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STUDY OF THE MMPP/GI/∞ QUEUEING SYSTEM WITH RANDOM CUSTOMERS' CAPACITIES

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Abstract: A queueing system with an infinite number of servers is considered. Customers arrive in the system according to a Markov Modulated Poisson Process (MMPP). Each customer carries a random quantity of work (capacity of the customer). In this study, service time does not depend on the customers' capacities; the latter are used just to fix some additional features of the system's evolution. It is shown that the joint probability distribution of the customers' number and total capacities in the system is two-dimensional Gaussian under the asymptotic condition of an infinitely growing service time. Simulation results allow determining the applicability area of the asymptotic result.

Keywords: infinite-server queueing system; random capacity of customers; Markov Modulated Poisson Process

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1 Introduction

Queueing systems represent a powerful mathematical tool for investigating the performance of a wide variety of real-life systems, ranging from telecommunication networks to financial markets, from computer architectures to supply chain management and airplane traffic control, just to cite a few. Analytical tractability of the corresponding models strongly depends on the nature of the underlying processes (Poisson arrivals have many nice features that strongly simplify the analysis) and on the system geometry.

Although physical resources are always finite, quite often it is easier to study queueing systems in which the corresponding parameters assume infinite values. For instance, the overflow probability is often used as an upper bound for the loss probability in finite-buffer queues and, indeed, asymptotic results are available even for strongly non-Markovian systems [1]. Moreover, infinite-server queueing systems may be applicable in case of models with a limited number of server devices as described in [2].

In this work, an infinite-server queueing system, fed by non-Poisson arrivals with random customers' capacities, is considered. Queues with random customers' capacities are useful for analysis and design issues in high-performance computer and communication systems, in which service time and customer volume are the independent quantities (see [3, 4] and references therein). For instance, in [3], performance analysis of LTE (Long Term Evolution) networks is carried out in terms of flow-level dynamics and the amount of required radio resources does not depend on the duration of the flow. Such queues are also important in modeling devices, where it is necessary to calculate a sufficient volume of buffer for data storing [5, 6]. The results for single-server queues with limited buffer and LIFO (last in, first out) service discipline were presented in [7], where algorithms for the calculation of stationary characteristics were derived.

A new trend in the study of queueing systems is the analysis of the systems with non-Poisson arrivals and nonexponential service time. So, in the works [2, 8–11], queues with renewal arrivals, Markovian Arrival processes (MAP), and MMPP are studied under various asymptotic conditions. The main contribution of this paper consists in extending such analysis, focusing on the properties of the bidimensional process describing the number of customers and the total capacity in the system when an infinite-server queue is fed by MMPP arrivals with random capacities and nonexponential service time distribution.

2 Matematical Model

Consider a queue with infinite number of servers (Fig. 1) and assume that customers arrive according to an MMPP. The input process is defined by its generator matrix $\mathbf{Q} = ||q_{ij}||$ of size $K \times K$ and the conditional rates $\lambda_1, \ldots, \lambda_K$, typically composed into the diagonal matrix $\mathbf{\Lambda} = \text{diag}\,\{\lambda_1, \ldots, \lambda_K\}$. Denote the underlying

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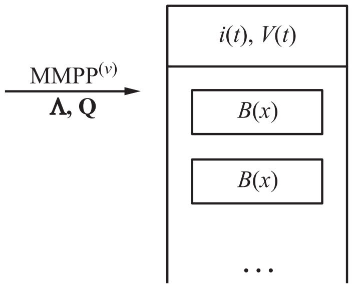

Figure 1 Queue MMPP/GI/∞ with random customers' capacities

Markov chain of the MMPP as $k(t) \in 1, 2, \ldots, K$. Let each customer has some random capacity $v > 0$ with distribution function $G(y)$. An arriving customer instantly occupies a server in the system and its service time has distribution function $B(x)$; when the service is completed, the customer leaves the system. Customers' capacities and service times are mutually independent and do not dependent on the epochs of customers' arrivals.

Denote by $i(t)$ and $V(t)$ the number of customers in the system at time t and their total capacity, respectively. Let us obtain the probabilistic characteristics of two-dimensional process $\{i(t), V(t)\}$. This process is not Markovian; therefore, the dynamic screening method has been used for its investigation.

Consider two time axes that are numbered as 0 and 1 (Fig. 2). Let axis 0 shows the epochs of customers' arrivals, while axis 1 corresponds to the screened process.

