Ha Huy Bang, Properties of functions in Orlicz spaces that depend on the geometry of their spectra, *Izvestiya: Mathematics*, 1997, Volume 61, Issue 2, 399–434

DOI: 10.1070/IM1997v061n02ABEH000120

Izvestiya: Mathematics **61**:2 399–434 ©1997 RAS(DoM) and LMS
Izvestiya RAN: Ser. Mat. **61**:2 163–198 UDC 517.9

Properties of functions in Orlicz spaces that depend on the geometry of their spectra

Kha Zui Bang

Abstract. We investigate the geometry of the spectra (the supports of the Fourier transforms) of functions belonging to the Orlicz space $L_\Phi(\mathbb{R}^n)$ and prove, in particular, that if $f \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$, and $f(x) \not\equiv 0$, then for any point in the spectrum of f there is a sequence of spectral points with non-zero components that converges to that point. It is shown that the behaviour of the sequence of Luxemburg norms of the derivatives of a function is completely characterized by its spectrum. A new method is suggested for deriving the Nikol'skii inequalities in the Luxemburg norm for functions with arbitrary spectra. The results are then applied to establish Paley–Wiener–Schwartz type theorems for cases that are not necessarily convex, and to study some questions in the theory of Sobolev–Orlicz spaces of infinite order that has been developed in recent years by Dubinskii and his students.

Entire functions of exponential type that are bounded on the real space $\mathbb{R}^n$ have some properties similar to those of trigonometric polynomials. Whereas the trigonometric polynomials are a suitable means for approximating periodic functions, the entire functions of exponential type can serve as a good tool for approximating non-periodic functions defined on n-dimensional space.

In this paper we study some properties of entire functions of exponential type (which, as functions of a real variable, belong to the Orlicz space) that depend on the geometry of their spectra (the supports of the Fourier transforms) and present some of their applications.

Let $f \in L_p(\mathbb{R}^n)$. Then its Fourier transform $\hat{f}(\xi)$ will in general be a distribution (if $p > 2$), and therefore the geometry of its spectrum is completely opaque. In § 1 we investigate the geometry of the spectra of functions in the Orlicz space $L_\Phi(\mathbb{R}^n)$ and prove, in particular, that if $f \in L_p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, and $f \not\equiv 0$, then for any point of the spectrum of f there is a sequence of spectral points of f with non-zero components that converges to that point. This investigation has applications in subsequent sections. In § 2 we study the behaviour of the sequence of norms $\|D^\alpha f\|_{(\Phi)}$, $\alpha \geqslant 0$, of the derivatives and show that it is completely determined by the spectrum of f. In § 3 a new method is applied to investigate the Nikol'skii inequalities in the norm of the Orlicz space for functions with arbitrary spectra. It should be noted that the Nikol'skii inequalities [1], [2], which play a significant

This work was carried out with the support of the Vietnam National Programme for the Natural Sciences.

AMS 1991 Mathematics Subject Classification. 26A99, 42B10.

part in the theory of functions and have extensive and important applications, have been studied in many papers (for example, see [1]–[5] and the references in [6]). In § 4 we apply the results of the foregoing sections to derive Paley–Wiener–Schwartz type theorems for cases that are not necessarily convex and to study some questions in the theory of Sobolev–Orlicz spaces of infinite order that has been developed in recent years by Dubinskii and his students.

The author would like to express his warmest gratitude to Correspondent Member of the Russian Academy of Sciences O. V. Besov for indicating an inaccuracy in an earlier proof of Theorem 1 of this paper.

§ 0. Preliminaries

Let $\Phi(t)\colon [0,+\infty) \to [0,+\infty]$ be an arbitrary Young function, that is, $\Phi(0)=0$, $\Phi(t) \geqslant 0$, $\Phi(t) \not\equiv 0$, and $\Phi(t)$ is convex.

The *complementary function* of $\Phi(t)$ is defined as

$$\overline{\Phi}(t) = \sup_{s\geqslant 0}\bigl\{ts - \Phi(s)\bigr\}$$

and is also a Young function. The definition of a Young function readily implies that $\Phi(t)/t$ does not decrease on $[0,+\infty)$ and so neither does $\Phi(t)$.

Furthermore, let G be a domain in $\mathbb{R}^n$ or a torus $\mathbb{T}^n$. We denote by $L_\Phi(G)$ the set of all functions $u(x)$ whose Luxemburg norms satisfy the inequality

$$\|u\|_{(\Phi)} = \inf\left\{\lambda > 0 : \int_G \Phi\left(\frac{|u(x)|}{\lambda}\right)\,dx \leqslant 1\right\} < \infty.$$

This defines $L_\Phi(G)$ as a Banach space. It is called an *Orlicz space*. The Luxemburg norm is equivalent to the Orlicz norm $\|\cdot\|_\Phi$, and we have

$$\|u\|_{(\Phi)} \leqslant \|u\|_\Phi \leqslant 2\|u\|_{(\Phi)}.$$

Recall that $\|\cdot\|_{(\Phi)} = \|\cdot\|_p$ if $1 \leqslant p < \infty$ and $\Phi(t) = t^p$, and $\|\cdot\|_{(\Phi)} = \|\cdot\|_\infty$ if $\Phi(t) = 0$ for $0 \leqslant t \leqslant 1$ and $\Phi(t) = \infty$ for $t > 1$ (see [7]–[10] for example).

Lemma 1 [10]**.** *Let $u \in L_\Phi(\mathbb{R}^n)$ and $v \in L_1(\mathbb{R}^n)$. Then*

$$\|u * v\|_\Phi \leqslant \|u\|_\Phi \|v\|_1.$$

Lemma 2 [10]**.** *Let $u \in L_\Phi(G)$ and $v \in L_{\overline{\Phi}}(G)$. Then*

$$\int_G |u(x)v(x)|\,dx \leqslant \|u\|_\Phi \|v\|_{\overline{\Phi}}.$$

Let Q be a domain in $\mathbb{R}^n$ and let $m \in \mathbb{Z}_+$. We denote by $W_{m,2}(Q)$ the Sobolev space, that is, the completion of $C^m(Q)$ with respect to the norm

$$\|f\|_{m,2} = \left(\sum_{|\alpha|\leqslant m} \|D^\alpha f\|^2_{L_2(Q)}\right)^{1/2},$$

and by $W^0_{m,2}(Q)$ the completion of $C_0^\infty(Q)$ with respect to this norm.

We set

$$H_{(s)} = \left\{ f \in S' : \|f\|_{(s)} = \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |Ff(\xi)|^2 \, d\xi \right)^{1/2} < \infty \right\}$$

for $s \in \mathbb{R}$. If $k \in \mathbb{Z}_+$, then we have the topological relation $H_{(k)} = W_{k,2}(\mathbb{R}^n)$ (see [11]–[13] for example).

The following notation will be used: F is the operator of Fourier transformation, $\mathrm{sp}(f) = \operatorname{supp} Ff$, $\Delta_\nu = \left\{\xi \in \mathbb{R}^n : |\xi_j| \leqslant \nu_j,\ j = 1, \dots, n\right\}$, $D = (D_1, \dots, D_n)$, $D_j = -i\frac{\partial}{\partial x_j}$, $\nu_j \geqslant 0$, $j = 1, \dots, n$, and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$. We assume that $\frac{0}{0} = 1$ and $\frac{\lambda}{0} = \infty$ for $\lambda > 0$.

§ 1. Spectrum geometry

In this section we study the spectrum geometry for functions belonging to the Orlicz space $L_\Phi(\mathbb{R}^n)$.

Theorem 1. *Let $\Phi(t) > 0$ for $t > 0$, $f \in L_\Phi(\mathbb{R}^n)$, $f(x) \not\equiv 0$, and let $\xi^0 \in \mathbb{R}^n$ be an arbitrary point. Then the support of the distribution Ff cannot belong to the hyperplanes $\xi_j = \xi_j^0$, $j = 1, \dots, n$.*

Proof. Let $\nu \geqslant 0$ be a vector such that $\xi^0 \in \Delta_\nu$ and let $\widehat{\varphi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\widehat{\varphi}$ is equal to unity in a neigbourhood of Δ_ν. Then, by virtue of Lemma 1, $F^{-1}\widehat{\varphi}\widehat{f} = \varphi * f \in L_\Phi(\mathbb{R}^n)$. Consequently, it suffices to prove the theorem only for functions with bounded spectra.

We set $\hat{h}(\xi) = \hat{f}(\xi - \xi^0)$. Then $h(x) = e^{i\xi^0 x} f(x)$ belongs to $L_\Phi(\mathbb{R}^n)$ and has bounded spectrum.

It remains to show that the support of the distribution $\hat{h}(\xi)$ cannot lie in the hyperplanes $\xi_j = 0$, $j = 1, \dots, n$. We prove this by contradiction. Suppose that the hyperplane $\xi_j = 0$, $j = 1, \dots, n$, contains the support of the distribution $\hat{h}(\xi)$. Set

$$G_j = \left\{\xi \in \mathbb{R}^n : \xi_i \neq 0, \quad i \in I \backslash \{j\}\right\}$$

for each $j = 1, \dots, n$, where $I = \{1, \dots, n\}$. It follows that G_j is open. The support of the distribution $\psi(\xi)\hat{h}(\xi)$ lies in the hyperplanes $\xi_j = 0$ for every function $\psi(\xi)$ belonging to $C_0^\infty(G_j)$. Hence, in view of a remark on Theorem 2.3.5 mentioned in [12], Example 5.1.2, we obtain

$$F^{-1}(\psi\hat{h})(x) = \sum_{l=0}^{N} g_l(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)(-ix_j)^l, \tag{1}$$

where N is the order of $\hat{h}(\xi)$ ($N < \infty$ because $\operatorname{supp}\hat{h}$ is compact) and $\hat{g}_l(\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_n)$, $0 \leqslant l \leqslant N$, is a distribution with a compact support.

Lemma 3 below implies that

$$C^{-1}\left\|F^{-1}(\psi\hat{h})\right\|_\infty \leqslant \left\|F^{-1}(\psi\hat{h})\right\|_{(\Phi)} = \|F^{-1}\psi * h\|_{(\Phi)} \leqslant 2\|F^{-1}\psi\|_1 \|h\|_{(\Phi)} < \infty$$

for some $C > 0$. Thus, (1) can hold only if $N = 0$. Therefore the function $F^{-1}(\psi\hat{h})(x)$ does not depend on x_j.

Next, by Lemma 1, we have $F^{-1}(\psi\hat{h})(x) \in L_\Phi(\mathbb{R}^n)$. Consequently, by definition,

$$\int_{\mathbb{R}^n} \big(|F^{-1}(\psi\hat{h})(x)|/\lambda\big)\,dx < \infty \tag{2}$$

for some $\lambda > 0$, whence it follows that

$$\Phi\big(|F^{-1}(\psi\hat{h})(x)|/2\lambda\big) \equiv 0. \tag{3}$$

Indeed, let

$$\Phi\big(|F^{-1}(\psi\hat{h})(x^0)|/2\lambda\big) > 0$$

for a point x^0. Since $\Phi(t)$ is non-decreasing and $F^{-1}(\psi\hat{h})(x)$ is continuous, we have

$$\Phi\big(|F^{-1}(\psi\hat{h})(x^0)|/2\lambda\big) \geqslant \delta$$

in a neighbourhood of x^0 for some $\delta > 0$. This contradicts (2) because $F^{-1}(\psi\hat{h})(x)$ does not depend on x_j.

From (3) and the assumption that $\Phi(t) > 0, \ \ t > 0$, we readily see that $F^{-1}(\psi\hat{h})(x) \equiv 0$. Whence, since $\psi(\xi) \in C_0^\infty(G_j)$ is arbitrary, we conclude that the support of $\hat{h}(\xi)$ must lie in the planes $\xi_i = \xi_j = 0, \ i,j \in I, \ i \neq j$.

We now set

$$G_{ij} = \big\{\xi \in \mathbb{R}^n : \xi_l \neq 0, \quad l \in I\backslash\{i,j\}\big\}$$

for $i,j \in I, \ \ i \neq j$. It follows that G_{ij} is open. Repeating the argument used in the case of G_j we can easily prove that $\psi(\xi)\hat{h}(\xi)$ is identically zero for any function $\psi \in C_0^\infty(G_{ij})$. Thus, we have shown that the support of $\hat{h}(\xi)$ lies in the planes $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = 0, \ i_1, i_2, i_3 \in I$.

Repeating the above argument a further $k-3$ times we see that $\hat{h}(\xi)$ is supported at the point $\xi_1 = \cdots = \xi_n = 0$, that is, $h(x)$ is a polynomial, which is possible only if $h(x) \equiv 0$. This contradicts the hypotheses of the theorem. Theorem 1 is proved.

Theorem 1 and its proof imply the following results.

Corollary 1. *Let $\Phi(t) > 0, \ t > 0$, and $f \in L_\Phi(\mathbb{R}^n)$. We assume that $f(x) \not\equiv 0$ and $\xi^0 \in \mathrm{sp}(f)$. Then $\mathrm{sp}(f)$ contains a sequence of spectral points with non-zero components that converges to ξ^0.*

Corollary 2. *Let $\Phi(t) > 0$ for $t > 0, \ f \in L_\Phi(\mathbb{R}^n), \ f(x) \not\equiv 0$, and let $\xi^0 \in \mathrm{sp}(f)$ be an arbitrary point. Then the hyperplanes $\xi_j = \xi_j^0, \ j = 1, \ldots, n$, cannot contain the support of the restriction of $\hat{f}(\xi)$ to any neighbourhood of ξ^0.*

Remark 1. The assumption that $\Phi(t) > 0, \ t > 0$, in the assertion of Theorem 1 cannot be dropped because otherwise $L_\Phi(\mathbb{R}^n)$ contains all constant functions.

Remark 2. Let $1 \leqslant p < \infty$ and let $f(x) \in L_p(\mathbb{R}^n), \ \ f(x) \not\equiv 0$. We assume that $\mathrm{sp}(f)$ is bounded. Then, by virtue of Theorem 1, the support of $\hat{f}(\xi)$ cannot belong to the hyperplanes $\xi_j = \xi_j^0, \ j = 1, \ldots, n$, where ξ^0 is an arbitrary point. At the same time, $\hat{f}(\xi)$ can have a sphere as support. Indeed, let $n=3$ and $f(x)=\frac{\sin|x|}{|x|}$. In this case it is known (see [14]) that

$$\operatorname{sp}(f)=\bigl\{\xi:|\xi|=1\bigr\},$$

and it can easily be proved that $f(x)\in L_p(\mathbb{R}^n)$ for any $p>3$.

Remark 3. Let $\Phi(t)>0$ for $t>0$, $f\in L_\Phi(\mathbb{R}^n)$, $f(x)\not\equiv 0$. Let $\xi^0\in\operatorname{sp}(f)$ be an arbitrary point. Then an orthogonal transformation of coordinates can be applied to prove that the support of the restriction of $\hat{f}(\xi)$ to an arbitrary neighbourhood of ξ^0 cannot belong to any set of finitely many hyperplanes.

§ 2. Behaviour of the sequence of norms of the derivatives

Let K be a compact set in $\mathbb{R}^n$ and let $\Phi(t)$ be an arbitrary Young function. We assume that

$$\mathfrak{M}_{K\Phi}=\bigl\{f(x)\in L_\Phi(\mathbb{R}^n):\operatorname{supp}Ff\subset K\bigr\}.$$

We begin by studying some properties of the spaces $\mathfrak{M}_{K\Phi}$.

