{*,m-j}(t) = \lambda^{j+1} h_{*,1}(t-(j+1)\Delta_{(p)}), \qquad j = 0, 1, \dots, m-2, \quad \lambda^m = 1. \tag{1.24}$$

By (1.16), the component $h_{*,1}(t)$ is a solution of the linear inhomogeneous equation

$$\dot h = -\mu_0 h + A_{0,p}(t)h + \sum_{k=1}^{s} A_{k,p}(t) h_{*,m-k+1}(t).$$

Since the functions $h_{*,m-k+1}(t)$ are expressed in terms of $h_{*,1}(t)$ by the following equalities (see (1.24)):

$$h_{*,m-k+1}(t) = \lambda^k h_{*,1}(t-k\Delta_{(p)}), \qquad k = 1, \dots, s,$$

the component $h_{*,1}(t)$ also satisfies a delay equation

$$\dot h = -\mu_0 h + \sum_{k=0}^{s} \lambda^k A_{k,p}(t) h(t-k\Delta_{(p)}). \tag{1.25}$$

Our construction shows that the equation (1.25) certainly has multiplier 1. We make the change $\exp(\mu_0 t)h \to h$ in this equation. Then the multiplier 1 is transformed to a multiplier $\exp(m\mu_0\Delta_{(p)}/p)$, and the equation (1.25) to the equation (1.17) with $\varkappa = \lambda \cdot \exp(\mu_0 \Delta_{(p)})$. Thus there necessarily exists a number l_0 such that

$$\nu_{l_0}(\varkappa)|_{\varkappa=\lambda\cdot\exp(\mu_0\Delta_{(p)})} = \exp\frac{m\mu_0\Delta_{(p)}}{p} = \nu.$$

The desired relations (1.19), (1.20) follow from this and the obvious equality $\nu^p = \varkappa^m$.

The arguments are similar in the case when the multiplier ν is multiple. Indeed, suppose that for a given multiplier there are exactly k_0 linearly independent Lyapunov–Floquet solutions. Then these solutions can be written in the matrix form $\exp(\mu_0 t)H(t)$, where the columns of the $mn \times k_0$ matrix $H(t)$ are linearly independent $T_{(p)}$-periodic vector-valued functions. Furthermore, since the system (1.16) is invariant under the changes (1.22), the following equality holds instead of (1.23):

$$\Lambda H(t+\Delta_{(p)}) = H(t)D \tag{1.26}$$

for some non-degenerate constant $k_0 \times k_0$ matrix D.

The property (1.26) enables us to reduce the justification of the formulae (1.19), (1.20) to the previous case. To do this, we fix an eigenvalue λ of the matrix D and denote the corresponding eigenvector by e. Then we easily see that (1.23) holds for the vector-valued function $h_*(t) = H(t)e$. Subsequent arguments are as above.

Thus we have established that every multiplier ν of the system (1.16) can be written in the form (1.20), where $\varkappa_0$ satisfies the equation (1.19) with number l_0. We now prove the converse. Suppose that the equation (1.19) with number $l = l_0$ admits a root $\varkappa = \varkappa_0 \neq 0$. Then the equation

$$\dot{h} = -\mu_0 h + \sum_{k=0}^{s} \varkappa_0^k \exp(-\mu_0 k\Delta_{(p)}) A_{k,p}(t) h(t - k\Delta_{(p)}) \tag{1.27}$$

with

$$\mu_0 = \frac{p}{m\Delta_{(p)}}(\ln \rho_0 + i\varphi_0), \qquad \rho_0 > 0, \quad 0 \leqslant \varphi_0 < 2\pi, \tag{1.28}$$

where $\rho_0 \exp(i\varphi_0) = \nu_{l_0}(\varkappa_0)$, admits a non-trivial $T_{(p)}$-periodic solution $h = \widetilde{h}(t)$. Furthermore, we put

$$\lambda = \varkappa_0 \exp(-\mu_0 \Delta_{(p)}) \tag{1.29}$$

and note that this value of λ satisfies the requirement $\lambda^m = 1$ (see (1.24)) because of (1.28) and the formula $\varkappa_0^m = [\nu_{l_0}(\varkappa_0)]^p$. It follows that the equations (1.27) and (1.25) coincide for this choice of λ.

To complete the proof, we consider the function $h_{*,1}(t) = \widetilde{h}(t)$ and define the remaining components $h_{*,j}(t)$, $j = 2, \dots, m$, by the equalities (1.24), (1.29). Using the relation (established above) between the equations (1.27) and (1.25), we obtain a Lyapunov–Floquet solution (1.21) of the original system (1.16) corresponding to the multiplier

$$\nu = \exp \frac{m\mu_0\Delta_{(p)}}{p} = \nu_{l_0}(\varkappa_0). \qquad \square$$

The lemmas established above give a general approach to the study of periodic solutions of travelling-wave type in the circular systems (1.10). Indeed, the problem of the existence of cycles of the form (1.13) reduces to finding a cycle $x_*(t, \Delta)$ of the auxiliary delay equation (1.11) and searching for roots of the equations (1.12). The problem of the stability of travelling waves is studied separately and, by Lemma 1.2, consists in analyzing the disposition of the roots of the equations (1.19). Although the system (1.19) generally contains a countable number of equations, the set of their non-zero roots is certainly finite (otherwise the finite-dimensional system (1.16) would have a countable number of distinct multipliers, which is impossible).

The analysis of the auxiliary equations (1.11), (1.17), which are basic in our approach, generally leads to non-local problems. But they are tractable in some cases of applicability of certain asymptotic methods. This happens in the case of the systems (1.4), (1.5) under conditions (1.9).

§ 2. Investigation of auxiliary equations

2.1. Analysis of an auxiliary non-linear equation. Consider the system (1.4) and make the change of variables

$$u_j = \exp x_j, \qquad j = 1, 2, \dots, m. \tag{2.1}$$

On account of (1.9), this system takes the form

$$\dot{x}_j = -1 + \frac{\alpha \exp(-x_j)}{\Omega(x_{j-1}, \dots, x_{j-s}, \varepsilon)}, \qquad j = 1, 2, \dots, m, \tag{2.2}$$

where $x_{-k} = x_{m-k}$, $k = 0, 1, \dots, s-1$, and the function Ω is defined by the formula

$$\Omega(y_1, \dots, y_s, \varepsilon) = 1 + \sum_{k=1}^{s} \delta_k \exp \frac{\gamma_k^0 y_k}{\varepsilon}. \tag{2.3}$$

We now pass from (2.2) to the corresponding auxiliary delay equation

$$\dot{x} = -1 + \frac{\alpha \exp(-x)}{\Omega(x(t-\Delta), \dots, (t-s\Delta), \varepsilon)}. \tag{2.4}$$

Our aim is to prove that this equation has a non-trivial periodic solution for every fixed value of the parameter $\Delta > 0$ and for all ϵ, $0 < \varepsilon \ll 1$.

The study of (2.4) simplifies since it admits a limit object as $\varepsilon \to 0$. Indeed, using the obvious equality

$$\lim_{\varepsilon \to 0} \frac{1}{\Omega(y_1, \dots, y_s, \varepsilon)} = R(y), \tag{2.5}$$

where $y_k = \text{const} \neq 0$, $k = 1, \dots, s$,

$$y = \max\{y_1, y_2, \dots, y_s\}, \qquad R(y) = \begin{cases} 0 & \text{for } y > 0, \\ 1 & \text{for } y < 0, \end{cases} \tag{2.6}$$

we see that the equation (2.4) passes to the following delay relay equation as $\varepsilon \to 0$:

$$\dot{x} = -1 + \alpha \exp(-x) R(y), \qquad y = \max\bigl\{x(t-\Delta), x(t-2\Delta), \dots, x(t-s\Delta)\bigr\}. \tag{2.7}$$

As in [15]–[17], we define the notion of a solution of (2.7) constructively. Fix a number

$$\sigma_0\colon 0 < \sigma_0 < \ln\biggl(1 + \frac{1}{\beta}\biggr), \tag{2.8}$$

where $\beta = \alpha - 1 > 0$ (see (1.9)), put

$$\theta(t) = \ln\bigl[1 + \beta(1 - \exp(-t))\bigr], \qquad t \in \biggl(-\ln\biggl(1 + \frac{1}{\beta}\biggr), +\infty\biggr), \tag{2.9}$$

and consider the set of functions

$$\begin{gathered} \varphi(t) \in C[-s\Delta - \sigma_0, -\sigma_0], \qquad \varphi(t) < 0 \quad \forall\, t \in [-s\Delta - \sigma_0, -\sigma_0], \\ \varphi(-\sigma_0) = \theta(-\sigma_0). \end{gathered} \tag{2.10}$$

We stress that the family (2.10) is well defined since, by (2.8), the point $t = -\sigma_0$ belongs to the domain of the function (2.9) and $\theta(-\sigma_0) < 0$.

We write $x_\varphi(t)$, $t \geqslant -\sigma_0$, for a solution of the equation (2.7) with an arbitrary initial function (2.10), and we shall construct this solution by the method of steps, successively considering time intervals of length Δ.