Let us introduce a function $S(t)$ (dynamic probability) that satisfies the condition $0 \leq S(t) \leq 1$. Let us assume that a customer, arriving in the system at time t, is screened to axis 1 with probability $S(t)$, and not screened with probability $1 - S(t)$.

Let the system be empty at moment t_0 and let us fix some arbitrary moment T in the future. $S(t)$ represents the probability that a customer arriving at time t will be serviced in the system by the moment T. It is easy to show [11] that $S(t) = 1 - B(T - t)$ for $t_0 \leq t \leq T$.

Denote by $n(t)$ and $W(t)$ the number of arrivals screened before the moment t on axis 1 and their total capacity, respectively. As it is shown in [9], the probability distribution of the number of customers in the system at the moment T coincides with the probability distribution of the number of screened arrivals on the axis:

$$P\{i(T) = m\} = P\{n(T) = m\}$$

for all $m = 0, 1, 2, \ldots$ It is easy to prove the same property for the extended process $\{i(t), V(t)\}$:

$$P\{i(T) = m, V(T) < z\} = P\{n(T) = m, W(T) < z\} \quad (1)$$

for all $m = 0, 1, 2, \ldots$ and $z \geq 0$. Let us use Eqs. (1) for the investigation of the process $\{i(t), V(t)\}$ via the analysis of the process $\{n(t), W(t)\}$.

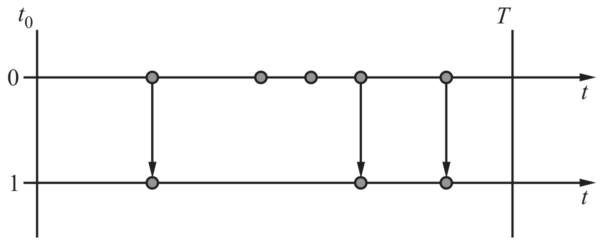

Figure 2 Screening of the customers' arrivals

3 Kolmogorov Differential Equations

Let us consider the three-dimensional Markovian process $\{k(t), n(t), W(t)\}$. Denoting the probability distribution of this process by $P(k, n, w, t) = P\{k(t) = k, n(t) = n, W(t) < w\}$ and taking into account the formula of total probability, one can write the following system of Kolmogorov differential equations:

$$\frac{\partial P(k, n, w, t)}{\partial t} = \lambda_k S(t) \left[\int_0^z P(k, n-1, w-y, t)\, dG(y) - P(k, n, w, t) \right] + \sum_v P(\nu, n, w, t) q_{\nu k}$$

for $k = 1, \ldots, K$; $n = 0, 1, 2, \ldots$; $w > 0$.

Let us introduce the partial characteristic function:

$$h(k, u_1, u_2, t) = \sum_{n=0}^{\infty} e^{ju_1 n} \int_0^{\infty} e^{ju_2 w} P(k, n, dw, t)$$

where $j = \sqrt{-1}$ is the imaginary unit. Then, one can write the following equations:

$$\frac{\partial h(k, u_1, u_2, t)}{\partial t} = h(k, u_1, u_2, t)\, \lambda_k S(t) \left(e^{ju_1} G^*(u_2) - 1 \right) + \sum_\nu h(\nu, n, w, t) q_{\nu k}$$

for $k = 1, \ldots, K$ where $G^*(u) = \int_0^\infty e^{juy} dG(y)$.

Let us rewrite this system in the matrix form:

$$\frac{\partial \mathbf{h}(u_1, u_2, t)}{\partial t} = \mathbf{h}(u_1, u_2, t) \left[\mathbf{\Lambda} S(t) \left(e^{ju_1} G^*(u_2) - 1 \right) + \mathbf{Q} \right] \quad (2)$$

with the initial condition

$$\mathbf{h}\left(u_1, u_2, t_0\right) = \mathbf{r} \tag{3}$$

where

$$\mathbf{h}\left(u_1, u_2, t\right) = \left[h\left(1, u_1, u_2, t\right), \ldots, h\left(K, u_1, u_2, t\right)\right]$$

and $\mathbf{r} = [r(1), \ldots, r(K)]$ represents the stationary distribution of the underlying Markov chain, i. e., vector $\mathbf{r}$ satisfies the following linear system:

$$\left.\begin{aligned} \mathbf{rQ} &= \mathbf{0}\,; \\ \mathbf{re} &= 1 \end{aligned}\right\} \tag{4}$$

where $\mathbf{e}$ is the column vector with all entries equal to 1.

4 Asymptotic Analysis

In general, the exact solution of Equation (2) is not available, but it may be found under asymptotic conditions. In this paper, the case of infinitely growing service time is considered.

