Sh. Bhaumik, S. K. Tiwari, General coupled fixed point theorem for a nonlinear contractive condition in a cone metric space, *Eurasian Math. J.*, 2018, том 9, номер 3, 25–32

DOI: 10.32523/2077-9879-2018-9-3-25-32

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 9, Number 3 (2018), 25 – 32

GENERAL COUPLED FIXED POINT THEOREM FOR A NONLINEAR CONTRACTIVE CONDITION IN A CONE METRIC SPACE

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Communicated by E.D. Nursultanov

Key words: coupled fixed point, cone metric space, fixed point theorem.

AMS Mathematics Subject Classification: 47H10, 54H25, 37C25, 55M20, 58C30.

Abstract. The existence and uniqueness of coupled fixed point theorem has been proved under various contractive condition in a cone metric space. The result is verified with the help of suitable example.

DOI: https://doi.org/10.32523/2077-9879-2018-9-3-25-32

1 Introduction

In 2006, the concept of a coupled fixed point was given by Bhaskar and Lakshmikantham in [2]. After that several authors proved various coupled and common coupled fixed point theorems in a partially ordered metric space, G-metric space, b-metric space, fuzzy metric space, cone metric space etc. Some of the works are noted in [7], [16], [19], [20].

In 2007, Huang and Zhang [9] introduced the concept of a cone metric space, where they generalized a metric space by replacing the set of real numbers with an ordered Banach space. Thus, the cone naturally induces a partial order in Banach space. Some of the works are noted in [12], [13], [17], [18] etc.

The aim of this paper is to establish the existence and uniqueness of coupled fixed point theorem satisfying some generalized contractive condition in a cone metric space. In this paper we do not impose the normality condition on the cone. The only assumption is that a cone P has a nonempty interior.

Definition 1. [9] Let E be a real Banach space and P be a subset of E. P is called a cone if
(i) P is nonempty, closed, and $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$; $x, y \in P \Longrightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Longrightarrow x = 0$.
Given a cone $P \subset E$, the partial ordering $\leq$ with respect to P is naturally defined for $x, y \in E$ by $x \leq y$ if and only if $x - y \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$, where int P denotes the interior of P.

A cone P is said to be normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \longrightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

A cone P is called regular if every increasing sequence which is bounded from above is convergent; that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$. Equivalently, a cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the sequel we always suppose that E is a real Banach space with a cone P in E such that int $P \neq \phi$ and $\leq$ is the partial ordering with respect to P.

Definition 2. [9] Let X be a nonempty set. Let a mapping $d : X \times X \longrightarrow E$ satisfy:
(i) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.
Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 3. [9] Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$.
(i) If for every $c \in E$ with $0 \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. This limit is denoted by $\lim_n x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

(ii) If for every $c \in E$ with $0 \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

(iii) If every Cauchy sequence in X is convergent in X, then X is called a complete cone metric space.

(iv) If P is a normal cone, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.

Lemma 1.1. *[4] If P is a normal cone in E, then the following statements hold.*
(i) If $x, y \in E$, $0 \leq x \leq y$ and $b \geq 0$, where b is a real number, then $0 \leq bx \leq by$.

(ii) If $x_n, y_n \in E$, $0 \leq x_n \leq y_n$ for $n \in \mathbb{N}$, $x, y \in E$ and $\lim_n \{x_n\} = x, \lim_n \{y_n\} = y$, then $0 \leq x \leq y$.

Lemma 1.2. *[4] Let P be a cone in E and $a, b, c \in E$.*
(i) If $a \leq b$ and $b \ll c$, then $a \ll c$.
(ii) If $a \ll b$ and $b \ll c$, then $a \ll c$.

Definition 4. [19] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if $x = F(x, y)$ and $y = F(y, x)$.

2 Main results

Theorem 2.1. *Let (X, d) be a complete cone metric space with a cone P having nonempty interior. Let $F : X \times X \longrightarrow X$ satisfy the inequality*

$$\begin{aligned} d(F(x,y), F(u,v)) \leq\ & l_1 max[d(x,u), d(x, F(x,y))] + l_2 max[d(y,v), d(y, F(y,x))] \\ & + l_3 max[d(u, F(x,y)), d(u, F(u,v))], \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$, where l_1, l_2, l_3 are non-negative real numbers such that $l_1 + l_2 + l_3 < 1$. Then $F(x, y)$ has a unique coupled fixed point in $X \times X$.