Consider the interval $t \in [-\sigma_0, \Delta-\sigma_0]$ and note that $\varphi(t-k\Delta)<0$, $k=1,\dots,s$, for these values of t. Therefore, in accordance with (2.6), (2.7), the function $x_\varphi(t)$ is a solution on this interval of the Cauchy problem $\dot{x} = -1+\alpha\exp(-x)$, $x|_{t=-\sigma_0} = \theta(-\sigma_0)$ and, therefore,

$$x_\varphi(t) = \theta(t). \tag{2.11}$$

Moving forward with step Δ, we can 'extend' the formula (2.11) in t as soon as we have

$$x_\varphi(t-k\Delta) < 0, \qquad k = 1, 2, \dots, s. \tag{2.12}$$

Thus this formula holds on the half-open interval $-\sigma_0 \leqslant t < \Delta$.

When $t = \Delta$, one of the conditions (2.12) fails for the first time, and a switching occurs. Namely, for $\Delta \leqslant t \leqslant 2\Delta$ the solution $x_\varphi(t)$ is defined from the Cauchy problem $\dot{x} = -1$, $x|_{t=\Delta} = \theta(\Delta)$, that is, by the equality

$$x_\varphi(t) = \theta(\Delta) + \Delta - t. \tag{2.13}$$

The formula (2.13) holds as soon as we have

$$y_\varphi(t) = \max\bigl\{x_\varphi(t-\Delta), x_\varphi(t-2\Delta), \dots, x_\varphi(t-s\Delta)\bigr\} > 0. \tag{2.14}$$

It follows from (2.11), (2.13) that the condition (2.14) holds on the interval $\Delta < t < s\Delta + t_0$, where

$$t_0 = \theta(\Delta) + \Delta > \Delta. \tag{2.15}$$

Indeed, if we assume *a priori* that (2.14) holds, then (2.11), (2.13) imply that the domains of positivity of the functions $x_\varphi(t-k\Delta)$, $k=1,\dots,s$, are the intervals $(k\Delta, k\Delta+t_0)$. Since the union of these intervals is equal to the interval $(s\Delta, s\Delta+t_0)$, we see that (2.14) does indeed hold on this interval.

As t passes through the value $t = s\Delta + t_0$, the function (2.14) changes sign and the next switching occurs. In this case by (2.7), (2.13), (2.15) we have the Cauchy problem $\dot{x} = -1+\alpha\exp(-x)$, $x|_{t=s\Delta+t_0} = -s\Delta$, whose solution is given by the formulae

$$\begin{gathered} x_\varphi(t) = \theta(t - T_0), \\ T_0 = (s+1)\Delta + \ln\bigl[1+\beta(1-\exp(-\Delta))\bigr] + \ln\biggl[1+\frac{1}{\beta}(1-\exp(-s\Delta))\biggr]. \end{gathered} \tag{2.16}$$

We discuss separately the domains of validity of the formulae (2.16). First we specify the choice of the parameter σ_0 in (2.10). In addition to (2.8), we adopt the requirement

$$\sigma_0 < \ln\biggl[1+\frac{1}{\beta}(1-\exp(-s\Delta))\biggr], \tag{2.17}$$

which means that $\sigma_0 < T_0 - s\Delta - t_0$. Recall that the relations (2.16) were obtained under the *a priori* hypothesis $y_\varphi(t) < 0$. By the equalities (2.13)–(2.16) and the condition (2.17), this hypothesis certainly holds on the time interval $s\Delta + t_0 < t \leqslant T_0 - \sigma_0$.

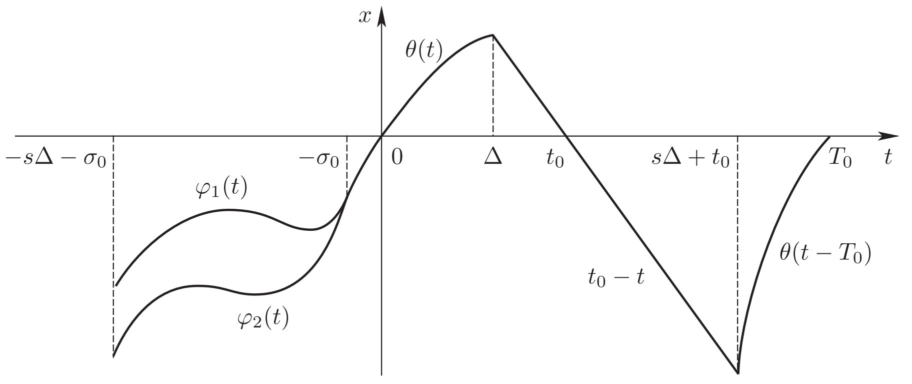

Figure 2

To conclude our construction of the solution $x_\varphi(t)$, we note that the condition (2.17) on σ_0 implies that the function $x_\varphi(t+T_0)$, $-s\Delta-\sigma_0\leqslant t\leqslant-\sigma_0$, belongs to the initial set of functions (2.10). This means that the whole process repeats when $t\geqslant T_0-\sigma_0$. Moreover, it follows from (2.11), (2.13), (2.16) that every solution $x_\varphi(t)$ with initial condition (2.10) coincides for all $t\geqslant-\sigma_0$ with the same T_0-periodic function (see Fig. 2)

$$x_0(t)=\begin{cases}\theta(t) & \text{for } 0\leqslant t\leqslant\Delta,\\ t_0-t & \text{for } \Delta\leqslant t\leqslant s\Delta+t_0,\\ \theta(t-T_0) & \text{for } s\Delta+t_0\leqslant t\leqslant T_0,\end{cases}\qquad x_0(t+T_0)\equiv x_0(t). \tag{2.18}$$

We now discuss the connection between periodic solutions of the equations (2.4) and (2.7). The following assertion is deduced from the general results in [15] on the C^1-closeness of the trajectories of the relay system and the relaxation system and from analogous results contained in [12]–[14], [16], [17].

Lemma 2.1. *Suppose that the parameter σ_0 satisfies the conditions (2.8), (2.17) and the other parameters satisfy (1.9). Then there is a sufficiently small $\varepsilon_0>0$ such that for all ε, $0<\varepsilon\leqslant\varepsilon_0$, the equation (2.4) has an exponentially orbitally stable cycle $x=x_*(t,\varepsilon)$, $x_*(-\sigma_0,\varepsilon)\equiv\theta(-\sigma_0)$ of period $T_*(\varepsilon)$, satisfying the limit equalities*

$$\lim_{\varepsilon\to0}\max_{0\leqslant t\leqslant T_*(\varepsilon)}|x_*(t,\varepsilon)-x_0(t)|=0,\qquad \lim_{\varepsilon\to0}T_*(\varepsilon)=T_0. \tag{2.19}$$

This lemma is proved (and similar assertions are thoroughly justified) in [12]–[17]. We restrict ourselves to the minimal results on the asymptotic behaviour of the cycle $x_*(t,\varepsilon)$. Besides the general properties (2.19) we need the following facts.

1) The equation $x_*(t,\varepsilon)=0$ has exactly two simple roots $\tau_j=\tau_j(\varepsilon)$, $j=1,2$, on the interval $-\sigma_0\leqslant t\leqslant T_0-\sigma_0/2$. They satisfy the asymptotic formulae

$$\tau_1=O(\varepsilon),\quad \tau_2=t_0+O(\varepsilon),\qquad \varepsilon\to0. \tag{2.20}$$

2) We have

$$\max_{-\sigma_0\leqslant t\leqslant T_0-\sigma_0/2}|x_*(t,\varepsilon)-x_0(t)|=O(\varepsilon),\qquad \max_{t\in\Sigma}|\dot x_*(t,\varepsilon)-\dot x_0(t)|=O(\varepsilon),$$
$$T_*(\varepsilon)=T_0+O(\varepsilon),\qquad \varepsilon\to 0, \tag{2.21}$$

where the set Σ is the closed interval $[-\sigma_0, T_0-\sigma_0/2]$ with the following open intervals deleted:

$$\bigl(\Delta+\tau_1(\varepsilon)-\sqrt{\varepsilon},\,\Delta+\tau_1(\varepsilon)+\sqrt{\varepsilon}\bigr),\qquad \bigl(s\Delta+\tau_2(\varepsilon)-\sqrt{\varepsilon},\,s\Delta+\tau_2(\varepsilon)+\sqrt{\varepsilon}\bigr),\qquad j=1,2.$$

2.2. Analysis of the auxiliary linear equation. In this subsection we study the asymptotic behaviour of the multipliers of the following linear equation analogous to (1.17):

$$\dot h=A_{0,*}(t,\varepsilon)h+\sum_{k=1}^{s}\varkappa^k A_{k,*}(t,\varepsilon)h(t-k\Delta). \tag{2.22}$$

Here the coefficients are given by the formulae

$$A_{0,*}(t,\varepsilon)=-\frac{\alpha\exp(-x_*(t,\varepsilon))}{\Omega_*(t,\varepsilon)}, \tag{2.23}$$

$$A_{k,*}(t,\varepsilon)=-\frac{\alpha\exp(-x_*(t,\varepsilon))}{\Omega_*^2(t,\varepsilon)}\,\frac{\delta_k\gamma_k^0}{\varepsilon}\exp\frac{\gamma_k^0x_*(t-k\Delta,\varepsilon)}{\varepsilon},\qquad k=1,\dots,s,$$
$$\Omega_*(t,\varepsilon)=1+\sum_{l=1}^{s}\delta_l\exp\frac{\gamma_l^0\,x_*(t-l\Delta,\varepsilon)}{\varepsilon}, \tag{2.24}$$

where $\varkappa$ is an arbitrary complex parameter.