Denote by

$$b_1 = \int\limits_0^\infty x dB(x) = \int\limits_0^\infty (1 - B(x))\, dx$$

the mean service time; then, the asymptotic condition is $b_1 \to \infty$.

Let us solve Problem (2)–(3) under such asymptotic condition and we obtain approximate solutions with different order of accuracy, named as "first-order asymptotic" $\mathbf{h}(u_1, u_2, t) \approx \mathbf{h}^{(1)}(u_1, u_2, t)$ and "second-order asymptotic" $\mathbf{h}(u_1, u_2, t) \approx \mathbf{h}^{(2)}(u_1, u_2, t)$.

4.1 First-order asymptotic analysis

Let us formulate and prove the following statement.

Lemma. *The first-order asymptotic characteristic function of the probability distribution of the process* $\{k(t), n(t), W(t)\}$ *has the form*:

$$\mathbf{h}^{(1)}(u_1, u_2, t) = \mathbf{r} \exp\left\{(ju_1\kappa_1 + ju_2\kappa_1 a_1)\int\limits_{t_0}^{t} S(v)\, dv\right\}$$

where $\kappa_1 = \mathbf{r\Lambda e}$ *and* $a_1 = \int_0^\infty y dG(y)$ *is the mean customer capacity*.

P r o o f. By performing the substitutions

$$\varepsilon = \frac{1}{b_1}\,; \quad \varepsilon t = \tau\,; \quad \varepsilon t_0 = \tau_0\,;$$

$$u_1 = \varepsilon x_1\,; \;\; u_2 = \varepsilon x_2\,; \;\; S(t) = S_1(\tau)\,;$$

$$\mathbf{h}(u_1, u_2, t) = \mathbf{f}_1(x_1, x_2, \tau, \varepsilon)$$

in expressions (2) and (3), one obtains

$$\varepsilon \frac{\partial \mathbf{f}_1(x_1, x_2, \tau, \varepsilon)}{\partial \tau} = \mathbf{f}_1(x_1, x_2, \tau, \varepsilon)\left[\mathbf{\Lambda} S_1(\tau)\left(e^{j\varepsilon x_1} G^*(\varepsilon x_2) - 1\right) + \mathbf{Q}\right] \tag{5}$$

with the initial condition

$$\mathbf{f}_1(x_1, x_2, \tau_0, \varepsilon) = \mathbf{r}\,. \tag{6}$$

Let us find the asymptotic solution of Problem (5)–(6) $\mathbf{f}_1(x_1, x_2, \tau) = \lim_{\varepsilon\to 0} \mathbf{f}_1(x_1, x_2, \tau, \varepsilon)$ in two steps.

Step 1. Let $\varepsilon \to 0$ in (5)–(6); then, one obtains the following system of equations:

$$\left\{\begin{aligned} &\mathbf{f}_1\left(x_1, x_2, \tau\right)\mathbf{Q} = \mathbf{0}\,; \\ &\mathbf{f}_1\left(x_1, x_2, \tau_0\right) = \mathbf{r}\,. \end{aligned}\right.$$

Taking into account (4), one can conclude that $\mathbf{f}_1(x_1, x_2, \tau)$ can be expressed as

$$\mathbf{f}_1(x_1, x_2, \tau) = \mathbf{r}\Phi_1(x_1, x_2, \tau) \tag{7}$$

where $\Phi_1(x_1, x_2, \tau)$ is some scalar function which satisfies the condition

$$\Phi_1(x_1, x_2, \tau_0) = 1\,. \tag{8}$$

Step 2. Let us multiply (5) by vector $\mathbf{e}$, substitute (7), divide the result by ε, and perform the asymptotic transition $\varepsilon \to 0$. Then, taking into account that $\mathbf{Qe} = \mathbf{0}$ and $\mathbf{re} = 1$, one obtains the following differential equation for the function $\Phi_1(x_1, x_2, \tau)$:

$$\frac{\partial \Phi_1(x_1, x_2, \tau)}{\partial \tau} = \Phi_1\left(x_1, x_2, \tau\right) S_1(\tau)\left(jx_1\kappa_1 + jx_2\kappa_1 a_1\right)\,. \tag{9}$$

The solution of Problem (8)–(9) is as follows:

$$\Phi_1(x_1, x_2, \tau) = \exp\left\{(jx_1\kappa_1 + jx_2\kappa_1 a_1)\int\limits_{\tau_0}^{\tau} S_1(v)\, dv\right\}.$$