Lemma 3. *The following continuous embeddings hold*:

$$\mathfrak{M}_{K1}\subset\mathfrak{M}_{K\Phi}\subset\mathfrak{M}_{K\infty},\tag{4}$$

where

$$\mathfrak{M}_{Kp}=\bigl\{f(x)\in L_p(\mathbb{R}^n):\operatorname{supp}Ff\subset K\bigr\},\qquad 1\leqslant p\leqslant\infty.$$

Proof. Let $\widehat{\varphi}\in C_0^\infty(\mathbb{R}^n)$ and let $\widehat{\varphi}$ be equal to unity in a neighbourhood of K. Assume that $f\in\mathfrak{M}_{K\Phi}$. Then $\hat{f}=\widehat{\varphi}\hat{f}$. By Lemma 1, this implies that

$$|\varphi*f(x)|=\left|\int\varphi(y)f(x-y)\,dy\right|\leqslant\|\varphi\|_{\overline{\Phi}}\|f\|_{(\Phi)}$$

for all $x\in\mathbb{R}^n$. Consequently,

$$\|f\|_\infty=\|\varphi*f\|_\infty\leqslant\|\varphi\|_{\overline{\Phi}}\|f\|_{(\Phi)}.$$

Thus, we have proved the right-hand embedding in (4), and it is possible to choose the constant

$$M_1=\inf\bigl\{\|\varphi\|_{\overline{\Phi}}:\widehat{\varphi}\in C_0^\infty(\mathbb{R}^n),\quad\widehat{\varphi}=1\quad\text{in a neighbourhood of}\quad K\bigr\}.$$

Now let $g\in\mathfrak{M}_{K1}$ and $K\subset\Delta_\nu$. Then the Nikol'skii inequality implies that

$$\|g\|_\infty\leqslant 2^n(\nu_1\dots\nu_n)\|g\|_1.$$

Set

$$H=\inf\left\{M>0:\Phi\left(\frac{1}{M}\right)\leqslant\frac{\|g\|_\infty}{\|g\|_1}\right\}.$$

In this case we have

$$\int \Phi\left(\frac{|g(x)|}{\|g\|_\infty(H+\epsilon)}\right) dx \leqslant \Phi\left(\frac{1}{H+\epsilon}\right)\frac{\|g\|_1}{\|g\|_\infty} \leqslant 1$$

for any $\epsilon > 0$, because it is clear that $\Phi(\lambda t) \leqslant \lambda\Phi(t)$ if $0 \leqslant \lambda \leqslant 1$. It follows that

$$\|g\|_{(\Phi)} \leqslant H\|g\|_\infty.$$

To prove that the embedding is continuous, we choose a function $\widehat{\varphi} \in C_0^\infty(\mathbb{R}^n)$ such that $\widehat{\varphi} = 1$ in a neighbourhood of K. Then the already proved right-hand embedding in (4) implies that

$$\|g\|_{(\Phi)} = \|g * \varphi\|_{(\Phi)} \leqslant 2\|g\|_1\|\varphi\|_\Phi < \infty.$$

Clearly, we have $\|\varphi\|_\Phi < \infty$. Therefore, the embedding constant can be defined as

$$2\inf\bigl\{\|\varphi\|_\Phi : \widehat{\varphi} \in C_0^\infty(\mathbb{R}^n), \quad \widehat{\varphi} = 1 \quad \text{in a neighbourhood of} \quad K\bigr\}.$$

Lemma 3 is proved.

Lemma 4. *Let $\Phi(t) > 0$ for $t > 0$. Then*

$$\lim_{|x|\to\infty} f(x) = 0 \tag{5}$$

for all $f \in \mathfrak{M}_{K\Phi}$.

Proof. Let $K \subset \Delta_\nu$. We prove the lemma by contradiction. Assume that there is a function $f \in \mathfrak{M}_{\nu\Phi}$, a constant $c > 0$, and a sequence of points $|x^m| \to \infty$ such that

$$|f(x^m)| \geqslant 2c, \qquad m = 1, 2, \dots . \tag{6}$$

We can suppose without loss of generality that

$$\int_{\mathbb{R}^n} \Phi\bigl(|f(x)|\bigr)\, dx < \infty \tag{7}$$

and that $|x_1^m| \to \infty,\ m \to \infty$. From Lemma 1, the relation

$$f(x) - f(x^m) = \int_{x_1^m}^{x_1} \frac{\partial}{\partial t_1} f(t)\, dt$$

and the Bernshtein–Nikol'skii inequality [2] it follows that

$$|f(x) - f(x^m)| \leqslant \nu_1\|f\|_\infty|x_1 - x_1^m| \tag{8}$$

for all $x \in \mathbb{R}^n$ and $m \geqslant 1$.

We set $r = c/\nu_1 \|f\|_\infty$; in this case (6) and (8) imply

$$|f(x)| \geqslant c \quad \text{for} \quad |x_1 - x_1^m| \leqslant r \quad \text{and} \quad m \geqslant 1. \tag{9}$$

On the other hand, it can be assumed without loss of generality that

$$x_1^{m+1} - x_1^m \geqslant r, \qquad m \geqslant 1.$$

Hence, from (7) and (9) we derive

$$\begin{aligned}\infty > \int_{\mathbb{R}^n} \Phi\big(|f(x)|\big)\,dx &\geqslant \sum_{m=1}^{\infty} \int_{|x-x^m|\leqslant r} \Phi\big(|f(x)|\big)\,dx \\ &\geqslant \sum_{m=1}^{\infty} \Phi(c) \operatorname{mes} B(x^m, r) = \sum_{m=1}^{\infty} \pi r^n \Phi(c) = \infty,\end{aligned}$$

where $B(x^m, r)$ is a ball of radius r with centre at the point x^m, which is impossible. Lemma 4 is proved.

Remark 4. Lemma 4 was proved by Plancherel and Pólya for $\Phi(t) = t^p$, $1 \leqslant p < \infty$, using a different method (see [2] for example). It does not hold if there is a point $t_0 > 0$ such that $\Phi(t_0) = 0$ because in this case $\mathfrak{M}_{K\Phi}$ contains all constant functions.

The key result in this section is the solution of the following problem. Let $f \in L_\Phi(\mathbb{R}^n)$ and let $\operatorname{sp}(f)$ be bounded. Then, clearly, $D^\alpha f(x) \in L_\Phi(\mathbb{R}^n)$ for all $\alpha \geqslant 0$. The question is how the sequence $\|D^\alpha f\|_{(\Phi)}$, $\alpha \geqslant 0$, behaves. It turns out that its properties are completely characterized by the spectrum of the function f. Namely, the following theorem is true.

Theorem 2. *Let $\Phi(t)$ be an arbitrary Young function and let $f(x) \in L_\Phi(\mathbb{R}^n)$. We assume that* $\operatorname{sp}(f)$ *is bounded. Then*

$$\lim_{|\alpha|\to\infty} \left(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\operatorname{sp}(f)} |\xi^\alpha| \right)^{1/|\alpha|} = 1. \tag{10}$$

Proof. We suppose that $f(x) \not\equiv 0$ and begin by proving that

$$\varliminf_{|\alpha|\to\infty} \big(\|D^\alpha f\|_{(\Phi)}/|\xi^\alpha|\big)^{1/|\alpha|} \geqslant 1 \tag{11}$$

for any point $\xi \in \operatorname{sp}(f)$.

Indeed, let $\xi^0 \in \operatorname{sp}(f)$ and $\xi_j^0 \neq 0$, $j = 1, \dots, n$. (By virtue of Theorem 1, such points exist if $\Phi(t) > 0$ for $t > 0$.) For convenience, let $\xi_j^0 > 0$, $j = 1, \dots, n$. Furthermore, fix an arbitrary number $\epsilon > 0$ such that $2\epsilon < \xi_j^0$, $j = 1, \dots, n$, and a domain G ($\xi^0 \in G$) in the cube $K = \{\xi : \xi_j^0 - \epsilon \leqslant \xi_j \leqslant \xi_j^0 + \epsilon,\ j = 1, \dots, n\}$ and choose functions $\hat{v}(\xi), \widehat{w}(\xi) \in C_0^\infty(G)$ such that $\xi^0 \in \operatorname{supp}(\hat{v}\hat{f})$ and $\langle \hat{v}\hat{f}, \widehat{w}\rangle \neq 0$.

Let $\psi \in C_0^\infty(G)$ and $\psi = 1$ in a neighbourhood of $\operatorname{supp}\widehat{w}$. Then

$$\begin{aligned}\langle \hat{v}\hat{f}, \widehat{w}\rangle &= \big\langle \psi(\xi)\xi^{-\alpha}\xi^{\alpha}\hat{v}(\xi)\hat{f}(\xi), \widehat{w}(\xi)\big\rangle = \big\langle \xi^{\alpha}\hat{v}(\xi)\hat{f}(\xi), \psi(\xi)\xi^{-\alpha}\widehat{w}(\xi)\big\rangle \\ &= \big\langle F^{-1}\big(\xi^{\alpha}\hat{v}(\xi)\hat{f}(\xi)\big), F\big(\psi(\xi)\xi^{-\alpha}\widehat{w}(\xi)\big)\big\rangle \\ &= \big\langle F^{-1}\big(\xi^{\alpha}\hat{v}(\xi)\hat{f}(\xi)\big), F\big(\xi^{-\alpha}\widehat{w}(\xi)\big)\big\rangle\end{aligned}$$

for any $\alpha \geqslant 0$.

Therefore, by Lemmas 1 and 2,

$$0 < |\langle \hat{v}\hat{f}, \widehat{w}\rangle| = |\langle D^{\alpha}(v * f), F\widehat{w}_{\alpha}\rangle| \leqslant 2\|v\|_1\|D^{\alpha}f\|_{(\Phi)}\|F\widehat{w}_{\alpha}\|_{(\overline{\Phi})}, \tag{12}$$

where $\widehat{w}_{\alpha}(\xi) = \xi^{-\alpha}\widehat{w}(\xi)$, $\alpha \geqslant 0$. We now prove that

$$\|F\widehat{w}_{\alpha}\|_{(\overline{\Phi})} \leqslant C(\xi^0 - 2\epsilon)^{-\alpha}, \qquad \alpha \geqslant 0, \tag{13}$$

where C is a constant not depending on α, and $\xi^0 - 2\epsilon = (\xi_1^0 - 2\epsilon, \dots, \xi_n^0 - 2\epsilon)$.

Indeed, we have

$$\begin{aligned}(-i)^{|\beta|}x^{\beta}F\widehat{\omega}_{\alpha}(x) &= (-i)^{|\beta|}\int_G x^{\beta}e^{-ix\xi}\xi^{-\alpha}\widehat{\omega}(\xi)\,d\xi = \int_G (D_{\xi}^{\beta}e^{-ix\xi})\big(\xi^{-\alpha}\widehat{\omega}(\xi)\big)\,d\xi \\ &= (-1)^{|\beta|}\int_G e^{-ix\xi}D^{\beta}\big(\xi^{-\alpha}\widehat{\omega}(\xi)\big)\,d\xi\end{aligned}$$

for any $\alpha, \beta \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$. Hence, in view of the Leibniz formula and the definition of the domain G, we obtain

$$\begin{aligned}&\sup_{x\in\mathbb{R}^n}|x^{\beta}F\widehat{\omega}_{\alpha}(x)| \\ &\qquad\leqslant \sum_{\gamma\leqslant\beta}\bigg\{\frac{\beta!}{\gamma!\,(\beta-\gamma)!}\prod_{k=1}^{n}\alpha_k\dots(\alpha_k+\gamma_k-1)\int_G|\xi^{-(\alpha+\gamma)}D^{\beta-\gamma}\widehat{\omega}(\xi)|\,d\xi\bigg\} \\ &\qquad\leqslant C_1(\xi^0-\epsilon)^{-\alpha}\sum_{\gamma\leqslant\beta}\frac{\beta!}{\gamma!\,(\beta-\gamma)!}\prod_{k=1}^{n}\alpha_k\dots(\alpha_k+\gamma_k-1)\end{aligned} \tag{14}$$

for $|\beta| \leqslant 2n$, where

$$C_1 = \max\bigg\{\int_G|\xi^{-\gamma}D^{\nu-\gamma}\widehat{\omega}(\xi)|\,d\xi : \gamma \leqslant \nu, \quad |\nu| \leqslant 2n\bigg\} < \infty.$$

It follows from (14) that

$$\sup_{x\in\mathbb{R}^n}|x^{\beta}\widehat{\omega}_{\alpha}(x)| \leqslant C_2(\xi^0 - 2\epsilon)^{-\alpha}$$

for all $|\beta| \leqslant 2n$ and $\alpha \geqslant 0$. This defines a constant C_3 such that

$$\sup_{x\in\mathbb{R}^n}(1+x_1^2)\dots(1+x_n^2)|F\widehat{\omega}_{\alpha}(x)| \leqslant C_3(\xi^0 - 2\epsilon)^{-\alpha} \tag{15}$$

for all $\alpha \geqslant 0$.

Next, let λ_0, $0 < \lambda_0 < \infty$, be such that $\overline{\Phi}(C_3/\lambda_0) \leqslant \pi^{-n}$. Then formula (15), the monotonicity of $\overline{\Phi}(t)$, and $\overline{\Phi}(\lambda t) \leqslant \lambda\overline{\Phi}(t)$, $0 \leqslant \lambda \leqslant 1$, imply

$$\int \overline{\Phi}\left(\frac{|F\widehat{\omega}_\alpha|}{\lambda_0(\xi^0 - 2\epsilon)^{-\alpha}}\right) dx \leqslant \int \overline{\Phi}\left(\frac{C_3}{\lambda_0(1+x_1^2)\dots(1+x_n^2)}\right) dx$$
$$\leqslant \overline{\Phi}\left(\frac{C_3}{\lambda_0}\right) \int \frac{dx}{(1+x_1^2)\dots(1+x_n^2)} \leqslant 1.$$

Therefore, by definition,

$$\|F\widehat{\omega}_\alpha\|_{(\overline{\Phi})} \leqslant \lambda_0(\xi^0 - 2\epsilon)^{-\alpha}$$

for all $\alpha \geqslant 0$. Thus, we have proved (13).

By combining (12) and (13), we derive

$$1 \leqslant \varliminf_{|\alpha|\to\infty} \left((\xi^0 - 2\epsilon)^{-\alpha}\|D^\alpha f\|_{(\Phi)}\right)^{1/|\alpha|}.$$

Hence, since $\epsilon > 0$ is arbitrary and

$$\left(\frac{(\xi^0 - 2\epsilon)^{-\alpha}}{(\xi^0)^{-\alpha}}\right)^{1/|\alpha|} \leqslant \max_{1\leqslant j\leqslant n} \frac{\xi_j^0}{\xi_j^0 - 2\epsilon},$$

we readily obtain (11).

We now prove (11) for the "zero" points. Let $\xi^0 \in \mathrm{sp}(f)$, $\xi^0 \neq 0$, $\xi_1^0 \dots \xi_n^0 = 0$, and, for convenience, let $\xi_j^0 > 0$, $j = 1, \dots, k$, $\xi_{k+1}^0 = \dots = \xi_n^0 = 0$ $(1 \leqslant k < n)$. We note that it suffices to prove (11) only for α such that $\alpha_{k+1} = \dots = \alpha_n = 0$, and in this case the proof is completely analogous to what was done above with a single modification in the choice of ϵ. Fix an arbitrary number $\epsilon > 0$ such that $2\epsilon < \min_{1\leqslant j\leqslant k} \xi_j^0$ and a domain G $(\xi^0 \in G)$ contained in the cube

$$K = \{\xi : \xi_j^0 - \epsilon \leqslant \xi_j \leqslant \xi_j^0 + \epsilon, \quad j = 1, \dots, n\}.$$

We next establish (10) for the case where $\Phi(t) > 0$, $t > 0$. First, let us prove by contradiction that

$$\varliminf_{|\alpha|\to\infty} \left(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\right)^{1/|\alpha|} \geqslant 1. \tag{16}$$

Assume that there is a subsequence I_1 such that

$$(I_1) \lim_{|\alpha|\to\infty} \left(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\right)^{1/|\alpha|} < 1, \tag{17}$$

where (I_1) symbolizes that the limit is taken only for the multi-indices $\alpha \in I_1$. Then there is a subsequence $I_2 \subset I_1$ and β_j, $0 \leqslant \beta_j \leqslant 1$, $j = 1, \dots, n$, such that $|\beta| = 1$ and

$$(I_2) \lim_{|\alpha|\to\infty} \frac{\alpha_j}{|\alpha|} = \beta_j, \qquad j = 1, \dots, n. \tag{18}$$

We now show that

$$\lim_{\gamma\to\beta}\sup_{\mathrm{sp}(f)}|\xi^\gamma| = \sup_{\mathrm{sp}(f)}|\xi^\beta| \tag{19}$$

if $\gamma \in \mathbb{R}^n_+$ and $\gamma \to \beta$.