The following assertion describes some properties of the coefficients (2.23), (2.24) which will be used later.

Lemma 2.2. *For all sufficiently small $\varepsilon>0$ we have bounds of the form*

$$\max_{-\sigma_0\leqslant t\leqslant T_*(\varepsilon)-\sigma_0}|A_{0,*}(t,\varepsilon)|\leqslant M_1,\qquad \max_{-\sigma_0\leqslant t\leqslant \Delta-\sigma_0}|A_{1,*}(t,\varepsilon)|\leqslant M_2\exp\frac{-q_1}{\varepsilon},$$
$$\max_{t\in\Sigma_1}|A_{1,*}(t,\varepsilon)|\leqslant M_3\exp\frac{-q_2}{\sqrt{\varepsilon}}; \tag{2.25}$$

$$\max_{-\sigma_0\leqslant t\leqslant T_*(\varepsilon)-\sigma_0}|A_{k,*}(t,\varepsilon)|\leqslant M_{k+2}\exp\frac{-q_{k+1}}{\varepsilon},\qquad k=2,\dots,s-1; \tag{2.26}$$

$$\max_{-\sigma_0\leqslant t\leqslant s\Delta-\sigma_0}|A_{s,*}(t,\varepsilon)|\leqslant M_{s+2}\exp\frac{-q_{s+1}}{\varepsilon},$$
$$\max_{t\in\Sigma_2}|A_{s,*}(t,\varepsilon)|\leqslant M_{s+3}\exp\frac{-q_{s+2}}{\sqrt{\varepsilon}}, \tag{2.27}$$

where

$$\Sigma_1=[-\sigma_0,T_*(\varepsilon)-\sigma_0]\setminus\bigl(\Delta+\tau_1(\varepsilon)-\sqrt{\varepsilon},\,\Delta+\tau_1(\varepsilon)+\sqrt{\varepsilon}\,\bigr),$$
$$\Sigma_2=[-\sigma_0,T_*(\varepsilon)-\sigma_0]\setminus\bigl(s\Delta+\tau_2(\varepsilon)-\sqrt{\varepsilon},\,s\Delta+\tau_2(\varepsilon)+\sqrt{\varepsilon}\,\bigr) \tag{2.28}$$

and the constants $M_k > 0$, $k = 1, \dots, s+3$, $q_k > 0$, $k = 1, \dots, s+2$, *are independent of* ε. *Moreover, we have the following asymptotic formulae as* $\varepsilon \to 0$:

$$\begin{aligned} \int_{\Delta+\tau_1(\varepsilon)-\sqrt{\varepsilon}}^{\Delta+\tau_1(\varepsilon)+\sqrt{\varepsilon}} A_{1,*}(t,\varepsilon)\,dt &= -\frac{1+1/\beta}{1+\beta(1-\exp(-\Delta))} + O(\varepsilon), \\ \int_{\Delta+\tau_1(\varepsilon)-\sqrt{\varepsilon}}^{\Delta+\tau_1(\varepsilon)+\sqrt{\varepsilon}} |A_{1,*}(t,\varepsilon)|\,dt &= \frac{1+1/\beta}{1+\beta(1-\exp(-\Delta))} + O(\varepsilon), \end{aligned} \tag{2.29}$$

$$\begin{aligned} \int_{s\Delta+\tau_2(\varepsilon)-\sqrt{\varepsilon}}^{s\Delta+\tau_2(\varepsilon)+\sqrt{\varepsilon}} A_{s,*}(t,\varepsilon)\,dt &= -(\beta+1)\exp(s\Delta) + O(\varepsilon), \\ \int_{s\Delta+\tau_2(\varepsilon)-\sqrt{\varepsilon}}^{s\Delta+\tau_2(\varepsilon)+\sqrt{\varepsilon}} |A_{s,*}(t,\varepsilon)|\,dt &= (\beta+1)\exp(s\Delta) + O(\varepsilon). \end{aligned} \tag{2.30}$$

Proof. First, note that the first bound in (2.25) is an obvious corollary of (2.23) and the general asymptotic properties (2.19) of the periodic solution $x_*(t,\varepsilon)$. Furthermore, by (2.19), (2.24) the coefficient $A_{k,*}(t,\varepsilon)$ admits the following estimate for all $t \in [-\sigma_0, T_*(\varepsilon)-\sigma_0]$ satisfying $x_0(t-k\Delta) \neq 0$ (that is, outside some fixed neighbourhoods of the points $t = k\Delta$ and $t = k\Delta + t_0$):

$$|A_{k,*}(t,\varepsilon)| \leqslant M \exp\frac{-q}{\varepsilon}, \qquad M, q = \text{const} > 0. \tag{2.31}$$

This estimate remains valid in a neighbourhood of the point $t = k\Delta$ provided that $x_0(k\Delta - l_0\Delta) > 0$ for some $l_0 \neq k$, $1 \leqslant l_0 \leqslant s$. Moreover, (2.31) holds in a neighbourhood of $t = k\Delta + t_0$ if there is a number $l_0 \neq k$, $1 \leqslant l_0 \leqslant s$, such that $x_0(k\Delta + t_0 - l_0\Delta) > 0$.

Combining these observations, we conclude that the bound (2.31) when $k=1$ and $s \geqslant 2$ holds in a neighbourhood of the point $t = \Delta + t_0$ since $x_0(\Delta + t_0 - l\Delta) > 0$ when $l = 2$. Furthermore, when $2 \leqslant k \leqslant s-1$, this bound holds on the whole closed interval $[-\sigma_0, T_*(\varepsilon) - \sigma_0]$ since $x_0(k\Delta - l\Delta) > 0$ when $l = k-1$ and $x_0(k\Delta + t_0 - l\Delta) > 0$ when $l = k+1$. Finally, when $k = s$ and $s \geqslant 2$, the bound holds in a neighbourhood of $t = s\Delta$ since $x_0(s\Delta - l\Delta) > 0$ for $l = s-1$. This automatically yields the second inequality in (2.25), the bounds (2.26), and the first inequality in (2.27).

To obtain the bounds in (2.25), (2.27) on the sets (2.28), we recall that the inequalities (2.31) with $k = 1$ and $k = s$ can be violated only in the neighbourhoods of the points $t = \Delta$ and $t = s\Delta + t_0$ respectively. Thus it suffices to consider only the suspicious sets $\Sigma_1 \cap [\Delta - \sigma_1, \Delta + \sigma_1]$ and $\Sigma_2 \cap [s\Delta + t_0 - \sigma_1, s\Delta + t_0 + \sigma_1]$, where σ_1 is any fixed number in the interval $(0, \min(t_0, T_0 - t_0))$. We now recall the finer asymptotic properties (2.20), (2.21) of the function $x_*(t,\varepsilon)$ and use them in (2.24). As a result, we see that

$$|A_{1,*}(t,\varepsilon)| \leqslant \frac{M}{\varepsilon}\exp\frac{-\gamma_1^0\,|x_0(t-\Delta)|}{\varepsilon}, \qquad t \in \Sigma_1 \cap [\Delta - \sigma_1,\ \Delta + \sigma_1];$$

$$|A_{s,*}(t,\varepsilon)| \leqslant \frac{M}{\varepsilon}\exp\frac{-\gamma_s^0\,|x_0(t-s\Delta)|}{\varepsilon}, \qquad t \in \Sigma_2 \cap [s\Delta + t_0 - \sigma_1,\ s\Delta + t_0 + \sigma_1],$$

where $M = \text{const} > 0$. Since $x_0(0) = 0$, $\dot{x}_0(0) > 0$, $x_0(t_0) = 0$, $\dot{x}_0(t_0) = -1$ (see (2.18)), the desired inequalities in (2.25), (2.27) follow in an obvious way.

We now pass to the asymptotic formulae (2.29), (2.30). For example, let us prove (2.29) (the case (2.30) is treated similarly). To do this, we introduce a new variable τ on the interval $\Delta+\tau_1-\sqrt{\varepsilon}\leqslant t\leqslant\Delta+\tau_1+\sqrt{\varepsilon}$ by the formula $\tau=(t-\tau_1-\Delta)/\varepsilon$. Then $x_*(t-\Delta,\varepsilon)/\varepsilon$ takes the form $x_*(\tau_1+\varepsilon\tau,\varepsilon)/\varepsilon$. Since $x_*(\tau_1,\varepsilon)=0$, this function can be written as

$$\frac{x_*(\tau_1+\varepsilon\tau,\varepsilon)}{\varepsilon}=\dot{x}_*(\tau_1+\varepsilon\overline{\tau},\varepsilon)\tau, \tag{2.32}$$

where $\overline{\tau}$ is such that $|\overline{\tau}|\leqslant|\tau|$. Using in (2.32) the asymptotic formulae (2.20), (2.21) and the obvious bound $|\overline{\tau}|\leqslant 1/\sqrt{\varepsilon}$, we conclude that

$$\frac{x_*(\tau_1+\varepsilon\tau,\varepsilon)}{\varepsilon}=\beta(1+(|\tau|+1)O(\varepsilon))\tau, \tag{2.33}$$

where the remainder term is uniform with respect to τ.