Substituting this expression into (7), one obtains

$$\mathbf{f}_1(x_1, x_2, \tau) = \mathbf{r}\exp\left\{(jx_1\kappa_1 + jx_2\kappa_1 a_1)\int\limits_{\tau_0}^{\tau} S_1(v)\, dv\right\}.$$

Therefore, one can write

$$\begin{aligned} \mathbf{h}^{(1)}(u_1, u_2, t) &= \mathbf{f}_1\left(x_1, x_2, \tau, \varepsilon\right) \approx \mathbf{f}_1\left(x_1, x_2, \tau\right) \\ &= \mathbf{r}\exp\left\{(jx_1\kappa_1 + jx_2\kappa_1 a_1)\int\limits_{\tau_0}^{\tau} S_1(v)dv\right\} \\ &= \mathbf{r}\exp\left\{(ju_1\kappa_1 + ju_2\kappa_1 a_1)\int\limits_{t_0}^{t} S(v)\, dv\right\}. \end{aligned}$$

Thus, the proof is complete.

4.2 Second-order asymptotic analysis

The main result is the following theorem.

Theorem. *The second-order asymptotic characteristic function of the probability distribution of the process* $\{k(t), n(t), W(t)\}$ *has the form*:

$$\mathbf{h}^{(2)}(u_1,u_2,t) = \mathbf{r}\exp\left\{(ju_1\kappa_1 + ju_2\kappa_1 a_1)\int_{t_0}^{t} S(v)\,dv + \frac{(ju_1)^2}{2}\left(\kappa_1\int_{t_0}^{t} S(v)\,dv + \kappa_2\int_{t_0}^{t} S^2(v)\,dv\right) + \frac{(ju_2)^2}{2}\left(\kappa_1 a_2\int_{t_0}^{t} S(v)\,dv + \kappa_2 a_1^2\int_{t_0}^{t} S^2(v)\,dv\right) + ju_1 ju_2\left(\kappa_1 a_1\int_{t_0}^{t} S(v)\,dv + \kappa_2 a_1\int_{t_0}^{t} S^2(v)\,dv\right)\right\}$$

where $\kappa_2 = 2\mathbf{g}(\mathbf{\Lambda} - \kappa_1\mathbf{I})\mathbf{e}$*;* $a_2 = \int_0^\infty y^2 dG(y)$*; and the row vector* $\mathbf{g}$ *satisfies the linear matrix system*

$$\begin{cases} \mathbf{gQ} = \mathbf{r}(\kappa_1\mathbf{I} - \mathbf{\Lambda})\,; \\ \mathbf{ge} = const\,. \end{cases}$$

P r o o f. Let $\mathbf{h}_2(x_1,x_2,t)$ be a vector function that satisfies the equation:

$$\mathbf{h}(u_1,u_2,t) = \mathbf{h}_2(u_1,u_2,t) \times \exp\left\{(ju_1\kappa_1 + ju_2\kappa_1 a_1)\int_{t_0}^{t} S(v)\,dv\right\}. \quad (10)$$

Substituting this expression into (2) and (3), one obtains the following problem:

$$\frac{\partial \mathbf{h}_2(u_1,u_2,t)}{\partial t} = \mathbf{h}_2(u_1,u_2,t)\left[(e^{ju_1}G^*(u_2) - 1)S(t)\mathbf{\Lambda} - (ju_1\kappa_1 + ju_2\kappa_1 a_1)S(t)\mathbf{I} + \mathbf{Q}\right] \quad (11)$$

with the initial condition

$$\mathbf{h}_2(u_1,u_2,t_0) = \mathbf{r} \quad (12)$$

where $\mathbf{I}$ is the identity matrix.

Let us make the substitutions:

$$\left.\begin{matrix} \varepsilon^2 = \dfrac{1}{b_1}\,; \quad \varepsilon^2 t = \tau\,; \quad \varepsilon^2 t_0 = \tau_0\,; \\ u_1 = \varepsilon x_1\,; \quad u_2 = \varepsilon x_2\,; \quad S(t) = S_1(t)\,; \\ \mathbf{h}_2(u_1,u_2,t) = \mathbf{f}_2(x_1,x_2,\tau,\varepsilon)\,. \end{matrix}\right\} \quad (13)$$