Fix an $h > 1$. Then there is an $\epsilon > 0$ such that $h\gamma \geqslant \beta$ for $\gamma \in \mathbb{R}^n_+$ and $|\gamma-\beta| \leqslant \epsilon$. Furthermore, let $|\xi| \leqslant M$ for all $\xi \in \mathrm{sp}(f)$. Then

$$|\xi^\gamma| = |\xi^{\gamma-\beta/h}|\,|\xi^\beta|^{1/h} \leqslant M^{|\gamma-\beta/h|}\sup_{\mathrm{sp}(f)}|\xi^\beta|^{1/h}$$

for $\xi \in \mathrm{sp}(f)$ and $\gamma \in \mathbb{R}^n_+$, $|\gamma-\beta| \leqslant \epsilon$. Consequently,

$$\overline{\lim_{\gamma\to\beta}}\sup_{\mathrm{sp}(f)}|\xi^\gamma| \leqslant M^{|\beta|(1-1/h)}\sup_{\mathrm{sp}(f)}|\xi^\beta|^{1/h}.$$

Letting h tend to 1 we derive the inequality

$$\overline{\lim_{\gamma\to\beta}}\sup_{\mathrm{sp}(f)}|\xi^\gamma| \leqslant \sup_{\mathrm{sp}(f)}|\xi^\beta|.$$

To obtain (19) it remains to prove that

$$\underline{\lim_{\gamma\to\beta}}\sup_{\mathrm{sp}(f)}|\xi^\gamma| \geqslant \sup_{\mathrm{sp}(f)}|\xi^\beta|. \tag{20}$$

Let $\xi^* \in \mathrm{sp}(f)$ be a point such that $|\xi^{*\beta}| = \sup_{\mathrm{sp}(f)}|\xi^\beta|$. Then by virtue of Theorem 1 the hyperplanes $\xi_j = 0$, $j = 1,\dots,n$, cannot contain the support of $\hat{f}(\xi)$. Hence, $|\xi^{*\beta}| > 0$. Next, the support of the restriction of $\hat{f}(\xi)$ to an arbitrary neighbourhood of the point ξ^* cannot lie in the hyperplanes $\xi_j = 0$, $j = 1,\dots,n$, either. Therefore, there is a sequence of points ${}_m\xi \in \mathrm{sp}(f)$, $m \geqslant 1$, such that ${}_m\xi_j \neq 0$, $j = 1,\dots,n$, for any $m \geqslant 1$, and ${}_m\xi \to \xi^*$ as $m \to \infty$. This implies

$$\sup_{\mathrm{sp}(f)}|\xi^\gamma| \geqslant |{}_m\xi^\gamma|$$

for any $m \geqslant 1$. It follows that

$$\underline{\lim_{\gamma\to\beta}}\sup_{\mathrm{sp}(f)}|\xi^\gamma| \geqslant \underline{\lim_{\gamma\to\beta}}|{}_m\xi^\gamma| = |{}_m\xi^\beta|.$$

Letting m tend to ∞, we derive (20) and, consequently, (19).

Now let $\lambda > 1$. Then there is a $k \geqslant 1$ such that $\lambda|{}_k\xi^\beta| \geqslant |\xi^{*\beta}|$. Hence, (18), (19), and (11) imply

$$\begin{aligned}(I_2)\underline{\lim_{|\alpha|\to\infty}}\Big(\|D^\alpha f\|_{(\Phi)}\Big/\sup_{\mathrm{sp}(f)}|\xi^\alpha|\Big)^{1/|\alpha|} &= (I_2)\underline{\lim_{|\alpha|\to\infty}}\|D^\alpha f\|^{1/|\alpha|}_{(\Phi)}/|\xi^{*\beta}| \\ \geqslant (I_2)\frac{1}{\lambda}\underline{\lim_{|\alpha|\to\infty}}\|D^\alpha f\|^{1/|\alpha|}_{(\Phi)}/|{}_k\xi^\beta| &= (I_2)\frac{1}{\lambda}\underline{\lim_{|\alpha|\to\infty}}\big(\|D^\alpha f\|_{(\Phi)}/|{}_k\xi^\alpha|\big)^{1/|\alpha|} \geqslant \frac{1}{\lambda}.\end{aligned}$$

This contradicts (17) as $\lambda \to 1$. We have thus proved (16).

Finally, let us show that

$$\overline{\lim_{|\alpha|\to\infty}}\left(\|D^\alpha f\|_{(\Phi)}\Big/\sup_{\mathrm{sp}(f)}|\xi^\alpha|\right)^{1/|\alpha|}\leqslant 1. \tag{21}$$

Fix a domain $G\supset \mathrm{sp}(f)$ and a function $\psi\in C_0^\infty(G)$ such that $\psi(\xi)$ is equal to unity in a neighbourhood of $\mathrm{sp}(f)$. We set $h_\alpha(\xi)=\psi(\xi)\xi^\alpha,\ \alpha\geqslant 0$. Then it follows from Hölder's inequality that

$$\begin{aligned}\|F^{-1}h_\alpha\|_1&=\int\left(|\hat h_\alpha(\xi)|^2\right)^{1/2}d\xi\\&\leqslant\left(\int|\hat h_\alpha(\xi)|^2\left(1+|\xi|^2\right)^s d\xi\right)^{1/2}\left(\int\left(1+|\xi|^2\right)^{-s}d\xi\right)^{1/2}\end{aligned}$$

for any $s>n/2$. Consequently,

$$\|F^{-1}h_\alpha\|_1\leqslant C'\|h_\alpha\|_{(s)}, \tag{22}$$

where $C'=C'(s)$ does not depend on h_α.

By combining (22) , the topological relation $H_{(k)}=W_{k,2}(\mathbb{R}^n)$, and

$$\|D^\alpha f\|_{(\Phi)}=\left\|F^{-1}\left(\psi(\xi)\xi^\alpha\right)*f\right\|_{(\Phi)}\leqslant 2\left\|F^{-1}\left(\psi(\xi)\xi^\alpha\right)\right\|_1\|f\|_{(\Phi)},$$

we derive the inequality

$$\|D^\alpha f\|_{(\Phi)}\leqslant C\left\|\psi(\xi)\xi^\alpha\right\|_{k,2}\|f\|_{(\Phi)},\qquad \alpha\geqslant 0, \tag{23}$$

where $k=\left[\frac{n}{2}\right]+1$ and C does not depend on f or α.

By the Leibniz formula we can find a constant $C_1=C_1(\psi,k)$ such that

$$\left\|\psi(\xi)\xi^\alpha\right\|_{k,2}\leqslant C_1|\alpha|^k\sup\left\{\sup_G|\xi^{\alpha-\gamma}|:\gamma\leqslant\alpha,\quad|\gamma|\leqslant k\right\},\qquad \alpha\geqslant 0. \tag{24}$$

On the other hand,

$$\lim_{|\alpha|\to\infty}\left(\sup\left\{\sup_G|\xi^{\alpha-\gamma}|:\gamma\leqslant\alpha,\quad|\gamma|\leqslant k\right\}\right)^{1/|\alpha|}\Big/\sup_G|\xi^\alpha|^{1/|\alpha|}=1. \tag{25}$$

We prove (25) by contradiction. Assume that there is a sequence I_1 and a number $\delta>1$ such that

$$\sup\left\{\sup_G|\xi^{\alpha-\gamma}|^{1/|\alpha|}:\gamma\leqslant\alpha,\quad|\gamma|\leqslant k\right\}\geqslant\delta\sup_G|\xi^\alpha|^{1/|\alpha|},\qquad \alpha\in I_1. \tag{26}$$

Then there is a subsequence $I_2 \subset I_1$, numbers β_j, $0 \leqslant \beta_j \leqslant 1$, $j = 1, \ldots, n$, and a multi-index γ^0, $|\gamma^0| \leqslant k$, such that $|\beta| = 1$ and

$$(I_2) \lim_{|\alpha|\to\infty} \frac{\alpha_j - \gamma_j^0}{|\alpha|} = \beta_j, \qquad j = 1, \ldots, n,$$

$$\sup \left\{ \sup_G |\xi^{\alpha-\gamma}|^{1/|\alpha|} : \gamma \leqslant \alpha, \quad |\gamma| \leqslant k \right\} = \sup_G |\xi^{\alpha-\gamma^0}|^{1/|\alpha|}$$

for all $\alpha \in I_2$. Therefore, arguing as in the proof of (19), we obtain

$$(I_2) \lim_{|\alpha|\to\infty} \sup_G |\xi^{\alpha-\gamma^0}|^{1/|\alpha|} = (I_2) \lim_{|\alpha|\to\infty} \sup_G |\xi^{\alpha}|^{1/|\alpha|} = \sup_G |\xi^{\beta}| > 0,$$

which contradicts (26). Relation (25) is proved.

Combining (23)–(25), we derive the inequality

$$\varlimsup_{|\alpha|\to\infty} \|D^\alpha f\|_{(\Phi)}^{1/|\alpha|} \Big/ \sup_G |\xi^\alpha|^{1/|\alpha|} \leqslant 1. \tag{27}$$

We now assume the contrary, namely, that (21) does not hold. Then there is a subsequence J, $\lambda > 1$ and β^j, $0 \leqslant \beta_j \leqslant 1$, $j = 1, \ldots, n$, such that $|\beta| = 1$ and

$$(J) \lim_{|\alpha|\to\infty} \|D^\alpha f\|_{(\Phi)}^{1/|\alpha|} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} = \lambda,$$

$$(J) \lim_{|\alpha|\to\infty} \frac{\alpha_j}{|\alpha|} = \beta_j, \qquad j = 1, \ldots, n.$$

Since (19) remains true if $\mathrm{sp}(f)$ is replaced by the set G (this can be proved in a similar way because G is open), it follows from (27) that

$$\sup_G |\xi^\beta| \Big/ \sup_{\mathrm{sp}(f)} |\xi^\beta| \geqslant \lambda$$

for any domain $G \supset \mathrm{sp}(f)$, which is impossible because $\sup_{\mathrm{sp}(f)} |\xi^\beta| > 0$. The proof of the first case is complete.

We next consider the other case, where $\Phi(t_0) = 0$ for some $t_0 > 0$. This turns out to be more complicated. We note that many of the facts that were used when proving the former case are false here (for example, relation (19)).

First, we prove that if $\sup_{\mathrm{sp}(f)} |\xi^\alpha| = 0$, then $D^\alpha f(x) \equiv 0$ (for the same α). Indeed, it can be assumed without loss of generality that $\alpha_j \neq 0$, $j = 1, \ldots, k$, and $\alpha_{k+1} = \cdots = \alpha_n = 0$ $(1 \leqslant k \leqslant n)$. Hence, the support of $\hat{f}(\xi)$ lies in the plane $\xi_j = 0$, $j \in \{1, \ldots, k\} = I$. It suffices to consider the case $\alpha_1 = \cdots = \alpha_k = 1$.

We show that if the support of $\xi^\alpha \psi(\xi) \hat{f}(\xi)$ lies in the plane $\xi_{i_1} = \cdots = \xi_{i_l} = 0$ for some $i_1, \ldots, i_l \in I$ and $\psi \in C_0^\infty(\mathbb{R}^n)$, then $D^\alpha F^{-1}\psi * f(x) \equiv 0$. Indeed, the support of $\xi^\alpha \psi(\xi) \hat{f}(\xi)$ is in the plane $\xi_1 = \cdots = \xi_l = 0$ (for brevity, assume that $i_j = j$, $j = 1, \ldots, l$). Therefore, taking into account a remark on Theorem 2.3.5 mentioned in [12], Example 5.1.2, we obtain

$$F^{-1}\big(\xi^\alpha \psi(\xi) \hat{f}(\xi)\big)(x) = \sum_{|\beta|\leqslant N} g_\beta(x'')(-ix')^\beta, \tag{28}$$

where N is the order of the distribution $\hat{f}(\xi)$ (here $N < \infty$ because $\operatorname{supp}\hat{f}$ is compact), $x' = (x_1, \dots, x_l)$, $x = (x', x'')$, $\beta \in \mathbb{Z}_+^l$, and $\hat{g}_\beta(\xi_{l+1}, \dots, \xi_n)$, $|\beta| \leqslant N$, is a distribution with compact support.

On the other hand, Lemmas 1 and 3 imply

$$\big\|F^{-1}\big(\xi^\alpha\psi(\xi)\hat{f}(\xi)\big)\big\|_\infty = \big\|F^{-1}\big(\xi^\alpha\psi(\xi)\big) * f\big\|_\infty \leqslant \big\|F^{-1}\big(\xi^\alpha\psi(\xi)\big)\big\|_1 \|f\|_\infty < \infty.$$

Consequently, we obtain

$$F^{-1}\big(\xi^\alpha\psi(\xi)\hat{f}(\xi)\big)(x) = D^\alpha F^{-1}\psi * f(x) = g_0(x'')$$

from (28).

Let $\gamma_1 = 0$, $\gamma_2 = \dots = \gamma_k = 1$, and $\gamma_{k+1} = \dots = \gamma_n = 0$. Then

$$D_{x_1} D^\gamma F^{-1}\psi * f(x) = g_0(x'').$$

Therefore,

$$D^\gamma F^{-1}\psi * f(x) = i x_1 g_0(x'') + t(x_2, \dots, x_n).$$

Hence, taking into account that $D^\gamma F^{-1}\psi * f \in L_\infty$ (which is obvious), we deduce that $g_0(x'') \equiv 0$, that is,

$$D^\alpha F^{-1}\psi * f(x) \equiv 0.$$

We now claim that the support of the distribution $\xi^\alpha\hat{f}(\xi)$ belongs to the plane $\xi_1 = \dots = \xi_k = 0$. Indeed, set

$$G_j = \big\{\xi \in \mathbb{R}^n : \xi_i \neq 0, \quad i \in I \backslash \{j\}\big\}$$

for each $j \in I$. Then G_j is open. For every $\varphi \in C_0^\infty(G_j)$ we choose a function $\psi(\xi)$ belonging to $C_0^\infty(G_j)$ such that $\psi = 1$ in a neighbourhood of $\operatorname{supp}\varphi$. Hence, the support of $\psi(\xi)\hat{f}(\xi)$ belongs to the hyperplane $\xi_j = 0$, and it follows from what we have proved that

$$\big\langle\xi^\alpha\hat{f}(\xi), \varphi(\xi)\big\rangle = \big\langle\xi^\alpha\psi(\xi)\hat{f}(\xi), \varphi(\xi)\big\rangle = \langle D^\alpha F^{-1}\psi * f, \widehat{\varphi}\rangle = 0.$$

Thus, we have proved that the support of $\xi^\alpha\hat{f}(\xi)$ lies in the planes $\xi_i = \xi_j = 0$, $i, j \in I$.

Set

$$G_{ij} = \big\{\xi \epsilon \mathbb{R}^n : \xi_l \neq 0, \quad l \in I \backslash \{i, j\}\big\}$$

for $i, j \in I$. Then G_{ij} is open. By repeating the arguments used in the proof for the case of G_j, it can readily be shown that

$$\big\langle\xi^\alpha\hat{f}(\xi), \varphi(\xi)\big\rangle = 0 \qquad \forall\varphi \in C_0^\infty(G_{ij}).$$

Hence, we have proved that the support of the distribution $\xi^\alpha\hat{f}(\xi)$ lies in the planes $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = 0$, $i_1, i_2, i_3 \in I$.

Repeating the above argument a further $k-3$ times, we see that the support of $\xi^\alpha\hat{f}(\xi)$ is contained in the plane $\xi_1 = \dots = \xi_k = 0$.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\psi = 1$ in a neighbourhood of $\mathrm{sp}(f)$. Then we have

$$\begin{aligned}\langle D^\alpha f, \widehat{\varphi}\rangle &= \big\langle \xi^\alpha \hat{f}(\xi), \varphi(\xi)\big\rangle = \big\langle \xi^\alpha \psi(\xi)\hat{f}(\xi), \varphi(\xi)\big\rangle \\ &= \langle D^\alpha F^{-1}\psi * f, \widehat{\varphi}\rangle = \langle 0, \widehat{\varphi}\rangle = 0\end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. On the other hand, it is known that $F\big(C_0^\infty(\mathbb{R}^n)\big)$ is dense in $L_1(\mathbb{R}^n)$. Therefore, it follows from the above relations and $D^\alpha f \in L_\infty(\mathbb{R}^n)$ that $D^\alpha f(x) \equiv 0$.