Proof. Let x_0 and y_0 be two arbitrary points in X.
Let

$$x_{2k+1} = F(x_{2k}, y_{2k}), \qquad y_{2k+1} = F(y_{2k}, x_{2k}).$$

and

$$x_{2k+2} = F(x_{2k+1}, y_{2k+1}), \qquad y_{2k+2} = F(y_{2k+1}, x_{2k+1}).$$

for $k = 0, 1, 2, 3, \cdots$. Then

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(F(x_{2k}, y_{2k}), F(x_{2k+1}, y_{2k+1})) \\ &\leq l_1 \max[d(x_{2k}, x_{2k+1}), d(x_{2k}, F(x_{2k}, x_{2k}))] \\ &+ l_2 \max[d(y_{2k}, y_{2k+1}), d(y_{2k}, F(y_{2k}, x_{2k}))] \\ &+ l_3 \max[d(x_{2k+1}, F(x_{2k}, y_{2k}), d(x_{2k+1}, F(x_{2k+1}, y_{2k+1}))] \\ &= l_1 \max[d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1})] + l_2 \max[d(y_{2k}, y_{2k+1}), d(y_{2k}, y_{2k+1})] \\ &+ l_3 \max[d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})] \\ &= l_1 d(x_{2k}, x_{2k+1}) + l_2 d(y_{2k}, y_{2k+1}) + l_3 d(x_{2k+1}, x_{2k+2}). \end{aligned} \tag{2.2}$$

Hence

$$d(x_{2k+1}, x_{2k+2}) \leq \frac{l_1}{1 - l_3} d(x_{2k}, x_{2k+1}) + \frac{l_2}{1 - l_3} d(y_{2k}, y_{2k+1}). \tag{2.3}$$

Similarly, we have

$$\begin{aligned} d(y_{2k+1}, y_{2k+2}) &= d(F(y_{2k}, x_{2k}), F(y_{2k+1}, x_{2k+1})) \\ &\leq l_1 \max[d(y_{2k}, y_{2k+1}), d(y_{2k}, F(y_{2k}, x_{2k}))] \\ &+ l_2 \max[d(x_{2k}, x_{2k+1}), d(x_{2k}, F(x_{2k}, y_{2k}))] \\ &+ l_3 \max[d(y_{2k+1}, F(y_{2k}, x_{2k}), d(y_{2k+1}, F(y_{2k+1}, x_{2k+1}))] \\ &= l_1 \max[d(y_{2k}, y_{2k+1}), d(y_{2k}, y_{2k+1})] + l_2 \max[d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1})] \\ &+ l_3 \max[d(y_{2k+1}, y_{2k+1}), d(y_{2k+1}, y_{2k+2})] \\ &= l_1 d(y_{2k}, y_{2k+1}) + l_2 d(x_{2k}, x_{2k+1}) + l_3 d(y_{2k+1}, y_{2k+2}). \end{aligned} \tag{2.4}$$

Hence

$$d(y_{2k+1}, y_{2k+2}) \leq \frac{l_1}{1 - l_3} d(y_{2k}, y_{2k+1}) + \frac{l_2}{1 - l_3} d(x_{2k}, x_{2k+1}). \tag{2.5}$$

Adding (2.3) and (2.5) we get

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) &\leq \frac{l_1 + l_2}{1 - l_3} d(x_{2k}, x_{2k+1}) + \frac{l_2 + l_2}{1 - l_3} d(y_{2k}, y_{2k+1}) \\ &\leq \frac{l_1 + l_2}{1 - l_3} [d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \\ &= r[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})], \end{aligned} \tag{2.6}$$

where

$$0 < r = \frac{l_1 + l_2}{1 - l_3} < 1.$$

Similarly to (2.6) we have

$$d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) \leq r[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})].$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\le r[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] \\ &\le r^2[d(x_{n-2}, x_{n-1}) + d(y_{n-2}, y_{n-1})] \\ &\cdots\cdots\cdots \\ &\le r^n[d(x_0, x_1) + d(y_0, y_1)] \end{aligned}$$

Now if we take

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = \beta_n$$

then,

$$\beta_n \le r\beta_{n-1} \le \cdots \le r^n \beta_0$$

For $m > n$ we have

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) &\le \beta_{m-1} + \beta_{m-2} + \cdots + \beta_n \\ &\le (r^{m-1} + r^{m-2} + \cdots + r^n)\beta_0 \\ &= r^n(1 + r + \cdots + r^{m-n-1})\beta_0 \\ &\le \frac{r^n}{1-r}\beta_0 \end{aligned} \tag{2.7}$$

Consequently,

$$\begin{aligned} d(x_n, x_m) &\le \frac{r^n}{1-r}\beta_0 \\ d(y_n, y_m) &\le \frac{r^n}{1-r}\beta_0 \end{aligned} \tag{2.8}$$

Let $0 \ll c$ be given. Choose a natural number N such that $\frac{r^n}{1-r}\beta_0 \ll c$ for $n > N$. Thus $d(x_n, x_m) \ll c$ and $d(y_n, y_m) \ll c$ for $m > n$. Therefore both the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