Using these notations, Problem (11)–(12) can be rewritten in the form

$$\varepsilon^2\frac{\partial \mathbf{f}_2(x_1,x_2,\tau,\varepsilon)}{\partial \tau} = \mathbf{f}_2(x_1,x_2,\tau,\varepsilon)\left[\mathbf{\Lambda}S_1(\tau)(e^{j\varepsilon x_1}G^*(\varepsilon x_2) - 1) - (j\varepsilon\kappa_1 x_1 + j\varepsilon\kappa_1 x_2 a_1)S_1(\tau)\mathbf{I} + \mathbf{Q}\right] \quad (14)$$

with the initial condition

$$\mathbf{f}_2(x_1,x_2,\tau_0,\varepsilon) = \mathbf{r}\,. \quad (15)$$

Let us find the asymptotic solution of this problem $\mathbf{f}_2(x_1,x_2,\tau) = \lim\limits_{\varepsilon\to 0}\mathbf{f}_2(x_1,x_2,\tau,\varepsilon)$ in three steps.

Step 1. Letting $\varepsilon \to 0$ in (14)–(15), one obtains the following system of equations:

$$\begin{cases} \mathbf{f}_2(x_1,x_2,\tau)\,\mathbf{Q} = \mathbf{0}\,; \\ \mathbf{f}_2(x_1,x_2,\tau_0) = \mathbf{r}\,. \end{cases}$$

Therefore, taking into account (4), one can write:

$$\mathbf{f}_2(x_1,x_2,\tau) = \mathbf{r}\Phi_2(x_1,x_2,\tau) \quad (16)$$

where $\Phi_2(x_1,x_2,\tau)$ is some scalar function which satisfies the condition

$$\Phi_2(x_1,x_2,\tau_0) = 1\,. \quad (17)$$

Step 2. Using (16), the function $\mathbf{f}_2(x_1,x_2,\tau)$ can be represented in the expansion form:

$$\mathbf{f}_2(x_1,x_2,\tau,\varepsilon) = \Phi_2(x_1,x_2,\tau)\left[\mathbf{r} + \mathbf{g}S_1(\tau)(j\varepsilon x_1 + j\varepsilon x_2 a_1)\right] + \mathbf{O}(\varepsilon^2) \quad (18)$$

where $\mathbf{g}$ is the row vector that satisfies the condition $\mathbf{ge} = const$ and $\mathbf{O}(\varepsilon^2)$ is the row vector of the second-order infinitesimals. Let us use substitution (18) and the expansion

$$e^{j\varepsilon x} = 1 + j\varepsilon x + O\left(\varepsilon^2\right)$$

in Eq. (14). Taking into account (4) and making the transition $\varepsilon \to 0$, one obtains the following matrix equation for the vector $\mathbf{g}$:

$$\mathbf{gQ} = \mathbf{r}\left(\kappa_1\mathbf{I} - \mathbf{\Lambda}\right)\,.$$

Step 3. Let us multiply Eq. (14) by vector $\mathbf{e}$ and use expression (18) and the second-order expansion:

$$e^{j\varepsilon x} = 1 + j\varepsilon x + \frac{(j\varepsilon x)^2}{2} + O\left(\varepsilon^3\right)\,.$$

After some transformations, using the notation

$$\kappa_2 = 2\mathbf{g}\left(\mathbf{\Lambda} - \kappa_1 \mathbf{I}\right)\mathbf{e}\,,$$

one obtains the following differential equation for the function $\Phi_2(x_1, x_2, \tau)$:

$$\frac{\partial \Phi_2(x_1,x_2,\tau)}{\partial \tau} = \Phi_2(x_1,x_2,\tau)\left[\frac{(jx_1)^2}{2}\left(\kappa_1 S_1(\tau) + \kappa_2 S_1^2(\tau)\right) + \frac{(jx_2)^2}{2}\left(\kappa_1 a_2 S_1(\tau) + \kappa_2 a_1^2 S_1^2(\tau)\right) + jx_1 jx_2\left(\kappa_1 a_1 S_1(\tau) + \kappa_2 a_1 S_1^2(\tau)\right)\right].$$

The solution of this equation with initial condition (17) is as follows:

$$\Phi_2(x_1,x_2,\tau) = \exp\left\{\frac{(jx_1)^2}{2}\left(\kappa_1\int_{\tau_0}^{\tau} S_1(v)\,dv + \kappa_2\int_{\tau_0}^{\tau} S_1^2(v)\,dv\right) + \frac{(jx_2)^2}{2}\left(\kappa_1 a_2\int_{\tau_0}^{\tau} S_1(v)\,dv + \kappa_2 a_1^2\int_{\tau_0}^{\tau} S_1^2(v)\,dv\right) + jx_1 jx_2\left(\kappa_1 a_1\int_{\tau_0}^{\tau} S_1(v)\,dv + \kappa_2 a_1\int_{\tau_0}^{\tau} S_1^2(v)\,dv\right)\right\}.$$