What has been proved implies that it suffices to establish (10) only for multi-indices $\alpha \geqslant 0$ satisfying $\sup_{\mathrm{sp}(f)} |\xi^\alpha| > 0$. Denote by P the set of these multi-indices.

We now prove by contradiction that

$$(P)\ \varliminf_{|\alpha|\to\infty} \Big(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\Big)^{1/|\alpha|} \geqslant 1. \tag{29}$$

Suppose that there is a subsequence $I \subset P$, a number $\lambda < 1$, and a vector $\beta \geqslant 0$, $|\beta| = 1$, such that

$$(I)\ \lim_{|\alpha|\to\infty} \Big(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\Big)^{1/|\alpha|} < \lambda, \tag{30}$$

$$(I)\ \lim_{|\alpha|\to\infty} \frac{\alpha}{|\alpha|} = \beta. \tag{31}$$

We note that

$$(I)\ \varliminf_{|\alpha|\to\infty} \sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} > 0. \tag{32}$$

For otherwise there is a subsequence $J \subset I$ such that

$$(J)\ \lim_{|\alpha|\to\infty} \sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} = 0. \tag{33}$$

Set

$$T_{i_1\ldots i_k} = \big\{\alpha \geqslant 0 : \alpha_{i_1} \neq 0, \ldots, \alpha_{i_k} \neq 0 \quad \text{and} \quad \alpha_j = 0 \quad \text{if} \quad j \notin \{i_1, \ldots, i_k\}\big\}$$

for an arbitrary k, $1 \leqslant k \leqslant n$, and $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Then there is a k, $1 \leqslant k \leqslant n$, and $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $J_{i_1\ldots i_k} = J \cap T_{i_1\ldots i_k}$ is unbounded. Consequently, it is clear that

$$(J_{i_1\ldots i_k})\ \varliminf_{|\alpha|\to\infty} \sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} \geqslant (J_{i_1\ldots i_k})\ \varliminf_{|\alpha|\to\infty} |\eta^\alpha|^{1/|\alpha|} > 0,$$

where η is an arbitrary point in $\mathrm{sp}(f)$ such that $\eta_{i_1} \neq 0, \ldots, \eta_{i_k} \neq 0$, which contradicts (33). Inequality (32) is proved.

Next, let ${}_\alpha\xi \in \mathrm{sp}(f) : |{}_\alpha\xi^\alpha| = \sup_{\mathrm{sp}(f)} |\xi^\alpha|$. Then ${}_\alpha\xi_{i_1} \neq 0, \dots, {}_\alpha\xi_{i_k} \neq 0$ for any $\alpha \in J_{i_1 \dots i_k}$, and it can be assumed without loss of generality that there is a subsequence such that

$$(J_{i_1 \dots i_k}) \lim_{|\alpha| \to \infty} {}_\alpha\xi = \xi^* \tag{34}$$

for some point $\xi^* \in \mathrm{sp}(f)$. Now let us consider the following two cases for this point ξ^*.

If $\xi^*_{i_j} \neq 0, \ \ j = 1, \dots, k$, then, obviously,

$$(J_{i_1 \dots i_k}) \lim_{|\alpha| \to \infty} |{}_\alpha\xi^\alpha|^{1/|\alpha|} = |\xi^{*\beta}| = (J_{i_1 \dots i_k}) \lim_{|\alpha| \to \infty} |\xi^{*\alpha}|^{1/|\alpha|}.$$

This together with $\xi^* \in \mathrm{sp}(f)$, (11), and (30) implies

$$\begin{aligned} 1 &\leqslant (J_{i_1 \dots i_k}) \lim_{|\alpha| \to \infty} \big(\|D^\alpha f\|_{(\Phi)} / |\xi^{*\alpha}|\big)^{1/|\alpha|} \\ &= (J_{i_1 \dots i_k}) \lim_{|\alpha| \to \infty} \Big(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\Big)^{1/|\alpha|} < \lambda < 1, \end{aligned}$$

which is impossible.

In the other case we assume without loss of generality that $\xi^*_{i_1} = \dots = \xi^*_{i_m} = 0$ and $\xi^*_{i_{m+1}} \neq 0, \dots, \xi^*_{i_k} \neq 0$ for some m, $1 \leqslant m \leqslant k$.

It follows from (32) and (34) that $\xi^* \neq 0$. Consequently, $m < k$. Furthermore, in view of (31), (32), (34), the definition of ${}_\alpha\xi$, and the hypothesis $\xi^*_{i_1} = \dots = \xi^*_{i_m} = 0$, we obtain $\beta_{i_1} = \cdots \beta_{i_m} = 0$. The inequalities

$$\begin{aligned} (J_{i_1 \dots i_k}) \lim_{|\alpha| \to \infty} \Big| {}_\alpha\xi_{i_{m+1}}^{\alpha_{i_{m+1}}} \cdots {}_\alpha\xi_{i_k}^{\alpha_{i_k}} \Big|^{1/|\alpha|} &= \Big| \xi^{*\,\beta_{i_{m+1}}}_{i_{m+1}} \cdots \xi^{*\,\beta_{i_k}}_{i_k} \Big| \\ &= (J_{i_1 \dots i_k}) \lim_{|\alpha| \to \infty} \Big| \xi^{*\,\alpha_{i_{m+1}}}_{i_{m+1}} \cdots \xi^{*\,\alpha_{i_k}}_{i_k} \Big|^{1/|\alpha|}, \end{aligned}$$

which are obvious, imply that there is a $\nu \in J_{i_1 \dots i_k}$ and an $N > 0$ such that

$$|{}_\alpha\xi_{i_l}| \leqslant \lambda^{-1} |{}_\nu\xi_{i_l}|, \qquad l = m+1, \dots, k, \tag{35}$$

for all $|\alpha| \geqslant N, \ \ \alpha \in J_{i_1 \dots i_k}$.

On the other hand, it follows from ${}_\nu\xi_{i_1} \neq 0, \ \dots\ , {}_\nu\xi_{i_k} \neq 0$ and

$$(J_{i_1 \dots i_k}) \lim_{|\alpha| \to \infty} {}_\alpha\xi_{i_j} = \xi^*_{i_j} = 0, \qquad j = 1, \dots, m,$$

that there is an $M > 0$ such that

$$|{}_\alpha\xi_{i_j}| \leqslant |{}_\nu\xi_{i_j}|, \qquad j = 1, \dots, m,$$

for all $|\alpha| \geqslant M, \ \ \alpha \in J_{i_1 \dots i_k}$. This together with (35) gives

$$|{}_\alpha\xi_{i_j}| \leqslant \lambda^{-1} |{}_\nu\xi_{i_j}|, \qquad j = 1, \dots, k,$$

for all $|\alpha| \geqslant \max\{M, N\}$, $\alpha \in J_{i_1 \dots i_k}$. Hence,

$$\sup_{\mathrm{sp}(f)} |\xi^\alpha|^{1/|\alpha|} = |_\alpha\xi^\alpha|^{1/|\alpha|} \leqslant \lambda^{-1} |_\nu\xi^\alpha|^{1/|\alpha|},$$

which together with (11) and (30) implies

$$\begin{aligned} 1 &\leqslant (J_{i_1 \dots i_k}) \varliminf_{|\alpha|\to\infty} \left(\|D^\alpha f\|_{(\Phi)} / |_\nu\xi^\alpha|\right)^{1/|\alpha|} \\ &\leqslant (J_{i_1 \dots i_k}) \lambda^{-1} \lim_{|\alpha|\to\infty} \left(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\right)^{1/|\alpha|} < 1. \end{aligned}$$

Thus, we have arrived at a contradiction. Inequality (29) is proved.

Finally, to complete the proof of the theorem, it remains to show that

$$(P) \varlimsup_{|\alpha|\to\infty} \left(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\right)^{1/|\alpha|} \leqslant 1. \tag{36}$$

We prove (36) by contradiction. Assume that there is a subsequence $I \subset P$, a number $h > 1$, and a vector $\beta \geqslant 0$, $|\beta| = 1$, such that

$$(I) \lim_{|\alpha|\to\infty} \left(\|D^\alpha f\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\right)^{1/|\alpha|} > h, \tag{37}$$

$$(I) \lim_{|\alpha|\to\infty} \frac{\alpha}{|\alpha|} = \beta. \tag{38}$$

Using the notation introduced earlier, we can assert that there is a k, $1 \leqslant k \leqslant n$, and $i_1, \dots, i_k \in \{1, \dots, n\}$ such that $I_{i_1 \dots i_k} = I \cap T_{i_1 \dots i_k}$ is unbounded.

We now have to delete the "bad" points from $\mathrm{sp}(f)$. Set

$$\begin{aligned} Q = \Big\{\eta \in \mathbb{R}^n :\ &\exists\{_m\xi\} \subset \mathrm{sp}(f), \quad {}_m\xi_j \neq 0, \\ &j \in \{i_1, \dots, i_k\}, \quad m \geqslant 1, \quad \lim_{m\to\infty} {}_m\xi = \eta\Big\}, \\ Q_\delta = \big\{x + y :\ &x \in Q, \quad |y| < \delta\big\}, \qquad \delta > 0, \end{aligned}$$

and $H = \mathbb{R}^n \backslash Q$. Then Q is closed and H and Q_δ are open.

Consequently, $\mathrm{sp}(f) \subset Q_\delta \cup H (= \mathbb{R}^n)$ implies

$$\hat{f}(\xi) = \varphi_\delta(\xi)\hat{f}(\xi) + \psi(\xi)\hat{f}(\xi), \qquad \varphi_\delta \in C_0^\infty(Q_\delta), \qquad \psi \in C_0^\infty(H).$$

Arguing as above, we can prove that $D^\alpha F^{-1}(\psi\hat{f})(x) \equiv 0$ for all $\alpha \in I_{i_1 \dots i_k}$. Hence, in view of (37), it follows that

$$(I_{i_1 \dots i_k}) \lim_{|\alpha|\to\infty} \left(\left\|D^\alpha F^{-1}(\varphi_\delta \hat{f})\right\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha|\right)^{1/|\alpha|} > h \tag{39}$$

for any $\delta > 0$.

On the other hand, repeating the argument used in the proof of (19), we derive the equation

$$(I_{i_1\dots i_k})\ \lim_{|\alpha|\to\infty} \sup_{Q_\delta} |\xi^\alpha|^{1/|\alpha|} = \sup_{Q_\delta} |\xi^\beta|. \tag{40}$$

Next, let ${}_m\theta \in Q_{1/m}$, that is, $|{}_m\theta^\beta| = \sup_{Q_{1/m}} |\xi^\beta|$, $m \geqslant 1$. Then there is a subsequence $\{m_k\}$ (to simplify the notation, we assume that $m_k = k$, $k \geqslant 1$) and a point $\theta^* \in Q$ such that ${}_m\theta \to \theta^*$, $m \to \infty$. Therefore

$$0 < \sup_Q |\xi^\beta| \leqslant \lim_{m\to\infty} |{}_m\theta^\beta| = |\theta^{*\beta}|.$$

Arguing as in the proof of (20) and taking (38) and the fact that $\theta^* \in Q$ into account, we obtain

$$|\theta^{*\beta}| \leqslant (I_{i_1\dots i_k})\ \underline{\lim}_{|\alpha|\to\infty} \sup_Q |\xi^\alpha|^{1/|\alpha|}. \tag{41}$$

Furthermore, noting that inequality (27) was proved for an arbitrary Young function, we conclude that

$$(I_{i_1\dots i_k})\ \overline{\lim}_{|\alpha|\to\infty} \left(\|D^\alpha F^{-1}(\varphi_{1/m}\hat{f})\|_{(\Phi)} \Big/ \sup_{Q_{1/m}} |\xi^\alpha| \right)^{1/|\alpha|} \leqslant 1 \tag{42}$$

for any $m \geqslant 1$.

We next fix an index $m \geqslant 1$ such that $|{}_m\theta^\beta| \leqslant h|\theta^{*\beta}|$. Combining (39)–(42), we obtain

$$\begin{aligned}
1 &\geqslant (I_{i_1\dots i_k})\ \overline{\lim}_{|\alpha|\to\infty} \left(\|D^\alpha F^{-1}(\varphi_{1/m}\hat{f})\|_{(\Phi)} \Big/ \sup_{Q_{1/m}} |\xi^\alpha| \right)^{1/|\alpha|} \\
&= (I_{i_1\dots i_k})\ \overline{\lim}_{|\alpha|\to\infty} \|D^\alpha F^{-1}(\varphi_{1/m}\hat{f})\|_{(\Phi)}^{1/|\alpha|} \Big/ |{}_m\theta^\beta| \\
&\geqslant (I_{i_1\dots i_k})\ \overline{\lim}_{|\alpha|\to\infty} h^{-1} \|D^\alpha F^{-1}(\varphi_{1/m}\hat{f})\|_{(\Phi)}^{1/|\alpha|} \Big/ |\theta^{*\beta}| \\
&\geqslant (I_{i_1\dots i_k})\ \overline{\lim}_{|\alpha|\to\infty} h^{-1} \left(\|D^\alpha F^{-1}(\varphi_{1/m}\hat{f})\|_{(\Phi)} \Big/ \sup_Q |\xi^\alpha| \right)^{1/|\alpha|} \\
&= (I_{i_1\dots i_k})\ \lim_{|\alpha|\to\infty} h^{-1} \left(\|D^\alpha F^{-1}(\varphi_{1/m}\hat{f})\|_{(\Phi)} \Big/ \sup_{\mathrm{sp}(f)} |\xi^\alpha| \right)^{1/|\alpha|} > 1,
\end{aligned}$$

which is impossible. The proof of Theorem 2 is complete.

Remark 5. Relation (10) shows that if the spectral points lying "far" from the origin are known, then it is possible to characterize the behaviour of the sequence $\|D^\alpha f\|_{(\Phi)}$, $\alpha \geqslant 0$, without calculating any derivatives. It should be noted that no constraint on the spectrum geometry is imposed here. The subtlty of these results consists in the behaviour of the sequence of norms $\|D^\alpha f\|_{(\Phi)}$, $|\alpha| \geqslant 0$, being studied in terms of the support of the Fourier transform of the function $f(x)$ itself, and, generally, this can have an arbitrary geometry.

Remark 6. Theorem 2 is also true in the case of fractional derivatives. Relation (10) is false if $\mathrm{sp}(f)$ is unbounded. At the same time, the following theorem holds.

Theorem 3. *Let $\Phi(t)$ be an arbitrary Young function and let $f(x) \in L_\Phi(\mathbb{R}^n)$. Suppose that $\operatorname{sp}(f)$ is bounded with respect to the variables $\xi_1, \dots, \xi_k$ $(1 \leqslant k \leqslant n)$. Then $D^\nu f(x) \in L_\Phi(\mathbb{R}^n)$ for all $\nu = (\nu_1, \dots, \nu_k, 0, \dots, 0) \in \mathbb{Z}_+^n$ and*

$$\lim_{|\nu|\to\infty} \left(\|D^\nu f\|_{(\Phi)} \Big/ \sup_{\operatorname{sp}(f)} |\xi^\nu| \right)^{1/|\nu|} = 1.$$

In the one-dimensional case, we have a stronger result (see [15]), whose proof differs substantially from the one in [16].

Theorem 4. *Let $0 = n_0 < n_1 < \cdots$ be a sequence of integers and let $\Phi(t)$ be an arbitrary Young function. We assume that $D^{n_k} f(x) \in L_\Phi(\mathbb{R})$, $k = 0, 1, \dots$. Then the limit*

$$d_f = \lim_{k\to\infty} \|D^{n_k} f\|_{(\Phi)}^{1/n_k}$$

always exists, and we have $d_f = \sigma_f = \sup\big\{|\xi| : \xi \in \operatorname{sp}(f)\big\}$.

Theorem 5. *Let K be an arbitrary compact set in $\mathbb{R}^n$ and let $\Phi(t)$ be an arbitrary Young function. Then for any $\epsilon > 0$ there is a constant C_ϵ such that*

$$\|D^\alpha f\|_{(\Phi)} \leqslant C_\epsilon (1+\epsilon)^{|\alpha|} \left(\sup_K |\xi^\alpha| \right) \|f\|_{(\Phi)}$$

for all $\alpha \geqslant 0$ and $f \in \mathfrak{M}_{K\Phi}$.

