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The dissipative property of a cubic non-linear Schrödinger equation

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Abstract. We study the large-time behaviour of solutions of the Cauchy problem for a non-linear Schrödinger equation. We consider the interaction between the resonance term and other types of non-linearity. We prove that solutions exist globally in time and find a large-time asymptotic representation for them. We show that the decay of solutions in the far region has the same order as in the linear case, while the solutions in the short-range region acquire an additional logarithmic decay, which is slower than in the case when there is no resonance term in the original equation.

Keywords: Schrödinger equation, cubic non-linearity, large-time asymptotics.

Dedicated to the blessed memory of my teacher
Il'ya Andreevich Shishmarev

§ 1. Introduction

This paper is devoted to studying the issues of global existence and large-time asymptotic behaviour of solutions of the Cauchy problem for a cubic non-linear Schrödinger equation in one space dimension:

$$\begin{gathered} iu_t + \frac{1}{2}u_{xx} = \mathcal{N}(u,\overline{u}), \qquad x \in \mathbb{R}, \quad t > 1, \\ u(1,x) = u_0(x), \qquad x \in \mathbb{R}, \end{gathered} \tag{1}$$

where the non-linearity is of the following form:

$$\mathcal{N}(u,\overline{u}) = \sum_{j=1}^{3} \lambda_j u^j \overline{u}^{3-j} = \lambda_1 u\overline{u}^2 + \lambda_2 |u|^2 u + \lambda_3 u^3$$

with coefficients $\lambda_1 = b \in [0,1)$, $\lambda_2 = 1$, $\lambda_3 = i\sqrt{3}\mu$, $0 < \mu < 1-b$. The more general case $\lambda_2 > 0$ may be reduced by scaling to the present case $\lambda_2 = 1$. Changing the variable by $u \to e^{i\theta}u$, we arrive at the case when $\lambda_1 = be^{-2i\theta}$ and $\lambda_3 = i\sqrt{3}\mu e^{2i\theta}$. We are interested in the interaction between the resonance term $|u|^2u$ and other types of non-linearity in the equation. The restriction $|\lambda_1| + |\lambda_3| < |\lambda_2|$ means that

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the resonance term $\lambda_2|u|^2u$ in a certain sense dominates the other non-resonance non-linearity types $\lambda_1 u\overline{u}^2$ and $\lambda_3 u^3$ in the equation (1).

The non-linear Schrödinger equation (1) with $\lambda_1 = \lambda_3 = 0$ was integrated by the inverse scattering method, and the asymptotic behaviour of its solutions as $t \to \infty$ has been studied (see [1], [2]). However, an explicit integration of a partial differential equation is possible only in very rare cases. Computations for large values of the time are quite difficult even for modern computers. Moreover, the asymptotic properties of solutions suggest one or another type of non-linearity and motivate the choice of models describing the physical processes.

The difficulties in the asymptotic study of solutions of the cubic non-linear Schrödinger equation (1) can be visualized by comparing the orders of decay in time of various terms in the equation. Starting with the familiar asymptotics as $t \to \infty$ of solutions of the linear Schrödinger equation (that is, (1) with $\mathcal{N}(u,\overline{u}) \equiv 0$)

$$u(t,x) = \frac{M}{\sqrt{it}}\widehat{u_0}\left(\frac{x}{t}\right) + O(t^{-\frac{3}{2}}),$$

where $M \equiv \exp\left(\frac{ix^2}{2t}\right)$, we assume that the solutions of (1) exhibit similar behaviour:

$$u(t,x) \approx \frac{M}{\sqrt{it}} w\left(t, \frac{x}{t}\right)$$

for some new unknown function w. We then obtain that the linear part of the equation is of the form $iu_t + \frac{1}{2}u_{xx} \approx \frac{M}{\sqrt{it}} iw_t$, and the non-linearity behaves like $\mathcal{N}(u,\overline{u}) \approx \lambda_1 i^{\frac{1}{2}} t^{-\frac{3}{2}} \overline{M}|w|^2\overline{w} + \lambda_2 i^{-\frac{1}{2}} t^{-\frac{3}{2}} M|w|^2 w + \lambda_3 i^{-\frac{3}{2}} t^{-\frac{3}{2}} M^3 w^3$. Thus we arrive at the following equation for w:

$$iw_t = i\lambda_1 t^{-1}\overline{M}^2|w|^2\overline{w} + \lambda_2 t^{-1}|w|^2 w - i\lambda_3 t^{-1} M^2 w^3 + o(t^{-1}).$$

Notice that the cubic terms in the last equation have a critical decay and, therefore, cannot be omitted in the first approximation. From the mathematical point of view, it would also be interesting to study the influence of various cubic non-linearities on the large-time asymptotics. The following results in this direction are currently known.

The non-existence of ordinary scattering states was proved in [3] for the non-linear Schrödinger equation (1) with $\lambda_1 = \lambda_3 = 0$, $\lambda_2 \neq 0$. Hence the solutions of the non-linear Schrödinger equation cannot be approximated by those of the linear Schrödinger equation (1) with $\mathcal{N}(u,\overline{u}) \equiv 0$. The asymptotics of solutions of the Cauchy problem for the equation (1) with $\lambda_1 = \lambda_3 = 0$, $\lambda_2 \neq 0$ was obtained in [4]–[6] (see also [7] and the literature cited therein). In particular, the following asymptotic representation of solutions was obtained:

$$u(t,x) = \frac{M}{\sqrt{it}} W_+\left(\frac{x}{t}\right) \exp\left(i\lambda_2 \left|W_+\left(\frac{x}{t}\right)\right|^2 \log t\right) + o\left(\frac{1}{\sqrt{t}}\right) \tag{2}$$

as $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$, where $W_+ \in \mathbf{L}^\infty$. Thus the presence of the resonance term $\lambda_2|u|^2u$ in (1) modifies the asymptotics of solutions by introducing an additional logarithmic oscillation of the main term when compared to the linear case. We notice that the resonance perturbation $\lambda_2|u|^2u$ is conservative in the sense that $iu_t + \frac{1}{2}u_{xx} = \lambda_2|u|^2u$ for the non-linear Schrödinger equation and the norm $\|u(t)\|_{\mathbf{L}^2}$ of a solution is preserved in time.

The asymptotics of solutions of the Cauchy problem for the non-linear Schrödinger equation with various cubic non-linearities including the derivative u_x of the solution as well as the resonance terms of the form $|u|^2u,\ |u_x|^2u,\ |u|^2u_x$ and others possessing the gauge property $\mathcal{N}(e^{i\theta}u) = e^{i\theta}\mathcal{N}(u)$ for all $\theta \in \mathbb{R}$, was studied in [8]–[14] (see the literature cited therein). It was proved that if the non-resonance non-linearities contain at least one derivative u_x of the solution, then the asymptotics is modified (as in (2)), but if there are no resonance terms, then the asymptotics of solutions is of a quasi-linear nature:

$$u(t,x) = \frac{M}{\sqrt{it}} W_+\left(\frac{x}{t}\right) + o\left(\frac{1}{\sqrt{t}}\right). \tag{3}$$

Various non-resonance cubic non-linearities (not containing u_x) in the Schrödinger equation were considered in [15]–[18]. It was shown that in the absence of resonance non-linearities they lead to a dissipative effect. In particular, the asymptotics of solutions of (1) with $\lambda_1 = \lambda_2 = 0,\ \lambda_3 = 1$ is of the form

$$u(t,x) = \frac{M}{\sqrt{it}}\widehat{u_0}\left(\frac{x}{t}\right)\left(1 + \frac{1}{\sqrt{3}}\left|\widehat{u_0}\left(\frac{x}{t}\right)\right|^2 \log\frac{t^2}{t+x^2}\right)^{-\frac{1}{2}} + O\left(t^{-\frac{1}{2}}\left(\log\frac{t^2}{t+x^2}\right)^{-\frac{1}{2}-\gamma}\right) \tag{4}$$

as $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$, where $\gamma > 0$ is small. The asymptotics (4) yields the following estimate for the decay:

$$\sup_{|x|\leqslant\sqrt{t}} |u(t,x)| \leqslant C\varepsilon t^{-\frac{1}{2}}(1 + \varepsilon^2 \log t)^{-\frac{1}{2}}.$$

Hence the solution attains a faster decay in the short-range region $|x| \leqslant \sqrt{t}$ when compared to the linear case. We notice that the resonance term $\lambda_2|u|^2u$ was excluded from consideration in the papers cited above. Thus we encounter the following question. What is the interaction between the resonance and non-resonance non-linearities in the one-dimensional Schrödinger equation (1)? The modified asymptotics (2) for solutions of the Cauchy problem (1) with odd initial perturbations u_0 was obtained in [19]. Note that the coefficients λ_1 and λ_3 do not occur in the main term of the asymptotic formula (2). Hence in the case of odd solutions (or when the non-resonance non-linearities contain at least one derivative), the nature of the main term of the asymptotics is determined by the resonance non-linearity $\lambda_2|u|^2u$.

In this paper we get rid of the condition that the initial data are odd and clarify the influence of the non-resonance non-linearities $\lambda_1 u\overline{u}^2$ and $\lambda_3 u^3$ on the asymptotics of solutions. We shall show that the solutions acquire an additional logarithmic decay in the region $|x| \leqslant \sqrt{t}$, but this decay is slower than in (4) (when there are no resonance interactions). Thus the dissipative character is introduced by the non-resonance non-linearities in the Schrödinger equation.

We now state the main result. As usual, we denote the Lebesgue space by $\mathbf{L}^p = \{s \in \mathbf{S}' \colon \|s\|_{\mathbf{L}^p} < \infty\}$ with the norm

$$\|s\|_{\mathbf{L}^p} = \left(\int_{\mathbb{R}} |s(x)|^p\,dx\right)^{1/p}, \quad 1 \leqslant p < \infty, \qquad \|s\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbb{R}} |s(x)|, \quad p = \infty.$$

We introduce the weighted Sobolev space

$$\mathbf{H}^{m,k} = \bigl\{s \in \mathbf{S}' \colon \|s\|_{\mathbf{H}^{m,k}} \equiv \|\langle x\rangle^k \langle i\partial\rangle^m s\|_{\mathbf{L}^2} < \infty\bigr\},$$

where $m, k \in \mathbb{R}$, $\langle x\rangle = \sqrt{1+x^2}$. The direct Fourier transform is given by

$$\mathcal{F}s = \widehat{s}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} s(x)\,dx.$$

Then $\mathcal{F}^{-1}s = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} s(\xi)\,d\xi$ is the inverse Fourier transform, and $\mathbf{H}^m = \mathbf{H}^{m,0}$ is the standard Sobolev space (we thus omit the superscript 0). Different positive constants will be denoted by the same letter C.

We shall use the technique of factorization of the linear Schrödinger group (see [20]),

$$\mathcal{U}(t) = e^{\frac{it}{2}\partial_x^2} = M(t)\mathcal{D}(t)\mathcal{V}(t)\mathcal{F},$$

where $M(t,x) = e^{\frac{i}{2t}x^2}$, $(\mathcal{D}(t)s)(x) = \frac{1}{\sqrt{it}} s\bigl(\frac{x}{t}\bigr)$ is the dilation operator and

$$\mathcal{V}(t) = \mathcal{F}M(t)\mathcal{F}^{-1} = \frac{\sqrt{it}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{it}{2}(\xi-y)^2}\,dy.$$

Put

$$\psi(t,\xi) = \varphi_\xi(t,\xi) + 2i \sum_{j=1,3} \chi_j \rho^j \overline{\rho}^{\,3-j} \frac{1}{\xi}\bigl(e^{\frac{it}{2}a_j\xi^2} - 1\bigr),$$

where $a_j = \frac{2j-4}{2j-3}$, $\chi_1 = b$, $\chi_3 = \mu$, $\rho(t) = \varphi(t,0)$, $\varphi(t) = \mathcal{F}\mathcal{U}(-t)u(t)$.

Theorem 1. *Suppose that $0 \leqslant b < 1$, $0 < \mu < 1-b$. Let the initial perturbation u_0 be such that $\|\varphi(1)\|_{\mathbf{L}^\infty} \leqslant \varepsilon_1$ and $\|\psi(1)\|_{\mathbf{L}^2} \leqslant \varepsilon_1^4$, where $\varepsilon_1 > 0$ is sufficiently small. Also assume that*

$$|\rho(1)| \geqslant \delta, \tag{5}$$

where $\delta = \varepsilon_1^{1+\nu}$ for a small $\nu > 0$. Then the Cauchy problem (1) has a unique solution $u \in \mathbf{C}([1,\infty);\mathbf{L}^2)$. Moreover, we have the decay estimates

$$C_1\delta t^{-\frac{1}{2}}(1+\varepsilon_1^4\log t)^{-\frac{1}{4}} \leqslant \sup_{|x|\leqslant\sqrt{t}} |u(t,x)| \leqslant C_2\varepsilon_1 t^{-\frac{1}{2}}(1+\delta^4\log t)^{-\frac{1}{4}},$$

$$\sup_{|x|>\sqrt{t}} |u(t,x)| \leqslant C\varepsilon_1 t^{-\frac{1}{2}}$$

for $t \geqslant 1$.

Remark 1. The initial perturbation can be chosen, for example, using the formulae $u_0 = \mathcal{U}(1)\mathcal{F}^{-1}\varphi(1)$ and

$$\varphi(1,\xi) = \frac{\delta}{1+\varepsilon_1^{12}\xi^2} - i\delta^3 \sum_{j\neq 2} \chi_j \int_\delta^1 e^{\frac{i\tau}{2}a_j\xi^2}\frac{d\tau}{\tau}.$$

Remark 2. In the region $|x| > \sqrt{t}$, the solution decays at the same rate as in the linear case. But in the short-range region $|x| \leqslant \sqrt{t}$ the solution acquires an additional logarithmic decay, which is slower than in the case (4) when there is no resonance term.

Remark 3. The hypothesis (5) in Theorem 1 excludes the case of odd initial perturbations considered in [19]. The assumption $0 \leqslant b < 1,\ 0 < \mu < 1-b$ is essential because otherwise we cannot guarantee that the solutions of the Cauchy problem (1) exist globally in time for arbitrary (even small) initial data.