Substituting this expression in formula (16) and performing the substitutions that are inverse to (13) and (10), one obtains

$$\mathbf{h}^{(2)}(u_1,u_2,t) = \mathbf{r}\exp\left\{(ju_1\kappa_1 + ju_2\kappa_1 a_1)\int_{t_0}^{t} S(v)\,dv + \frac{(ju_1)^2}{2}\left(\kappa_1\int_{t_0}^{t} S(v)\,dv + \kappa_2\int_{t_0}^{t} S^2(v)\,dv\right) + \frac{(ju_2)^2}{2}\left(\kappa_1 a_2\int_{t_0}^{t} S(v)\,dv + \kappa_2 a_1^2\int_{t_0}^{t} S^2(v)\,dv\right) + ju_1 ju_2\left(\kappa_1 a_1\int_{t_0}^{t} S(v)\,dv + \kappa_2 a_1\int_{t_0}^{t} S^2(v)\,dv\right)\right\}$$

for the asymptotic characteristic function of the process $\{k(t), n(t), W(t)\}$. The proof is complete.

Corollary. Assuming $t = T$ and $t_0 \to -\infty$ and using Eqs. (1), one obtains the steady-state characteristic function of the process under study $\{i(t), V(t)\}$:

$$h(u_1,u_2) = \exp\left\{(ju_1\kappa_1 b_1 + ju_2\kappa_1 a_1 b_1) + \frac{(ju_1)^2}{2}(\kappa_1 b_1 + \kappa_2 b_2) + \frac{(ju_2)^2}{2}\left(\kappa_1 a_2 b_1 + \kappa_2 a_1^2 b_2\right) + ju_1 ju_2(\kappa_1 a_1 b_1 + \kappa_2 a_1 b_2)\right\} \quad (19)$$

where

$$b_1 = \int_0^\infty (1 - B(v))\,dv\,;\quad b_2 = \int_0^\infty (1 - B(v))^2\,dv\,.$$

From the form of the characteristic function (19), it is clear that the probability distribution of the two-dimensional process $\{i(t), V(t)\}$ is asymptotically Gaussian with vector of means

$$\mathbf{a} = \begin{bmatrix} \kappa_1 b_1 & \kappa_1 a_1 b_1 \end{bmatrix}$$

and covariance matrix

$$\mathbf{K} = \begin{bmatrix} \sigma_1^2 & K_{12} \\ K_{12} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \kappa_1 b_1 + \kappa_2 b_2 & \kappa_1 a_1 b_1 + \kappa_2 a_1 b_2 \\ \kappa_1 a_1 b_1 + \kappa_2 a_1 b_2 & \kappa_1 a_2 b_1 + \kappa_2 a_1^2 b_2 \end{bmatrix}.$$

Therefore, the correlation coefficient is given by

$$r = \frac{K_{12}}{\sigma_1\sigma_2} = \frac{\kappa_1 a_1 b_1 + \kappa_2 a_1 b_2}{\sqrt{\kappa_1 b_1 + \kappa_2 b_2}\,\sqrt{\kappa_1 a_2 b_1 + \kappa_2 a_1^2 b_2}}\,.$$

5 Numerical Example

Result (19) is obtained under the asymptotic condition $b_1 \to \infty$. Therefore, it may be used just as an approximation when b_1 is large enough. To test its practical applicability, the present authors considered several numerical examples, varying all the system parameters (including the distributions of the service time and of the customer capacity). Since all the different simulation sets led to similar results, for sake of brevity, in the following, just one of them is discussed in detail. In particular, let us assume that the input MMPP is characterized by the matrices:

$$\mathbf{Q} = \begin{bmatrix} -0.8 & 0.4 & 0.4 \\ 0.3 & -0.6 & 0.3 \\ 0.4 & 0.4 & -0.8 \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}.$$

Table 1 Kolmogorov distances between simulation results and asymptotic values for the number of customers in the system

N	Δ
1	0.265
10	0.039
15	0.032
20	**0.027**
25	**0.025**
50	**0.017**
100	**0.012**

Table 2 Kolmogorov distances between simulation results and asymptotic values for the total capacity in the system

N	Δ
1	0.355
10	0.033
15	**0.025**
20	**0.021**
25	**0.019**
50	**0.013**
100	**0.010**

Hence, the fundamental rate of arrivals is $\kappa_1 = \mathbf{r}\boldsymbol{\Lambda}\mathbf{e} = 1$ customers per time unit. Let us also assume that customers' capacities have uniform distribution in the range [0; 1] and service time has gamma distribution with shape and inverse scale parameters $\alpha = 1.5$ and $\beta = \alpha/N$, respectively. So, when $N \to \infty$, one obtains the asymptotic condition of an infinite growing service time ($b_1 = \alpha/\beta = N \to \infty$).