Proof. We construct a family

$$T_\alpha(f) = D^\alpha f(x) / (1+\epsilon)^{|\alpha|} \sup_K |\xi^\alpha|, \qquad \alpha \geqslant 0,$$

of continuous linear operators in $\mathfrak{M}_{K\Phi}$. Then, by virtue of Theorem 2, the set $\big\{T_\alpha(f) : \alpha \geqslant 0\big\}$ is bounded in $\mathfrak{M}_{K\Phi}$ for any function $f \in \mathfrak{M}_{K\Phi}$. Consequently, by the Banach–Steinhaus theorem, it is equicontinuous. Theorem 5 is proved.

We now consider the corresponding results for periodic functions.

Let $\mathbb{T}^n$ be an n-dimensional torus. Denote by $L_\Phi(\mathbb{T}^n)$ the space of functions u that are 2π-periodic with respect to each of the variables and such that

$$|||u|||_{(\Phi)} = \inf\left\{ \lambda > 0 : \int_{\mathbb{T}^n} \Phi\big(|u(x)|/\lambda\big)\, dx \leqslant 1 \right\} < \infty.$$

The following results can be proved by the method of expansion into Fourier series.

Theorem 6. *Let $g \in L_\Phi(\mathbb{T}^n)$ and $g(x) \not\equiv 0$. We assume that $\operatorname{sp}(g)$ is bounded. Then*

$$\lim_{|\alpha|\to\infty} \left(|||D^\alpha g|||_{(\Phi)} \Big/ \sup_{k\in\operatorname{sp}(g)} |k^\alpha| \right)^{1/|\alpha|} = 1,$$

where $k \in \mathbb{Z}^n$.

Theorem 7. *Let I be an unbounded set of multi-indices and let $\Phi(t)$ be an arbitrary Young function. Suppose that the generalized derivatives $D^\alpha f(x)$ of a function $f \in L_\Phi(\mathbb{T}^n)$ belong to $L_\Phi(\mathbb{T}^n)$ for all $\alpha \in I$. Then*

$$\varliminf_{|\alpha|\to\infty} \bigl(\||D^\alpha f\||_{(\Phi)}/|k^\alpha|\bigr)^{1/|\alpha|} \geqslant 1$$

for an arbitrary point $k \in \operatorname{sp}(f)$. This inequality is exact.

§ 3. Nikol'skii's inequality in the Luxemburg norm

An important series of papers on classes of functions of several variables by Nikol'skii and his students relates, in particular, to Nikol'skii's inequality for trigonometric polynomials and entire functions of exponential type. This inequality makes it possible to develop methods in the approximation theory of functions of several variables with the aid of which relations can be established between the differential properties of a function in one Lebesgue metric and those in another.

Now recall Nikol'skii's inequality. Let $1 \leqslant p \leqslant q \leqslant \infty$. Then we have

$$\|t_m\|_{q,2\pi} \leqslant 2^n \biggl(\prod_{j=1}^n m_j\biggr)^{1/p-1/q} \|t_m\|_{p,2\pi}$$

for trigonometric polynomials of the form

$$t_m(x) = \sum_{j_1=-m_1}^{m_1} \cdots \sum_{j_n=-m_n}^{m_n} c_{(j_1,\dots,j_n)} \exp\bigl(i(j_1x_1 + \dots + j_nx_n)\bigr)$$

and

$$\|f\|_q \leqslant 2^n \biggl(\prod_{j=1}^n \nu_j\biggr)^{1/p-1/q} \|f\|_p$$

for entire functions of an exponential type ν.

These inequalities have attracted the attention of many mathematicians, for example, Zygmund [17], Ibragimov [3]–[5], Nessel and Wilmes [18], [19], Triebel [20], [21], Burenkov [6], and so on.

The Nikol'skii inequalities for symmetric spaces were considered in [22]–[24].

In this section we make an attempt to establish Nikol'skii's inequality for Orlicz norms. This is a complicated problem, if only because of the difficulty associated with (explicitly) comparing Young functions. Note that, by our definition, Orlicz spaces are not always symmetric.

Definition 1. A function $A(t)\colon [0,+\infty) \to [0,+\infty]$ is said to be *quasi-convex* if

$$A(\lambda t) \leqslant \lambda A(t), \qquad 0 \leqslant \lambda \leqslant 1, \qquad t \geqslant 0. \tag{43}$$

Clearly, all Young functions are quasi-convex.

A quasi-convex function $A(t)$ is said to be *trivial* if $A(t) = +\infty$ for all $t > 0$. It is obvious that $\lim_{t\to 0} A(t) = 0$ if $A(t)$ is a non-trivial quasi-convex function.

Definition 2. Let $\Phi(t)$ and $\Psi(t)$ be Young functions. We say that $\Psi(t)$ *majorizes* $\Phi(t)$ if there is a non-trivial quasi-convex function $A(t)$ such that

$$\Phi(t) \leqslant \Psi\big(A(t)\big), \qquad t \geqslant 0. \tag{44}$$

Example 1. Let $1 \leqslant p \leqslant q < \infty$ and let $\Phi(t) = t^q$ and $\Psi(t) = t^p$ for $t \geqslant 0$. Then

$$\Phi(t) = (t^{q/p})^p = \Psi(t^{q/p}), \qquad t \geqslant 0.$$

Consequently, $\Psi(t)$ majorizes $\Phi(t)$ (here $A(t) = t^{q/p}$).

Let $C_{K\Phi}$ be the exact constant in the inequality

$$\|f\|_\infty \leqslant C_{K\Phi}\|f\|_{(\Phi)} \qquad \forall f \in \mathfrak{M}_{K\Phi}.$$

Then we have the following theorem.

Theorem 8. *Let $\Psi(t)$ majorize $\Phi(t)$. Then*

$$\|f\|_{(\Phi)} \leqslant \frac{A(C_{K\Phi})}{C_{K\Phi}}\|f\|_{(\Psi)} \tag{45}$$

for all $f \in \mathfrak{M}_{K\Psi}$, where $A(t)$ satisfies (44).

Proof. Let $f \in \mathfrak{M}_{K\Psi}$. First, we prove that

$$\int \Psi\bigg(A\bigg(\frac{|f(x)|}{M}\bigg)\bigg)\,dx \leqslant 1 \tag{46}$$

for some $M > 0$.

To see this, take a sufficiently large $M > 0$ such that

$$C_{K\Psi}^{-1}A\big(C_{K\Psi}\big(\|f\|_{(\Psi)} + 1\big)/M\big) \leqslant 1.$$

Then (43) and the inequality $|f(x)| \leqslant C_{K\Psi}\|f\|_{(\Psi)}$ imply

$$\begin{aligned}
\Psi\bigg(A\bigg(\frac{|f(x)|}{M}\bigg)\bigg) &= \Psi\bigg(A\bigg(\frac{|f(x)|}{C_{K\Psi}\big(\|f\|_{(\Psi)}+1\big)}\,\frac{C_{K\Psi}\big(\|f\|_{(\Psi)}+1\big)}{M}\bigg)\bigg)\\
&\leqslant \Psi\bigg(\frac{|f(x)|}{C_{K\Psi}\big(\|f\|_{(\Psi)}+1\big)}A\bigg(\frac{C_{K\Psi}\big(\|f\|_{(\Psi)}+1\big)}{M}\bigg)\bigg)\\
&\leqslant \Psi\bigg(\frac{|f(x)|}{\|f\|_{(\Psi)}+1}\bigg).
\end{aligned}$$

Whence, we conclude that $f \in \mathfrak{M}_{K\Phi}$ and then (46) holds because

$$\int \Psi\bigg(\frac{|f(x)|}{\|f\|_{(\Psi)}+1}\bigg)\,dx \leqslant 1.$$

Fix an arbitrary $\lambda > 0$ such that

$$\int \Psi\left(A\left(\frac{|f(x)|}{\lambda}\right)\right) dx \leqslant 1.$$

Then $\lambda \geqslant \|f\|_{(\Phi)}$. Since

$$|f(x)| \leqslant C_{K\Phi}\|f\|_{(\Phi)} \leqslant \lambda C_{K\Phi},$$

it follows that

$$\Psi\left(A\left(\frac{|f(x)|}{\lambda}\right)\right) = \Psi\left(A\left(\frac{|f(x)|}{\lambda C_{K\Phi}}C_{K\Phi}\right)\right) \leqslant \Psi\left(\frac{|f(x)|}{\lambda C_{K\Phi}}A(C_{K\Phi})\right).$$

Therefore,

$$\begin{aligned}\|f\|_{(\Phi)} &\leqslant \inf\left\{\lambda > 0 : \int \Psi\left(A\left(\frac{|f(x)|}{\lambda}\right)\right) dx \leqslant 1\right\} \\ &\leqslant \inf\left\{\lambda > 0 : \int \Psi\left(\frac{|f(x)|}{\lambda C_{K\Phi}}A(C_{K\Phi})\right) dx \leqslant 1\right\} = \frac{A(C_{K\Phi})}{C_{K\Phi}}\|f\|_{(\Psi)}.\end{aligned}$$

The theorem is proved.

Remark 7. If it is only known that

$$\|f\|_\infty \leqslant C\|f\|_{(\Phi)} \qquad \forall f \in \mathfrak{M}_{K\Phi},$$

then the proof of Theorem 8 implies

$$\|f\|_{(\Phi)} \leqslant \frac{A(C)}{C}\|f\|_{(\Psi)} \tag{47}$$

for all $f \in \mathfrak{M}_{K\Psi}$. Hence, it follows from property (43) that

$$\frac{A(C_{K\Phi})}{C_{K\Phi}} \leqslant \frac{A(C)}{C}.$$

Remark 8. Let us consider the case $\Phi(t) = t^q$, $\Psi(t) = t^p$, $1 \leqslant p < q < \infty$. Here, according to results of Nikol'skii [1], [2] and Ibragimov [3]–[5], we have

$$\begin{aligned}&\|f\|_\infty \leqslant 2^n(\nu_1 \dots \nu_n)^{1/q}\|f\|_q, \\ &\|f\|_\infty \leqslant \left(\left(\frac{s}{\pi}\right)^n \nu_1 \dots \nu_n\right)^{1/q} \|f\|_q\end{aligned}$$

for all $f \in \mathfrak{M}_{\nu p}$, where s is the smallest integer greater than or equal to $q/2$. Hence, by (47), we obtain

$$\|f\|_q \leqslant \left(2^n(\nu_1 \dots \nu_n)^{1/q}\right)^{q/p-1}\|f\|_p, \tag{48}$$

$$\|f\|_q \leqslant \left(\left(\left(\frac{s}{\pi}\right)^n \nu_1 \dots \nu_n\right)^{1/q}\right)^{q/p-1} \|f\|_p. \tag{49}$$

It turns out that the constant in (48) is less than the corresponding constant in Nikol'skii's inequality if $q < 2p$, while the constant in (49) coincides with Ibragimov's constant [3]–[5].

Definition 3. Let $\Phi(t)$ and $\Psi(t)$ be Young functions and let $C > 0$. We say that $\Psi(t)$ *C-majorizes* $\Phi(t)$ if there is a non-trivial quasi-convex function $A(t)$ and a number $C^* > C$ such that

$$\Phi(t) \leqslant \Psi\big(A(t)\big), \qquad 0 \leqslant t < C^*.$$

The following stronger result holds.

Theorem 9. *Let $\Psi(t)$ $C_{K\Phi}$-majorize $\Phi(t)$. Then* (45) *holds.*

Proof. Let $f \in \mathfrak{M}_{K\Psi}$. Repeating the first part of the proof of Theorem 8, we obtain $f \in \mathfrak{M}_{K\Phi}$.

Next, choose an $\epsilon_0 > 0$ such that

$$C^*\big(\|f\|_{(\Phi)} - \epsilon_0\big) = C_{K\Phi}\|f\|_{(\Phi)}.$$

Then the definition of $C_{K\Phi}$ implies that

$$|f(x)| < C^*\big(\|f\|_{(\Phi)} - \epsilon\big), \qquad x \in \mathbb{R}^n,$$

for all $0 < \epsilon < \epsilon_0$. It follows that

$$\Phi\left(\frac{|f(x)|}{\|f\|_{(\Phi)} - \epsilon}\right) \leqslant \Psi\left(A\left(\frac{|f(x)|}{\|f\|_{(\Phi)} - \epsilon}\right)\right), \qquad x \in \mathbb{R}^n,$$

for any $0 < \epsilon < \epsilon_0$. Therefore, since

$$\int \Phi\left(\frac{|f(x)|}{\|f\|_{(\Phi)} - \epsilon}\right) dx > 1,$$

which is implied by the definition of $\|f\|_{(\Phi)}$, we obtain

$$\int \Psi\left(A\left(\frac{|f(x)|}{\|f\|_{(\Phi)} - \epsilon}\right)\right) dx > 1$$

for any $0 < \epsilon < \epsilon_0$. Consequently,

$$\|f\|_{(\Phi)} \leqslant \inf\left\{\lambda > 0 : \int \Psi\left(A\left(\frac{|f(x)|}{\lambda}\right)\right) dx \leqslant 1\right\}.$$

The rest of the argument is as in Theorem 8. The theorem is proved.

The following more exact result can be established in like manner.

Theorem 10. *Let $\Psi(t)$ C_f-majorize $\Phi(t)$. We assume that $f \in L_\Psi(\mathbb{R}^n)$ and that* $\operatorname{sp}(f)$ *is bounded. Then*

$$\|f\|_{(\Phi)} \leqslant \frac{A(C_f)}{C_f}\|f\|_{(\Psi)},$$

where $C_f = \|f\|_\infty / \|f\|_{(\Phi)}$.

To derive further Nikol'skii inequalities, we introduce the notion of the order of a quasi-convex function $A(t)$.

Definition 4. Let $C > 0$. The *C-order* of a quasi-convex function $A(t)$ is defined as the supremum of all numbers $p \geqslant 0$ such that

$$A(\lambda C) \leqslant \lambda^p A(C)$$

for all $0 \leqslant \lambda \leqslant 1$, and is denoted $\operatorname{ord} A$.

The fact that $A(t)$ is quasi-convex implies $\operatorname{ord} A \geqslant 1$. It is quite clear that we have $A(\lambda C) \leqslant \lambda^{\operatorname{ord} A} A(C)$ for all $0 \leqslant \lambda \leqslant 1$.

Theorem 11. *Let $\Psi(t)$ $C_{f\Phi}$-majorize $\Phi(t)$. Assume that $f \in L_\Psi(\mathbb{R}^n)$ and that $\operatorname{sp}(f)$ is bounded. Then*

$$\|f\|_{(\Phi)} \leqslant \frac{A^{1/\operatorname{ord} A}(C_{f\Phi})}{C_{f\Phi}} C_{f\Psi_1}^{\operatorname{ord} A - 1} \|f\|_{(\Psi)}, \tag{50}$$

where $C_{f\Phi} = \|f\|_\infty / \|f\|_{(\Phi)}$, $C_{f\Psi_1} = \|f\|_\infty / \|f\|_{(\Psi_1)}$, $\Psi_1(t) = \Psi(t^{\operatorname{ord} A})$, and $\operatorname{ord} A$ *is the $C_{f\Phi}$-order of $A(t)$.*

Proof. As was shown in the proof of Theorem 8, $f \in L_\Phi(\mathbb{R}^n)$. On the other hand, the assumption of the theorem implies that

$$\Phi(t) \leqslant \Psi\big(A(t)\big), \qquad 0 \leqslant t < C^*, \tag{51}$$

where $C_{f\Phi} < C^*$ is some number.