At the final step, we substitute in (2.24) for $k=1$ the relation (2.33) along with the formulae

$$\exp\frac{\gamma_l^0 x_*(t-l\Delta,\varepsilon)}{\varepsilon}=O\biggl(\exp\frac{-q}{\varepsilon}\biggr),\qquad q=\mathrm{const}>0,\quad l=2,\dots,s,$$

which hold for the values of t in question. As a result, we see that

$$\begin{aligned}&\biggl|\varepsilon A_{1,*}(\tau_1+\Delta+\varepsilon\tau,\varepsilon)+\frac{\alpha\delta_1\gamma_1^0\exp(-\theta(\Delta))\exp(\gamma_1^0\beta\tau)}{(1+\delta_1\exp(\gamma_1^0\beta\tau))^2}\biggr|\\&\qquad\leqslant M\varepsilon(|\tau|+1)^2\exp(-\gamma_1^0\beta|\tau|),\qquad -\frac{1}{\sqrt{\varepsilon}}\leqslant\tau\leqslant\frac{1}{\sqrt{\varepsilon}}.\end{aligned} \tag{2.34}$$

We use the bound (2.34) directly to calculate the integrals in (2.29). Thus we get the required asymptotic formulae

$$\begin{aligned}\int_{\Delta+\tau_1(\varepsilon)-\sqrt{\varepsilon}}^{\Delta+\tau_1(\varepsilon)+\sqrt{\varepsilon}}A_{1,*}(t,\varepsilon)\,dt&=-\int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}}\frac{\alpha\delta_1\gamma_1^0\exp(-\theta(\Delta))\exp(\gamma_1^0\beta\tau)}{(1+\delta_1\exp(\gamma_1^0\beta\tau))^2}\,d\tau+O(\varepsilon)\\&=-\frac{\alpha\exp(-\theta(\Delta))}{\beta}+O(\varepsilon)=-\frac{1+1/\beta}{1+\beta(1-\exp(-\Delta))}+O(\varepsilon),\\\int_{\Delta+\tau_1(\varepsilon)-\sqrt{\varepsilon}}^{\Delta+\tau_1(\varepsilon)+\sqrt{\varepsilon}}|A_{1,*}(t,\varepsilon)|\,dt&=\int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}}\frac{\alpha\delta_1\gamma_1^0\exp(-\theta(\Delta))\exp(\gamma_1^0\beta\tau)}{(1+\delta_1\exp(\gamma_1^0\beta\tau))^2}\,d\tau+O(\varepsilon)\\&=\frac{1+1/\beta}{1+\beta(1-\exp(-\Delta))}+O(\varepsilon).\qquad\square\end{aligned}$$

Another auxiliary assertion concerns the asymptotic behaviour of the solution $\widetilde{h}(t,\varkappa,\varepsilon)$ of the equation (2.22) with an arbitrary $\varkappa$-independent initial function $h_0(t)$ in the space (over the field of complex numbers)

$$C_0=\bigl\{h_0(t)\in C[-s\Delta-\sigma_0,-\sigma_0]\colon h_0(-\sigma_0)=0\bigr\}. \tag{2.35}$$

The norm in (2.35) is given by the formula

$$\|h_0\|=\max_{-s\Delta-\sigma_0\leqslant t\leqslant-\sigma_0}|h_0(t)|.$$

Lemma 2.3. *For every $r>0$ one can find positive constants $M=M(r)$, $\varepsilon_0=\varepsilon_0(r)$, $q=q(r)$ such that the following bound holds for all ϵ, $0<\varepsilon\leqslant\varepsilon_0$, $h_0(t)\in C_0$ and $\varkappa\in B(r)$, where $B(r):=\{\varkappa\in\mathbb{C}\colon |\varkappa|\leqslant r\}$:*

$$\max_{-\sigma_0\leqslant t\leqslant T_*(\varepsilon)-\sigma_0}\biggl(|\widetilde{h}(t,\varkappa,\varepsilon)|+\biggl|\frac{\partial\widetilde{h}}{\partial\varkappa}(t,\varkappa,\varepsilon)\biggr|\biggr)\leqslant M\exp\frac{-q}{\varepsilon}\|h_0\|. \tag{2.36}$$

Proof. We first consider the interval $-\sigma_0\leqslant t\leqslant\Delta-\sigma_0$, on which the solution $\widetilde{h}(t,\varkappa,\varepsilon)$ is given by the explicit formula

$$\widetilde{h}(t,\varkappa,\varepsilon)=\int_{-\sigma_0}^{t}\exp\biggl(\int_{\tau}^{t}A_{0,*}(\sigma,\varepsilon)\,d\sigma\biggr)\biggl[\sum_{k=1}^{s}\varkappa^k A_{k,*}(\tau,\varepsilon)h_0(\tau-k\Delta)\biggr]d\tau, \qquad -\sigma_0\leqslant t\leqslant\Delta-\sigma_0. \tag{2.37}$$

This formula yields that $\widetilde{h}(t,\varkappa,\varepsilon)$ is analytic in $\varkappa\in B(r)$. Furthermore, by (2.25)–(2.27), the coefficient $A_{0,*}(t,\varepsilon)$ is bounded on this time interval and all the coefficients $A_{k,*}(t,\varepsilon)$, $k=1,\dots,s$, are exponentially small (they admit bounds of the form (2.31)). Therefore we conclude that

$$\max_t\biggl(|\widetilde{h}|+\biggl|\frac{\partial\widetilde{h}}{\partial\varkappa}\biggr|\biggr)\leqslant M\exp\frac{-q}{\varepsilon}\|h_0\|, \tag{2.38}$$

where the maximum is taken over $t\in[-\sigma_0,\Delta-\sigma_0]$, and $M,q>0$ are certain universal constants.

We use the method of steps to extend the bound (2.38) to the interval $[\Delta-\sigma_0, s\Delta-\sigma_0]$ of the variable t. Divide it into the intervals $[l\Delta-\sigma_0,(l+1)\Delta-\sigma_0]$, $l=1,\dots,s-1$, and assume that the required inequality holds for $t\in[-\sigma_0,l\Delta-\sigma_0]$. Then on the lth interval $[l\Delta-\sigma_0,(l+1)\Delta-\sigma_0]$ we have the following explicit formula for $\widetilde{h}(t,\varkappa,\varepsilon)$ in analogy with (2.37):

$$\begin{aligned}\widetilde{h}(t,\varkappa,\varepsilon)&=\widetilde{h}(l\Delta-\sigma_0,\varkappa,\varepsilon)\exp\biggl(\int_{l\Delta-\sigma_0}^{t}A_{0,*}(\sigma,\varepsilon)\,d\sigma\biggr)\\&\quad+\int_{l\Delta-\sigma_0}^{t}\exp\biggl(\int_{\tau}^{t}A_{0,*}(\sigma,\varepsilon)\,d\sigma\biggr)\biggl[\sum_{k=1}^{l}\varkappa^k A_{k,*}(\tau,\varepsilon)\widetilde{h}(\tau-k\Delta,\varkappa,\varepsilon)\\&\quad+\sum_{k=l+1}^{s}\varkappa^k A_{k,*}(\tau,\varepsilon)h_0(\tau-k\Delta)\biggr]d\tau, \qquad t\in[l\Delta-\sigma_0,\ (l+1)\Delta-\sigma_0].\end{aligned} \tag{2.39}$$

To analyze the right-hand side of (2.39), we take into account that the functions $\widetilde{h}(t-k\Delta,\varkappa,\varepsilon)$, $k=1,\dots,l$, and their derivatives with respect to $\varkappa$ are exponentially small on the interval $t\in[-\sigma_0,l\Delta-\sigma_0]$ by our assumption (see (2.38)). Combining all the properties (2.25)–(2.30), we obtain that the coefficients of the equation (2.22) are integrally bounded:

$$\int_{-\sigma_0}^{T_*(\varepsilon)-\sigma_0}\biggl(\sum_{k=0}^{s}|A_{k,*}(t,\varepsilon)|\biggr)dt\leqslant M, \qquad M=\mathrm{const}>0. \tag{2.40}$$

The group of coefficients $A_{k,*}(t,\varepsilon)$, $k = l+1,\dots,s$, satisfies bounds of the form (2.31) for $t \in [l\Delta - \sigma_0, (l+1)\Delta - \sigma_0]$. Combining all the facts listed above, we see that the required inequality (2.38) holds at the next step.

For $t \in [s\Delta - \sigma_0, T_*(\varepsilon) - \sigma_0]$ we consider the equation (2.22) with the completely recycled initial condition $\widetilde{h}(t,\varkappa,\varepsilon)$, $-\sigma_0 \leqslant t \leqslant s\Delta - \sigma_0$, satisfying the required estimate (2.38). We use the method of steps to obtain this estimate on the remaining interval of the variable t. To do this, we divide this interval into pieces of the form $[s\Delta - \sigma_0 + k\Delta, s\Delta - \sigma_0 + (k+1)\Delta]$, $k = 0,1,\dots,k_0$, and $[s\Delta - \sigma_0 + (k_0+1)\Delta, T_*(\varepsilon) - \sigma_0]$, where $k_0 = \lfloor T_*(\varepsilon)/\Delta - s - 1 \rfloor$ and $\lfloor \cdot \rfloor$ is the integer part. Furthermore, writing the analogue of (2.39) at the kth step, we first establish the analyticity of $\widetilde{h}(t,\varkappa,\varepsilon)$. Then, using the integral boundedness property (2.40), we extend the required estimate (2.38) one step further. □

We now proceed directly to the asymptotic calculation of multipliers of the equation (2.22). Consider the monodromy operator $U(\varkappa,\varepsilon)$ of this equation. It acts on the space $C[-s\Delta - \sigma_0, -\sigma_0]$ (over the field of complex numbers) by the rule

$$U(\varkappa,\varepsilon)\,h_0 = h(t + T_*(\varepsilon),\varkappa,\varepsilon), \qquad -s\Delta - \sigma_0 \leqslant t \leqslant -\sigma_0, \tag{2.41}$$

where $h(t,\varkappa,\varepsilon)$, $-\sigma_0 \leqslant t \leqslant T_*(\varepsilon) - \sigma_0$, is a solution of (2.22) with initial function $h_0(t)$, $-s\Delta - \sigma_0 \leqslant t \leqslant -\sigma_0$. Let $\nu_l(\varkappa,\varepsilon)$, $l \in \mathbb{N}$, be the eigenvalues of the operator (2.41) indexed in decreasing order of absolute value.