Theorem 1 describes a very interesting property of equation (1). Introducing a new unknown function $\varphi(t) = \mathcal{F}\mathcal{U}(-t)u(t)$, we get the following ordinary differential equation for $\rho(t) = \varphi(t,0)$:

$$i\rho' = t^{-1}\sum_{j=1}^{3}\chi_j\rho^j\overline{\rho}^{3-j} + R_1(t),$$

where $\chi_1 = b,\ \chi_2 = 1,\ \chi_3 = \mu$, and $R_1(t)$ stands for the remainder. In the case when $R_1(t) \equiv 0$, we can pass to the polar coordinates $\rho = re^{-i\theta}$ and get the system

$$r' = -t^{-1}r^3(\mu-b)\sin(2\theta), \qquad \theta' = t^{-1}r^2\big(1+(\mu+b)\cos(2\theta)\big). \tag{6}$$

The system (6) has a first integral $r^2(1+(\mu+b)\cos(2\theta))^{-\frac{\mu-b}{\mu+b}} = C$. This means that the solutions of (6) oscillate for $|\mu+b| < 1$ and can either grow or decay in time in the case when $|\mu+b| \geqslant 1$. Thus the higher-order terms in the remainder $R_1(t)$ have an essential influence on the stability of the solutions. In what follows we distinguish the fifth-order terms in $R_1(t)$ and prove (in Lemma 3) that $\rho(t)$ satisfies the following ordinary differential equation:

$$i\rho' = t^{-1}\sum_{j=1}^{3}\chi_j\rho^j\overline{\rho}^{3-j} - t^{-1}\sum_{l=1}^{5}\omega_l\rho^l\overline{\rho}^{5-l} + R_2(t), \tag{7}$$

where $\omega_1 = -\frac{2\pi}{3}b^2,\ \omega_2 = -\mu\big(\frac{\pi}{3} + i\ln 3\big),\ \omega_3 = -4b\mu\big(\frac{\pi}{3} - i\ln 3\big),\ \omega_4 = -\frac{2\pi}{3}b,$ $\omega_5 = -6i\mu^2\ln\frac{3+\sqrt{5}}{6},\ a_j = \frac{2j-4}{2j-3}$, and $R_2(t)$ stands for the remainder. In Lemma 4 we show that the solutions $\rho(t)$ of (7) decay logarithmically in time provided that $0 \leqslant b < 1,\ 0 < \mu < 1-b$. We note that in the non-resonance case $\lambda_2 = 0$ of equation (1) we get an equation

$$i\rho' = t^{-1}\sum_{j=1,3}\chi_j\rho^j\overline{\rho}^{3-j} + R_1(t),$$

which has decaying solutions. This explains why we had no need to study the higher-order terms of R_1 in [15]–[18]. We also note that our method is inapplicable in the case when $\varphi(t,0)=0$. Hence the existence of a global-in-time solution and the calculation of its asymptotics remains an open question when $\widehat{u_0}(0)=0$.

In [19] we used the change of function $u(t,x)=t^{-\frac{1}{2}}Ev(t,\xi)$, where $E=e^{\frac{it}{2}\xi^2}$ and $\xi=\frac{x}{t}$. The new function $v=\mathcal{F}M\mathcal{U}(-t)u$ satisfies the equation

$$i\partial_t v+\frac{1}{2t^2}\partial_\xi^2 v=\frac{1}{t}\sum_{j=1}^{3}E^{2j-4}\lambda_j v^j\overline{v}^{\,3-j}.$$

While estimating the derivative v_ξ, we got the secular terms $i(2j-4)\xi E^{2j-4}\lambda_j v^j\overline{v}^{\,3-j}$ for $j\neq 2$ because of the rapidly oscillating factor E^{2j+2}. To exclude these terms, we used a transformation similar to the normal form transformation of Shatah [21]. We introduced the operator

$$\mathcal{I}(v)=v_\xi-2t\xi\sum_{j=1,3}A_jE^{2j-4}\lambda_j v^j\overline{v}^{\,3-j},$$

where $A_j=(1+(2j-3)it\xi^2)^{-1}$. Since the solution was odd, we were able to estimate the $\mathbf{L}^2$-norm of $\mathcal{I}(v)$ in terms of $\|v_\xi\|_{\mathbf{L}^2}$. By contrast, in the present paper we make the transformation $\varphi(t)=\mathcal{F}\mathcal{U}(-t)u(t)$ and write down the equation for $\rho(t)=\varphi(t,0)$ with non-linear terms up to and including the fifth order. In Lemma 4 we prove that $\rho(t)$ decays logarithmically in time. In § 2 we get a bound for the derivative $\partial_\xi\varphi(t,\xi)$ of the new function. Then in § 3 we estimate the difference $\varphi(t,\xi)-\varphi(t,0)$ and write down the equation (7) for $\rho(t)=\varphi(t,0)$. Solving this equation, we prove the following two-sided bound in Lemma 4:

$$C_1\delta(1+\varepsilon_1^4\log t)^{-\frac{1}{4}}\leqslant|\rho(t)|\leqslant C_2\varepsilon_1(1+\delta^4\log t)^{-\frac{1}{4}}.$$

Theorem 1 is proved in § 3. In § 4 we prove auxiliary identities and estimates.

§ 2. A bound for the derivative

As in [20], we use factorization for the free Schrödinger group

$$\mathcal{U}(t)=e^{\frac{it}{2}\partial_x^2}=M(t)\mathcal{D}(t)\mathcal{V}(t)\mathcal{F},$$

where $M(t,x)=e^{\frac{i}{2t}x^2}$ and $(\mathcal{D}(t)s)(x)=e^{-i\frac{\pi}{4}\operatorname{sign}t}\frac{1}{\sqrt{|t|}}s\bigl(\frac{x}{t}\bigr)$ is the dilation operator,

$$\mathcal{V}(t)=\mathcal{F}M(t)\mathcal{F}^{-1}=e^{i\frac{\pi}{4}\operatorname{sign}t}\frac{\sqrt{|t|}}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{it}{2}(\xi-y)^2}\,dy.$$

We also have $\mathcal{F}\mathcal{U}(-t)=i\mathcal{V}(-t)\overline{E}(t)\mathcal{D}\bigl(\frac{1}{t}\bigr)$, where $E(t,\xi)=e^{\frac{it}{2}\xi^2}$. Here we have used the commutation relation $\mathcal{D}\bigl(\frac{1}{t}\bigr)M(t)=E(t)\mathcal{D}\bigl(\frac{1}{t}\bigr)$. Then we obtain

$$\mathcal{F}M\mathcal{U}(-t)\biggl(i\partial_t+\frac{1}{2}\partial_x^2\biggr)=\mathcal{L}\mathcal{F}M\mathcal{U}(-t),$$

where $\mathcal{L}=i\partial_t+\frac{1}{2t^2}\partial_\xi^2$ and

$$\mathcal{F}M\mathcal{U}(-t)u^\beta\overline{u}^\alpha=i\overline{E}(t)\mathcal{D}\biggl(\frac{1}{t}\biggr)u^\beta\overline{u}^\alpha=i^{\frac{1+\alpha-\beta}{2}}t^{-1}E^{\beta-\alpha-1}v^\beta\overline{v}^{\,\alpha}.$$

Here $v = \mathcal{F}M\mathcal{U}(-t)u = i\overline{E}\mathcal{D}(\frac{1}{t})u$. Multiplying (1) by $\mathcal{F}M\mathcal{U}(-t)$, we get

$$\mathcal{L}v = t^{-1}\sum_{j=1}^{3} i^{2-j}\lambda_j E^{2j-4}v^j\overline{v}^{3-j}. \tag{8}$$

Acting by the operator $\mathcal{V}(-t)$ on equation (8), we find the equation

$$i\partial_t\varphi = t^{-1}\sum_{j=1}^{3} i^{2-j}\lambda_j \mathcal{V}(-t)(E^{2j-4}v^j\overline{v}^{3-j}) \tag{9}$$

for the new unknown function

$$\varphi(t) = \mathcal{F}\mathcal{U}(-t)u(t) = \mathcal{V}(-t)v(t) = e^{-i\frac{\pi}{4}\operatorname{sign} t}\frac{\sqrt{|t|}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{it}{2}(\xi-y)^2}v(t,y)\,dy.$$

Using the identity

$$\begin{aligned}\mathcal{V}(-t)E^{\rho-1}s &= e^{-i\frac{\pi}{4}\operatorname{sign} t}\frac{\sqrt{|t|}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{it}{2}(\xi^2-2y\xi+y^2\rho)}s(y)\,dy\\ &= e^{i\frac{\pi}{4}(\operatorname{sign}\rho+\operatorname{sign}(t\rho)-\operatorname{sign} t)}E^{1-\frac{1}{\rho}}\mathcal{D}(\rho)\mathcal{V}(-t\rho)s\end{aligned} \tag{10}$$

for $\rho \neq 0$, we obtain an equation for φ:

$$i\partial_t\varphi = t^{-1}\sum_{j=1}^{3}\theta_j E^{a_j}\mathcal{D}(2j-3)\mathcal{V}((3-2j)t)v^j\overline{v}^{3-j}, \tag{11}$$

where $\theta_j = i^{2-j}\lambda_j e^{i\frac{\pi}{4}(\operatorname{sign}(2j-3)+\operatorname{sign}((2j-3)t)-\operatorname{sign} t)}$, $a_j = \frac{2j-4}{2j-3}$.

We now state the local existence theorem for solutions of the Cauchy problem for equation (11) (see [7]).

Theorem 2. *Let $\varphi_0 \in \mathbf{H}^1$ be the initial perturbation. Then for some $T > 1$ there is a unique solution $\varphi \in \mathbf{C}([1,T];\mathbf{H}^1)$ of the Cauchy problem for equation (11) with the initial perturbation $\varphi(1) = \varphi_0$.*

We introduce the self-model variable $\widetilde{\xi} = \xi\sqrt{t}$. Define the norms

$$\|\varphi\|_{\mathbf{X}_T} = \sup_{t\in[1,T]}\big(\|\varphi(t)\|_{\mathbf{L}^\infty} + Q^{\frac{1}{4}}(t)\|\langle\widetilde{\xi}\rangle^{-\gamma}\varphi(t)\|_{\mathbf{L}^\infty}\big),$$

$$\|\varphi\|_{\mathbf{Z}_T} = \sup_{t\in[1,T]} Q^{\frac{3}{4}}(t)\|\langle\widetilde{\xi}\rangle^{-\gamma}(\varphi-\rho)\|_{\mathbf{L}^\infty}, \qquad \|\psi\|_{\mathbf{Y}_T} = \sup_{t\in[1,T]} t^{-\frac{1}{4}}Q^{\frac{5}{4}}(t)\|\psi(t)\|_{\mathbf{L}^2},$$

where $Q(t) = 1 + \delta^4\log t$, $\delta = \varepsilon_1^{1+\nu}$, $\varepsilon = \varepsilon_1^{1-\nu}$, $\nu > 0$ is small so that $\delta < \varepsilon_1 < \varepsilon$. We also assume that $\gamma > 0$ is small. Put $\rho(t) = \varphi(t,0)$ and

$$\psi = \varphi_\xi - \sum_{j\neq 2} a_j\chi_j\rho^j\overline{\rho}^{3-j}\xi\int_0^t E^{a_j}\,d\tau \tag{12}$$

for $\chi_j = \lambda_j\frac{i^{2-j}}{\sqrt{|2j-3|}}e^{i\frac{\pi}{4}(\operatorname{sign}(2j-3)-1)}$. By our choice of $\lambda_1 = b$, $\lambda_2 = 1$, $\lambda_3 = i\mu\sqrt{3}$ we have $\chi_1 = b$, $\chi_2 = 1$, $\chi_3 = \mu$.

Lemma 1. *Let the initial perturbation* $\varphi_0 \in \mathbf{H}^1$ *be such that* $\|\varphi_0\|_{\mathbf{L}^\infty} \leqslant \varepsilon_1$ *and* $\|\psi_0\|_{\mathbf{L}^2} \leqslant \varepsilon_1^4$, *where* $\varepsilon_1 > 0$ *is sufficiently small. Suppose that the solution* $\varphi \in \mathbf{C}([1,T];\mathbf{H}^1)$ *of* (11) *satisfies the estimates* $\|\varphi\|_{\mathbf{Z}_T} \leqslant \varepsilon^3$, $\|\varphi\|_{\mathbf{X}_T} \leqslant \varepsilon$. *Then* $\|\psi\|_{\mathbf{Y}_T} < \varepsilon^4$.

Proof. Assume that the inequality $\|\psi\|_{\mathbf{Y}_T} < \varepsilon^4$ does not hold. Since the solution is continuous, there is a maximal $\widetilde{T} \in (1,T]$ such that $\|w\|_{\mathbf{Y}_{\widetilde{T}}} \leqslant \varepsilon^4$. Differentiating (11), we get

$$\partial_t\varphi_\xi = \sum_{j\neq 2}\theta_j a_j \xi E^{a_j}\mathcal{D}(2j-3)\mathcal{V}((3-2j)t)v^j\overline{v}^{3-j} + R_1, \tag{13}$$

where

$$R_1 = -it^{-1}\sum_{j=1}^{3}\theta_j E^{a_j}\mathcal{D}(2j-3)\mathcal{V}((3-2j)t)\partial_\xi(v^j\overline{v}^{3-j}).$$

We represent the first term on the right-hand side of (13) in the form

$$\theta_j a_j\xi E^{a_j}\mathcal{D}(2j-3)\mathcal{V}((3-2j)t)v^j\overline{v}^{3-j} = a_j\chi_j\rho^j\overline{\rho}^{3-j}\xi E^{a_j} + \theta_j a_j \xi E^{a_j}\mathcal{D}(2j-3)\mathcal{V}((3-2j)t)g_j,$$

where $\chi_j = \theta_j\frac{e^{-i\frac{\pi}{4}\operatorname{sign}(2j-3)}}{\sqrt{|2j-3|}}$ and $g_j = v^j\overline{v}^{3-j} - \rho^j\overline{\rho}^{3-j}$. Write the equation (13) in the following way:

$$\partial_t\varphi_\xi = \sum_{j\neq 2} a_j\chi_j\rho^j\overline{\rho}^{3-j}\xi E^{a_j} + R_1 + R_2, \tag{14}$$

where

$$R_2 = \sum_{j\neq 2}\theta_j a_j\xi E^{a_j}\mathcal{D}(2j-3)\mathcal{V}((3-2j)t)g_j = \sum_{j\neq 2}\nu_j\widetilde{\xi}\int_{\mathbb{R}} e^{itS_j}g_j(t,y)\,dy.$$

Here $S_j = a_j\frac{\xi^2}{2} - \frac{3-2j}{2}\big(\frac{\xi}{2j-3} - y\big)^2 = \frac{\xi^2}{2} - \xi y + \frac{2j-3}{2}y^2$ and $\nu_j = \frac{\theta_j a_j}{\sqrt{2\pi}}e^{i\frac{\pi}{2}\operatorname{sign}(3-2j)}$. Rewrite the first term on the right-hand side of (14) in the form

$$\sum_{j\neq 2} a_j\chi_j\rho^j\overline{\rho}^{3-j}\xi E^{a_j} = \partial_t\bigg(\sum_{j\neq 2}a_j\chi_j\rho^j\overline{\rho}^{3-j}\xi\int_0^t E^{a_j}\,d\tau\bigg) - \sum_{j\neq 2}a_j\chi_j(\rho^j\overline{\rho}^{3-j})_t\xi\int_0^t E^{a_j}\,d\tau.$$