The goal is to find a lower bound of parameter N for the applicability of the approximation (19). To this aim, series of simulation experiments have been carried out for increasing values of N and the asymptotic distributions have been compared with the empiric ones by using the Kolmogorov distance [12, 13]

$$\Delta = \sup_x |F(x) - A(x)|$$

as an accuracy measure. Here, $F(x)$ is the cumulative distribution function built on the basis of simulation results and $A(x)$ is the Gaussian approximation based on (19).

Let us consider the marginal distributions of the customers' number and the total capacity in the system.

In the first case, the asymptotic values of mean and variance are equal to N and $1.144N$, respectively, and the corresponding values of the Kolmogorov distance for increasing values of parameter N are presented in Table 1. Similarly, for the total capacity in the system, mean and variance are equal to $0.5N$ and $0.369N$, respectively, and Table 2 shows the Kolmogorov distance.

One can notice that the asymptotic results become more accurate when the parameter N increases. Fig-

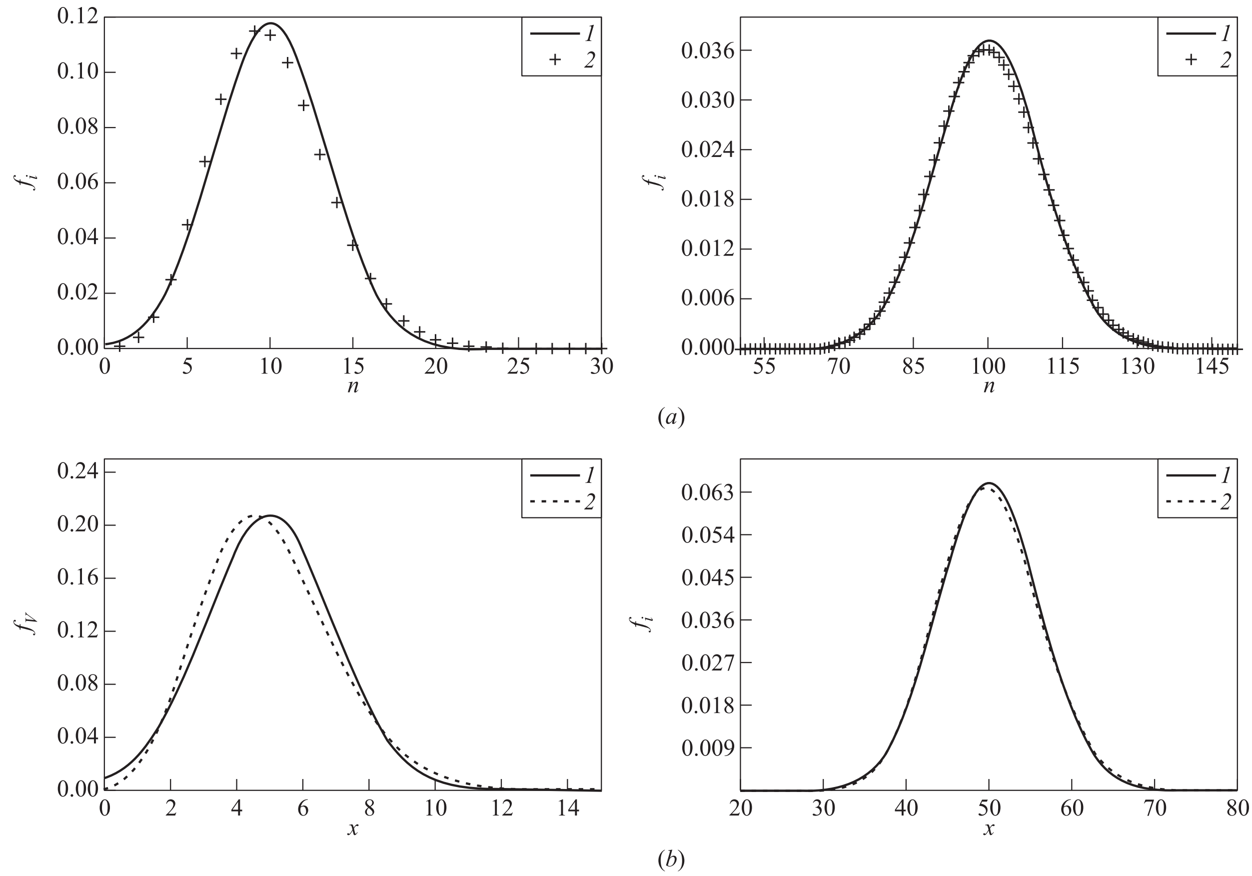

Figure 3 Distributions of the number of customers (*a*) and of the total capacity (*b*) for different values of N: left column — $N = 10$; right column — $N = 100$; *1* — theoretical results; and *2* — simulation

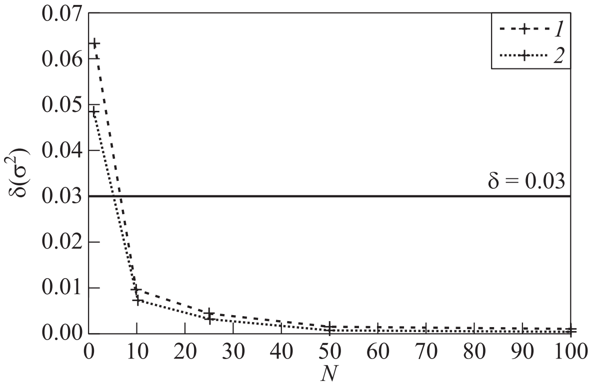

Figure 4 Relative error for the variance of the number of customers $i(t)$ (*1*) and the total capacity $V(t)$ (*2*)

ure 3 compares the asymptotic approximations with the empirical results for the number of customers and the total capacity in the system.

As typically done in the literature [12], let us suppose that an approximation is applicable if its Kolmogorov distance is less than 0.03. Hence, one can conclude that the asymptotic results are applicable for values of the parameter N equal to 15 or more (marked by boldface in Tables 1 and 2).

Then, let us compare the asymptotic value of some characteristics of the queueing system with the corresponding empirical characteristics, using the relative error

$$\delta = \frac{|d-a|}{d}$$

where d denotes the value constructed on the basis of simulation results and a is obtained from (19).