Lemma 2.4. *For every $r > 0$ there are constants $\varepsilon_0 = \varepsilon_0(r) > 0$, $M = M(r) > 0$, $q = q(r) > 0$, such that for all ε, $0 < \varepsilon \leqslant \varepsilon_0$, and $\varkappa \in B(r)$ we have*

$$\sup_{l \geqslant 2} |\nu_l(\varkappa,\varepsilon)| \leqslant M \exp\frac{-q}{\varepsilon}. \tag{2.42}$$

As $\varepsilon \to 0$, the multiplier $\nu_1(\varkappa,\varepsilon)$ admits the following asymptotic representation uniformly in $\varkappa \in B(r)$:

$$\nu_1(\varkappa,\varepsilon) = [(\omega_1+1)\varkappa - \omega_1][(\omega_2+1)\varkappa^s - \omega_2] + O(\varepsilon), \tag{2.43}$$

where

$$\omega_1 = \frac{\beta\exp(-\Delta)}{1+\beta(1-\exp(-\Delta))}, \qquad \omega_2 = \frac{\exp(-s\Delta)}{\beta + 1 - \exp(-s\Delta)}. \tag{2.44}$$

Proof. We fix any positive r and assume that the parameter $\varkappa$ in (2.22) belongs to the set $B(r)$. Consider the finite-dimensional operator

$$V(\varkappa,\varepsilon)\,h_0 = h_0(-\sigma_0)h_*(t + T_*(\varepsilon),\varkappa,\varepsilon), \qquad -s\Delta - \sigma_0 \leqslant t \leqslant -\sigma_0, \tag{2.45}$$

where $h_*(t,\varkappa,\varepsilon)$ is a solution of (2.22) on the interval $-\sigma_0 \leqslant t \leqslant T_*(\varepsilon) - \sigma_0$ with the initial function $h_* \equiv 1$, $-s\Delta - \sigma_0 \leqslant t \leqslant -\sigma_0$.

To describe the relation between the operators (2.41) and (2.45), we consider the function

$$\widetilde{h}(t,\varkappa,\varepsilon) = h(t,\varkappa,\varepsilon) - h_0(-\sigma_0)h_*(t,\varkappa,\varepsilon) \tag{2.46}$$

and note that for $t \in [-\sigma_0, T_*(\varepsilon) - \sigma_0]$ it is also a solution of (2.22). Since we obviously have $\widetilde{h}(t,\varkappa,\varepsilon) = h_0(t) - h_0(-\sigma_0) \in C_0$ for $-s\Delta - \sigma_0 \leqslant t \leqslant -\sigma_0$, where

C_0 is the space (2.35), we can apply the bound (2.36) to (2.46). It follows from this bound that the operator $W(\varkappa,\varepsilon)=U(\varkappa,\varepsilon)-V(\varkappa,\varepsilon)$ satisfies the inequality

$$\|W(\varkappa,\varepsilon)\|_{C[-s\Delta-\sigma_0,-\sigma_0]\to C[-s\Delta-\sigma_0,-\sigma_0]} + \left\|\frac{\partial}{\partial\varkappa}W(\varkappa,\varepsilon)\right\|_{C[-s\Delta-\sigma_0,-\sigma_0]\to C[-s\Delta-\sigma_0,-\sigma_0]} \leqslant M\exp\frac{-q}{\varepsilon}, \tag{2.47}$$

where the universal constants M, $q>0$ depend only on the choice of r.

At the next step of the proof we study the spectral properties of the operator (2.45). Note that its spectrum consists of two points: an eigenvalue $\nu=\nu_*(\varkappa,\varepsilon)$, where $\nu_*(\varkappa,\varepsilon)=h_*(T_*(\varepsilon)-\sigma_0,\varkappa,\varepsilon)$, and an eigenvalue $\nu=0$ of infinite multiplicity. Below we shall prove the following asymptotic equalities (which hold uniformly in $\varkappa\in B(r)$) for $\nu_*(\varkappa,\varepsilon)$ as $\varepsilon\to 0$:

$$\begin{gathered}\nu_*(\varkappa,\varepsilon)=[(\omega_1+1)\varkappa-\omega_1][(\omega_2+1)\varkappa^s-\omega_2]+O(\varepsilon),\\ \frac{\partial\nu_*}{\partial\varkappa}(\varkappa,\varepsilon)=(\omega_1+1)[(\omega_2+1)\varkappa^s-\omega_2]+(\omega_2+1)s\varkappa^{s-1}[(\omega_1+1)\varkappa-\omega_1]+O(\varepsilon),\end{gathered} \tag{2.48}$$

where ω_1, ω_2 are the constants in (2.44).

To justify the relations (2.48), we must know the asymptotic behaviour of the solution $h_*(t,\varkappa,\varepsilon)$. Therefore we consider the equation (2.22) with initial condition $h\equiv 1$, $-s\Delta-\sigma_0\leqslant t\leqslant -\sigma_0$, and integrate it over the time interval $-\sigma_0\leqslant t\leqslant T_*(\varepsilon)-\sigma_0$ by the method of steps, taking into account that the coefficients $A_{1,*}(t,\varepsilon)$, $A_{s,*}(t,\varepsilon)$ form families of δ-function type (see (2.25)–(2.30), (2.34)) while the coefficients $A_{k,*}(t,\varepsilon)$, $k=2,\dots,s-1$, are exponentially small (see (2.26)) and $A_{0,*}(t,\varepsilon)$ satisfies the following asymptotic formula (uniformly in $t\in\Sigma_1\cap\Sigma_2$) because of (2.21), (2.23):

$$A_{0,*}(t,\varepsilon)=A_0(t)+O(\varepsilon),\qquad \varepsilon\to 0,$$

where

$$A_0(t)=\begin{cases}-\alpha\exp(-x_0(t)) & \text{for}\ \ t\in[-\sigma_0,\Delta)\cup\left(s\Delta+t_0,T_0-\dfrac{\sigma_0}{2}\right],\\ 0 & \text{for}\ \ t\in(\Delta,s\Delta+t_0).\end{cases}$$

As a result we obtain, first,

$$\max_{-\sigma_0\leqslant t\leqslant T_*(\varepsilon)-\sigma_0}\left(|h_*(t,\varkappa,\varepsilon)|+\left|\frac{\partial h_*}{\partial\varkappa}(t,\varkappa,\varepsilon)\right|\right)\leqslant M,\qquad M=\text{const}>0, \tag{2.49}$$

and second, uniformly in $t\in\Sigma_1\cap\Sigma_2$ and $\varkappa\in B(r)$,

$$h_*(t,\varkappa,\varepsilon)=h(t,\varkappa)+O(\varepsilon),\qquad \frac{\partial h_*}{\partial\varkappa}(t,\varkappa,\varepsilon)=\frac{\partial h}{\partial\varkappa}(t,\varkappa)+O(\varepsilon), \tag{2.50}$$

where Σ_1, Σ_2 are the sets (2.28) and $h(t,\varkappa)$, $t\geqslant-\sigma_0$, is a solution of the impulsive Cauchy problem

$$\dot h=A_0(t)h,\qquad h|_{t=-\sigma_0}=1, \tag{2.51}$$

$$\begin{gathered}h(\Delta+0)-h(\Delta-0)=-\varkappa\frac{1+1/\beta}{1+\beta(1-\exp(-\Delta))}h(0),\\ h(s\Delta+t_0+0)-h(s\Delta+t_0-0)=-\varkappa^s(\beta+1)\exp(s\Delta)h(t_0).\end{gathered} \tag{2.52}$$

The properties (2.48) obviously follow from (2.50) and the equality $h(T_0-\sigma_0,\varkappa) = [(\omega_1+1)\varkappa-\omega_1][(\omega_2+1)\varkappa^s-\omega_2]$, which is verified by integration of the system (2.51), (2.52) (we omit the corresponding obvious calculations).