Furthermore, using the identity $e^{itS_j} = H_j(\partial_t(te^{itS_j}) + S_{jy}\partial_y e^{itS_j})$, where $H_j = (1+it(S_j+S_{jy}^2))^{-1}$ and

$$S_j + S_{j\eta}^2 = \frac{1}{2}\bigg((4j-5)(2j-3)\Big(y-\frac{\xi}{2j-3}\Big)^2 + \frac{2j-4}{2j-3}\xi^2\bigg) \geqslant C(\xi^2+y^2)$$

for $j = 0, 1, 3$, we find that

$$R_2 = \sum_{j\neq 2} \nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} g_j \, dy = \partial_t \Big(\sum_{j\neq 2} \nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} t H_j g_j \, dy \Big) - \sum_{j\neq 2} \nu_j \xi \int_{\mathbb{R}} e^{itS_j} \big(t\partial_t(\sqrt{t} H_j g_j) + \sqrt{t}\partial_y (S_{jy} H_j g_j) \big) \, dy.$$

Then the equation (14) takes the form

$$w_t = R_1 + R_3 + R_4,$$

where

$$w = \varphi_\xi - \sum_{j\neq 2} a_j \chi_j \rho^j \overline{\rho}^{3-j} \xi \int_0^t E^{a_j} \, d\tau - \sum_{j\neq 2} \nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} t H_j g_j \, dy,$$

$$R_3 = - \sum_{j\neq 2} a_j \chi_j (\rho^j \overline{\rho}^{3-j})_t \xi \int_0^t E^{a_j} \, d\tau,$$

$$R_4 = - \sum_{j\neq 2} \nu_j \xi \int_{\mathbb{R}} e^{itS_j} \big(t\partial_t(\sqrt{t} H_j g_j) + \sqrt{t}\partial_y (S_{jy} H_j g_j) \big) \, dy.$$

Notice that for $a > 0$ we have

$$\begin{aligned} \mathcal{V}(t)\xi \int_0^t E^a \, d\tau &= (\xi t - i\partial_\xi) \int_0^1 \mathcal{V}(t) E^{az} \, dz = (\xi t - i\partial_\xi) \int_0^1 E^{\frac{az}{1+az}} \frac{dz}{\sqrt{|1+az|}} \\ &= -\xi t \int_0^1 e^{\frac{it}{2}\xi^2 \frac{az}{1+az}} z \partial_z \frac{1}{\big(1 + \frac{it}{2}\xi^2 \frac{az}{(1+az)^2}\big)(1+az)^{\frac{3}{2}}} \, dz + \frac{C\xi t e^{\frac{it}{2}\xi^2 \frac{a}{1+a}}}{1 + \frac{it}{2}\xi^2 \frac{a}{(1+a)^2}} \\ &= O(\xi t \langle \widetilde{\xi} \rangle^{-2+\gamma}). \end{aligned}$$

To estimate R_3, we use the bound $|\int_0^t E^a \, d\tau| \leqslant Ct\langle \widetilde{\xi} \rangle^{-2}$ for $a > 0$. Then

$$\|R_3\|_{\mathbf{L}^2} \leqslant Ct^{-1} |\rho|^5 \Big\| \xi \int_0^t E^{a_j} \, d\tau \Big\|_{\mathbf{L}^2} \leqslant C\varepsilon^5 t^{-\frac{3}{4}} Q^{-\frac{5}{4}}(t).$$

Since $v_\xi = \mathcal{V}(t)\varphi_\xi$, we have

$$v_\xi = \mathcal{V}(t) w + \sum_{j\neq 2} a_j \chi_j \rho^j \overline{\rho}^{3-j} \mathcal{V}(t) \xi \int_0^t E^{a_j} \, d\tau + \sum_{j\neq 2} \nu_j \mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} t \widetilde{\xi} H_j g_j \, dy.$$

Thus we must consider terms of the form $\mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} \widetilde{\xi} H_j s \, dy$ and $\mathcal{V}(t) \times \int_{\mathbb{R}} e^{itS_j} \widetilde{\xi} S_{jy} H_j s \, dy$. We have

$$\mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} \widetilde{\xi} H_j s \, dy = \int_{\mathbb{R}} P_j(\widetilde{\xi}, \widetilde{y}) s(t,y) \, dy,$$

where

$$P_j(\widetilde{\xi}, \widetilde{y}) = -\frac{2e^{i\frac{\pi}{4}}}{3\sqrt{2\pi}} \overline{E} \partial_{\widetilde{\xi}} \int_{\mathbb{R}} \frac{e^{i\widetilde{x}(\widetilde{\xi} - \widetilde{y}) + \frac{i}{2}(2j-3)\widetilde{y}^2} d\widetilde{x}}{\big(\widetilde{x} - \frac{4j-5}{3}\widetilde{y}\big)^2 + \frac{2}{9}((5-4j)(2-j)\widetilde{y}^2 - 3i)}.$$

By Cauchy's theorem, $\int_{\mathbb{R}} \frac{e^{ixz}dz}{(z-a)^2+b^2} = \frac{\pi}{b} e^{iax-b|x|}$ for $x \in \mathbb{R}$, $\operatorname{Re} b > 0$. Therefore,

$$P_j(\widetilde{\xi}, \widetilde{y}\,) = -\frac{2\sqrt{\pi} e^{i\frac{\pi}{4}}}{\Omega} \overline{E} e^{\frac{i}{2}(2j-3)\widetilde{y}^{\,2}} \partial_{\widetilde{\xi}}\, e^{i\frac{4j-5}{3}\widetilde{y}(\widetilde{\xi}-\widetilde{y}\,) - \frac{\sqrt{2}}{3}|\widetilde{\xi}-\widetilde{y}|\Omega},$$

where $\Omega = \sqrt{(5-4j)(2-j)\widetilde{y}^{\,2} - 3i}$. Hence we get the estimate

$$|P_j(\widetilde{\xi}, \widetilde{y}\,)| \leqslant Ce^{-C|\widetilde{\xi}-\widetilde{y}|\langle \widetilde{y}\,\rangle}.$$

It is also necessary to consider the kernel

$$\mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} it\widetilde{\xi} S_{jy} H_j s\, dy \\ = -\mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} \widetilde{\xi} H_{jy} s\, dy + \overline{E} \int_{\mathbb{R}} s(t,y) \partial_y (e^{\frac{i}{2}(2j-3)\widetilde{y}^{\,2}} P_j(\widetilde{\xi}, \widetilde{y}\,))\, dy.$$

Since $\langle \widetilde{\xi}\,\rangle^{\alpha} e^{-C|\widetilde{\xi}-\widetilde{y}|\langle \widetilde{y}\,\rangle} \leqslant \langle \widetilde{y}\,\rangle^{\alpha} e^{-C|\widetilde{\xi}-\widetilde{y}|\langle \widetilde{y}\,\rangle}$, we find the bounds

$$\left\| \langle \widetilde{\xi}\,\rangle^{\alpha} \mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} \widetilde{\xi} H_j s\, dy \right\|_{\mathbf{L}^2} \leqslant C \left\| \langle \widetilde{\xi}\,\rangle^{\alpha} \int_{\mathbb{R}} |s(t,y)| e^{-C|\widetilde{\xi}-\widetilde{y}|\langle \widetilde{y}\,\rangle}\, dy \right\|_{\mathbf{L}^2} \\ \leqslant Ct^{-\frac{1}{2}} \|\langle \widetilde{\xi}\,\rangle^{\alpha-1} s\|_{\mathbf{L}^2}$$

for $\alpha \in [0,1]$ and

$$\left\| \mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} t\widetilde{\xi} S_{jy} H_j s\, dy \right\|_{\mathbf{L}^2} \\ \leqslant Ct^{\frac{1}{2}} \left\| \int_{\mathbb{R}} \frac{|s(t,y)|\, dy}{\widetilde{\xi}^{\,2} + \widetilde{y}^{\,2}} \right\|_{\mathbf{L}^2} + Ct^{\frac{1}{2}} \left\| \int_{\mathbb{R}} |s(t,y)|\, |\widetilde{y}| e^{-C|\widetilde{\xi}-\widetilde{y}|\langle \widetilde{y}\,\rangle}\, dy \right\|_{\mathbf{L}^2} \leqslant C\|s(t,\xi)\|_{\mathbf{L}^2}.$$

Then we get

$$\|R_1\|_{\mathbf{L}^2} \leqslant Ct^{-1} \||v|^2 v_{\xi}\|_{\mathbf{L}^2} \leqslant Ct^{-1} \|v\|^2_{\mathbf{L}^\infty} \|\mathcal{V}(t) w\|_{\mathbf{L}^2} \\ + Ct^{-1} \sum_{j\neq 2} |\rho|^3 \|\langle \widetilde{\xi}\,\rangle^{-\gamma} v\|^2_{\mathbf{L}^\infty} \left\| \langle \widetilde{\xi}\,\rangle^{2\gamma} \mathcal{V}(t) \xi \int_0^t E^{a_j}\, d\tau \right\|_{\mathbf{L}^2} \\ + C \sum_{j\neq 2} \|\langle \widetilde{\xi}\,\rangle^{-\gamma} v\|^2_{\mathbf{L}^\infty} \left\| \langle \widetilde{\xi}\,\rangle^{2\gamma} \mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} \widetilde{\xi} H_j g_j\, dy \right\|_{\mathbf{L}^2} \\ \leqslant Ct^{-1} \|v\|^2_{\mathbf{L}^\infty} \|w\|_{\mathbf{L}^2} + Ct^{-\frac{3}{4}} |\rho|^3 \|\langle \widetilde{\xi}\,\rangle^{-\gamma} v\|^2_{\mathbf{L}^\infty} \\ + Ct^{-\frac{1}{2}} \|\langle \widetilde{\xi}\,\rangle^{-\gamma} v\|^4_{\mathbf{L}^\infty} \|v-\varphi\|_{\mathbf{L}^2} \leqslant C\varepsilon^5 t^{-\frac{3}{4}} Q^{-\frac{5}{4}}(t)$$

because

$$\|\langle \widetilde{\xi}\,\rangle^{2\gamma-1} g_j\|_{\mathbf{L}^2} \leqslant \|\langle \widetilde{\xi}\,\rangle^{-\gamma} v\|^2_{\mathbf{L}^\infty} (\|v-\varphi\|_{\mathbf{L}^2} + \|\langle \widetilde{\xi}\,\rangle^{5\gamma-1}\|_{\mathbf{L}^2} \|\langle \widetilde{\xi}\,\rangle^{-\gamma} (\varphi-\rho)\|_{\mathbf{L}^\infty}), \\ \|v-\varphi\|_{\mathbf{L}^2} = \|(\mathcal{V}(t)-1)\varphi\|_{\mathbf{L}^2} = \|\mathcal{F}(M(t)-1)\mathcal{F}^{-1}\varphi\|_{\mathbf{L}^2} \\ \leqslant Ct^{-\frac{1}{2}} \|x\mathcal{F}^{-1}\varphi\|_{\mathbf{L}^2} = Ct^{-\frac{1}{2}} \|\varphi_{\xi}\|_{\mathbf{L}^2} = Ct^{-\frac{1}{2}} \|v_{\xi}\|_{\mathbf{L}^2}.$$

We now consider R_4. Using the equation $v_t = \frac{i}{2t^2} v_{\xi\xi} - i\mathcal{L}v$ and integrating by parts in R_4, we get

$$\begin{aligned} R_4 = &- \sum_{j\neq 2} \nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} \big(\sqrt{t}\partial_t(\sqrt{t}H_j) + \partial_y(S_{jy}H_j)\big) g_j \, dy \\ &+ \sum_{j\neq 2} it\nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} H_j \big(jv^{j-1}\overline{v}^{\,3-j}\mathcal{L}v - (3-j)v^j \overline{v}^{\,2-j}\overline{\mathcal{L}v}\big) \, dy \\ &+ \sum_{j\neq 2} \frac{i}{2t} \nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} H_j \big(j(v^{j-1}\overline{v}^{\,3-j})_y v_y - (3-j)(v^j\overline{v}^{\,2-j})_y \overline{v_y}\big) \, dy \\ &- \sum_{j\neq 2} \frac{i}{2t} \nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} (itS_{jy}H_j + H_{jy}) \big(jv^{j-1}\overline{v}^{\,3-j} v_y - (3-j)v^j\overline{v}^{\,2-j}\overline{v_y}\big) \, dy \\ &- \sum_{j\neq 2} \nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} S_{jy} H_j \partial_y g_j \, dy + \sum_{j\neq 2} \nu_j \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} H_j t\partial_t(\rho^j \overline{\rho}^{\,3-j}) \, dy. \end{aligned} \tag{15}$$

For the first term we write

$$\begin{aligned} &\mathcal{V}(t)\widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} \big(\sqrt{t}\partial_t(\sqrt{t}H_j) + \partial_y(S_{jy}H_j)\big) g_j \, dy \\ &\quad = \Big(2j - \frac{7}{2}\Big)\mathcal{V}(t)\widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} H_j g_j \, dy + \mathcal{V}(t)\widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} H_j^2 g_j \, dy \\ &\qquad - (4j-5)it\mathcal{V}(t)\widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} S_{jy}^2 H_j^2 g_j \, dy. \end{aligned}$$

Here the first term can be estimated as follows:

$$\Big\| \mathcal{V}(t)\widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} H_j g_j \, dy \Big\|_{\mathbf{L}^2} \leqslant Ct^{-\frac{1}{2}} \|\langle \widetilde{\xi} \rangle^{-1} g_j\|_{\mathbf{L}^2},$$

and the second term admits the bound

$$\Big\| \mathcal{V}(t)\widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} H_j^2 g_j \, dy \Big\|_{\mathbf{L}^2} \leqslant C \Big\| \int_{\mathbb{R}} (\langle \widetilde{\xi} \rangle + \langle \widetilde{y} \rangle)^{-3} |g_j| \, dy \Big\|_{\mathbf{L}^2} \leqslant Ct^{-\frac{1}{2}} \|\langle \widetilde{\xi} \rangle^{-1} g_j\|_{\mathbf{L}^2}.$$

We integrate by parts in the last term of (15):

$$\begin{aligned} &\Big\| it\widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} S_{jy}^2 H_j^2 g_j \, dy \Big\|_{\mathbf{L}^2} \leqslant C \Big\| \int_{\mathbb{R}} (\langle \widetilde{\xi} \rangle + \langle \widetilde{y} \rangle)^{-3} |g_j| \, dy \Big\|_{\mathbf{L}^2} \\ &\quad + Ct^{-\frac{1}{2}} \Big\| \int_{\mathbb{R}} (\langle \widetilde{\xi} \rangle + \langle \widetilde{y} \rangle)^{-2} |\partial_y g_j| \, dy \Big\|_{\mathbf{L}^2} \leqslant Ct^{-\frac{1}{2}} \|\langle \widetilde{\xi} \rangle^{-1} g_j\|_{\mathbf{L}^2} + Ct^{-1} \|\partial_\xi g_j\|_{\mathbf{L}^2}. \end{aligned}$$