From the completeness of X, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

Now we prove that $F(x,y) = x$ and $F(y,x) = y$. Substituting $x_{2k+1} = F(x,y)$ and $y_{2k+1} = F(y,x)$ in (2.6), we obtain

$$d(F(x,y), x_{2k+2}) + d(F(y,x), y_{2k+2}) \le r[d(x_{2k}, F(x,y)) + d(y_{2k}, F(y,x))]$$

Letting $k \longrightarrow \infty$ we obtain

$$d(F(x,y), x) + d(F(y,x), y) \le r[d(x, F(x,y)) + d(y, F(y,x))]$$

or

$$(1-r)[d(F(x,y), x) + d(F(y,x), y)] \le 0.$$

Since $0 < r < 1$

$$d(F(x,y), x) + d(F(y,x), y) \le 0.$$

Since $d(F(x,y), x) \ge 0$, $d(F(y,x), y) \ge 0$, it follows that

$$d(F(x,y),x) = 0 \text{ and } d(F(y,x),y) = 0,$$

Hence

$$F(x,y) = x \text{ and } F(y,x) = y.$$

In order to prove the uniqueness let (x',y') be another point of $X \times X$ such that $x' = F(x',y')$ and $y' = F(y',x')$. Then

$$\begin{aligned} d(x,x') &= d(F(x,y),F(x',y')) \\ &\leq l_1\max[d(x,x'),d(x,F(x,y))] + l_2\max[d(y,y'),d(y,F(y,x))] \\ &+ l_3\max[d(x',F(x,y)),d(x',F(x',y'))] \\ &= l_1\max[d(x,x'),d(x,x)] + l_2\max[d(y,y'),d(y,y)] \\ &+ l_3\max[d(x',x),d(x',x')] \\ &= l_1 d(x,x') + l_2 d(y,y') + l_3 d(x',x). \end{aligned}$$

Hence

$$d(x,x') \leq \frac{l_2}{1-l_1-l_3} d(y,y'). \tag{2.9}$$

Similarly, we can prove that

$$d(y,y') \leq \frac{l_2}{1-l_1-l_3} d(x,x'). \tag{2.10}$$

Adding (2.9) and (2.10) we get

$$d(x,x') + d(y,y') \leq \frac{l_2}{1-l_1-l_3}[d(x,x') + d(y,y')],$$

or

$$(1 - \frac{l_2}{1-l_1-l_3})[d(x,x') + d(y,y')] \leq 0,$$

which implies

$$d(x,x') + d(y,y') = 0,$$

hence $\quad d(x,x') + d(y,y') = 0 \iff x = x' \quad$ and $\quad y = y'$. □

If we take $l_3 = 0$ in Theorem 2.1. we get the following results as corollaries.

Corollary 2.1. *Let (X,d) be a complete cone metric space with a cone P having nonempty interior. Let $F : X \times X \longrightarrow X$ satisfy the inequality*

$$d(F(x,y),F(u,v)) \leq l_1 max[d(x,u),d(x,F(x,y))] + l_2 max[d(y,v),d(y,F(y,x))]$$

for all $x,y,u,v \in X$, where l_1, l_2, are non-negative real numbers such that $l_1 + l_2 < 1$.

Then $F(x,y)$ has a unique coupled fixed point in $X \times X$.

Corollary 2.2. *Let (X,d) be a complete cone metric space with a cone P having nonempty interior. Let $F : X \times X \longrightarrow X$ be such that $max[d(x,u),d(x,F(x,y))] = d(x,u)$ and $max[d(y,v),d(y,F(y,x))] = d(y,v)$ and satisfy the inequality*

$$d(F(x,y),F(u,v)) \leq l_1 d(x,u) + l_2 d(y,v)$$

for all x, y, u, $v \in X$, where l_1, l_2, are non-negative real numbers such that $l_1 + l_2 < 1$.

Then $F(x,y)$ has a unique coupled fixed point in $X \times X$.

Corollary 2.3. *Let (X, d) be a complete cone metric space with a cone P having nonempty interior. Let $F : X \times X \longrightarrow X$ be such that $max[d(x,u), d(x, F(x,y))] = d(x, F(x,y))$ and $max[d(y,v), d(y, F(y,x))] = d(y, F(y,x))$ and satisfy the inequality*

$$d(F(x,y), F(u,v)) \leq l_1 d(x, F(x,y)) + l_2 d(y, F(y,x))$$

for all $x, y, u, v \in X$, where l_1, l_2, are non-negative real numbers such that $l_1 + l_2 < 1$.

Then $F(x,y)$ has a unique coupled fixed point in $X \times X$.

Acknowledgments

The author is very thankful for helpful suggestions and corrections made by the referees who reviewed this paper.

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Received: 30.10.2017