We now consider the initial operator U and note from the equalities $U=V+W$ and $(\nu I-U)^{-1}=(I-(\nu I-V)^{-1}W)^{-1}(\nu I-V)^{-1}$ that every $\nu\in\mathbb{C}$ satisfying the inequality

$$\|(\nu I-V)^{-1}W\|_{C[-s\Delta-\sigma_0,-\sigma_0]\to C[-s\Delta-\sigma_0,-\sigma_0]}<1 \tag{2.53}$$

belongs to its resolvent set. We recall that W admits the bound (2.47). Concerning the operator $(\nu I-V)^{-1}$, we deduce from the equalities

$$(\nu I-V)^{-1}h_0=\frac{h_0(t)}{\nu}+c_*h_*(t+T_*(\varepsilon),\varkappa,\varepsilon), \qquad -s\Delta-\sigma_0\leqslant t\leqslant-\sigma_0,$$
$$c_*=\frac{h_0(-\sigma_0)}{\nu(\nu-\nu_*(\varkappa,\varepsilon))}$$

and the bound (2.49) that

$$\|(\nu I-V)^{-1}\|_{C[-s\Delta-\sigma_0,-\sigma_0]\to C[-s\Delta-\sigma_0,-\sigma_0]}\leqslant\frac{M(1+|\nu|)}{|\nu||\nu-\nu_*(\varkappa,\varepsilon)|} \qquad \forall\,\nu\in\mathbb{C},\qquad \nu\neq0,\nu_*(\varkappa,\varepsilon), \tag{2.54}$$

where $M=\mathrm{const}>0$.

At the last step of the proof of Lemma 2.4 we combine the bounds (2.47), (2.54) with the asymptotic representations (2.48). As a result, we see that every point $\nu\in\mathbb{C}$ in the set

$$\mathbb{C}\setminus\{O_1\cup O_2\}, \tag{2.55}$$

where

$$O_1=\Bigl\{\nu\colon |\nu|<\exp\frac{-q_1}{\varepsilon}\Bigr\}, \qquad O_2=\Bigl\{\nu\colon |\nu-\nu_*(\varkappa,\varepsilon)|<\exp\frac{-q_2}{\varepsilon}\Bigr\} \tag{2.56}$$

and the constants $q_j>0$, $j=1,2$, are appropriately small, satisfies the condition (2.53) and, therefore, is regular for the operator $U(\varkappa,\varepsilon)$. Hence the spectrum of this operator lies in the complement of (2.55), that is, in the discs (2.56). The required relations (2.42) and (2.43) follow from this and (2.48) in an obvious way. □

We make another useful observation. Let $\varkappa_k$, $k=0,1,\dots,s$, be the roots of the equation

$$[(\omega_1+1)\varkappa-\omega_1][(\omega_2+1)\varkappa^s-\omega_2]=0 \tag{2.57}$$

and suppose that the parameter $\varkappa$ varies over the set

$$B_\delta(r)=B(r)\setminus\bigcup_{k=0}^{s}\{\varkappa\in\mathbb{C}\colon |\varkappa-\varkappa_k|<\delta\}, \tag{2.58}$$

where $\delta > 0$. Then the multiplier $\nu_1(\varkappa, \varepsilon)$ of the equation (2.22) is simple. Moreover, it depends analytically on $\varkappa$ and, besides (2.43), admits the following asymptotic representation as $\varepsilon \to 0$:

$$\frac{\partial \nu_1}{\partial \varkappa}(\varkappa, \varepsilon) \\ = (\omega_1 + 1)[(\omega_2 + 1)\varkappa^s - \omega_2] + (\omega_2 + 1)s\varkappa^{s-1}[(\omega_1 + 1)\varkappa - \omega_1] + O(\varepsilon) \tag{2.59}$$

uniformly with respect to $\varkappa \in B_\delta(r)$.

Indeed, when $\varkappa \in B_\delta(r)$ the left-hand side of (2.57) is non-zero and, therefore (see (2.48)), the eigenvalue $\nu = \nu_*(\varkappa, \varepsilon)$ of the operator (2.45) is simple. Under the perturbation of V by the term W (which is analytic in $\varkappa$) of order $\exp(-q/\varepsilon)$, $q = \text{const} > 0$, the eigenvalue $\nu = \nu_*(\varkappa, \varepsilon)$ becomes a simple eigenvalue $\nu = \nu_1(\varkappa, \varepsilon)$ which depends analytically on $\varkappa$ and satisfies

$$\nu_1(\varkappa, \varepsilon) = \nu_*(\varkappa, \varepsilon) + O\biggl(\exp\frac{-q}{\varepsilon}\biggr),$$
$$\frac{\partial \nu_1}{\partial \varkappa}(\varkappa, \varepsilon) = \frac{\partial \nu_*}{\partial \varkappa}(\varkappa, \varepsilon) + O\biggl(\exp\frac{-q}{\varepsilon}\biggr), \qquad \varepsilon \to 0.$$

The required asymptotic equality (2.59) follows from this and (2.48).

To complete our preparatory constructions, we state another assertion. Taking into account (2.43) and a further analysis of equations of the form (1.19), we shall use the following lemma.

Lemma 2.5. *For every positive integer p satisfying the requirement*

$$p < \frac{m}{s+1} \tag{2.60}$$

and for all values of the parameters

$$\omega_1, \omega_2\colon \quad \omega_j > 0, \quad j = 1, 2, \quad \omega_1 + s\omega_2 < \frac{m}{p} - s - 1, \tag{2.61}$$

the equation

$$P(\varkappa) := [(\omega_1 + 1)\varkappa - \omega_1]^p[(\omega_2 + 1)\varkappa^s - \omega_2]^p - \varkappa^m = 0 \tag{2.62}$$

has a simple root 1, *and its other roots split into two sets*: $\Gamma_1 \subset \{\varkappa \in \mathbb{C}\colon |\varkappa| < 1\}$ *and* $\Gamma_2 \subset \{\varkappa \in \mathbb{C}\colon |\varkappa| > 1\}$. *The set* Γ_1 *contains* $p(s+1)$ *elements. The set* Γ_2 *consists of* $m - p(s+1) - 1$ *roots and is empty when* $m = p(s+1) + 1$.

Proof. Fix an arbitrary positive integer p and real numbers ω_1, ω_2 satisfying the requirements (2.60), (2.61). Consider the polynomial

$$P_\mu(\varkappa) = [(\mu\omega_1 + 1)\varkappa - \mu\omega_1]^p[(\mu\omega_2 + 1)\varkappa^s - \mu\omega_2]^p - \varkappa^m, \tag{2.63}$$

where $\mu \in (0, 1]$. It is clear from the explicit formula for this polynomial that $P_\mu(1) \equiv 0$. Moreover, by (2.60), (2.61) we have

$$\frac{d}{d\varkappa} P_\mu(\varkappa)\Big|_{\varkappa=1} = p(\mu\omega_1 + 1) + ps(\mu\omega_2 + 1) - m \leqslant p(\omega_1 + 1) + ps(\omega_2 + 1) - m < 0.$$

To study the other roots of (2.63), we begin with the case when $\mu \ll 1$.

Note that when $\mu \ll 1$ the polynomial $P_\mu(\varkappa)$ has exactly $p(s+1)$ roots tending to zero as $\mu \to 0$. The remaining roots (to be denoted by $\varkappa_l(\mu)$, $l = 1, \dots, m-p(s+1)-1$), other than 1, become equal to $\exp(2\pi i l/(m-p(s+1)))$, $l = 1, \dots, m-p(s+1)-1$, when $\mu = 0$. We easily verify that

$$\frac{d}{d\mu}|\varkappa_l(\mu)|^2\bigg|_{\mu=0} = \frac{2p}{m-p(s+1)}\biggl(1-\cos\frac{2\pi l}{m-p(s+1)}\biggr)\omega_1 + \frac{2p}{m-p(s+1)}\biggl(1-\cos\frac{2\pi l s}{m-p(s+1)}\biggr)\omega_2 > 0.$$

This proves the lemma for the roots of (2.63) in the case when $0 < \mu \ll 1$.

We now suppose that the equation $P_{\mu_0}(\varkappa) = 0$ with some $\mu_0 \in (0,1]$ admits a root $\varkappa_0 = \exp(i\psi_0)$, $\psi_0 \geqslant 0$. Then we necessarily have

$$\begin{aligned}1 = |\varkappa_0|^{2m} &= |(\mu_0\omega_1+1)\varkappa_0 - \mu_0\omega_1|^{2p}|(\mu_0\omega_2+1)\varkappa_0^s - \mu_0\omega_2|^{2p}\\ &= [1+2\mu_0\omega_1(\mu_0\omega_1+1)(1-\cos\psi_0)]^p[1+2\mu_0\omega_2(\mu_0\omega_2+1)(1-\cos(s\psi_0))]^p.\end{aligned}$$

Clearly, it follows that $\cos\psi_0 = 1$ and $\varkappa_0 = 1$.

We write m_j, $j = 1, 2$, for the number of roots of the equation $P_\mu(\varkappa) = 0$ in the sets $\{\varkappa \in \mathbb{C}\colon |\varkappa| < 1\}$ and $\{\varkappa \in \mathbb{C}\colon |\varkappa| > 1\}$ respectively. The analysis above shows that the numbers m_j remain the same as μ varies. Thus when $\mu = 1$ they remain as they were in the case $0 < \mu \ll 1$, that is, $m_1 = p(s+1)$, $m_2 = m - p(s+1) - 1$. □

§ 3. Final results

3.1. The existence and stability theorems for travelling waves. In this subsection we apply the auxiliary constructions of § 2 to study the initial system (2.2). We recall that the problem of the existence of its travelling waves (1.13) reduces to finding periodic solutions (with periods $m\Delta/p$, $p \in \mathbb{N}$) of the auxiliary equation (2.4). Therefore in what follows we denote the periodic solution of (2.4) specified in Lemma 2.1 and its period by $x_*(t,\varepsilon,\Delta)$ and $T_*(\varepsilon,\Delta)$ respectively, to stress their dependence on Δ. We similarly denote the period of the function (2.18) by $T_0(\Delta)$.