Hence,

$$\begin{aligned} &\Big\| \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} \big(\sqrt{t}\partial_t(\sqrt{t}H_j) + \partial_y(S_{jy}H_j)\big) g_j \, dy \Big\|_{\mathbf{L}^2} \\ &\qquad \leqslant Ct^{-\frac{1}{2}} \|\langle \widetilde{\xi} \rangle^{-1} g_j\|_{\mathbf{L}^2} + Ct^{-1} \|\partial_\xi g_j\|_{\mathbf{L}^2} \leqslant C\varepsilon^5 t^{-\frac{3}{4}} Q^{-\frac{5}{4}}(t). \end{aligned}$$

To estimate the second term in (15), we use the inequality

$$\left\| \langle \widetilde{\xi}\, \rangle^{\alpha} \mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} \widetilde{\xi} H_j s \, dy \right\|_{\mathbf{L}^2} \leqslant C t^{-\frac{1}{2}} \| \langle \widetilde{\xi}\, \rangle^{\alpha - 1} s(t, \xi) \|_{\mathbf{L}^2}$$

and deduce from (8) that

$$\left\| t \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} H_j \big(j v^{j-1} \overline{v}^{3-j} \mathcal{L} v - (3-j) v^j \overline{v}^{2-j} \overline{\mathcal{L} v} \big) \, dy \right\|_{\mathbf{L}^2} \leqslant C t^{-\frac{1}{2}} \| \langle \widetilde{\xi}\, \rangle^{-1} v^5(t, \xi) \|_{\mathbf{L}^2} \leqslant C \varepsilon^5 t^{-\frac{3}{4}} Q^{-\frac{5}{4}}(t).$$

Then we obtain from (15) that

$$\|R_4\|_{\mathbf{L}^2} \leqslant C \varepsilon^5 t^{-\frac{3}{4}} Q^{-\frac{5}{4}}(t) + C \varepsilon t^{-1} \| \langle \widetilde{\xi}\, \rangle^{-1} \|_{\mathbf{L}^2} \| v_\xi \|_{\mathbf{L}^2}^2 + C t^{-1} \| |v|^2 v_y \|_{\mathbf{L}^2} + C t^{-\frac{1}{2}} |\rho|^5 \| \langle \widetilde{\xi}\, \rangle^{-1} \|_{\mathbf{L}^2} \leqslant C \varepsilon^5 t^{-\frac{3}{4}} Q^{-\frac{5}{4}}(t).$$

Thus we get $\frac{d}{dt} \|w\|_{\mathbf{L}^2} \leqslant C \varepsilon^5 t^{-\frac{3}{4}} Q^{-\frac{5}{4}}(t)$. Integrating this with respect to time, we have

$$\|w\|_{\mathbf{L}^2} \leqslant \varepsilon_1^4 + C \varepsilon^5 t^{\frac{1}{4}} Q^{-\frac{5}{4}}(t) < \frac{\varepsilon^4}{2} t^{\frac{1}{4}} Q^{-\frac{5}{4}}(t)$$

for $t \in [1, \widetilde{T}]$. We also have

$$\begin{aligned} \|\psi - w\|_{\mathbf{L}^2} &\leqslant C \left\| \widetilde{\xi} \int_{\mathbb{R}} e^{itS_j} t H_j g_j \, dy \right\|_{\mathbf{L}^2} = C t \left\| \mathcal{V}(t) \int_{\mathbb{R}} e^{itS_j} \widetilde{\xi} H_j g_j \, dy \right\|_{\mathbf{L}^2} \\ &\leqslant C t^{\frac{1}{2}} \| \langle \widetilde{\xi}\, \rangle^{-\gamma} v \|_{\mathbf{L}^\infty}^2 \big(\|v - \varphi\|_{\mathbf{L}^2} + \| \langle \widetilde{\xi}\, \rangle^{5\gamma - 1} \|_{\mathbf{L}^2} \| \langle \widetilde{\xi}\, \rangle^{-\gamma} (\varphi - \rho) \|_{\mathbf{L}^\infty} \big) \\ &\leqslant C \varepsilon^5 t^{\frac{1}{4}} Q^{-\frac{5}{4}}(t). \end{aligned}$$

This contradicts the assumption made at the beginning of the proof. Hence the estimate stated in the lemma holds for all $t \in [1, T]$. □

§ 3. Bounds in the uniform metric

We first estimate the difference $\varphi(t, \xi) - \rho(t)$.

Lemma 2. *Let the initial perturbation* $\varphi_0 \in \mathbf{H}^1$ *be such that* $\|\varphi_0\|_{\mathbf{L}^\infty} \leqslant \varepsilon_1$ *and* $\|\psi_0\|_{\mathbf{L}^2} \leqslant \varepsilon_1^4$, *where* $\varepsilon_1 > 0$ *is sufficiently small. Suppose that the solution* $\varphi \in \mathbf{C}([1, T]; \mathbf{H}^1)$ *of* (11) *is such that* $\|\varphi\|_{\mathbf{X}_T} \leqslant \varepsilon$ *and* $\|\psi\|_{\mathbf{Y}_T} \leqslant \varepsilon^4$. *Then* $\|\varphi\|_{\mathbf{Z}_T} \leqslant \varepsilon^3$.

Proof. We define a primitive for all $\xi \neq 0$ by the formulae

$$\partial_\xi^{-1} f = \begin{cases} -\displaystyle\int_\xi^\infty f(\xi) \, d\xi & \text{for} \quad \xi > 0, \\ \displaystyle\int_{-\infty}^\xi f(\xi) \, d\xi & \text{for} \quad \xi < 0. \end{cases}$$

We put

$$G_b(\widetilde{\xi}\,) = -e^{i \frac{\pi}{4} \operatorname{sign} b} \frac{\sqrt{|b|}}{\sqrt{2\pi}} \, \partial_{\widetilde{\xi}}^{-1} e^{-\frac{ib}{2} \widetilde{\xi}^2}$$

for $b \neq 0$. Since $G_b(+0) - G_b(-0) = e^{i\frac{\pi}{4}\operatorname{sign} b}\frac{\sqrt{|b|}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{ib}{2}z^2}\,dz = 1$, integration by parts in the integral $\mathcal{V}(bt)$ yields that

$$\begin{aligned}\mathcal{V}(bt)s &= e^{i\frac{\pi}{4}\operatorname{sign}(bt)}\frac{\sqrt{|bt|}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{ibt}{2}y^2} s(t,\xi-y)\,dy \\ &= s(t,\xi) - \int_{\mathbb{R}} G_b(\widetilde{y})s_\xi(t,\xi-y)\,dy. \end{aligned} \tag{16}$$

In particular,

$$q(t,\xi) = v(t,\xi) - v(t,0) - \int_{\mathbb{R}}\big(G_{-1}(\widetilde{\xi}-\widetilde{y}) - G_{-1}(-\widetilde{y})\big)v_y(t,y)\,dy. \tag{17}$$

For all $x, y > 0$ we get

$$|G_b(x) - G_b(y)| = \left|\frac{1}{\sqrt{2\pi}}\int_y^x e^{\frac{ib}{2}\widetilde{\xi}^2}\,d\widetilde{\xi}\right| \leqslant C|x-y|.$$

On the other hand, for $x > 0$ we have

$$|G_b(x)| \leqslant C\left|\int_x^\infty \partial_\eta\left(\frac{\eta e^{\frac{ib}{2}\eta^2}}{1+ib\eta^2}\right)d\eta\right| + C\left|\int_x^\infty \frac{\eta^2 e^{\frac{ib}{2}\eta^2}}{(1+ib\eta^2)^2}\,d\eta\right| \leqslant C\langle x\rangle^{-1}.$$

Hence we obtain that $|G_b(x) - G_b(y)| \leqslant C|x-y|^\gamma(\langle x\rangle^{\gamma-1} + \langle y\rangle^{\gamma-1})$ for all $x, y > 0$. This yields the bound

$$\begin{aligned}\|G_b(\widetilde{\xi}-\widetilde{y}) - G_b(-\widetilde{y})\|_{\mathbf{L}^2_y} &\leqslant Ct^{-\frac14}\left(\int_{|\widetilde{y}|\geqslant|\widetilde{\xi}|}|G_b(|\widetilde{\xi}|-\widetilde{y}) - G_b(-\widetilde{y})|^2 d\widetilde{y}\right)^{\frac12} \\ &\quad + Ct^{-\frac14}\left(\int_0^{|\widetilde{\xi}|}|G_b(|\widetilde{\xi}|-\widetilde{y}) - G_b(-\widetilde{y})|^2\,d\widetilde{y}\right)^{\frac12} \leqslant Ct^{-\frac14}|\widetilde{\xi}|^\gamma, \end{aligned} \tag{18}$$

so that by (17) we find that

$$\begin{aligned}|q(t,\xi) - (v(t,\xi) - v(t,0))| &\leqslant C\|G_{-1}(\widetilde{\xi}-\widetilde{y}) - G_{-1}(-\widetilde{y})\|_{\mathbf{L}^2_y}\|v_y\|_{\mathbf{L}^2} \\ &\leqslant C\varepsilon^3|\widetilde{\xi}|^\gamma Q^{-\frac34}(t). \end{aligned} \tag{19}$$

We now rewrite the right-hand side of (11) using (16). This yields that

$$\mathcal{V}((3-2j)t)v^j\overline{v}^{3-j} = v^j\overline{v}^{3-j} - \int_{\mathbb{R}} G_{3-2j}(\widetilde{y})\partial_\xi v^j\overline{v}^{3-j}(t,\xi-y)\,dy. \tag{20}$$

Substituting (20) into (11), we get

$$i\partial_t q(t,\xi) = \big(R_5(t,\xi) - R_5(t,0)\big) - \big(R_6(t,\xi) - R_6(t,0)\big), \tag{21}$$

where

$$R_5(t,\xi) = t^{-1}\sum_{j=1}^{3}\theta_j E^{a_j}\mathcal{D}(2j-3)v^j\overline{v}^{3-j}(t,\xi),$$

$$R_6(t,\xi) = t^{-1}\sum_{j=1}^{3}\theta_j E^{a_j}\mathcal{D}(2j-3)\int_{\mathbb{R}} G_{3-2j}(\widetilde{y})\partial_\xi v^j\overline{v}^{3-j}(t,\xi-y)\,dy.$$

Since $|v(t,0)| \leqslant Q^{-\frac{1}{4}}(t)\|v\|_{\mathbf{X}_T} \leqslant \varepsilon Q^{-\frac{1}{4}}(t)$, we have

$$\begin{aligned}|R_5(t,\xi)-R_5(t,0)| &\leqslant Ct^{-1}\sum_{j=1}^{3}\big|\mathcal{D}(2j-3)v^j\overline{v}^{3-j}(t,\xi)-v^j\overline{v}^{3-j}(t,0)\big| \\ &\qquad + Ct^{-1}\sum_{j\neq 2}|(E^{a_j}-1)v^j\overline{v}^{3-j}(t,0)| \\ &\leqslant C\varepsilon^2 t^{-1}\sum_{j=1}^{3}|\mathcal{D}(2j-3)q(t,\xi)| + C\varepsilon^3 t^{-1}|\widetilde{\xi}|^{\gamma}Q^{-\frac{3}{4}}(t).\end{aligned}$$

In view of (18) we obtain

$$\begin{aligned}&|R_6(t,\xi)-R_6(t,0)| \\ &\leqslant Ct^{-1}\sum_{j=1}^{3}\left|\mathcal{D}(2j-3)\int_{\mathbb{R}}\big(G_{3-2j}(\widetilde{\xi}-\widetilde{y})-G_{3-2j}(-\widetilde{y})\big)\partial_y v^j\overline{v}^{3-j}(t,y)\,dy\right| \\ &\quad + Ct^{-1}\sum_{j\neq 2}|E^{a_j}-1|\left|\int_{\mathbb{R}}G_b(\widetilde{y})\partial_y v^j\overline{v}^{3-j}(t,y)\,dy\right| \leqslant C\varepsilon^5 t^{-1}|\widetilde{\xi}|^{\gamma}Q^{-\frac{3}{4}}(t).\end{aligned}$$

Hence it follows from (21) that

$$|\partial_t q(t,\xi)| \leqslant C\varepsilon^2 t^{-1}\sum_{j=1}^{3}|\mathcal{D}(2j-3)q(t,\xi)| + C\varepsilon^3 t^{-1}|\widetilde{\xi}|^{\gamma}Q^{-\frac{3}{4}}(t). \tag{22}$$

We now prove the lemma by contradiction. Since $q(t)$ is continuous, there is a maximal interval of time $[1,\widetilde{T}]$, $\widetilde{T}\in(1,T]$, such that $|\varphi(t,\xi)-\rho(t)|\leqslant C\varepsilon^3|\widetilde{\xi}|^{\gamma}Q^{-\frac{3}{4}}(t)$ for all $t\in[1,\widetilde{T}]$, $\xi\in\mathbb{R}$. Then we obtain from (22) that $|\partial_t q(t,\xi)|\leqslant C\varepsilon^3 t^{\frac{\gamma}{2}-1}|\xi|^{\gamma}Q^{-\frac{3}{4}}(t)$. Integrating this with respect to time, we see that

$$\begin{aligned}|q(t,\xi)| &\leqslant |q(1,\xi)| + C\varepsilon^3|\xi|^{\gamma}\int_1^t \tau^{\frac{\gamma}{2}-1}Q^{-\frac{3}{4}}(\tau)\,d\tau \\ &\leqslant \varepsilon_1|\xi|^{\gamma} + C\varepsilon^3|\xi|^{\gamma}t^{\frac{\gamma}{4}} + C\varepsilon^3|\xi|^{\gamma}t^{\frac{\gamma}{2}}Q^{-\frac{3}{4}}(t) < C\varepsilon^3 t^{\frac{\gamma}{2}-1}|\xi|^{\gamma}Q^{-\frac{3}{4}}(t)\end{aligned}$$

for all $t\in[1,\widetilde{T}]$. We arrive at a contradiction. □

We introduce the notation

$$\omega_1 = -\frac{2\pi}{3}b^2, \qquad \omega_2 = -\mu\left(\frac{\pi}{3}+i\ln 3\right),$$
$$\omega_3 = -4b\mu\left(\frac{\pi}{3}-i\ln 3\right), \qquad \omega_4 = -\frac{2\pi}{3}b, \qquad \omega_5 = -6i\mu^2\ln\frac{3+\sqrt{5}}{6}.$$

Lemma 3. *Let $\varphi\in\mathbf{C}([1,T];\mathbf{H}^1)$ be a solution of (11) such that $\|\varphi\|_{\mathbf{X}_T}\leqslant\varepsilon$ and $\|\psi\|_{\mathbf{Y}_T}\leqslant\varepsilon^4$. Then the function $\rho(t)=\varphi(t,0)$ satisfies the equation*

$$i\rho' = t^{-1}\sum_{j=1}^{3}\chi_j\rho^j\overline{\rho}^{3-j} - t^{-1}\sum_{l=1}^{5}\omega_l\rho^l\overline{\rho}^{5-l} + O(t^{-1}\varepsilon^7 Q^{-\frac{7}{4}}) \tag{23}$$

for all $t\in[1,T]$.