In more detail, the mean values of the processes $i(t)$ and $V(t)$ are very close (with $\delta < 10^{-5}$ for all N) and the relative errors of the variance decreases with N as shown in Fig. 4.

Finally, Table 3 shows the relative error for the correlation coefficient.

Table 3 Relative error for the correlation coefficient

N	δ
1	$\mathbf{60 \cdot 10^{-4}}$
10	$\mathbf{11 \cdot 10^{-4}}$
15	$\mathbf{7 \cdot 10^{-4}}$
20	$\mathbf{5 \cdot 10^{-4}}$
25	$\mathbf{4 \cdot 10^{-4}}$
50	$\mathbf{1 \cdot 10^{-4}}$
100	$\mathbf{0.8 \cdot 10^{-4}}$

6 Concluding Remarks

In the paper, the queue with MMPP arrivals, infinite number of servers, and nonexponential service time is considered. Moreover, random customers' capacities, independent of their service time, are assumed. The analysis is performed under the asymptotic condition of an infinitely growing service time. It is shown that two-dimensional probability distribution of customers' number and total capacity in the system is two-dimensional Gaussian under this asymptotic condition. Numerical results show that asymptotic results have enough accuracy for the marginal distributions of number of customers and of the total capacity in the system when the service rate exceeds the fundamental rate of arrivals by at least 15 times.

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ИССЛЕДОВАНИЕ СИСТЕМЫ МАССОВОГО ОБСЛУЖИВАНИЯ MMPP/GI/∞ С ТРЕБОВАНИЯМИ СЛУЧАЙНОГО ОБЪЕМА*

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Аннотация: Проведено исследование системы массового обслуживания с неограниченным числом приборов. Заявки поступают в систему в виде марковски-модулированного пуассоновского потока. Каждая заявка несет в себе произвольное количество данных (объем заявки). В этом исследовании время обслуживания не зависит от объема заявок. Показано, что совместное распределение вероятностей числа заявок в системе и их суммарного объема является двумерным гауссовским при асимптотическом условии растущего времени обслуживания. Имитационное моделирование и численные эксперименты позволили определить область применимости асимптотического результата.

Ключевые слова: бесконечнолинейная система массового обслуживания; случайный объем заявок; MMPP-поток

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