Furthermore, let us choose an $\epsilon_0 > 0$ such that

$$C^*\big(\|f\|_{(\Phi)} - \epsilon_0\big) = C_{f\Phi} \|f\|_{(\Phi)}.$$

Then the definition of $C_{f\Phi}$ implies

$$|f(x)| < C^*\big(\|f\|_{(\Phi)} - \epsilon\big), \qquad x \in \mathbb{R}^n,$$

for all ϵ, $0 < \epsilon < \epsilon_0$. This together with (51) yields

$$\Phi\left(\frac{|f(x)|}{\|f\|_{(\Phi)} - \epsilon}\right) \leqslant \Psi\left(A\left(\frac{|f(x)|}{\|f\|_{(\Phi)} - \epsilon}\right)\right), \qquad x \in \mathbb{R}^n,$$

for any ϵ, $0 < \epsilon < \epsilon_0$. Therefore, since

$$\int \Phi\left(\frac{|f(x)|}{\|f\|_{(\Phi)} - \epsilon}\right) dx > 1 \qquad \forall \epsilon > 0,$$

we obtain

$$\int \Psi\left(A\left(\frac{|f(x)|}{\|f\|_{(\Phi)} - \epsilon}\right)\right) dx > 1$$

for every ϵ, $0 < \epsilon < \epsilon_0$. It follows that

$$\|f\|_{(\Phi)} \leqslant \inf\left\{\lambda > 0 : \int \Psi\left(A\left(\frac{|f(x)|}{\lambda}\right)\right) dx \leqslant 1\right\}. \tag{52}$$

We now fix an arbitrary $\lambda > 0$ such that

$$\int \Psi\Big(A\Big(\frac{|f(x)|}{\lambda}\Big)\Big)\, dx \leqslant 1.$$

Then it follows from (52) that $\lambda \geqslant \|f\|_{(\Phi)}$. Consequently,

$$A\Big(\frac{|f(x)|}{\lambda}\Big) = A\Big(\frac{|f(x)|}{\lambda C_{f\Phi}} C_{f\Phi}\Big) \leqslant \Big(\frac{|f(x)|}{\lambda C_{f\Phi}}\Big)^{\operatorname{ord} A} A(C_{f\Phi}).$$

Hence, taking (52) into account, we obtain

$$\begin{aligned}
\|f\|_{(\Phi)} &\leqslant \inf\Big\{\lambda > 0 : \int \Psi\Big(A\Big(\frac{|f(x)|}{\lambda}\Big)\Big)\, dx \leqslant 1\Big\}\\
&\leqslant \inf\Big\{\lambda > 0 : \int \Psi\Big(\Big(\frac{|f(x)|}{\lambda C_{f\Phi}}\Big)^{\operatorname{ord} A} A(C_{f\Phi})\Big)\, dx \leqslant 1\Big\}\\
&= \inf\Big\{\lambda > 0 : \int \Psi\Big(\Big(\frac{|f(x)|A^{1/\operatorname{ord} A}(C_{f\Phi})}{\lambda C_{f\Phi}}\Big)^{\operatorname{ord} A}\Big)\, dx \leqslant 1\Big\}\\
&= \frac{A^{1/\operatorname{ord} A}(C_{f\Phi})}{C_{f\Phi}} \inf\Big\{\lambda > 0 : \int \Psi_1\Big(\frac{|f(x)|}{\lambda}\Big)\, dx \leqslant 1\Big\},
\end{aligned}$$

because $\Psi_1(t) = \Psi(t^{\operatorname{ord} A})$ is a Young function and $\|\beta f\|_{(\Psi_1)} = |\beta|\, \|f\|_{(\Psi_1)}$ for any β.

Thus, we have proved that

$$\|f\|_{(\Phi)} \leqslant \frac{A^{1/\operatorname{ord} A}(C_{f\Phi})}{C_{f\Phi}} \|f\|_{(\Psi_1)}.$$

Hence, on applying Theorem 10 for the pair of functions $\Psi_1(t)$ and $\Psi(t)$, we obtain

$$\|f\|_{(\Phi)} \leqslant \frac{A^{1/\operatorname{ord} A}(C_{f\Phi})}{C_{f\Phi}} C_{f\Psi_1}^{\operatorname{ord} A - 1} \|f\|_{(\Psi)}.$$

The theorem is proved.

Let us consider the case $\Phi(t) = t^q,\ \ \Psi(t) = t^p,\ \ 1 \leqslant p < q < \infty$. Here we have $A(t) = t^{q/p}$, and the C-order of $A(t)$ is equal to q/p for all $C > 0$. In this situation the constant in inequality (50) is equal to $C_{fq}^{q/p}$, because in this case we have $\Psi_1(t) \equiv \Phi(t)$, where $C_{fq} = \|f\|_\infty / \|f\|_q$.

It should be noted that, in contrast to the well-known Nikol'skii inequalities, the constants in Theorems 10 and 11 depend on the function itself (and not on its class).

We next consider Nikol'skii's inequality for trigonometric polynomials.

Let K be a compact subset in $\mathbb{R}^n$. Denote by $\mathcal{P}_{K\Phi}$ the space of all functions $f \in L_\Phi(\mathbb{T}^n)$ such that $\operatorname{sp}(f) \subset K$. In this case, it is easy to prove that there is a continuous embedding $\mathcal{P}_{K\Phi} \subset \mathcal{P}_{K\infty}$, where $\mathcal{P}_{K\infty}$ is the space of all bounded periodic functions with spectrum on K.

Arguing as above, we can easily establish the following results.

Theorem 12. *Let $\Psi(t)$ $C^*_{K\Phi}$-majorize $\Phi(t)$. Then*

$$|||f|||_{(\Phi)} \leqslant \frac{A(C^*_{K\Phi})}{C^*_{K\Phi}} |||f|||_{(\Psi)}$$

for all $f \in \mathcal{P}_{K\Psi}$, where $C^*_{K\Phi}$ is the exact constant in the inequality

$$|||f|||_\infty \leqslant C|||f|||_{(\Phi)}.$$

Theorem 13. *Let $\Psi(t)$ $C^*_{f\Phi}$-majorize $\Phi(t)$. Assume that $f \in L_\Psi(\mathbb{T}^n)$ and that* $\operatorname{sp}(f)$ *is bounded. Then*

$$|||f|||_{(\Phi)} \leqslant \frac{A(C^*_{f\Phi})}{C^*_{f\Phi}} |||f|||_{(\Psi)},$$

where $C^*_{f\Phi} = |||f|||_\infty / |||f|||_{(\Phi)}$.

Theorem 14. *Let $\Psi(t)$ $C^*_{f\Phi}$-majorize $\Phi(t)$ and let $f \in L_\Psi(\mathbb{T}^n)$. We suppose that* $\operatorname{sp}(f)$ *is bounded. Then*

$$|||f|||_{(\Phi)} \leqslant \frac{A^{1/\operatorname{ord} A}(C^*_{f\Phi})}{C^*_{f\Phi}} C^{\operatorname{ord} A - 1}_{f\Psi_1} |||f|||_{(\Psi)},$$

where $C^*_{f\Phi} = |||f|||_\infty / |||f|||_{(\Phi)}$, $C^*_{f\Psi_1} = |||f|||_\infty / |||f|||_{(\Psi_1)}$, $\Psi_1(t) = \Psi_1(t^{\operatorname{ord} A})$, and $\operatorname{ord} A$ *is the $C^*_{f\Phi}$-order of $A(t)$.*

§ 4. Some applications

Let $0 \leqslant \lambda_\alpha \leqslant \infty$ for $\alpha \in \mathbb{Z}^n_+$ and let $G\{\lambda_\alpha\} = \bigcap_{\alpha \geqslant 0} \{\xi \in \mathbb{R}^n : |\xi^\alpha| \leqslant \lambda_\alpha\}$.

Definition 5. We call $G\{\lambda_\alpha\}$ the *set generated by the number sequence* $\{\lambda_\alpha\}$.

Obviously, $G\{\lambda_\alpha\}$ is closed, $(r_1\xi_1, \dots, r_n\xi_n) \in G\{\lambda_\alpha\}$ if $\xi \in G\{\lambda_\alpha\}$ and $|r_j| \leqslant 1$, $j = 1, \dots, n$, and we have

$$G\{\lambda_\alpha\} = G\Big\{ \sup_{\xi \in G\{\lambda_\alpha\}} |\xi^\alpha| \Big\}.$$

The set $G\{\lambda_\alpha\}$ is compact if, for example, $\lambda_\alpha < \infty \quad \forall \alpha \geqslant 0$.

We note that $G\{\lambda_\alpha\}$ can be non-convex. For example, let $n = 2$ and

$$\lambda_{(i,j)} = 2^{|i-j|} \qquad \forall i, j \in \mathbb{Z}_+.$$

The set

$$G\{\lambda_{(i,j)}\} = \big\{(x, y) \in \mathbb{R}^2 : |xy| \leqslant 1, \quad |x| \leqslant 2, \quad |y| \leqslant 2\big\},$$

which is called the *cross of the hyperbola*, is non-convex.

Let $K \subset \mathbb{R}^n$. If we set $g(K) = G\{\sup_K |\xi^\alpha|\}$, then $K \subset g(K)$. We call $g(K)$ the *g-hull* of K.

Definition 6. A set K is said to *possess the g-property* if $K = g(K)$.

It is clear that any set $G\{\lambda_\alpha\}$ generated by a number sequence possesses the g-property and, obviously, vice versa.

Lemma 5. *Let I be a family of indices and let $K_i = g(K_i)$, $i \in I$. Then $\bigcap_{i\in I} K_i$ also possesses the g-property.*

Proof. Let $x \in g\left(\bigcap_{i\in I} K_i\right)$ and let $j \in I$. Then

$$|x^\alpha| \leqslant \sup\Big\{|\xi^\alpha| : \xi \in \bigcap_{i\in I} K_i\Big\} \leqslant \sup\big\{|\xi^\alpha| : \xi \in K_j\big\}$$

for any $\alpha \in \mathbb{Z}^n_+$. Hence, $x \in g(K_j) = K_j$. Therefore $x \in \bigcap_{i\in I} K_j$.

The following question arises: does every compact set K such that

$$x \in K, \qquad -1 \leqslant \lambda_j \leqslant 1, \qquad j = 1, \dots, n \quad \Longrightarrow \quad (\lambda_1 x_1, \dots, \lambda_n x_n) \in K \tag{53}$$

possess the g-property? The answer turns out to be negative. Indeed, let K be a subset of $G = \big\{(x,y) \in \mathbb{R}^2 : |xy| \leqslant 1, \quad |x| \leqslant 2, \quad |y| \leqslant 2\big\}$ such that K possesses property (53), $K \neq G$, and K includes the points $\left(\frac{1}{2}, 2\right)$ and $\left(2, \frac{1}{2}\right)$. Such sets K obviously exist, for example, $K = \big\{|x| \leqslant 2, \quad |y| \leqslant \frac{1}{2}\big\} \cup \big\{|x| \leqslant \frac{1}{2}, |y| \leqslant 2\big\}$. It follows that

$$\sup_K |x^i y^j| = 2^{|i-j|} = \sup_G |x^i y^j|$$

for all integers $i, j \geqslant 0$. Consequently, K does not possess the g-property.

Lemma 6. *Every symmetric convex compact set possesses the g-property.*

Proof. Let K be a symmetric convex compact set in $\mathbb{R}^n$. It can easily be seen that K satisfies (53). Furthermore, let $y \notin K$. Then there is a vector $a \in \mathbb{R}^n$ such that

$$ay > \sup_{x\in K} ax, \tag{54}$$

where $a\xi = a_1\xi_1 + \dots + a_n\xi_n$, by the convexity of K. From (53), we have

$$\sup_{x\in K} ax = \sup_{x\in K} \big(|a_1x_1| + \dots + |a_nx_n|\big). \tag{55}$$

It also follows from (53) that $y \notin K$ if and only if $\big(|y_1|, \dots, |y_n|\big) \notin K$. Hence, it suffices to consider only the case $y \geqslant 0$ for which, by virtue of (54) and (55), there is a vector a such that $a_j \geqslant 0$ and $a_j = 0$ if $y_j = 0$, $j = 1, \dots, n$.

For convenience, we assume that $y_j > 0$, $j = 1, \dots, n$.

Let $x \in K_+ = \{x \in K : x \geqslant 0\}$. In this case, a well-known classical inequality implies

$$\begin{aligned}\left(\frac{ax}{ay}\right)^{ay} &= \left(\frac{a_1y_1x_1/y_1 + \dots + a_ny_nx_n/y_n}{a_1y_1 + \dots + a_ny_n}\right)^{a_1y_1 + \dots + a_ny_n} \\ &\geqslant \left(\frac{x_1}{y_1}\right)^{a_1y_1} \dots \left(\frac{x_n}{y_n}\right)^{a_ny_n}.\end{aligned}$$

Therefore, in view of (54), we obtain

$$1 > \sup_{x\in K_+} \left(\frac{x_1}{y_1}\right)^{a_1y_1} \dots \left(\frac{x_n}{y_n}\right)^{a_ny_n}.$$

On approximating a_jy_j by rational numbers, we can write

$$1 > \sup_{x\in K_+} \left(\frac{x_1}{y_1}\right)^{p_1/q_1} \dots \left(\frac{x_n}{y_n}\right)^{p_n/q_n},$$

where $p_j \geqslant 0$ and $q_j > 0$, $j = 1, \dots, n$, are integers. Consequently, it follows from (53) that

$$y^\alpha > \sup_{x\in K_+} x^\alpha = \sup_{x\in K} |x^\alpha|,$$

where $\alpha_j = q_1 \dots q_n p_j/q_j$, $j = 1, \dots, n$. This means that $y \notin g(K)$ for any $y \notin K$. The proof is complete.

We now prove the non-convex version of the Paley–Wiener–Schwartz theorem characterizing the relationship between the behaviour of the sequence of norms of the derivatives of a function and the support of its Fourier transform.

It is clear that the set $G\{\lambda_\alpha\}$ generated by a number sequence does not change under the replacement $\lambda_{m\beta} \to \lambda_\beta^m$ if there is an $m \geqslant 1$ and a $\beta \geqslant 0$ such that $\lambda_\beta^m < \lambda_{m\beta}$. Hence, it can always be assumed in the definition of $G\{\lambda_\alpha\}$ that

$$\lambda_\alpha^m \geqslant \lambda_{m\alpha} \qquad \forall m \geqslant 1, \qquad \alpha \geqslant 0. \tag{56}$$

Definition 7. A sequence $\{\lambda_\alpha\}$ is said to be *regular* if $\{\lambda_\alpha\}$ satisfies condition (56).

Theorem 15. *Let $\Phi(t)$ be an arbitrary Young function and let $f \in L_\Phi(\mathbb{R}^n)$. Assume that $G\{\lambda_\alpha\}$ is bounded and that $\{\lambda_\alpha\}$ is a regular sequence. In this case we have* $\operatorname{sp}(f) \subset G\{\lambda_\alpha\}$ *if and only if the following condition holds*:

$$\varlimsup_{|\alpha|\to\infty} \Big(\|D^\alpha f\|_{(\Phi)} \big/ \lambda_\alpha\Big)^{1/|\alpha|} \leqslant 1. \tag{57}$$

Proof. Let $\operatorname{sp}(f) \subset G\{\lambda_\alpha\}$. Then

$$\sup_{\operatorname{sp}(f)} |\xi^\alpha| \leqslant \lambda_\alpha, \qquad \alpha \geqslant 0.$$

Consequently, by Theorem 2, we obtain (57).

Conversely, let (57) hold. We note that inequality (11) also holds in the case when $\operatorname{sp}(f)$ is unbounded. Therefore, (11), (57), and the fact that the sequence $\{\lambda_\alpha\}$ is regular readily imply that $\operatorname{sp}(f)$ is bounded. Whence, in view of Theorem 2,

$$\varlimsup_{|\alpha|\to\infty} \Big(\sup_{\operatorname{sp}(f)} |\xi^\alpha| \big/ \lambda_\alpha\Big)^{1/|\alpha|} \leqslant 1.$$

Consequently, given an arbitrary $\epsilon > 0$, there is an index $N < \infty$ such that

$$\sup_{\mathrm{sp}(f)} |\xi^\alpha| \leqslant (1+\epsilon)^{|\alpha|}\lambda_\alpha, \qquad |\alpha| \geqslant N.$$

On the other hand, since the sequence $\{\lambda_\alpha\}$ is regular, we have

$$\sup_{\mathrm{sp}(f)} |\xi^\alpha| \leqslant (1+\epsilon)^{|\alpha|}\lambda_\alpha$$

for all $\alpha \geqslant 0$. It follows that

$$\mathrm{sp}(f) \subset (1+\epsilon)G\{\lambda_\alpha\}.$$

Letting ϵ tend to 0, we obtain $\mathrm{sp}(f) \subset G\{\lambda_\alpha\}$. The theorem is proved.