Proof. By (16) we have

$$v(t,0) = \mathcal{V}(t)\varphi = \rho(t) + \int_{\mathbb{R}} G_1(\widetilde{\xi})\varphi_\xi(t,\xi)\,d\xi. \tag{24}$$

Since $ita_k\xi E^{a_k z} = \partial_\xi\big(\frac{E^{a_k z}-1}{z}\big)$, the change of variable $\tau = tz$ in (12) yields that

$$\varphi_\xi = -\sum_{k\neq 2} i\chi_k\rho^k\overline{\rho}^{3-k}\partial_\xi\int_0^1 (E^{a_k z}-1)\,\frac{dz}{z} + \psi. \tag{25}$$

Substituting (25) into (24), we have

$$v(t,0) = \rho(t) - \sum_{k\neq 2} i\chi_k\Phi_{a_k}\rho^k\overline{\rho}^{3-k} + R_7, \tag{26}$$

where $\Phi_a = \int_{\mathbb{R}} G_1(\widetilde{\xi})\partial_\xi\int_0^1 (E^{az}-1)\,\frac{dz}{z}\,d\xi$ and $R_7 = \int_{\mathbb{R}} G_1(\widetilde{\xi})\psi(t,\xi)\,d\xi$. We see from the definition that the function $G_b(\widetilde{y})$ decays at infinity as $|\widetilde{y}|^{-1}$. Hence the remainder R_7 can be estimated as follows:

$$|R_7| \leqslant C\|\langle\widetilde{y}\rangle^{-1}\|_{\mathbf{L}^2}\|\psi\|_{\mathbf{L}^2} \leqslant C\varepsilon^5 Q^{-\frac{5}{4}}(t).$$

Taking $\xi = 0$ in (11), we get

$$i\rho' = t^{-1}\sum_{j=1}^{3}\chi_j\mathcal{V}((3-2j)t)v^j\overline{v}^{3-j}\Big|_{\xi=0} \tag{27}$$

for $\rho(t) = \varphi(t,0)$. Using (16), we obtain for the right-hand side of (27) that

$$\begin{aligned}\mathcal{V}((3-2j)t)v^j\overline{v}^{3-j}|_{\xi=0} &= v^j\overline{v}^{3-j}(t,0) + j\rho^{j-1}\overline{\rho}^{3-j}\int_{\mathbb{R}} G_{3-2j}(\widetilde{y})v_y\,dy \\ &\quad + (3-j)\rho^j\overline{\rho}^{2-j}\int_{\mathbb{R}} G_{3-2j}(\widetilde{y})\overline{v_y}\,dy + R_8\end{aligned} \tag{28}$$

since $G_b(-\widetilde{\xi}) = -G_b(\widetilde{\xi})$, where

$$\begin{aligned}R_8 = j\int_{\mathbb{R}} G_{3-2j}(\widetilde{y})(v^{j-1}\overline{v}^{3-j}(t,y) - \rho^{j-1}\overline{\rho}^{3-j})v_y\,dy \\ + (3-j)\int_{\mathbb{R}} G_{3-2j}(\widetilde{y})(v^j\overline{v}^{2-j}(t,y) - \rho^j\overline{\rho}^{2-j})\overline{v_y}\,dy.\end{aligned}$$

In view of (19) we obtain that $|v(t,\xi) - v(t,0)| \leqslant C\varepsilon^3|\widetilde{\xi}|^\gamma Q^{-\frac{3}{4}}(t)$. Therefore, using (26), we have

$$|R_8| \leqslant C\varepsilon^4 Q^{-1}(t)\int_{\mathbb{R}}\langle\widetilde{y}\rangle^{-1}(1+|\widetilde{y}|^\gamma)|v_y|\,dy \leqslant C\varepsilon^7 Q^{-\frac{7}{4}}(t).$$

We also obtain from (26) that $v^j\overline{v}^{3-j}(t,0) = \rho^j\overline{\rho}^{3-j} - \Omega_j - \overline{\Omega_{3-j}} + O(\varepsilon^7 Q^{-\frac{7}{4}})$, where $\Omega_j = j\sum_{k\neq 2} i\chi_k\Phi_{a_k}\rho^{j+k-1}\overline{\rho}^{6-j-k}$. Note that

$$G_{-b}(\widetilde{\xi}) = -e^{i\frac{\pi}{4}\operatorname{sign} b}\frac{\sqrt{|tb|}}{\sqrt{2\pi}}\partial_{\widetilde{\xi}}^{-1}e^{-\frac{ib}{2}\widetilde{\xi}^2} = \overline{G_b(\widetilde{\xi})},$$

whence

$$\int_{\mathbb{R}} G_{3-2j}(\widetilde{\xi})\overline{v_\xi}\,d\xi = \overline{\int_{\mathbb{R}} G_{3-2(3-j)}(\widetilde{\xi})v_\xi\,d\xi}.$$

Thus we find from (28) that

$$\mathcal{V}((3-2j)t)v^j\overline{v}^{3-j}\big|_{\xi=0} = \rho^j\overline{\rho}^{3-j} - \Omega_j - \overline{\Omega_{3-j}} + W_j + \overline{W_{3-j}} + O(\varepsilon^7 Q^{-\frac{7}{4}}), \quad (29)$$

where $W_j = j\rho^{j-1}\overline{\rho}^{3-j}\int_{\mathbb{R}} G_{3-2j}(\widetilde{\xi})v_\xi\,d\xi$. In view of (25) we have

$$v_\xi = \mathcal{V}(t)\varphi_\xi = -\sum_{k\neq 2} i\chi_k\rho^k\overline{\rho}^{3-k}\partial_\xi\mathcal{V}(t)\int_0^1 (E^{a_k z}-1)\,\frac{dz}{z} + \mathcal{V}(t)\psi.$$

Using the identity (10) with $\rho = 1 - a_k z$, we obtain that

$$\mathcal{V}(t)E^{a_k z} = E^{\frac{a_k z}{1-a_k z}}\,\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-a_k z))}}{\sqrt{|1-a_k z|}}.$$

Therefore,

$$v_\xi = -\sum_{k\neq 2} i\chi_k\rho^k\overline{\rho}^{3-k}\partial_\xi\int_0^1 \big(E^{\frac{a_k z}{1-a_k z}}-1\big)\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-a_k z))}\,dz}{z\sqrt{|1-a_k z|}} + \mathcal{V}(t)\psi. \quad (30)$$

Hence

$$W_j = -\sum_{k\neq 2} ij\chi_k\Psi_{3-2j,a_k}\rho^{j+k-1}\overline{\rho}^{6-j-k} + O(\varepsilon^7 Q^{-\frac{7}{4}}),$$

where

$$\Psi_{b,a} = \int_{\mathbb{R}} G_b(\widetilde{\xi})\partial_\xi\,d\xi\int_0^1 (E^{\frac{az}{1-az}}-1)\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-az))}\,dz}{z\sqrt{|1-az|}}.$$

Using (27) and (29), we thus get

$$\begin{aligned} i\rho' = {} & t^{-1}\sum_{j=1}^{3}\chi_j\rho^j\overline{\rho}^{3-j} - t^{-1}\sum_{j=1}^{3}\sum_{k\neq 2}\chi_j A_{j,k}\rho^{j+k-1}\overline{\rho}^{6-j-k} \\ & - t^{-1}\sum_{j=1}^{2}\sum_{k\neq 2}\chi_{3-j}\overline{A_{j,k}}\rho^{6-j-k}\overline{\rho}^{j+k-1} + O(t^{-1}\varepsilon^7 Q^{-\frac{7}{4}}), \end{aligned} \quad (31)$$

where $A_{j,k} = ij\chi_k(\Phi_{a_k} + \Psi_{3-2j,a_k})$. Note that

$$\sum_{j=1}^{3}\sum_{k\neq 2}\chi_j A_{j,k}\rho^{j+k-1}\overline{\rho}^{6-j-k} + \sum_{j=1}^{2}\sum_{k\neq 2}\chi_{3-j}\overline{A_{j,k}}\rho^{6-j-k}\overline{\rho}^{j+k-1} = \sum_{l=1}^{5}\omega_l\rho^l\overline{\rho}^{5-l},$$

where

$$\omega_l = \sum_{\substack{k=1,3\\ l-2\leqslant k\leqslant l}}\chi_{l+1-k}A_{l+1-k,k} + \sum_{\substack{k=1,3\\ 3-l\leqslant k\leqslant 5-l}}\chi_{k+l-3}\overline{A_{6-k-l,k}}.$$

Hence we get (23) with the coefficients ω_l being calculated in Lemma 7 below. □

We pass to the polar coordinates $\rho = re^{-i\theta}$ in (23). Then

$$r' - ir\theta' = -t^{-1}r^3 \sum_{j=1}^{3} i\chi_j e^{(2-j)2i\theta} + t^{-1}r^5 \sum_{l=1}^{5} i\omega_l e^{(3-l)2i\theta} + H,$$

where $H = O(t^{-1}\varepsilon^6 Q^{-\frac{7}{4}})$. Separating the real and imaginary parts, we get

$$r' = t^{-1}r^3 F_1(\theta) + t^{-1}r^5 K_1(\theta) + rf_1, \qquad \theta' = t^{-1}r^2 F_2(\theta) + t^{-1}r^4 K_2(\theta) + f_2,$$

where

$$F_1(\theta) = -\operatorname{Re}\Bigl(\sum_{j=1,3} i\chi_j e^{(2-j)2i\theta}\Bigr), \qquad F_2(\theta) = \lambda_2 + \operatorname{Im}\Bigl(\sum_{j=1,3} i\chi_j e^{(2-j)2i\theta}\Bigr),$$

$$K_1(\theta) = \operatorname{Re}\sum_{l=1}^{5} i\omega_l e^{(3-l)2i\theta}, \qquad K_2(\theta) = -\operatorname{Im}\sum_{l=1}^{5} i\omega_l e^{(3-l)2i\theta},$$

$$f_1 = \frac{1}{r}\operatorname{Re} H, \qquad f_2 = -\frac{1}{r}\operatorname{Im} H.$$

We introduce a new function $z = r^2 \exp\bigl(-2\int \frac{F_1(\theta')}{F_2(\theta')}\,d\theta'\bigr)$. Then $z' = -z^2 K_3(\theta)\theta'' + F_3$, where

$$K_3(\theta) = -2\exp\Bigl(2\int \frac{F_1(\theta')}{F_2(\theta')}\,d\theta'\Bigr)\Bigl(\frac{K_1(\theta)}{F_2(\theta)} - \frac{F_1(\theta)}{F_2^2(\theta)}K_2(\theta)\Bigr)$$

and $F_3 = z^2(t^{-1}r^4K_2 + f_2)K_3 + 2z(f_1 - \frac{F_1(\theta)}{F_2(\theta)}f_2)$. Hence we get a system

$$z' = -z^2K_3\theta'' + F_3, \qquad \theta'' = t^{-1}zF_2\exp\Bigl(2\int \frac{F_1(\theta')}{F_2(\theta')}\,d\theta'\Bigr) + t^{-1}r^4K_2 + f_2.$$

By our choice of $\lambda_1 = b$, $\lambda_2 = 1$ and $\lambda_3 = i\mu\sqrt{3}$, we have $\chi_1 = b$, $\chi_2 = 1$, $\chi_3 = \mu$ and $F_1(\theta) = -(a-2b)\sin(2\theta)$, $F_2(\theta) = 1 + a\cos(2\theta)$, where $a = b + \mu$. Hence,

$$2\int \frac{F_1(\theta')}{F_2(\theta')}\,d\theta' = \Bigl(1 - \frac{2b}{a}\Bigr)\ln(1 + a\cos(2\theta)).$$

Thus we have

$$z' = -z^2K_3\theta'' + F_3, \qquad \theta'' = t^{-1}z(1 + a\cos(2\theta))^{2-\frac{2b}{a}} + t^{-1}r^4K_2 + f_2, \tag{32}$$

where

$$K_3(\theta) = -\frac{2K_1(\theta)}{(1 + a\cos(2\theta))^{\frac{2b}{a}}} - \frac{2(a-2b)\sin(2\theta)K_2(\theta)}{(1 + a\cos(2\theta))^{1+\frac{2b}{a}}}.$$

Here

$$K_1(\theta) = \frac{2\pi}{3}b^2\sin(4\theta) + 6\mu^2\Bigl(\ln\frac{3+\sqrt{5}}{6}\Bigr)\cos(4\theta) + \frac{\pi}{3}(\mu - 2b)\sin(2\theta) + (\ln 3)\mu\cos(2\theta) - 4b\mu\ln 3,$$

$$K_2(\theta) = \frac{2\pi}{3}b^2\cos(4\theta) + 6\mu^2\Bigl(\ln\frac{3+\sqrt{5}}{6}\Bigr)\sin(4\theta) + \frac{\pi}{3}(\mu + 2b)\cos(2\theta) - (\ln 3)\mu\sin(2\theta) + \frac{4\pi}{3}b\mu.$$

In the following lemma we study the system of ordinary differential equations (32) whose initial perturbation $z(1) = |\rho(1)|^2(1 + a\cos(2\theta(1)))^{1-\frac{2b}{a}}$ satisfies the inequalities $\frac{\delta^2}{2}(1+a)^{1-\frac{2b}{a}} < z(1) < 2\varepsilon_1^2(1-a)^{1-\frac{2b}{a}}$.