We first study the problem of the existence of a periodic solution with period $T = m\Delta/p$ for the relay equation (2.7). To do this, we consider the equation

$$T_0(\Delta) = \frac{m\Delta}{p} \tag{3.1}$$

for the seeking the phase shift $\Delta > 0$, assuming that the positive integers m and p are related by the conditions

$$\frac{m}{s+1+\beta+s/\beta} < p < \frac{m}{s+1}. \tag{3.2}$$

Using the second formula in (2.16), we rewrite (3.1) in the equivalent form

$$\Phi(\Delta) := \exp\biggl[\biggl(\frac{m}{p}-s-1\biggr)\Delta\biggr] - \bigl[1+\beta(1-\exp(-\Delta))\bigr]\biggl[1+\frac{1}{\beta}(1-\exp(-s\Delta))\biggr] = 0. \tag{3.3}$$

When the inequalities (3.2) hold, the equation (3.3) has the unique solution $\Delta = \widehat{\Delta}_{(p)}$ on the semi-axis $\Delta > 0$, and we have

$$\Phi'(\Delta)\big|_{\Delta=\widehat{\Delta}_{(p)}} > 0. \tag{3.4}$$

Indeed, the explicit formula for the function $\Phi(\Delta)$ shows that it is a solution of a certain linear homogeneous fifth-order differential equation with constant coefficients in the variable $\Delta \in \mathbb{R}$ and the characteristic roots of this equation are real. Therefore $\Phi(\Delta)$ admits at most four zeros on the whole axis $\Delta \in \mathbb{R}$. Since the following properties hold by (3.2):

$$\Phi(0) = 0, \qquad \Phi'(0) = \frac{m}{p} - s - 1 - \beta - \frac{s}{\beta} < 0,$$
$$\lim_{\Delta\to+\infty} \Phi(\Delta) = +\infty, \qquad \lim_{\Delta\to-\infty} \Phi(\Delta) = -\infty,$$

we see that the number of zeros is odd and each of the semi-axes $\Delta > 0$ and $\Delta < 0$ contains at least one of them. Hence the function $\Phi(\Delta)$ has exactly three zeros, and the root $\Delta = \widehat{\Delta}_{(p)} > 0$ of (3.3) is uniquely determined and automatically satisfies the required inequality (3.4).

Thus we have established that for $\Delta = \widehat{\Delta}_{(p)}$ the periodic solution (2.18) of the equation (2.7) has the desired period $m\widehat{\Delta}_{(p)}/p$. Consider the periodic solution $x_*(t,\varepsilon,\Delta)$ of (2.4), along with the corresponding equation

$$T_*(\varepsilon,\Delta) = \frac{m\Delta}{p}. \tag{3.5}$$

We recall from (2.21) that the period $T_*(\varepsilon,\Delta)$ admits an asymptotic representation $T_*(\varepsilon,\Delta) = T_0(\Delta) + O(\varepsilon)$ uniformly with respect to Δ in any compact set $K \subset (0,+\infty)$. Since the root $\Delta = \widehat{\Delta}_{(p)}$ of the equation (3.1) is simple, it follows that the equation (3.5) has at least one root $\Delta = \widehat{\Delta}_{(p)}(\varepsilon)$ such that

$$\widehat{\Delta}_{(p)}(\varepsilon) = \widehat{\Delta}_{(p)} + O(\varepsilon). \tag{3.6}$$

Summarizing the constructions above, we arrive at the following assertion.

Theorem 3.1. *Let m, p be positive integers satisfying the conditions (3.2). Then there is a sufficiently small $\varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon \leqslant \varepsilon_0$, the system (2.2), (2.3) admits a cycle (a travelling wave)*

$$C_p\colon \quad x_j = \widehat{x}_{(p)}(t + (j-1)\widehat{\Delta}_{(p)}(\varepsilon), \varepsilon), \qquad j = 1, 2, \dots, m, \tag{3.7}$$

where $\widehat{x}_{(p)}(t,\varepsilon) = x_*(t,\varepsilon,\Delta)|_{\Delta=\widehat{\Delta}_{(p)}(\varepsilon)}$ and $\widehat{\Delta}_{(p)}(\varepsilon)$ is the root (3.6) of (3.5).

We now pass to the problem of the stability of the cycle (3.7).

Theorem 3.2. *Under the hypotheses of the previous theorem, the cycle (3.7) with a fixed number p is exponentially orbitally stable when $m = p(s+1)+1$ and unstable otherwise.*

Proof. We recall that all the multipliers ν of the cycle (3.7) are given by equations of the form (1.19), (1.20), where $\nu_l(\varkappa) = \nu_l(\varkappa,\varepsilon)$, $l \geqslant 1$, are the multipliers of the auxiliary linear equation (2.22) for $\Delta = \widehat{\Delta}_{(p)}(\varepsilon)$. Hence we have the following bound for the monodromy operator $\mathscr{V}(\varepsilon)$ of the system (1.16) corresponding to the cycle (3.7):

$$\|\mathscr{V}(\varepsilon)\|_{\mathbb{R}^m\to\mathbb{R}^m} \leqslant M, \qquad M = \text{const} > 0. \tag{3.8}$$

This bound follows from the properties (2.25)–(2.30) of the coefficients (2.23), (2.24). Therefore in the equation

$$[\nu_l(\varkappa,\varepsilon)]^p = \varkappa^m, \qquad l \in \mathbb{N}, \tag{3.9}$$

using the obvious inequality $|\nu| \leqslant \|\mathscr{V}(\varepsilon)\|$ and a relation of the form (1.20), we can restrict ourselves to the values $\varkappa \in \mathbb{C}$ with $|\varkappa| \leqslant r$, where $r = (M+1)^{p/m}$, M being the constant in (3.8).

Thus, under the condition $\varkappa \in B(r)$, we substitute $\Delta = \widehat{\Delta}_{(p)}(\varepsilon)$ in the equation (2.22) and use Lemma 2.4. It will be clear from the analysis below that the complete set of multipliers of the cycle (3.7) can be constructed from the roots of (3.9) with $l = 1$. Hence there is no need to consider other values of l.

Using the asymptotic representation (2.43), we conclude that when $l = 1$, the equation in question can be written in the form

$$[(\widehat{\omega}_1+1)\varkappa - \widehat{\omega}_1]^p[(\widehat{\omega}_2+1)\varkappa^s - \widehat{\omega}_2]^p = \varkappa^m + O(\varepsilon), \tag{3.10}$$

where $\widehat{\omega}_1$, $\widehat{\omega}_2$ are the constants (2.44) for $\Delta = \widehat{\Delta}_{(p)}$. Note that the quantities m, p, $\widehat{\omega}_1$, $\widehat{\omega}_2$ satisfy the requirements (2.60), (2.61) by (3.2), (3.4). Using this and Lemma 2.5, we conclude that when $\varepsilon = 0$ the equation (3.10) has a simple root $\varkappa = 1$, and the other roots (to be denoted by $\widehat{\varkappa}_j$, $j = 1, \dots, m-1$) do not lie on the unit circle.

In the case when $\varepsilon > 0$ we consider the equation (3.10) for $\varkappa \in B_\delta(r)$, where $B_\delta(r)$ is the set (2.58) with $\varkappa_k$, $k = 0, 1, \dots, s$, being the roots of the equation (2.57) for $\omega_1 = \widehat{\omega}_1$, $\omega_2 = \widehat{\omega}_2$. The parameter $\delta > 0$ is chosen small enough to guarantee the inclusions $\varkappa = \widehat{\varkappa}_j \in B_\delta(r)$, $j = 1, \dots, m-1$, $\varkappa = 1 \in B_\delta(r)$. We emphasize that for $\varkappa = \varkappa_k$, $k = 0, 1, \dots, s$, the left-hand side of (3.10) becomes equal to zero while the right-hand side is certainly non-zero. Hence the equation (3.10) with sufficiently small δ has no roots in any of the balls $\{\varkappa \in \mathbb{C} \colon |\varkappa - \varkappa_k| \leqslant \delta\}$ and, therefore, we can discard these balls and consider only the set $B_\delta(r)$. On this set, the multiplier $\nu_1(\varkappa,\varepsilon)$ is simple and depends analytically on $\varkappa$ and, besides (2.43), satisfies the asymptotic equality (2.59).

We note that when $\varepsilon > 0$ the equation (3.10) still admits the solution $\varkappa = 1$ since the equation (2.22) certainly has multiplier 1 for $\varkappa = 1$, $\Delta = \widehat{\Delta}_{(p)}(\varepsilon)$. In this case it is the linearization of the equation (2.4) at the cycle $x = \widehat{x}_{(p)}(t,\varepsilon)$. By the analyticity (mentioned above) of the function $\nu_1(\varkappa,\varepsilon)$, the equation (3.10) for $\omega_1 = \widehat{\omega}_1$ and $\omega_2 = \widehat{\omega}_2$ differs from (2.62) by an analytic term of order ε (in the C^1-metric with respect to $\varkappa$). Hence the number of its roots in $B_\delta(r)$ is equal to m, as in the case $\varepsilon = 0$, and the root $\varkappa = 1$ is simple. The remaining $m-1$ roots of this equation tend to the roots $\widehat{\varkappa}_j$, $j = 1, \dots, m-1$, of the equation (2.62) as $\varepsilon \to 0$.

Let us summarize. It follows from the analysis above that all the multipliers $\nu_j(\varepsilon)$, $j = 1, \dots, m-1$, of the cycle (3.7), except for the simple multiplier 1, tend to the limits $\widehat{\varkappa}_j^{m/k}$, $j = 1, \dots, m-1$, as $\varepsilon \to 0$. Therefore (see Lemma 2.5) their absolute values are all smaller than 1 in the case when $m = p(s+1)+1$, and there is at least one multiplier with absolute value larger than 1 in the case when $m \neq p(s+1)+1$. $\square$

3.2. Conclusion. Note that the number of values of p satisfying the inequalities (3.2) grows unrestrictedly as $m \to +\infty$. Hence so does the number of coexisting cycles (3.7). But they are all unstable in the case when $m \neq p_0(s+1)+1 \ \forall p_0 \in \mathbb{N}$. The only possible stable cycle is that with number $p = p_0$ when $m = p_0(s+1)+1$. This cycle always exists since the requirements (3.2) hold automatically for the pair of numbers $m = p_0(s+1)+1$, $p = p_0$.

Thus the theory developed above does not answer the question of the attractors of the system (2.2) when $m \neq p_0(s+1)+1 \ \forall p_0 \in \mathbb{N}$. To clarify the situation in this case, we use the results of a numerical experiment which was carried out for the system

$$\dot{x}_j = -1 + \frac{\alpha \exp(-x_j)}{1+\exp(\gamma_1 x_{j-1}) + \exp(\gamma_2 x_{j-2}) + \exp(\gamma_3 x_{j-3})}, \qquad j = 1, 2, \dots, m, \tag{3.11}$$

where $x_0 = x_m$, $x_{-1} = x_{m-1}$, $x_{-2} = x_{m-2}$, with the values of the parameters $\alpha = 5.5$, $\gamma_1 = \gamma_2 = \gamma_3 = 100$ and for $5 \leqslant m \leqslant 9$.

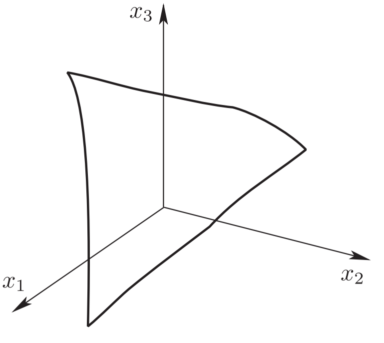

Figure 3

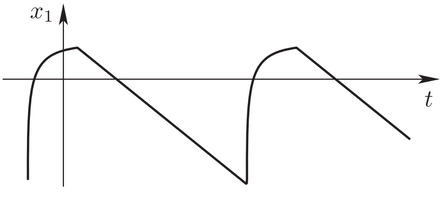

Figure 4

When $m = 5$, according to our theory, the cycle (3.7) of the system (3.11) with number $p = 1$ is stable. It seems to be the only attractor (we were not able to find any others). The projection of this cycle to the subspace (x_1, x_2, x_3) is shown in Fig. 3, and the graph of its component x_1 as a function on t is shown in Fig. 4.

When $m = 6$ the stable cycles are not travelling waves. There are several such cycles, all obtained from one another by cyclic permutations of the coordinates x_j, $j = 1, \dots, 6$. The projection of one such cycle to the subspace (x_1, x_2, x_3) is shown in Fig. 5.

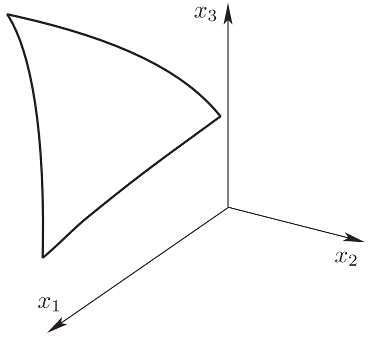

Figure 5

The most interesting situation is realized for $m = 7$. Then the system (3.11) admits a stable cycle which is a travelling wave but not a cycle (3.7). Note that only the simplest travelling waves of the system (2.2) were constructed above. These waves correspond to a cycle $x_0(t)$ of the relay equation (2.7) such that $x_0(t) < 0$ on the time interval $-s\Delta - \sigma_0 \leqslant t \leqslant -\sigma_0$ of length $s\Delta$. However, equation (2.7) also admits cycles with several sign changes on any interval of length $s\Delta$. These cycles generate more complicated travelling waves of the system (2.2). One such wave was detected numerically for the system (3.11) with $m = 7$. Its projection to the subspace (x_1, x_2, x_3) is shown in Fig. 6, and the graph of the component x_1 is shown in Fig. 7.

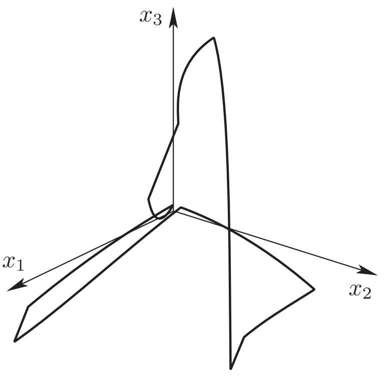

Figure 6

The situation in the remaining cases $m = 8$ and $m = 9$ is as follows. When $m = 8$ the attractors of (3.11) turn out to be equilibrium states passing to each other under cyclic permutations of the coordinates. When $m = 9$, the cycle (3.7)

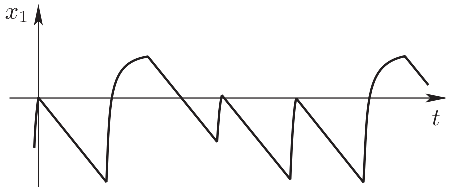

Figure 7

with number $p = 2$ is stable. We note that no new phenomena occur under the further growth of m: all types of attractors described above reappear one by one. For example, when $m = 10$, the stable cycles are not travelling waves. When $m = 11$, the attractor is a more complicated wave similar to the one shown in Figs. 6, 7, and so on.

The following system deserves separate study:

$$\dot{x}_j = -1 + \frac{\alpha \exp(-x_j)}{1 + \exp(\gamma\, x_{j-1})}, \qquad j = 1, 2, \dots, m, \quad x_0 = x_m. \tag{3.12}$$

It is obtained by the change of variables (2.1) from the multidimensional repressilator

$$\dot{u}_j = -u_j + \frac{\alpha}{1 + u_{j-1}^{\gamma}}, \qquad j = 1, 2, \dots, m, \quad u_0 = u_m. \tag{3.13}$$

Note that the system (1.3) mentioned above is a particular case of (3.13).

It follows from our theoretical constructions that if $\alpha > 1$, $\gamma \gg 1$ and $m = 2p_0+1$, then the system (3.12) admits a stable travelling wave (3.7) with number $p = p_0$. But if $m = 2p_0$, then numerical analysis shows that its attractors are equilibrium states.

To conclude, we note that all the results obtained above for the system (1.4) hold without change for the system (1.5). Indeed, under the conditions (1.9) it is transformed by the changes (2.1) to a system of the form (2.2), where the function $\Omega(y_1, \dots, y_s, \varepsilon)$ is given by the following analogue of (2.3):

$$\Omega(y_1, \dots, y_s, \varepsilon) = \prod_{k=1}^{s} \biggl(1 + \delta_k \exp \frac{\gamma_k^0 y_k}{\varepsilon}\biggr). \tag{3.14}$$

The relations (2.5), (2.6) are easily seen to hold for (3.14) and, therefore, we are still dealing with the same relay equation (2.7).

The situation for the system (1.6) is somewhat more complicated. Here, under the conditions (1.9), (2.1) we obtain a system (2.2) with the function

$$\Omega(y_1, \dots, y_s, \varepsilon) = 1 + \exp\biggl(\sum_{k=1}^{s} \frac{\gamma_k^0 y_k}{\varepsilon}\biggr),$$

and the corresponding delay relay equation is of the form

$$\dot{x} = -1 + \alpha \exp(-x) R(y), \qquad y = \sum_{k=1}^{s} \gamma_k^0 x(t - k\Delta), \tag{3.15}$$

where $R(y)$ is the function in (2.6).

We now consider the system (1.7) and note that on account of the relations (1.9), (2.1) it passes to the system

$$\dot{x}_j = -1 + \sum_{k=1}^{s} \frac{\alpha_k \exp(-x_j)}{1 + \delta_k \exp(\gamma_k^0 x_{j-k}/\varepsilon)}, \qquad j = 1, 2, \dots, m,$$

where $x_{-k} = x_{m-k}$, $k = 0, 1, \dots, s-1$. The delay relay equation can now be written in the form

$$\dot{x} = -1 + \sum_{k=1}^{s} \alpha_k \exp(-x) R(y_k)\big|_{y_k = x(t-k\Delta)}. \tag{3.16}$$

The problem of travelling waves for the systems (1.6), (1.7) is related, first of all, to finding periodic solutions of the relay equations (3.15), (3.16). The study of these equations is a separate and still-unsolved problem. Another open problem is to find periodic solutions of (2.7) with several sign changes on any interval of length $s\Delta$. We recall that the existence of such solutions is confirmed by numerical analysis.

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