We next apply the above results to the theory of Sobolev–Orlicz spaces of infinite order to resolve some problems arising in the study of non-linear differential equations of infinite order with coefficients of arbitrary rate of growth. The theory of spaces of infinite order was introduced by Dubinskii and studied by him and also by Balashova, Chan Dyk Van, Klenina, Konyaev, Kobilov, Umarov, Agadzhanov, Groshev, and the author of the present paper, among others. The following questions have been considered: non-triviality, theory of traces, relationship with boundary-value problems, geometric properties, and so on (for example, see [25]–[27] and the references there). It should be noted that the theory of infinite-order function spaces differs from that of finite-order spaces if only in the fact that the question of the existence of a non-zero element (that is, the question of non-triviality) is not at all simple in the former. The positive answer to this question plays a key role in the theory of boundary-value problems for infinite-order differential equations. The problem of finding solutions to boundary-value problems for infinite-order equation is meaningful if the corresponding energy spaces are non-trivial (see [28]–[31] for example).

Let I be an unbounded set of integer indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \geqslant 0$, where $j = 1, \dots, n$, and let $\Phi_\alpha(t)$, $\alpha \in I$, be an arbitrary Young function. Then

$$W^\infty L\{\Phi_\alpha, \mathbb{R}^n\} = \left\{ f(x) \in S' : \sum_{\alpha \in I} \|D^\alpha f\|_{(\Phi_\alpha)} < \infty \right\}$$

is called a *Sobolev–Orlicz space of infinite order.*

Let us derive a non-triviality condition for $W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$. We assume that $0 \in I$ and $\Phi_0(t) > 0$, $t > 0$, because otherwise $W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$ is non-trivial. An application of Theorem 2 yields the following theorem [31].

Theorem 16. *The space $W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$ is non-trivial if and only if there exist numbers $C, q > 0$ such that*

$$\sum_{\alpha \in I} \Phi_\alpha(C q^{|\alpha|}) < \infty. \tag{58}$$

We now describe the properties of functions belonging to $W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$.

Definition 8. Assume that condition (58) holds. Denote by C_Φ the union of all points $\xi \in \mathbb{R}^n$ such that

$$\sum_{\alpha\in I} \Phi_\alpha\Bigl(C_\xi \sup_{x\in G_\xi} |x^\alpha|\Bigr) < \infty \tag{59}$$

for some domain $G_\xi \ni \xi$ and a number $C_\xi > 0$. Clearly, G_Φ is open, non-empty, and symmetric with respect to the origin.

Theorem 17. *Assume that condition* (58) *holds. Then*

$$F^{-1}\bigl[C_0^\infty(G_\Phi)\bigr] \subset W^\infty L\{\Phi_\alpha, \mathbb{R}^n\},$$

and $W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$ *does not contain any function* $F^{-1}g(x)$ *if* $g \in C_0^\infty(\mathbb{R}^n)$ *and* $\operatorname{supp} g \cap (\mathbb{R}^n\backslash\overline{G}_\Phi) \neq \varnothing$.

Proof. Let $\varphi(\xi) \in C_0^\infty(G_\Phi)$. Then for any point $\xi \in \operatorname{supp}\varphi$ there is a bounded domain $G_\xi \ni \xi$ satisfying condition (59). Since $\operatorname{supp}\varphi$ is compact, we can find an index $M < \infty$, a number $C > 0$, and bounded domains G_j such that

$$\operatorname{supp}\varphi \subset \bigcup_{j=1}^M G_j$$

and

$$\sum_{\alpha\in I} \Phi_\alpha\Bigl(C \sup_{G_j} |\xi^\alpha|\Bigr) < \infty, \qquad j = 1,\dots,M. \tag{60}$$

Next, let us define

$$G^* = \bigl\{(t_1\xi_1,\dots,t_n\xi_n) : \xi \in G, \quad -1 \leqslant t_j \leqslant 1, \quad j = 1,\dots,n\bigr\}$$

for every domain G. Then G^* is also a domain and

$$\sup_G |\xi^\alpha| = \sup_{G^*} |\xi^\alpha| \tag{61}$$

for all $\alpha \geqslant 0$. By virtue of the compactness of $\operatorname{supp}\varphi$, we have

$$\operatorname{supp}\varphi \subset \lambda \bigcup_{j=1}^M G_j^* \tag{62}$$

for some λ, $0 < \lambda < 1$.

Let $\psi(x) = (F^{-1}\varphi)(x)$. Then $\psi \in \mathfrak{M}_{K1}$, where for brevity we set $K = \operatorname{supp}\varphi$. We assume for convenience that $C = 1$.

We now prove that $\psi \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$. First,

$$\begin{aligned}
|x^\beta D^\alpha \psi(x)| &\leqslant \int_K |D^\beta\bigl(\xi^\alpha\varphi(\xi)\bigr)|\, d\xi \\
&\leqslant \sum_{\gamma\leqslant\beta,\alpha} \frac{\beta!}{\gamma!\,(\beta-\alpha)!} \prod_{k=1}^n \alpha_k \dots (\alpha_k - \gamma_k + 1) \int_K |\xi^{\alpha-\gamma} D^{\beta-\gamma}\varphi(\xi)|\, d\xi
\end{aligned}$$

for any $\alpha, \beta \geqslant 0$. Hence, taking

$$\prod_{k=1}^{n} \alpha_k \dots (\alpha_k - \gamma_k + 1) \leqslant |\alpha|^{|\gamma|}, \qquad \sum_{\gamma \leqslant \beta} \frac{\beta!}{\gamma!\,(\beta-\gamma)!} = 2^{|\beta|}$$

into account, we obtain

$$|x^\beta D^\alpha \psi(x)| \leqslant 2^{2n} |\alpha|^{2n} C_{2n} \max\left\{\sup_K |\xi^{\alpha-\gamma}| : \gamma \leqslant \alpha, \quad |\gamma| \leqslant 2n\right\}$$

for all $x \in \mathbb{R}^n$, $\alpha \geqslant 0$, and $|\beta| \leqslant 2n$, where

$$C_{2n} = \max\left\{ \int_K |D^{\beta-\gamma}\varphi(\xi)|\, d\xi : \gamma \leqslant \beta, \quad |\beta| \leqslant 2n\right\}.$$

On the other hand, (25) implies

$$\lim_{|\alpha|\to\infty} \left(\max\left\{\sup_K |\xi^{\alpha-\gamma}| : \gamma \leqslant \alpha, \quad |\gamma| \leqslant 2n\right\}\right)^{1/|\alpha|} \Big/ \sup_K |\xi^\alpha|^{1/|\alpha|} = 1.$$

Consequently, for any $\epsilon > 0$ there is a C_ϵ such that

$$(1+x_1^2)\dots(1+x_n^2)|D^\alpha\psi(x)| \leqslant 2^{2n} C_{2n} C_\epsilon |\alpha|^{2n} (1+\epsilon)^{|\alpha|} \sup_K |\xi^\alpha| \tag{63}$$

for all $x \in \mathbb{R}^n$ and $\alpha \geqslant 0$.

From (62) and (63) we obtain

$$(1+x_1^2)\dots(1+x_n^2)|D^\alpha\psi(x)| \leqslant 2^{2n} C_{2n} C_\epsilon |\alpha|^{2n} (1+\epsilon)^{|\alpha|} \lambda^{|\alpha|} \sup_{G^*} |\xi^\alpha|$$

for $G^* = \bigcup_{j=1}^M G_j^*$.

Let $\epsilon > 0$ satisfy $(1+\epsilon)\lambda < 1$ and let $(1+\epsilon)\lambda = \lambda_1 q$, where $0 < \lambda_1$ and $q < 1$. Then

$$\lim_{|\alpha|\to\infty} |\alpha|^{2n} (1+\epsilon)^{|\alpha|} \lambda_1^{|\alpha|} = 0$$

implies that

$$|D^\alpha\psi(x)| \leqslant (1+x_1^2)^{-1} \dots (1+x_n^2)^{-1} q^{|\alpha|} \sup_{G^*} |\xi^\alpha|$$

for all $|\alpha| \geqslant N_1$ and $x \in \mathbb{R}^n$. It follows that

$$\begin{aligned}
\|D^\alpha\psi\|_{(\Phi_\alpha)} &= \inf\left\{\gamma > 0 : \int_{\mathbb{R}^n} \Phi_\alpha\left(\frac{|D^\alpha\psi(x)|}{\gamma}\right) dx \leqslant 1\right\} \\
&\leqslant \inf\left\{\gamma > 0 : \int_{\mathbb{R}^n} \Phi_\alpha\left(q^{|\alpha|} \sup_{G^*} |\xi^\alpha| \Big/ \gamma(1+x_1^2)\dots(1+x_n^2)\right) dx \leqslant 1\right\} \\
&\leqslant \inf\left\{\gamma > 0 : \Phi_\alpha\left(q^{|\alpha|} \sup_{G^*} |\xi^\alpha| \Big/ \gamma\right) \int_{\mathbb{R}^n} (1+x_1^2)^{-1} \dots (1+x_n^2)^{-1}\, dx \leqslant 1\right\} \\
&= \inf\left\{\gamma > 0 : \pi^n \Phi_\alpha\left(q^{|\alpha|} \sup_{G^*} |\xi^\alpha| \Big/ \gamma\right) \leqslant 1\right\}
\end{aligned} \tag{64}$$

for $|\alpha| \geqslant N_1$.

On the other hand, (60) and (61) imply

$$\sum_{\alpha\in I}\Phi_\alpha\left(\sup_{G^*}|\xi^\alpha|\right)\leqslant\sum_{\alpha\in I}\max_{1\leqslant j\leqslant M}\Phi_\alpha\left(\sup_{G_j}|\xi^\alpha|\right)\leqslant\sum_{j=1}^{M}\sum_{\alpha\in I}\Phi_\alpha\left(\sup_{G_j}|\xi^\alpha|\right)<\infty.$$

Consequently, there is an index $N_2<\infty$ such that

$$\Phi_\alpha\left(\sup_{G^*}|\xi^\alpha|\right)\leqslant\pi^{-n},\qquad |\alpha|\geqslant N_2.$$

Therefore

$$\pi^n\Phi_\alpha\left(q^{|\alpha|}\sup_{G^*}|\xi^\alpha|\Big/q^{|\alpha|}\right)\leqslant 1,\qquad |\alpha|\geqslant N_2.$$

This inequality and (64) imply that

$$\|D^\alpha\psi\|_{(\Phi_\alpha)}\leqslant q^{|\alpha|}\tag{65}$$

for all $|\alpha|\geqslant N_0=\max\{N_1,N_2\}$.

But inequality (63) clearly shows that $D^\alpha\psi(x)\in L_\Phi(\mathbb{R}^n)$ for any Young function $\Phi(t)$ and for all $\alpha\geqslant 0$. Hence, in view of (65), we have $\psi(x)\in W^\infty L\{\Phi_\alpha,\mathbb{R}^n\}$. The first part of Theorem 17 is proved.

We now prove the second part. Let $g(\xi)\in C_0^\infty(\mathbb{R}^n)$ be a function such that $\operatorname{supp} g\cap(\mathbb{R}^n\backslash\overline{G}_\Phi)=\varnothing$. Then, since $\mathbb{R}^n\backslash\overline{G}_\Phi$ is open, it follows that there is a point $\xi\in\operatorname{supp} g,\ \xi_j\neq 0,\ j=1,\dots,n$, and a neighbourhood $U_\xi\subset\mathbb{R}^n\backslash\overline{G}_\Phi,\ \xi\in U_\xi$.

But $\xi\notin\overline{G}_\Phi$ implies that there is an $r<1$ such that

$$\sum_{\alpha\in I}\Phi_\alpha\left(Cr^{|\alpha|}\sup_{G_\xi}|x^\alpha|\right)=\infty\tag{66}$$

for any $C<\infty$ and any domain $G_\xi\ni\xi$, because otherwise $\forall r<1\ \ \exists C_r,\ \ \exists G_\xi\ni\xi$

$$\sum_{\alpha\in I}\Phi_\alpha\left(C_r r^{|\alpha|}\sup_{G_\xi}|x^\alpha|\right)<\infty,$$

that is,

$$\sum_{\alpha\in I}\Phi_\alpha\left(C_r\sup_{rG_\xi}|x^{|\alpha|}|\right)<\infty.$$

We note that $\xi\in rG_\xi$ for some $r<1$. Fix such an r and set $Q=rG_\xi$. Then we obtain

$$\sum_{\alpha\in I}\Phi_\alpha\left(C_r\sup_{Q}|x^\alpha|\right)<\infty,$$

and therefore $\xi\in G_\Phi$, which is impossible. Relation (66) is proved.

We next prove by contradiction that $f = F^{-1}g \notin W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$. Assume that $f \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$ and also that $\sum_{\alpha\in I} \|D^\alpha f\|_{(\Phi_\alpha)} \leqslant 1$, which does not restrict the generality. Set

$$f_h(x) = \frac{1}{\operatorname{mes} B(0,h)} \int_{B(0,h)} f(x+t)\,dt, \qquad h > 0.$$

Jensen's inequality implies that

$$\Phi_\alpha\big(|D^\alpha f_h(x)|\big) \leqslant \frac{1}{\operatorname{mes} B(0,h)} \int_{B(0,h)} \Phi_\alpha\big(|D^\alpha f(x+t)|\big)\,dt.$$

On the other hand, it can be assumed that all the functions $\Phi_\alpha(t)$, $\alpha \in I$, are continuous on the left. For if some $\Phi(t)$ is not continuous on the left, then there is a point $t_0 > 0$ such that

$$\lim_{t\to t_0-} \Phi(t) < \Phi(t_0) \leqslant \infty, \qquad \Phi(t) = \infty \quad \text{for} \quad t > t_0.$$

We set

$$\Psi(t) = \begin{cases} \Phi(t), & t \neq t_0, \\ \lim_{t\to t_0-} \Phi(t), & t = t_0. \end{cases}$$

Then $\Psi(t)$ is a Young function continuous on the left, and we have

$$\|\cdot\|_{(\Psi)} = \|\cdot\|_{(\Phi)}.$$

Hence, $\Phi(t)$ can be replaced by $\Psi(t)$.

Therefore

$$\int \Phi_\alpha\big(|D^\alpha f(x)|\big)\,dx \leqslant \|D^\alpha f\|_{(\Phi_\alpha)}$$

implies

$$\sum_{\alpha\in I} \Phi_\alpha\big(\|D^\alpha f_h\|_\infty\big) \leqslant \frac{1}{\operatorname{mes} B(0,h)}, \tag{67}$$

because $\|D^\alpha f\|_{(\Phi_\alpha)} \leqslant 1$ (see [7]–[9]). It is clear from the corresponding definition that $f_h(x)$ converges weakly in S' ($f_h \to f$ as $h \to 0$). This implies the weak convergence of $\hat{g}_h$ in S' ($\hat{g}_h \to \hat{f}$). It follows that the point ξ defined above belongs to $\operatorname{supp} \hat{f}_h$ for sufficiently small $h > 0$, because $\xi \in \operatorname{supp} \hat{f} = \operatorname{supp} g$.

Fix a sufficiently small $h > 0$ such that $\xi \in \operatorname{supp} \hat{f}_h$. Then (11) implies

$$\underline{\lim}_{|\alpha|\to\infty} \big(\|D^\alpha f_h\|_\infty / |\xi^\alpha|\big)^{1/|\alpha|} \geqslant 1.$$

Hence, for any $\epsilon > 0$ there is a $C_\epsilon > 0$ such that

$$\|D^\alpha f_h\|_\infty \geqslant C_\epsilon (1-\epsilon)^{|\alpha|} |\xi^\alpha|, \qquad \alpha \in I. \tag{68}$$

On the other hand, $\xi_j \neq 0, \quad j = 1, \dots, n$, implies that for any $\lambda > 1$ there is a neighbourhood G of ξ such that

$$\sup_{x \in G} |x^\alpha| \leqslant \lambda^{|\alpha|} |\xi^\alpha| \tag{69}$$

for all $\alpha \geqslant 0$.