Lemma 4. *Suppose that there is a solution $z \in \mathbf{C}^1([1,T])$ of the system (32) whose initial perturbation $z(1)$ satisfies the inequalities $\frac{\delta^2}{2}(1+a)^{1-\frac{2b}{a}} < z(1) < 2\varepsilon_1^2(1-a)^{1-\frac{2b}{a}}$. Then there are constants C_1 and C_2 such that $C_1 z(1)\Theta^{-\frac{1}{2}}(t) < z(t) < C_2 z(1)\Theta^{-\frac{1}{2}}(t)$ for $t \in [1,T]$, where $\Theta(t) = 1 + \sigma z^2(1)\log t$ and the constant $\sigma > 0$ is defined in Lemma 8 below.*

Proof. The proof is by contradiction. Since the solution $z(t)$ is continuous, there is a maximal interval of time $[1,\widetilde{T}]$, $\widetilde{T} > 1$, such that

$$C_1 z(1)\Theta^{-\frac{1}{2}}(t) \leqslant z(t) \leqslant C_2 z(1)\Theta^{-\frac{1}{2}}(t) \tag{33}$$

for all $t \in [1,\widetilde{T}]$. Dividing the first equation of the system (32) by z^2 and integrating with respect to t, we get $\int_1^t \frac{z'}{z^2}\,d\tau = \int_1^t K_3\theta'\,d\tau - \int_1^t z^{-2}F_3\,d\tau$. It follows that

$$\frac{1}{z(t)} = \frac{1}{z(1)} + \int_{\theta(1)}^{\theta(t)} K_3(\theta)\,d\theta - \int_1^t z^{-2}F_3\,d\tau.$$

Hence

$$z(t) = z(1)\biggl(1 + z(1)\biggl(\int_{\theta(1)}^{\theta(t)} K_3(\theta)\,d\theta - \int_1^t z^{-2}F_3\,d\tau\biggr)\biggr)^{-1}. \tag{34}$$

By Lemma 8 (see § 4 below) we have

$$\int_{\theta(1)}^{\theta(t)} K_3(\theta)\,d\theta = \sigma\theta(t) + K_5(t), \tag{35}$$

where $\sigma > 0$ and $K_5(t) = K_4(\theta(t)) - \sigma\theta(1) - K_4(\theta(1))$ is a bounded function. Substituting (35) into (34), we find that

$$z(t) = \frac{z(1)}{1 + z(1)(\sigma\theta(t) + F_4)}, \tag{36}$$

where $F_4 = K_5(t) - \int_1^t z^{-2}F_3\,d\tau$. Substituting (36) into the second equation of the system (32), we get

$$\frac{\frac{1}{z(1)} + \sigma\theta}{(1 + a\cos(2\theta))^{2-\frac{2b}{a}}}\theta'' = t^{-1} + F_5, \tag{37}$$

where $F_5 = \frac{1}{z}(t^{-1}r^4K_2 + f_2)(1 - zF_4)(1 + a\cos(2\theta))^{\frac{2b}{a}-2} - t^{-1}zF_4$. Integrating (37) with respect to t, we obtain

$$\int_{\theta(1)}^{\theta(t)} \frac{\frac{1}{z(1)} + \sigma\theta}{(1 + a\cos(2\theta))^{2-\frac{2b}{a}}}\,d\theta = \log t + \int_1^t F_5\,d\tau. \tag{38}$$

In view of (33) we have

$$\begin{aligned}\left|\int_1^t z^{-2}F_3\,d\tau\right| &\leqslant C\varepsilon_1^4\int_1^t \Theta^{-1}(\tau)\tau^{-1}\,d\tau + C\varepsilon^6\delta^{-3}\int_1^t \Theta^{\frac34}(\tau)Q^{-\frac74}\tau^{-1}\,d\tau\\ &\leqslant C\varepsilon^{\nu}z(1)\int_1^t \Theta^{-\frac12}(\tau)\tau^{-1}\,d\tau\\ &= \frac{C\varepsilon^{\nu}}{z(1)}\int_0^{\sigma z^2(1)\log t}(1+y)^{-\frac12}\,dy \leqslant C\varepsilon^{\nu}z(1)\Theta^{-\frac12}(t)\log t.\end{aligned}$$

Thus we can now estimate the remainder in (36):

$$|F_4| \leqslant |K_5(t)| + \left|\int_1^t z^{-2}F_3\,d\tau\right| \leqslant C + C\varepsilon^{\nu}z(1)\Theta^{-\frac12}(t)\log t.$$

Using (33), we can also estimate the remainder in (38):

$$\begin{aligned}&\left|\int_1^t F_5\,d\tau\right| \leqslant C\int_1^t \bigl(z(1)\Theta^{-\frac12}(\tau) + \varepsilon^6\delta^{-3}\Theta^{\frac34}(\tau)Q^{-\frac74}\bigr)\tau^{-1}\,d\tau\\ &\quad + Cz(1)\int_1^t \Theta^{-\frac12}(\tau)(1+\varepsilon^{\nu}z(1)\Theta^{-\frac12}(\tau)\log\tau)\tau^{-1}\,d\tau \leqslant C\varepsilon^{\nu}\int_1^t \tau^{-1}\,d\tau \leqslant C\varepsilon^{\nu}\log t.\end{aligned}$$

Therefore (38) yields the two-sided estimate

$$\begin{aligned}\frac{h(t)}{(1+a)^{2-\frac{2b}{a}}}\left(\frac{\sigma}{2}h(t) + \frac{1}{z(1)} + \sigma\theta(1)\right) &\leqslant \log t + \int_1^t F_5\,d\tau\\ &\leqslant \frac{h(t)}{(1-a)^{2-\frac{2b}{a}}}\left(\frac{\sigma}{2}h(t) + \frac{1}{z(1)} + \sigma\theta(1)\right)\end{aligned}$$

for $h(t) = \theta(t) - \theta(1)$. Thus we have $S_1 \leqslant \frac{\sigma}{2}h^2 + Bh \leqslant S_2$, where $B = \frac{1}{z(1)} + \sigma\theta(1)$, $S_1 = (1-a)^{2-\frac{2b}{a}}(1 - C\varepsilon^{\nu})\log t$ and $S_2 = (1+a)^{2-\frac{2b}{a}}(1 + C\varepsilon^{\nu})\log t$. We denote the roots of the equation $\frac{\sigma}{2}h^2 + Bh = S_1$ by $h_2 = \frac{1}{\sigma}\bigl(-B - \sqrt{B^2 + 2\sigma S_1}\bigr)$ and $h_3 = \frac{1}{\sigma}\bigl(-B + \sqrt{B^2 + 2\sigma S_1}\bigr)$. We also denote the roots of the equation $\frac{\sigma}{2}h^2 + Bh = S_2$ by $h_1 = \frac{1}{\sigma}\bigl(-B - \sqrt{B^2 + 2\sigma S_2}\bigr)$ and $h_4 = \frac{1}{\sigma}\bigl(-B + \sqrt{B^2 + 2\sigma S_2}\bigr)$. Since $h_3(1) = h(1) = h_4(1) = 0$, we obtain by continuity that $h_3(t) \leqslant h(t) \leqslant h_4(t)$. Furthermore, estimating

$$h_4 \leqslant 3(1+a)^{2-\frac{2b}{a}}\Theta^{-\frac12}(t)z(1)\log t, \qquad h_3 \geqslant (1-a)^{2-\frac{2b}{a}}\Theta^{-\frac12}(t)z(1)\log t,$$

we have

$$\theta(t) \leqslant \theta(1) + 3(1+a)^{2-\frac{2b}{a}}\Theta^{-\frac12}(t)z(1)\log t, \tag{39}$$

$$\theta(t) \geqslant \theta(1) + (1-a)^{2-\frac{2b}{a}}\Theta^{-\frac12}(t)z(1)\log t. \tag{40}$$

In view of (39), (40) and (36) we then find that

$$\begin{aligned}z(t) &\leqslant \frac{z(1)}{1 - Cz(1) + ((1-a)^{2-\frac{2b}{a}} - C\varepsilon^{\nu})\Theta^{-\frac12}(t)\sigma z^2(1)\log t}\\ &\leqslant \frac{z(1)}{(1-a)^{2-\frac{2b}{a}} - C\varepsilon^{\nu}}\Theta^{-\frac12}(t)\end{aligned}$$

and $z(t) > \frac{z(1)}{3(1+a)^{2-\frac{2b}{a}}+C\varepsilon^{\nu}}\Theta^{-\frac{1}{2}}(t)$ for $t \in [1, \widetilde{T}]$. We have arrived at a contradiction. Hence the estimates stated in the lemma hold for all $t \in [1, T]$. □

In the following lemma we estimate the $\mathbf{X}_T$-norm of φ.

Lemma 5. *Let the initial perturbation* $\varphi_0 \in \mathbf{H}^1$ *be such that* $\|\varphi_0\|_{\mathbf{L}^\infty} \leqslant \varepsilon_1$ *and* $\|\psi_0\|_{\mathbf{L}^2} \leqslant \varepsilon_1^4$, *where* $\varepsilon_1 > 0$ *is sufficiently small. Suppose that the solution* $\varphi \in \mathbf{C}([1,T]; \mathbf{H}^1)$ *of* (11) *is such that* $\|\varphi\|_{\mathbf{Z}_T} \leqslant \varepsilon^3$ *and* $\|\psi\|_{\mathbf{Y}_T} \leqslant \varepsilon^4$. *Then*

$$\|\varphi\|_{\mathbf{X}_T} < \varepsilon. \tag{41}$$

Proof. Assume that (41) does not hold. Since φ is continuous, there is a maximal time $\widetilde{T} \in (1, T]$ such that

$$\|\varphi(t)\|_{\mathbf{L}^\infty} + Q^{\frac{1}{4}}(t)\|\langle\widetilde{\xi}\,\rangle^{-\gamma}\varphi(t)\|_{\mathbf{L}^\infty} \leqslant \varepsilon$$

for all $t \in [1, \widetilde{T}]$. Using Lemma 2, we see that $\langle\widetilde{\xi}\,\rangle^{-\gamma}|\varphi(t,\xi) - \rho(t)| \leqslant \varepsilon^3 Q^{-\frac{3}{4}}(t)$ for $t \in [1, \widetilde{T}]$. Then it follows from Lemma 4 that

$$\langle\widetilde{\xi}\,\rangle^{-\gamma}|\varphi(t,\xi)| \leqslant \langle\widetilde{\xi}\,\rangle^{-\gamma}|\rho(t)| + \langle\widetilde{\xi}\,\rangle^{-\gamma}|\varphi(t,\xi) - \rho(t)| \leqslant C\varepsilon_1 Q^{-\frac{1}{4}}(t) + \varepsilon^3 Q^{-\frac{3}{4}}(t).$$

Thus

$$Q^{\frac{1}{4}}(t)\|\langle\widetilde{\xi}\,\rangle^{-\gamma}\varphi(t)\|_{\mathbf{L}^\infty} \leqslant C\varepsilon_1 < \frac{\varepsilon}{4} \tag{42}$$

for all $t \in [1, \widetilde{T}]$. As above, for $|\widetilde{\xi}\,| \leqslant \langle\log t\rangle$ we obtain that

$$|\varphi(t,\xi)| \leqslant |\rho(t)| + |\varphi(t,\xi) - \rho(t)| \leqslant C\varepsilon_1 + C\langle\widetilde{\xi}\,\rangle^{\gamma}\varepsilon^3 Q^{-\frac{3}{4}}(t) < \frac{\varepsilon}{4}. \tag{43}$$

Thus, to prove the bound $\|\varphi(t)\|_{\mathbf{L}^\infty} < \frac{\varepsilon}{2}$, it suffices to consider the case when $t\xi^2 \geqslant \langle\log t\rangle^2$. By (16) we have

$$\begin{aligned}|\varphi(t,\xi) - v(t,\xi)| &\leqslant C\int_{\mathbb{R}}\langle\sqrt{t}(\xi-\eta)\rangle^{-1}|w(t,\eta)|\,d\eta \\ &\quad + C\sqrt{t}\int_{\mathbb{R}}\langle\sqrt{t}(\xi-\eta)\rangle^{-1}\langle\sqrt{t}\eta\rangle^{-1}|v^3(t,\eta)|\,d\eta \\ &\leqslant C\varepsilon^4 Q^{-\frac{5}{4}}(t) + C\varepsilon^3 Q^{-\frac{3}{4}}(t)\sqrt{t}\int_{\mathbb{R}}\langle\sqrt{t}(\xi-\eta)\rangle^{-1}\langle\sqrt{t}\eta\rangle^{-1+3\gamma}\,d\eta,\end{aligned}$$

where the second term admits the following estimate for all $\widetilde{\xi}$:

$$\begin{aligned}&\sqrt{t}\int_{\mathbb{R}}\langle\sqrt{t}(\xi-\eta)\rangle^{-1}\langle\sqrt{t}\eta\rangle^{3\gamma-1}\,d\eta \\ &\quad = \int_{|\widetilde{\eta}|\leqslant\frac{|\widetilde{\xi}\,|}{2}}\langle\widetilde{\xi}-\widetilde{\eta}\,\rangle^{-1}\langle\widetilde{\eta}\,\rangle^{3\gamma-1}d\widetilde{\eta} + \int_{|\widetilde{\eta}|>\frac{|\widetilde{\xi}\,|}{2}}\langle\widetilde{\xi}-\widetilde{\eta}\,\rangle^{-1}\langle\widetilde{\eta}\,\rangle^{3\gamma-1}d\widetilde{\eta} \leqslant C\langle\widetilde{\xi}\,\rangle^{4\gamma-1}.\end{aligned}$$

Hence

$$|\varphi(t,\xi) - v(t,\xi)| \leqslant C\varepsilon^4 Q^{-\frac{5}{4}}(t) + C\varepsilon^3\langle\widetilde{\xi}\,\rangle^{4\gamma-1}Q^{-\frac{3}{4}}(t). \tag{44}$$

Using (20) and (11), we obtain for $t\xi^2 \geqslant \langle \log t\rangle^2$ that

$$i\partial_t \varphi(t,\xi) = t^{-1}\sum_{j=1}^{3}\theta_j E^{a_j}\mathcal{D}(2j-3)v^j\overline{v}^{3-j}(t,\xi) + R_6,$$

where

$$R_6 = t^{-1}\sum_{j=1}^{3}\theta_j E^{a_j}\mathcal{D}(2j-3)\int_{\mathbb{R}} G_{3-2j}(\widetilde{y})\partial_\xi v^j\overline{v}^{3-j}(t,\xi-y)\,dy.$$

Since $|G_{3-2j}(\widetilde{\xi})| \leqslant C\langle\widetilde{\xi}\rangle^{-1}$ and

$$\left|\partial_\xi \int_0^1 \left(E^{\frac{a_k z}{1-a_k z}} - 1\right)\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-a_k z))}}{z\sqrt{|1-a_k z|}}\,dz\right| \leqslant Ct^{\frac{1}{2}}\langle\widetilde{\xi}\rangle^{-1},$$

we obtain from (30) that

$$\begin{aligned}|R_6| &\leqslant Ct^{-1}\langle\widetilde{\xi}\rangle^{-\frac{3}{4}}|\rho|^3\|\langle\widetilde{\xi}\rangle^{-\gamma}v(t)\|^2_{\mathbf{L}^\infty} + C\varepsilon^2 t^{-\frac{5}{4}}\|\psi\|_{\mathbf{L}^2}\\ &\leqslant C\varepsilon^5 t^{-1}\langle\log t\rangle^{-\frac{3}{4}}Q^{-\frac{5}{4}}(t) + C\varepsilon^6 t^{-1}Q^{-\frac{5}{4}}(t)\end{aligned}$$

for $t\xi^2 \geqslant \langle\log t\rangle^2$. Then, by (42) and (44), we have

$$\begin{aligned}i\partial_t\varphi(t,\xi) &= t^{-1}\sum_{j=1}^{3}\theta_j E^{a_j}\mathcal{D}(2j-3)\varphi^j\overline{\varphi}^{3-j}(t,\xi)\\ &\qquad + O(\varepsilon^5 t^{-1}\langle\log t\rangle^{-\frac{3}{4}}Q^{-\frac{3}{4}}(t)) + O(\varepsilon^6 t^{-1}Q^{-\frac{5}{4}}(t)).\end{aligned}$$

Multiplying this by $e^{i\int_1^t |\varphi|^2\frac{d\tau}{\tau}}$, we find that

$$\begin{aligned}i\partial_t\varphi_1(t,\xi) &= t^{-1}e^{i\int_1^t|\varphi|^2\frac{d\tau}{\tau}}\sum_{j=1,3}\theta_j E^{a_j}\mathcal{D}(2j-3)\varphi^j\overline{\varphi}^{3-j}(t,\xi)\\ &\qquad + O(\varepsilon^5 t^{-1}\langle\log t\rangle^{-\frac{3}{4}}Q^{-\frac{3}{4}}(t)) + O(\varepsilon^6 t^{-1}Q^{-\frac{5}{4}}(t))\end{aligned} \tag{45}$$

for the new unknown function $\varphi_1(t,\xi) = e^{i\int_1^t|\varphi|^2\frac{d\tau}{\tau}}\varphi(t,\xi)$. We consider the equation (45) for $t \geqslant t_1$, where $t_1 = \max(1,t_2)$ and t_2 is such that $\frac{t_2}{\langle\log t_2\rangle^2} = \frac{1}{\xi^2}$. Using the identity $E^{a_j} = \frac{2}{a_j i\xi^2}\partial_t E^{a_j}$, we find that

$$\begin{aligned}&t^{-1}e^{i\int_1^t|\varphi|^2\frac{d\tau}{\tau}}\sum_{j=1,3}\theta_j E^{a_j}\mathcal{D}(2j-3)\varphi^j\overline{\varphi}^{3-j}(t,\xi)\\ &\quad = -i\partial_t\left(it^{-1}e^{i\int_1^t|\varphi|^2\frac{d\tau}{\tau}}\sum_{j=1,3}\frac{2\theta_j}{a_j i\xi^2}E^{a_j}\mathcal{D}(2j-3)\varphi^j\overline{\varphi}^{3-j}(t,\xi)\right) + R_9,\end{aligned} \tag{46}$$

where

$$R_9 = \sum_{j=1,3}\frac{2\theta_j}{a_j i\xi^2}E^{a_j}\partial_t(t^{-1}e^{i\int_1^t|\varphi|^2\frac{d\tau}{\tau}}\mathcal{D}(2j-3)\varphi^j\overline{\varphi}^{3-j}(t,\xi)).$$

Since $|R_9| \leqslant C\varepsilon^3 t^{-1}\langle \log t\rangle^{-\frac{5}{4}} + C\varepsilon^6 t^{-1} Q^{-\frac{5}{4}}(t)$ in the region $t\xi^2 \geqslant \langle \log t\rangle^2$, an application of (46) to (45) yields the equation

$$i\partial_t \varphi_2 = O\big(\varepsilon^3 t^{-1}\langle \log t\rangle^{-\frac{5}{4}}\big) + O\big(\varepsilon^6 t^{-1} Q^{-\frac{5}{4}}(t)\big) \tag{47}$$

for the new function

$$\varphi_2(t,\xi) = \varphi_1(t,\xi) + it^{-1} e^{i\int_1^t |\varphi|^2 \frac{d\tau}{\tau}} \sum_{j=1,3} \frac{2\theta_j}{a_j i\xi^2} E^{a_j} \mathcal{D}(2j-3)\varphi^j \overline{\varphi}^{3-j}(t,\xi).$$

Integrating (47) over $t \geqslant t_1$, we obtain from (43) and (44) that

$$|\varphi_2(t)| \leqslant |\varphi_2(t_1)| + C\varepsilon^{1+\nu} \leqslant C\varepsilon_1 + C\varepsilon^{1+\nu} < \frac{\varepsilon}{4}$$

for all $t \in [1, \widetilde{T}]$. Hence $|\varphi(t,\xi)| < \frac{\varepsilon}{4}$ for $t\xi^2 \geqslant \langle \log t\rangle^2$. Thus $\|\varphi(t)\|_{\mathbf{L}^\infty} < \frac{\varepsilon}{2}$. This contradicts our assumption. Thus the bound (41) holds for all $t \in [1, T]$. □

Proof of Theorem 1. Lemma 1 establishes that a priori estimates for the norms $\|\varphi\|_{\mathbf{Z}_T}$ and $\|\varphi\|_{\mathbf{X}_T}$ imply an estimate for $\|\psi\|_{\mathbf{Y}_T}$. In its turn, Lemma 2 yields an estimate for $\|\varphi\|_{\mathbf{Z}_T}$ provided that we know estimates for $\|\varphi\|_{\mathbf{X}_T}$ and $\|\psi\|_{\mathbf{Y}_T}$. Finally, it is proved in Lemma 5 that estimates for $\|\varphi\|_{\mathbf{Z}_T}$ and $\|\psi\|_{\mathbf{Y}_T}$ yield an a priori estimate for $\|\varphi\|_{\mathbf{X}_T}$. Thus, using the standard bootstrap, we obtain from the local existence of solutions (which was established in Theorem 2) that there is a global-in-time solution $\varphi \in \mathbf{C}([1,\infty); \mathbf{H}^1)$ of the Cauchy problem for the equation (9), and this solution satisfies the bound $\|\varphi\|_{\mathbf{X}_\infty} + \|\psi\|_{\mathbf{Y}_\infty} < 2\varepsilon$. To prove the decay estimate for the solution, we use the formula $u(t,x) = t^{-\frac{1}{2}} Ev(t,\xi)$, $\xi = \frac{x}{t}$, whence $\sup_{|x|\leqslant\sqrt{t}} |u(t)| = t^{-\frac{1}{2}} \sup_{|\xi|\leqslant 1/\sqrt{t}} |Ev(t,\xi)|$. Then we see from (44) that

$$\begin{aligned}\Big| \sup_{|x|\leqslant\sqrt{t}} |u(t)| - t^{-\frac{1}{2}}|\rho(t)| \Big| &\leqslant Ct^{-\frac{1}{2}} \sup_{|\xi|\leqslant\frac{1}{\sqrt{t}}} |v(t,\xi) - \varphi(t,\xi)| + Ct^{-\frac{1}{2}} |\varphi(t,\xi) - \rho(t)| \\ &\leqslant C\varepsilon^2 t^{-\frac{1}{2}} Q^{-\frac{1}{4}}(t)\end{aligned}$$

as $t \to \infty$. By Lemma 5 this proves the first bound in the theorem. The second bound is obtained as follows:

$$|u(t,x)| = t^{-\frac{1}{2}} |Ev(t,\xi)| \leqslant t^{-\frac{1}{2}} \|v\|_{\mathbf{L}^\infty} \leqslant C\varepsilon t^{-\frac{1}{2}}. \qquad \square$$

§ 4. Appendix

We start with an explicit calculation of the integrals

$$\Phi_a = \int_{\mathbb{R}} G_1(\widetilde{\xi})\partial_\xi \int_0^1 (E^{az} - 1)\,\frac{dz}{z}\, d\xi,$$

$$\Psi_{b,a} = \int_{\mathbb{R}} G_b(\widetilde{\xi})\partial_\xi \int_0^1 (E^{\frac{az}{1-az}} - 1) \frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-az))}\, dz}{z\sqrt{|1-az|}}\, d\xi$$

for $a > 0$, $|b| \geqslant 1$.

Lemma 6. *The following equalities hold*:

$$\Phi_a = \begin{cases} -2\ln\left(\dfrac{1+\sqrt{1-a}}{2}\right) & \text{\emph{for}} \quad 0 < a \leqslant 1, \\ -\ln\dfrac{a}{4} + 2i\arctan\sqrt{a-1} & \text{\emph{for}} \quad a > 1, \end{cases}$$

and $\Psi_{b,a} = \Phi_h - \Phi_a$, *where* $h = \frac{b+1}{b}a,\ a > 0,\ |b| \geqslant 1$.

Proof. By the identity

$$\int_0^\infty e^{-\frac{i}{2}\xi^2 a}\,d\xi = \frac{\sqrt{2\pi}}{2\sqrt{|a|}}e^{-i\frac{\pi}{4}\operatorname{sign} a} \tag{48}$$

we obtain that

$$\Phi_a = \int_{\mathbb{R}} G_1(\widetilde{\xi})\partial_\xi \int_0^1 (E^{az}-1)\,\frac{dz}{z}\,d\xi = \int_0^a \left(\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-z))}}{\sqrt{|1-z|}} - 1\right)\frac{dz}{z}.$$

We have

$$\int \frac{dz}{z\sqrt{|1-z|}} = \begin{cases} \ln\left(\dfrac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right) & \text{for} \quad 0 < z < 1, \\ 2\arctan\sqrt{z-1} & \text{for} \quad z > 1. \end{cases}$$

Therefore

$$\Phi_a = \begin{cases} -2\ln\left(\dfrac{1+\sqrt{1-a}}{2}\right) & \text{for} \quad 0 < a \leqslant 1, \\ -\ln\dfrac{a}{4} + 2i\arctan\sqrt{a-1} & \text{for} \quad a > 1. \end{cases}$$

In view of (48) we get

$$\Psi_{b,a} = \int_{\mathbb{R}} G_b(\widetilde{\xi})\partial_\xi \int_0^1 (E^{\frac{az}{1-az}}-1)\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-az))}\,dz}{z\sqrt{|1-az|}}\,d\xi = \Lambda_{b,a} - \Phi_a,$$

$$\Lambda_{b,a} = \int_0^1 \left(\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-az)+\operatorname{sign} b+\operatorname{sign}(\frac{az}{1-az}-b))}}{\sqrt{|1-hz|}} - 1\right)\frac{dz}{z}.$$

We have

$$\Lambda_{b,a} = \int_0^h \left(\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-gz)-\operatorname{sign} b(\operatorname{sign}(1-z)(1-gz)-1))}}{\sqrt{|1-z|}} - 1\right)\frac{dz}{z},$$

where $g = \frac{a}{h} = \frac{b}{b+1}$. Notice that $\Lambda_{-1,a} = 0$. To prove the second equality in the lemma, we use the following identity:

$$1 - \operatorname{sign}(1-gz) - \operatorname{sign} b(\operatorname{sign}(1-z)(1-gz)-1) = 1 - \operatorname{sign}(1-z) \tag{49}$$

for $0 < z < h$. Then

$$\begin{aligned} \Lambda_{b,a} &= \int_0^h \left(\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-gz)-\operatorname{sign} b(\operatorname{sign}(1-z)(1-gz)-1))}}{\sqrt{|1-z|}} - 1\right)\frac{dz}{z} \\ &= \int_0^h \left(\frac{e^{i\frac{\pi}{4}(1-\operatorname{sign}(1-z))}}{\sqrt{|1-z|}} - 1\right)\frac{dz}{z} = \Phi_h. \qquad \square \end{aligned}$$

We now calculate the coefficients

$$\omega_l = \sum_{\substack{k=1,3\\ l-2\leqslant k\leqslant l}} \chi_{l+1-k} A_{l+1-k,k} + \sum_{\substack{k=1,3\\ 3-l\leqslant k\leqslant 5-l}} \chi_{k+l-3}\overline{A_{6-k-l,k}},$$

where $A_{j,k} = ij\chi_k(\Phi_{a_k} + \Psi_{3-2j,a_k})$, $\chi_j = \lambda_j \frac{i^{2-j}}{\sqrt{|2j-3|}} e^{i\frac{\pi}{4}(\operatorname{sign}(2j-3)-1)}$, $a_j = \frac{2j-4}{2j-3}$.

Lemma 7. *The following equalities hold*:

$$\omega_1 = -\frac{2\pi}{3}b^2, \qquad \omega_2 = -\mu\left(\frac{\pi}{3} + i\ln 3\right),$$

$$\omega_3 = -4b\mu\left(\frac{\pi}{3} - i\ln 3\right), \qquad \omega_4 = -\frac{2\pi}{3}b, \qquad \omega_5 = -6i\mu^2 \ln\frac{3+\sqrt{5}}{6}.$$

Proof. By Lemma 6 we have $\Phi_{a_k} + \Psi_{b_j,a_k} = \Phi_{h_{j,k}}$, where $h_{j,k} = \frac{b_j+1}{b_j}a_k$, $b_j = 3-2j$. Since $a_1 = 2$ and $a_3 = \frac{2}{3}$, we have $h_{1,1} = 4$, $h_{1,3} = \frac{4}{3}$, $h_{2,k} = 0$, $h_{3,1} = \frac{4}{3}$, $h_{3,3} = \frac{4}{9}$. Moreover, $\chi_1 = \lambda_1$, $\chi_2 = \lambda_2$, $\chi_3 = -\frac{i}{\sqrt{3}}\lambda_3$. Therefore we find by Lemma 6 that $\Phi_{h_{1,1}} = \frac{2\pi i}{3}$, $\Phi_{h_{1,3}} = \Phi_{h_{3,1}} = \ln 3 + \frac{\pi i}{3}$, $\Phi_{h_{3,3}} = 2\ln\frac{6}{3+\sqrt{5}}$, $\Phi_{h_{2,k}} = 0$. We thus get $A_{2,k} = 0$, $A_{1,1} = -\frac{2\pi}{3}\lambda_1$, $A_{1,3} = \frac{1}{\sqrt{3}}\lambda_3\left(\ln 3 + \frac{\pi i}{3}\right)$, $A_{3,1} = 3i\lambda_1\left(\ln 3 + \frac{\pi i}{3}\right)$, $A_{3,3} = 2\sqrt{3}\lambda_3 \ln\frac{6}{3+\sqrt{5}}$, whence

$$\omega_1 = \chi_1 A_{1,1} + \chi_1\overline{A_{2,3}} = -\frac{2\pi}{3}\lambda_1^2,$$

$$\omega_2 = \chi_2 A_{2,1} + \chi_2\overline{A_{1,3}} = \lambda_2\overline{\lambda_3}\left(\frac{1}{\sqrt{3}}\ln 3 - \frac{\pi i}{3\sqrt{3}}\right),$$

$$\omega_3 = \chi_3 A_{3,1} + \chi_1 A_{1,3} + \chi_1\overline{A_{2,1}} = \frac{4}{\sqrt{3}}\lambda_1\lambda_3\left(\ln 3 + \frac{\pi i}{3}\right),$$

$$\omega_4 = \chi_2 A_{2,3} + \chi_2\overline{A_{1,1}} = -\frac{2\pi}{3}\overline{\lambda_1}\lambda_2, \qquad \omega_5 = \chi_3 A_{3,3} = -2i\lambda_3^2 \ln\frac{6}{3+\sqrt{5}}.$$

By our choice of $\lambda_1 = b$, $\lambda_2 = 1$, $\lambda_3 = i\mu\sqrt{3}$ we have $\chi_1 = b$, $\chi_2 = 1$, $\chi_3 = \mu$. Thus we arrive at the result of the lemma. □

In the following lemma we consider

$$K_3(\theta) = -\frac{2K_1(\theta)}{(1+a\cos(2\theta))^{\frac{2b}{a}}} - \frac{2(a-2b)\sin(2\theta)K_2(\theta)}{(1+a\cos(2\theta))^{1+\frac{2b}{a}}},$$

where

$$K_1(\theta) = \frac{2\pi}{3}b^2\sin(4\theta) + 6\mu^2\left(\ln\frac{3+\sqrt{5}}{6}\right)\cos(4\theta) + \frac{\pi}{3}(\mu-2b)\sin(2\theta) + (\ln 3)\mu\cos(2\theta) - 4b\mu\ln 3,$$

$$K_2(\theta) = \frac{2\pi}{3}b^2\cos(4\theta) + 6\mu^2\left(\ln\frac{3+\sqrt{5}}{6}\right)\sin(4\theta) + \frac{\pi}{3}(\mu+2b)\cos(2\theta) - (\ln 3)\mu\sin(2\theta) + \frac{4\pi}{3}b\mu.$$

Lemma 8. *Suppose that $\mu > 0$, $b \geqslant 0$ and $\mu + b < 1$. Then there is a constant $\sigma > 0$ such that*

$$\int_{\theta(1)}^{\theta(t)} K_3(\theta')\, d\theta' = \sigma\theta + K_4(\theta),$$

where $K_4(\theta)$ is a bounded function.

Proof. We write $K_3(\theta) = 2\mu K_m(\theta) + K_r(\theta)$, where

$$K_m(\theta) = \frac{6\mu\big(-\ln\frac{3+\sqrt{5}}{6}\big)\cos(4\theta) + (\ln 3)(-\cos(2\theta)) + 4b\ln 3}{(1 + a\cos(2\theta))^{\beta-1}} + \frac{(\mu - b)\big(\ln 3 + 12\mu\big(-\ln\frac{3+\sqrt{5}}{6}\big)\cos(2\theta)\big)\sin^2(2\theta)}{(1 + a\cos(2\theta))^{\beta}},$$

$$K_r(\theta) = -\frac{2\big(\frac{2\pi}{3}b^2\sin(4\theta) + \frac{\pi}{3}(\mu - 2b)\sin(2\theta)\big)}{(1 + a\cos(2\theta))^{\beta-1}} - \frac{2(a - 2b)\sin(2\theta)\big(\frac{2\pi}{3}b^2\cos(4\theta) + \frac{\pi}{3}(\mu + 2b)\cos(2\theta) + \frac{4\pi}{3}b\mu\big)}{(1 + a\cos(2\theta))^{\beta}}.$$

Here $a = b + \mu$, $\beta = 1 + \frac{2b}{a}$. It is easily seen that $\int_{\pi/2}^{\pi} K_r(\theta)\, d\theta = -\int_0^{\pi/2} K_r(\theta)\, d\theta$ and $\int_{\pi/2}^{\pi} K_m(\theta)\, d\theta = \int_0^{\pi/2} K_m(\theta)\, d\theta$. Hence $\int_0^{\pi} K_r(\theta)\, d\theta = 0$. We shall now show that $\int_0^{\pi/2} K_m(\theta)\, d\theta = \frac{\pi}{4\mu}\sigma > 0$. This will prove the lemma with some bounded function $K_4(\theta)$ since the function $K_3(\theta)$ is periodic with period π. In the case when $b \geqslant \mu > 0$, we write

$$\begin{aligned}2\int_0^{\pi/2} K_m(\theta)\, d\theta &= 6\mu\Big(-\ln\frac{3+\sqrt{5}}{6}\Big)\int_0^{\pi}\frac{\cos(2x)}{(1 + a\cos x)^{\beta-1}}\, dx \\ &\quad + 12\mu(b - \mu)\Big(-\ln\frac{3+\sqrt{5}}{6}\Big)\int_0^{\pi}\frac{\sin^2 x(-\cos x)}{(1 + a\cos x)^{\beta}}\, dx \\ &\quad + \int_0^{\pi}\frac{(\ln 3)(-\cos x) + 4b\ln 3}{(1 + a\cos x)^{\beta-1}}\, dx - \int_0^{\pi}\frac{(b - \mu)(\ln 3)\sin^2 x}{(1 + a\cos x)^{\beta}}\, dx,\end{aligned}$$

whence by the identity $\big(1 - \frac{1}{a}\cos x\big)(1 + a\cos x) - \sin^2 x = -\frac{1-a^2}{a}\cos x$ we get

$$\begin{aligned}2\int_0^{\pi/2} K_m(\theta)\, d\theta &= 6\mu\Big(-\ln\frac{3+\sqrt{5}}{6}\Big)\int_0^{\pi}\frac{\cos(2x)}{(1 + a\cos x)^{\beta-1}}\, dx \\ &\quad + 12\mu(b - \mu)\Big(-\ln\frac{3+\sqrt{5}}{6}\Big)\int_0^{\pi}\frac{\sin^2 x(-\cos x)}{(1 + a\cos x)^{\beta}}\, dx \\ &\quad + \frac{2\mu}{a}(\ln 3)\int_0^{\pi}\frac{(-\cos x)\, dx}{(1 + a\cos x)^{\beta-1}} + (3b + \mu)(\ln 3)\int_0^{\pi}\frac{dx}{(1 + a\cos x)^{\beta-1}} \\ &\quad + (b - \mu)(\ln 3)\frac{1 - a^2}{a}\int_0^{\pi}\frac{(-\cos x)}{(1 + a\cos x)^{\beta}}\, dx > 0.\end{aligned}$$

Consider the case when $1 > \mu > b \geqslant 0$. We use the inequality

$$\frac{\sin^2 x}{2(1+a\cos x)} = \frac{1-\cos^2 x}{2(1+a\cos x)} \leqslant 1.$$

Then

$$\begin{aligned}
2\int_0^{\pi/2} K_m(\theta)\,d\theta &\geqslant \int_0^{\pi} \frac{6\mu\big(-\ln\frac{3+\sqrt{5}}{6}\big)\cos(2x) + (\ln 3)(-\cos x)}{(1+a\cos x)^{\beta-1}}\,dx \\
&\quad + \int_0^{\pi} \frac{(\mu-b)\big(\big(1+\frac{2b}{\mu-b}\big)\ln 3 + \mu\big(-12\ln\frac{3+\sqrt{5}}{6}\big)\cos x\big)\sin^2 x}{(1+a\cos x)^{\beta}}\,dx \\
&\geqslant (\mu-b)\mu\bigg(-12\ln\frac{3+\sqrt{5}}{6}\bigg)\int_0^{\pi} \frac{(g+\cos x)\sin^2 x}{(1+a\cos x)^{\beta}}\,dx,
\end{aligned}$$

where $g = \frac{a}{\mu(\mu-b)}q$, $q = \frac{\ln 3}{\left(-12\ln\frac{3+\sqrt{5}}{6}\right)}$. If $g \geqslant 1$, then $2\int_0^{\pi/2} K_m(\theta)\,d\theta > 0$. Hence it remains to consider the case when $g < 1$, that is, $b < (\mu-b)\frac{\mu-q}{2q} < (\mu-b)\frac{1-q}{2q}$, which implies that $b < \mu\frac{1-q}{1+q} < \frac{1}{5}\mu \leqslant \frac{1}{4}a$ and, therefore, $\beta < \frac{3}{2}$. We have

$$\begin{aligned}
2\int_0^{\pi/2} K_m(\theta)\,d\theta &= 6\mu\bigg(-\ln\frac{3+\sqrt{5}}{6}\bigg)I_1 + (\ln 3)I_2 + 4b(\ln 3)I_3 \\
&\quad + (\mu-b)(\ln 3)I_4 - 12\mu(\mu-b)\bigg(-\ln\frac{3+\sqrt{5}}{6}\bigg)I_5,
\end{aligned}$$

where

$$I_1 = \int_0^{\pi} \frac{\cos(2x)}{(1+a\cos x)^{\beta-1}}\,dx, \qquad I_2 = \int_0^{\pi} \frac{(-\cos x)}{(1+a\cos x)^{\beta-1}}\,dx,$$
$$I_3 = \int_0^{\pi} \frac{dx}{(1+a\cos x)^{\beta-1}}, \qquad I_4 = \int_0^{\pi} \frac{\sin^2 x}{(1+a\cos x)^{\beta}}\,dx,$$
$$I_5 = \int_0^{\pi} \frac{(-\cos x)\sin^2 x}{(1+a\cos x)^{\beta}}\,dx.$$

By Taylor's theorem we find that

$$\begin{aligned}
&(1-at)^{1-\beta} - (1+at)^{1-\beta} \\
&\quad = 2(\beta-1)at + \frac{(\beta-1)\beta}{2}a^2t^2\big((1-\xi_1)^{-\beta-1} - (1+\xi_2)^{-\beta-1}\big) \geqslant 2(\beta-1)at
\end{aligned}$$

for $\beta \geqslant 1$, $t > 0$, $0 < a < 1$. By making the change of variable $t = \cos x$, we get

$$\begin{aligned}
I_1 &= \int_0^{\pi} \frac{\cos(2x)}{(1+a\cos x)^{\beta-1}}\,dx = \int_0^1 \frac{((1-at)^{1-\beta} - (1+at)^{1-\beta})(2t^2-1)}{\sqrt{1-t^2}}\,dt \\
&\geqslant 2(\beta-1)a\int_0^1 \frac{(2t^2-1)t}{\sqrt{1-t^2}}\,dt = \frac{2}{3}(\beta-1)a \geqslant 0.
\end{aligned}$$

We similarly have

$$I_2 = \int_0^\pi \frac{(-\cos x)}{(1+a\cos x)^{\beta-1}}\,dx = \int_0^1 \frac{\big((1-at)^{1-\beta}-(1+at)^{1-\beta}\big)t}{\sqrt{1-t^2}}\,dt$$
$$\geqslant 2(\beta-1)a\int_0^1 \frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{\pi(\beta-1)a}{2}.$$

Again by Taylor's theorem we have

$$(1+at)^\beta + (1-at)^\beta$$
$$= 2+\beta(\beta-1)a^2t^2 + \frac{\beta(\beta-1)(\beta-2)}{6}a^3t^3\big((1+\xi_1)^{\beta-3}-(1-\xi_2)^{\beta-3}\big)$$
$$\geqslant 2(1-t^2)+(2+\beta(\beta-1)a^2)t^2$$

for $\beta\in(1,2)$. Therefore,

$$I_3 = \int_0^\pi \frac{dx}{(1+a\cos x)^{\beta-1}} = \int_0^1 \frac{(1+at)^{1-\beta}+(1-at)^{1-\beta}}{\sqrt{1-t^2}}\,dt$$
$$\geqslant \int_0^1 \frac{2+\beta(\beta-1)a^2t^2}{\sqrt{1-t^2}}\,dt \geqslant \pi,$$
$$I_4 = \int_0^\pi \frac{\sin^2 x\,dx}{(1+a\cos x)^\beta} \geqslant 2\int_0^1 \frac{(1-t^2)^{\frac{3}{2}}}{(1-a^2t^2)^\beta}\,dt + (2+\beta(\beta-1)a^2)\int_0^1 \frac{\sqrt{1-t^2}t^2}{(1-a^2t^2)^\beta}\,dt$$
$$\geqslant \frac{3}{8}\pi + (2+\beta(\beta-1)a^2)\int_0^1 \frac{\sqrt{1-t^2}t^2}{(1-a^2t^2)^\beta}\,dt.$$

Finally, by Taylor's theorem, we obtain

$$(1+at)^\beta-(1-at)^\beta = 2\beta at - \frac{\beta(\beta-1)}{2}a^2t^2\big((1-\xi_1)^{\beta-2}-(1+\xi_2)^{\beta-2}\big) \leqslant 2\beta at$$

for $\beta\in(1,2)$. Hence we have

$$I_5 = \int_0^\pi \frac{(-\cos x)\sin^2 x}{(1+a\cos x)^\beta}\,dx = \int_0^1 ((1+at)^\beta-(1-at)^\beta)\frac{\sqrt{1-t^2}t}{(1-a^2t^2)^\beta}\,dt$$
$$\leqslant 2\beta a\int_0^1 \frac{\sqrt{1-t^2}t^2}{(1-a^2t^2)^\beta}\,dt.$$

Thus for $1>\mu>b\geqslant 0$ we get

$$2\int_0^{\pi/2} K_m(\theta)\,d\theta \geqslant (\ln 3)\Big(\frac{2}{3}b\mu k + 5\pi b + \frac{3}{8}\pi(\mu-b)\Big)$$
$$+ 2(\mu-b)(\ln 3)(1+\beta ba-\mu\beta ak)\int_0^1 \frac{\sqrt{1-t^2}t^2}{(1-a^2t^2)^\beta}\,dt,$$

where

$$k = \frac{-12\ln\frac{3+\sqrt{5}}{6}}{\ln 3}.$$

Using the bound

$$\int_0^1 \frac{\sqrt{1-t^2}t^2}{(1-a^2t^2)^\beta}\,dt \leqslant \int_0^1 \frac{t}{(1-a^2t^2)^{\beta-\frac12}}\,dt \leqslant \frac{1}{2a^2}\left(\frac32-\beta\right)^{-1},$$

we find that

$$2\int_0^{\pi/2} K_m(\theta)\,d\theta \geqslant \frac23(\ln 3)k\mu b + \frac{(\mu-b)\ln 3}{a\left(\frac32-\beta\right)}\left(1+\frac{3}{16}\pi-k\right)a + \frac{(\mu-b)\ln 3}{a\left(\frac32-\beta\right)}\left(\beta(1+k)+5\pi\frac{\frac32-\beta}{\mu-b}a-\frac{3\pi}{4}-2k\right)b>0$$

since $b<\frac{\mu}{5}$. □

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