Let us choose a $\lambda > 1$ and an $\epsilon > 0$ such that $\lambda^{-1}(1-\epsilon) \geqslant r$. Then (67)–(69) imply

$$\begin{aligned}\sum_{\alpha \in I} \Phi_\alpha \left(C_\epsilon r^{|\alpha|} \sup_G |\xi^\alpha| \right) &\leqslant \sum_{\alpha \in I} \Phi_\alpha \left(C_\epsilon \lambda^{-|\alpha|} (1-\epsilon)^{|\alpha|} \sup_G |\xi^\alpha| \right) \\ &\leqslant \sum_{\alpha \in I} \Phi_\alpha \big(C_\epsilon (1-\epsilon)^{|\alpha|} |\xi^\alpha| \big) \leqslant \sum_{\alpha \in I} \Phi_\alpha \big(\|D^\alpha f_h\|_\infty \big) \leqslant \frac{1}{\operatorname{mes} B(0,h)},\end{aligned}$$

which is impossible by virtue of (66). The theorem is proved.

Theorem 18. *Assume that condition* (58) *holds. The space* $W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$ *contains all functions* $f \in L_1(\mathbb{R}^n)$ *with* $\operatorname{sp}(f) \subset G_\Phi$*, but does not contain any function* $g \in L_1(\mathbb{R}^n)$ *with* $\operatorname{sp}(g) \cap (\mathbb{R}^n \backslash \overline{C}_\Phi) \neq \varnothing$*. Moreover, if* $\Phi_\alpha(t) > 0, \quad t > 0$*, for an* $\alpha \in I$*, then* $\operatorname{sp}(g) \cap (\mathbb{R}^n \backslash \overline{C}_\Phi) = \varnothing$ *for any* $g \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$*. Otherwise the latter fact is not true.*

Proof. Let $f \in L_1(\mathbb{R}^n)$ and $\operatorname{sp}(f) \subset G_\Phi$. Choose a function $\widehat{\varphi}(\xi) \in C_0^\infty(G_\Phi)$ such that $\widehat{\varphi}(\xi)$ is equal to unity in a neighbourhood of $\operatorname{sp}(f)$. Then the inequalities

$$\|D^\alpha f\|_{(\Phi_\alpha)} = \|f * D^\alpha \varphi\|_{(\Phi_\alpha)} \leqslant 2 \|f\|_1 \|D^\alpha \varphi\|_{(\Phi_\alpha)}, \qquad \alpha \in I,$$

and Theorem 17 readily imply $f \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$, because $\varphi \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$.

The proof that $W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$ does not contain any function $g \in L_1(\mathbb{R}^n)$ with $\operatorname{sp}(g) \cap (\mathbb{R}^n \backslash \overline{C}_\Phi) \neq \varnothing$ is perfectly analogous to the argument in the proof of the second part of Theorem 17 (by Corollary 1, if $\xi \in \operatorname{sp}(g)$ is an arbitrary point, then $\operatorname{sp}(g)$ contains a sequence of points with non-zero components that converges to ξ).

Furthermore, let $\Phi_\beta(t) > 0, \quad t > 0$, for a $\beta \in I$, and $g(x) \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$. It suffices to consider only the case $g(x) \not\equiv 0$, when the inequalities $\|D^\beta g\|_{(\Phi_\beta)} < \infty$, $\|g\|_{(\Phi_0)} < \infty$, and Theorem 1 imply the following:

if $D^\beta g(x) \equiv 0$, then $\operatorname{sp}(g) \subset \{0\}$, whence $\operatorname{sp}(g) \cap (\mathbb{R}^n \backslash \overline{C}_\Phi) = \varnothing$;

if $D^\beta g(x) \not\equiv 0$, then the support of the restriction of $\hat{g}(\xi)$ to an arbitrary neighbourhood of any point belonging to $\operatorname{sp}(g)$ does not lie in the hyperplanes $\xi_j = 0$, $j = 1, \dots, n$.

Hence, the desired assertion can be proved by analogy with the proof of the second part of Theorem 17.

We conclude by constructing a counterexample. Let $n \geqslant 2$ and

$$I_j = \left\{ \Big(\underbrace{m, \dots, m}_{j-1}, m^2, m, \dots, m \Big) : m \geqslant 0 \right\}, \qquad I = \bigcup_{j=1}^n I_j.$$

We assume that $\Phi_\alpha(t) = 0$ for $0 \leqslant t \leqslant 1$ and $\Phi_\alpha(t) = \infty$ for $t > 1$ and all $\alpha \in I$ (that is, for $\|\cdot\|_{(\Phi_\alpha)} = \|\cdot\|_\infty$). Then it can easily be seen that $G_\Phi = M$, where $M = \{\xi \in \mathbb{R}^n : |\xi_j| < 1,\ j = 1, \dots, n\}$. For if $x \notin \overline{M}$, then there is an index j, $1 \leqslant j \leqslant n$, such that $|x_j| > 1$. We consider a neighbourhood U_x and an arbitrary number C, $0 < C < \infty$. It follows that there is a $\theta > 0$ such that

$$\sup_{\xi \in U_x} |\xi^\alpha| \geqslant \theta^{(n-1)m} |x_j|^{m^2}, \qquad \alpha = \Big(\underbrace{m, \dots, m}_{j-1}, m^2, m, \dots, m\Big)$$

for all $\alpha \in I_j$. Hence, since $\theta^{n-1}|x_j|^m \to \infty$ as $m \to \infty$, we obtain $\Phi_\alpha\left(C \sup_{\xi \in U_x} |\xi^\alpha|\right) = \infty$ for sufficiently large $|\alpha|$, $\alpha \in I_j$. This means that $x \notin G_\Phi$.

Moreover, the inequality $|x_j| > 1$ implies that there is a neighbourhood V of x such that $|y_j| > 1$ for any point $y \in V$. Consequently, $V \cap G_\Phi = \varnothing$. Therefore, $x \notin \overline{G}_\Phi$. We have thus proved that

$$\overline{G}_\Phi \subset \overline{M} = \{\xi \in \mathbb{R}^n : |\xi_j| \leqslant 1, \quad j = 1, \dots, n\}.$$

On the other hand, the choice of the functions $\Phi_\alpha(t)$ implies that $M \subset G_\Phi$. Hence, $G_\Phi = M$ because G_Φ is always open.

Finally, note that any function

$$\varphi(x) = \varphi(x_1, \dots, x_{n-1}) = F^{-1}\widehat{\varphi}(\xi), \qquad \widehat{\varphi}(\xi) \in C_0^\infty(\mathbb{R}^{n-1}),$$

belongs to $W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$, because $D^\alpha \varphi(x) \equiv 0$ for all $\alpha \in I$, $|\alpha| > 0$. This means that the relation $\operatorname{sp}(g) \cap (\mathbb{R}^n \backslash \overline{C}_\Phi) = \varnothing$ is false. The theorem is proved.

Remark 9. We have shown that the relation $\operatorname{sp}(g) \cap (\mathbb{R}^n \backslash \overline{C}_\Phi) = \varnothing$ can be false for $n \geqslant 2$ if for any $\alpha \in I$ there is a $t_\alpha > 0$ such that $\Phi_\alpha(t_\alpha) = 0$. A stronger result holds for $n = 1$.

Theorem 19. *Suppose that the corresponding condition* (58) *holds. Then we have* $\operatorname{sp}(g) \cap (\mathbb{R}^1 \backslash \overline{C}_\Phi) = \varnothing$ *for any* $g \in W^\infty L\{\Phi_m, \mathbb{R}^1\}$.

Proof. We prove the theorem by contradiction. Assume that there is a function $g(x) \in W^\infty L\{\Phi_m, \mathbb{R}^1\}$ and a point $\xi \in \operatorname{sp}(g)$ such that $\xi \in \mathbb{R}^1 \backslash \overline{G}_\Phi$. Then we have $G_\Phi = (-b, b)$, $0 < b < \infty$, and $|\xi| > b$.

Define

$$g_h(x) = \frac{1}{h} \int_0^h g(x+t)\, dt, \qquad h > 0.$$

Then, as was shown in the proof of Theorem 17, we have

$$\sum_{m \in I} \Phi_m\big(\|D^m g_h\|_\infty\big) \leqslant \frac{1}{h} < \infty,$$

because it can be assumed that $\sum_{m \in I} \|D^m g\|_{(\Phi_m)} \leqslant 1$. Next, the weak convergence $\hat{g}_h \to \hat{g}$ in S' implies that $\xi \in \operatorname{supp} \hat{g}_h$ for sufficiently small $h > 0$. Fix such an $h > 0$. Then (11) implies

$$\varliminf_{m \to \infty} \|D^m g_h\|_\infty^{1/m} \geqslant |\xi|.$$

For the sake of definiteness, assume that $\xi > 0$. Then we can find ϵ and C with $0 < \epsilon,\ C < \infty$ such that $\xi - \epsilon > b$ and

$$\|D^m g_h\|_\infty \geqslant C(\xi - \epsilon)^m, \qquad m \geqslant 0.$$

Therefore,

$$\sum_{m \in I} \Phi_m\bigl(C(\xi - \epsilon)^m\bigr) < \infty,$$

which is impossible, because $\xi - \epsilon > b$ and $G_\Phi = (-b, b)$. The theorem is proved.

Theorem 20. *Assume that condition* (58) *holds. Then*

$$\overline{G}_\Phi \subset \cup\bigl\{\mathrm{sp}(g) : g \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}\bigr\}.$$

Proof. Theorem 17 implies

$$G_\Phi \subset \cup\bigl\{\mathrm{sp}(g) : g \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}\bigr\}.$$

Furthermore, let $\xi^0 \in \overline{G}_\Phi \backslash G_\Phi$. Then there is a subsequence $\{\xi^k\} \subset G_\Phi$ such that $\xi^k \to \xi^0$ as $k \to \infty$ and $\xi^k \neq \xi^l,\ \ k \neq l$. Suppose that $\epsilon_k > 0$ and $k \geqslant 1$ are such that $B(\xi^k, \epsilon_k) \cap B(\xi^l, \epsilon_l) = \varnothing$ and $B(\xi^k, \epsilon_k) \subset G_\Phi,\ \ k \geqslant 1$. Choose functions $\widehat{\varphi}_k(\xi) \in C_0^\infty\bigl(B(\xi^k, \epsilon_k)\bigr)$ and numbers $\gamma_k > 0$ such that $\widehat{\varphi}_k(\xi^k) \neq 0,\ \ k \geqslant 1$, and

$$\sum_{k=1}^\infty \gamma_k |||\varphi_k||| < \infty.$$

Then $\psi(x) = \sum_{k=1}^\infty \gamma_k \varphi_k(x) \in W^\infty L\{\Phi_\alpha, \mathbb{R}^n\}$ and $\xi^0 \in \mathrm{sp}(\psi)$. The theorem is proved.

Bibliography

[1] S. M. Nikol'skii, "Inequalities for entire function of finite degree and their application in the theory of differentiable functions of several variables", *Trudy Mat. Inst. Steklov* **38** (1951), 244-278. (Russian)

[2] S. M. Nikol'skii, *Approximation of functions of several variables and embedding theorems*, Nauka, Moscow 1977. (Russian)

[3] I. I. Ibragimov, *Extremal properties of entire functions of a finite degree*, ELM, Baku 1962. (Russian)

[4] I. I. Ibragimov, "Extremal properties in a class of entire functions of finite degree", *Izv. Akad. Nauk SSSR. Ser. Mat.* **23** (1959), 243–256. (Russian)

[5] I. I. Ibragimov, "Some extremal problems in a class of entire functions of a finite degree", Studies of modern problems in approximation theory, Azerbaijan SSSR Academy of Sciences Press, Baku 1965, pp. 212–219. (Russian)

[6] V. I. Burenkov, "Embedding theorems and extensions for classes of differentiable functions of several variables defined throughout the space", Advances in Science, Mathematical Analysis, USSR Academy of Sciences, VINITI Press, Moscow 1966, pp. 71–155. (Russian)

[7] M. A. Krasnosel'skii and Ya. B. Rutitskii, *Convex functions and Orlicz spaces*, Fizmatgiz, Moscow 1958. (Russian)

[8] R. Adams, *Sobolev spaces*, Academic Press, New York 1975.

[9] W. Luxemburg, *Banach function spaces: Thesis*, Technische Hogeschool te Delft, The Netherlands 1955.

[10] R. O'Neil, "Fractional integration in Orlicz space. I", *Trans. Amer. Math. Soc.* **115** (1965), 300–328.

[11] S. L. Sobolev, *Some applications of functional analysis in mathematical physics*, Siberian Division of the USSR Academy of Sciences Press, Novosibirsk 1962. (Russian)

[12] L. Hörmander, *The analysis of linear partial differential operators*, vol. 1, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo 1983; Russian transl., Mir, Moscow 1986.

[13] V. S. Vladimirov, *Generalized functions in mathematical physics*, Nauka, Moscow 1976. (Russian)

[14] I. M. Gel'fand and G. E. Shilov, *Generalized functions. Issue* 2. *Spaces of test functions and distributions*, Fizmatgiz, Moscow 1958. (Russian)

[15] Ha Huy Bang and M. Marimoto, "The sequence of Luxemburg norms of derivatives", *Tokyo J. Math.* **17**:1 (1994), 141–147.

[16] Ha Huy Bang, "A property of infinitely differentiable functions", *Proc. Amer. Math. Soc.* **108**:1 (1990), 73–76.

[17] G. Szegö and A. Zygmund, "On certain mean values of polynomials", *J. Analyse Math.* **3** (1953), 225–244.

[18] R. J. Nessel and G. Wilmes, "Nikolskii-type inequalities in connection with regular spectral measures", *Acta Math.* **33** (1979), 169–182.

[19] R. J. Nessel and G. Wilmes, "Nikolskii-type inequalities for trigonometric polynomials and entire functions of exponential type", *J. Austral. Math. Soc.* **25** (1978), 7–18.

[20] H. Triebel, "General function spaces. II: Inequalities of Plancherel–Polya–Nikolskii type, L_p-spaces of analytic functions: $0 < p \leqslant \infty$", *J. Approximation Theory.* **19** (1977), 154–175.

[21] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel-Boston-Stuttgart 1983.

[22] V. A. Rodin, "Jackson and Nikol'skii inequalities for trigonometric polynomials in a symmetric space", Proceedings of the 7th Winter School (Dragobych, 1974), Moscow 1976, 133–139. (Russian)

[23] V. I. Ovchinnikov, "Interpolation theorems implied by the Grothendieck inequality", *Funktsional. Anal. i Prilozhen.* **10**:4 (1976), 45–54; English transl., *Functional Anal. Appl.* **10** (1976), 287–294.

[24] M. Z. Berkolaiko and V. I. Ovchinnikov, "Inequalities for entire functions of exponential type in symmetric spaces", *Trudy Mat. Inst. Steklov* **161** (1983), 3–17; English transl. in *Proc. Steklov Inst. Math.* **1984**, no. 3.

[25] Yu. A. Dubinskii, "Sobolev spaces of infinite order", *Uspekhi Mat. Nauk* **46**:6 (1991), 97–131; English transl., *Russian Math. Surveys* **46**:6 (1991), 107–147.

[26] Ju. Dubinskij, *Sobolev spaces of infinite order and differential equations*, Reidel, Dordrecht 1986.

[27] Tran Duc Van and Dinh Nho Hao, *Differential operators of infinite order with real arguments and their applications*, World Scientific, Singapore 1994.

[28] Yu. A. Dubinskii, "Sobolev spaces of infinite order and the behaviour of problems as the order of the equations increases indefinitely", *Mat. Sb.* **98**:2 (1975), 163–184; English transl. in *Math. USSR-Sb.* **27** (1975).

[29] Yu. A. Dubinskii, "Non-triviality of Sobolev spaces in the case of a complete Euclidean space and a torus", *Mat. Sb.* **100**:3 (1976), 436–446; English transl. in *Math. USSR-Sb.* **29** (1976).

[30] Chan Dyk Van, *Non-linear differential equations and function spaces of infinite order*, Byelorus. State Univ. Press, Minsk 1983. (Russian)

[31] Kha Zui Bang, "Criteria for non-triviality of classes and Sobolev–Orlicz spaces of infinite order in a complete Euclidean space", *Sibirsk. Mat. Zh.* **31**:1 (1990), 208–213; English transl. in *Siberian Math. J.* **31** (1990).

Institute of Mathematics, Hanoi, Vietnam

Received by 20/JUN/95
Translated by V. M. VOLOSOV
Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX