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SMOOTHNESS SPACES RELATED TO MORREY SPACES – A SURVEY. II

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Abstract. We continue the discussion of different strategies of introducing smoothness spaces related to Morrey spaces.

1 Introduction

This is a continuation of the survey [93]. In [93] we have discussed various possibilities of introducing smoothness spaces related to Morrey spaces. All together we have considered eight scales of function spaces, namely

$$B^{s,\tau}_{p,q}, \quad F^{s,\tau}_{p,q}, \quad \mathcal{N}^s_{p,q,u}, \quad \mathcal{E}^s_{p,q,u}, \quad N^s_{p,q,u}, \quad E^s_{p,q,u}, \quad B^{s,\tau}_{p,q,\mathrm{unif}} \qquad \text{and} \qquad F^{s,\tau}_{p,q,\mathrm{unif}}$$

(all definitions are recalled in the Appendix at the end of this paper). Our main concern are the Nikol'skii-Besov type spaces $B^{s,\tau}_{p,q}$ and the Lizorkin-Triebel type spaces $F^{s,\tau}_{p,q}$. In the literature also the Nikol'skii-Besov-Morrey spaces $\mathcal{N}^s_{p,q,u}$ as well as the Lizorkin-Triebel-Morrey spaces $\mathcal{E}^s_{p,q,u}$ play an important role. The scales $N^s_{p,q,u}, E^s_{p,q,u}, B^{s,\tau}_{p,q,\mathrm{unif}}$ and $F^{s,\tau}_{p,q,\mathrm{unif}}$ represent certain local versions of these classes and will be of minor importance in our survey. To give the reader a rough orientation we recall the following relations:

- Let $s \in \mathbb{R}$, $0 < p \leqslant u < \infty$ and $0 < q \leqslant \infty$. Then we have

$$\mathcal{E}^s_{u,q,p} = F^{s,\frac{1}{p}-\frac{1}{u}}_{p,q}$$

in the sense of equivalent quasi-norms.

- Let $s \in \mathbb{R}$ and $0 < p \leqslant u \leqslant \infty$. Then we have

$$\mathcal{N}^s_{u,\infty,p} = B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty}$$

in the sense of equivalent quasi-norms.

- Let $s \in \mathbb{R}$, $0 < p \leqslant u < \infty$ and $0 < q < \infty$. Then we have the continuous embeddings
$$\mathcal{N}^{s}_{u,q,p} \hookrightarrow B^{s,\frac{1}{p}-\frac{1}{u}}_{p,q} . \tag{1.1}$$
If $p < u$, then the embedding is strict.

We refer to [89], see also [93]. As a consequence we have
$$\mathcal{E}^{s_0}_{p_0,q_0,u_0} \in \{F^{s,\tau}_{p,q} : \quad s \in \mathbb{R},\ 0 < p < \infty,\ 0 < q \leqslant \infty,\ \tau \geqslant 0\}$$
and
$$\mathcal{N}^{s_0}_{p_0,\infty,u_0} \in \{B^{s,\tau}_{p,\infty} : \quad s \in \mathbb{R},\ 0 < p \leqslant \infty,\ \tau \geqslant 0\}$$
for all admissible values of s_0, p_0, u_0 and q_0, whereas
$$\mathcal{N}^{s_0}_{p_0,q_0,u_0} \notin \{B^{s,\tau}_{p,q} : \quad s \in \mathbb{R},\ 0 < p \leqslant \infty,\ 0 < q \leqslant \infty,\ \tau \geqslant 0\}$$
for all admissible values of s_0, p_0, u_0 and $0 < q_0 < \infty$. The differences between the Nikol'skii-Besov type scale $B^{s,\tau}_{p,q}$ and the Nikol'skii-Besov-Morrey scale $\mathcal{N}^{s}_{p,q,u}$ $(q < \infty)$ will be discussed in certain detail. With this respect also the relation of the scale $\mathcal{E}^{s}_{p,q,u}$ to the scale $\mathcal{N}^{s}_{p,q,u}$ is of interest. Recall that the Morrey space $\mathcal{M}^{p}_{u}$ coincides with $\mathcal{E}^{0}_{p,2,u}$, $1 < u \leqslant p < \infty$, in the sense of equivalent norms, see [58] and [82].

- Let $0 < u \leqslant p < \infty$, $0 < q, q_0, q_1 \leqslant \infty$ and $s \in \mathbb{R}$. Then
$$\mathcal{N}^{s}_{p,\min(q,u),u} \hookrightarrow \mathcal{E}^{s}_{p,q,u} \hookrightarrow \mathcal{N}^{s}_{p,\infty,u} . \tag{1.2}$$
The embedding $\mathcal{E}^{s}_{p,q_0,u} \hookrightarrow \mathcal{N}^{s}_{p,q_1,u}$ implies $q_1 = \infty$.

- Let $1 \leqslant u \leqslant p < \infty$. It holds
$$\mathcal{N}^{0}_{p,\min(q,u),u} \hookrightarrow \mathcal{M}^{p}_{u} \hookrightarrow \mathcal{N}^{0}_{p,\infty,u} . \tag{1.3}$$

We refer to [79] and [93, Lemma 5].

Furthermore, in addition to the above scales of function spaces we shall use many times the notation $\mathcal{B}^{s,\tau}_{p,q}$, see Definition 5. However, we have
$$\mathcal{B}^{s,\frac{1}{u}-\frac{1}{p}}_{u,q} = \mathcal{N}^{s}_{p,q,u} , \qquad 0 < u \leqslant p \leqslant \infty , \tag{1.4}$$
and always
$$\mathcal{B}^{s,\tau}_{p,q} \hookrightarrow B^{s,\tau}_{p,q} ,$$
see Remark 13 and formula (23) in [93]. Beside the basic relations between these different types of smoothness spaces related to Morrey spaces we investigated in [93] the boundedness of pseudo-differential operators, descriptions by wavelets, characterizations by differences, pointwise multipliers, diffeomorphisms and traces. Here, in Part II, we shall concentrate on interpolation (with fixed p), Gagliardo-Nirenberg type inequalities and embeddings. Whereas the collected assertions in Part I have been essentially known, Part II contains some new material and is, for this reason, written with proofs.

We repeat from Part I one comment concerning the used notation. The situation in the literature is a little bit chaotic. At least in some cases there is no common well-accepted notation. Not only the letter for certain parameters is changing but also its position. The reader should always have a look to the used definition by comparing the results within this survey with others.

The paper is organized as follows. In Section 2 we discuss interpolation of $F^{s,\tau}_{p,q}$ and $B^{s,\tau}_{p,q}$ with fixed p and fixed τ (essentially real interpolation). In the next Section 3 we prove Gagliardo-Nirenberg type inequalities with respect to the scales $F^{s,\tau}_{p,q}$ and $B^{s,\tau}_{p,q}$. These results will be employed in Section 4 for deriving some embeddings. Section 5 is devoted to some further approaches to smoothness spaces related to Morrey spaces. In the final Section 6 we collect some open problems. Here, in the second part of this section, we add also a few comments on possible generalizations. In the Appendix at the end of this paper we recall the definitions of the function spaces under consideration here. There also some notational conventions are collected (the same as in Part I).

2 Interpolation Nikol'skii-Besov and Lizorkin-Triebel type spaces for fixed p and fixed τ

Interpolation theory has established as one of the most important tools in the theory of function spaces during the last fifty years. Distributed in the literature one can find a number of results concerning interpolation of Morrey spaces and interpolation of smoothness spaces related to Morrey spaces, but no systematic treatment. It is not our aim to give such a treatment here, we only summarize and supplement the theory from our point of view. For the basics of real and complex interpolation of Banach and quasi-Banach spaces we refer to [4] and [97].

It is not difficult to describe the real interpolation spaces $(\mathcal{N}^{s_0}_{p,q_0,u}, \mathcal{N}^{s_1}_{p,q_1,u})_{\theta,q}$ by using the method of retractions and coretraction. However, since we want to determine the spaces $(B^{s_0,\tau}_{p,q_0}, B^{s_1,\tau}_{p,q_1})_{\theta,q}$, we follow a different strategy. Clearly,

$$(\mathcal{N}^{s_0}_{u,q_0,p}, \mathcal{N}^{s_1}_{u,q_1,p})_{\theta,q} \hookrightarrow (B^{s_0,\frac{1}{p}-\frac{1}{u}}_{p,q_0}, B^{s_1,\frac{1}{p}-\frac{1}{u}}_{p,q_1})_{\theta,q}$$

see (1.1). We proceed by making use of the deep relations between approximation spaces and interpolation spaces. First, we recall some notions from approximation spaces.

2.1 Approximation spaces

We follow Pietsch [72], but see also [24, Section 7.9]. Let $(A_n)_n$ denote a sequence of subsets of the quasi-Banach space X such that the following conditions are satisfied

(i) $A_1 \subset A_2 \subset \ldots \subset X$;

(ii) $\lambda A_n \subset A_n$ for all scalars λ and all $n \in \mathbb{N}$;

(iii) $A_m + A_n \subset A_{m+n}$ for all $m, n \in \mathbb{N}$.

We put $A_0 := \{0\}$. Then we define the associated best approximation. Let $f \in X$. Then we put

$$a_n\big(f, X\big) := \inf \{\|f - g\|_X : \quad g \in A_n\} \,, \qquad n \in \mathbb{N}_0 \,.$$

Obviously

$$a_0(f, X) = \| f \|_X \geqslant a_1(f, X) \geqslant a_2(f, X) \geqslant \ldots \geqslant 0 \,.$$

We are interested in approximation spaces relative to the numbers a_n, $n \in \mathbb{N}_0$. Let $\mathcal{A}^s_q(X, (A_n)_n)$ be the collection of all elements $f \in X$, such that

$$\big\| f \,|\mathcal{A}^s_q(X, (A_n)_n)\big\| := \begin{cases} \Big(\sum\limits_{n=0}^{\infty} \Big[(n+1)^s \, a_n(f, X)\Big]^q \dfrac{1}{n+1}\Big)^{1/q}, & \text{if } \ 0 < q < \infty \,, \\ \sup\limits_{n=0,1,\ldots} \ (n+1)^s \, a_n(f, X) \,, & \text{if } \ q = \infty \,, \end{cases}$$

where $s > 0$. There is a useful equivalent characterization by taking into account only the numbers a_{2^n}, $n \in \mathbb{N}_0$. Indeed, by means of the monotonicity of the numbers a_n, it follows

$$\big\| f \,|\mathcal{A}^s_q(X, (A_n)_n)\big\| \asymp \begin{cases} a_0(f, X) + \Big(\sum\limits_{n=0}^{\infty} \Big[2^{ns} \, a_{2^n}(f, X)\Big)^q\Big)^{1/q}, & \text{if } \ 0 < q < \infty \,, \\ a_0(f, X) + \sup\limits_{n=0,1,\ldots} \ 2^{ns} \, a_{2^n}(f, X) \,, & \text{if } \ q = \infty \,. \end{cases}$$

Later on we shall need some assertions on real interpolation of these scales. For the basics in real interpolation we refer to [4, 97].

Bernstein and Jackson inequalities

Let $(A_n)_n$ denote a sequence of subsets of the quasi-Banach space X with the properties (i)-(iii). Let Y be a quasi-Banach space satisfying $Y \hookrightarrow X$. We say that the Jackson inequality is satisfied if there exist a constant c_1 and a positive number r such that

$$a_n(f, X) \leqslant c_1 \, n^{-r} \, \|f\|_Y \tag{2.1}$$

holds for all $f \in Y$ and all $n \in \mathbb{N}$. We say that the Bernstein inequality is satisfied if there exists a constant c_2 such that

$$\|f\|_Y \leqslant c_2 \, n^r \, \|f\|_X \tag{2.2}$$

holds for all $f \in A_n$ and all n.

Under those circumstances the following is known, see [24, Theorem 7.9.1].

Proposition 2.1. *If both the Jackson and the Bernstein inequalities hold for the spaces* X *and* Y*, then for* $0 < s < r$ *and* $0 < q \leqslant \infty$ *we have*

$$\mathcal{A}^s_q(X, (A_n)_n) = (X, Y)_{s/r,q}$$

in the sense of equivalent quasi-norms.

Some other useful properties are collected in the following proposition.

Proposition 2.2. *Let X be a quasi-Banach space and let $(A_n)_n$ be a sequence of subsets of X with the properties (i)-(iii). Let $0 < \theta < 1$.*

(i) *Let $0 < q, q_0 \leqslant \infty$ and $s > 0$. Then it holds*

$$\big(X, \mathcal{A}^s_{q_0}(X,(A_n)_n)\big)_{\theta,q} = \mathcal{A}^{\theta s}_q(X,(A_n)_n)\,. \tag{2.3}$$

(ii) *Let $0 < q, q_0, q_1 \leqslant \infty$, $s_0, s_1 > 0$ and $s_0 \neq s_1$. Then, with $s := (1-\theta)\, s_0 + \theta\, s_1$, it holds*

$$\big(\mathcal{A}^{s_0}_{q_0}(X,(A_n)_n), \mathcal{A}^{s_1}_{q_1}(X,(A_n)_n)\big)_{\theta,q} = \mathcal{A}^{s}_q(X,(A_n)_n)\,. \tag{2.4}$$

Remark 1. These classical assertions can be found in, e.g., [71, 6, 23, 24]. Some interesting generalizations can be found in Luther [55] and Luther, Almira [56].

2.2 Best approximation in Nikol'skii-Besov type spaces

Now we return to the concrete situation we are interested in. For $X := B^{0,\tau}_{p,q_0}$ we define

$$A_n := \{g \in B^{0,\tau}_{p,q_0} :\ \mathrm{supp}\ \mathcal{F}g\ \subset B(0,n)\} \qquad n \in \mathbb{N}\,. \tag{2.5}$$

Let $Y := B^{r,\tau}_{p,q_1}$, $r > 0$, $0 \leqslant \tau \leqslant 1/p$. First, we shall study the corresponding Jackson and Bernstein inequalities. Let $\psi \in C_0^\infty$ be a smooth cut-off function as defined in (7.1). For $f \in B^{r,\tau}_{p,q_1}$ we put

$$g_n(x) := \mathcal{F}^{-1}[\psi(2^{-n+2}\xi)\, \mathcal{F}f(\xi)](x)\,, \qquad x \in \mathbb{R}^d, \quad n \in \mathbb{N}_0.$$

Employing some basic inequalities for pseudo-differential operators, see Theorem 5.1 in [108] or Theorem 2 in [93], combined with the fact that

$$\mathcal{F}^{-1}[\psi(2^{-n+2}\xi)\, \mathcal{F}p(\xi)](x) = p(x) \qquad \text{for all polynomials} \quad p \quad \text{and for all} \quad n\,,$$

we derive $g_n \in A_{2^n}$, $n \in \mathbb{N}_0$. Furthermore, by working with the associated smooth dyadic decomposition of unity, see (7.2), we obtain the identity

$$f - g_n = \sum_{\ell=n-1}^{\infty} \mathcal{F}^{-1}[\varphi_\ell(\xi)\, \mathcal{F}f(\xi)]\,, \qquad n \geqslant 2\,, \qquad f \in \mathcal{S}'\,. \tag{2.6}$$

Let $\varepsilon := \min(1,p,q)$. Then, using Lemma 7.1 in the Appendix, we find

$$\begin{aligned} a_{2^n}(f, B^{0,\tau}_{p,q_0})^\varepsilon &\leqslant \|f - g_n\|^\varepsilon_{B^{0,\tau}_{p,q_0}} = \Big\| \sum_{\ell=n-1}^{\infty} \mathcal{F}^{-1}[\varphi_\ell(\xi)\, \mathcal{F}f(\xi)] \Big\|^\varepsilon_{B^{0,\tau}_{p,q_0}} \\ &\leqslant \sum_{\ell=n-1}^{\infty} \|\, \mathcal{F}^{-1}[\varphi_\ell(\xi)\, \mathcal{F}f(\xi)]\, \|^\varepsilon_{B^{0,\tau}_{p,q_0}}\,, \qquad n \geqslant 2\,. \end{aligned} \tag{2.7}$$

Employing again some basic estimates for pseudo-differential operators, see Theorem 5.1 in [108] or Theorem 2 in [93], we conclude in case $\ell \geqslant 1$

$$\|\, \mathcal{F}^{-1}[\varphi_\ell(\xi)\, \mathcal{F}f(\xi)]\, \|_{B^{0,\tau}_{p,q_0}}$$

$$
\begin{aligned}
&= \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{\substack{\ell-1\leqslant j\leqslant \ell+1\\ j\geqslant \max(j_Q,0)}} \left[\int_Q |\mathcal{F}^{-1}[\varphi_j(\xi)\,\varphi_\ell(\xi)\mathcal{F}f(\xi)](x)|^p\,dx\right]^{q/p}\right\}^{1/q}\\
&\leqslant c_1\, 2^{-(\ell-1)r} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \sup_{\substack{\ell-1\leqslant j\leqslant \ell+1\\ j\geqslant \max(j_Q,0)}} \left[\int_Q (2^{jr}|\mathcal{F}^{-1}[\varphi_j(\xi)\,\varphi_\ell(\xi)\mathcal{F}f(\xi)](x)|)^p\,dx\right]^{1/p}\right\}\\
&= c_1\, 2^{-(\ell-1)r}\, \|\, \mathcal{F}^{-1}[\varphi_\ell(\xi)\,\mathcal{F}f(\xi)]\,\|_{B^{r,\tau}_{p,\infty}}\\
&\leqslant c_2\, 2^{-\ell r}\, \|\, f\,\|_{B^{r,\tau}_{p,q_1}}\,,
\end{aligned}
\tag{2.8}
$$

(we used Theorem 5.1 in [108] with respect to $B^{r,\tau}_{p,q_1}$ in the last step). Let $a_\ell(x,D)$ denote the operator $f\mapsto \mathcal{F}^{-1}[\varphi_\ell(\xi)\,\mathcal{F}f(\xi)]$. The constant c_2 in (2.8) may be chosen independent from ℓ since (7.2) and Theorem 5.1 in [108] yield

$$
\begin{aligned}
\|a_\ell(x,D)|B^{r,\tau}_{p,\infty}\to B^{r,\tau}_{p,\infty}\| &\lesssim \max_{|\alpha|,|\beta|\leqslant M}\ \sup_{x,\xi}\,(1+|\xi|)^{-\mu-|\alpha|+|\beta|}\,|\partial_x^\alpha\partial_\xi^\beta\,\varphi_\ell(\xi)|\\
&\lesssim \max_{|\beta|\leqslant M}\ \sup_{\xi}\,(1+|\xi|)^{|\beta|}\,2^{-\ell|\beta|}\,|(\partial_\xi^\beta\,\varphi_1)(2^{-\ell+1}\xi)|\\
&\lesssim \max_{|\beta|\leqslant M}\ \sup_{2^\ell\leqslant|\xi|\leqslant 3\,2^\ell}\,(1+|\xi|)^{|\beta|}\,2^{-\ell|\beta|}\\
&\leqslant c_3<\infty\,,
\end{aligned}
\tag{2.9}
$$

where $M:=M(p,\tau)$. Here we used the obvious fact that $\mathcal{F}^{-1}[\varphi_\ell(\xi)\,\mathcal{F}p(\xi)]=0$ if $\ell\geqslant 1$ and p is any polynomial. Inserting this together with (2.8) into (2.7) we have proved the Jackson inequality

$$
a_{2^n}(f,B^{0,\tau}_{p,q_0})\leqslant c\,2^{-nr}\,\|\,f\,\|_{B^{r,\tau}_{p,q_1}}\,,\qquad n\geqslant 2\,,\quad \tau\leqslant 1/p\,,
\tag{2.10}
$$

where c is independent of f and n. A simple monotonicity argument yields (2.10) for all $n\in\mathbb{N}$ (with modified c if necessary). Now we study the Bernstein inequality. Let $f\in A_{2^n}$. Then

$$
\begin{aligned}
\|f\|_{B^{r,\tau}_{p,q_1}} &= \sup_{Q\in\mathcal{Q}}\frac{1}{|Q|^\tau}\left\{\sum_{\max(j_Q,0)\leqslant j\leqslant n+1}\left[\int_Q|2^{jr}\,\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)](x)|^p\,dx\right]^{q_1/p}\right\}^{1/q_1}\\
&\leqslant \left(\sum_{\max(j_Q,0)\leqslant j\leqslant n+1} 2^{jrq_1}\right)^{1/q_1} \qquad\qquad (2.11)\\
&\quad\times \sup_{Q\in\mathcal{Q}}\frac{1}{|Q|^\tau}\left\{\sup_{\max(j_Q,0)\leqslant j\leqslant n+1}\left[\int_Q|\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)](x)|^p\,dx\right]^{1/p}\right\}\\
&\leqslant c\,2^{(n+1)r}\,\|f\|_{B^{0,\tau}_{p,q_0}} \qquad\qquad (2.12)
\end{aligned}
$$

holds for all $f\in A_{2^n}$ and all n with c independent of these. Again a monotonicity argument yields the general case, i.e., the Bernstein inequality for all A_n, $n\in\mathbb{N}$. As a consequence of Proposition 2.1 and an obvious lifting argument we obtain the following result.

Lemma 2.1. *Let $0 < p \leqslant \infty$, $0 < q, q_0, q_1 \leqslant \infty$, $s_0 \in \mathbb{R}$, $0 < s < r$ and $0 \leqslant \tau \leqslant 1/p$. Then we have*

$$\mathcal{A}^s_q(B^{s_0,\tau}_{p,q_0}, (A_n)_n) = (B^{s_0,\tau}_{p,q_0}, B^{s_0+r,\tau}_{p,q_1})_{s/r,q}$$

in the sense of equivalent quasi-norms.

Next we turn to the study of the approximation spaces $\mathcal{A}^s_q(B^{s_0,\tau}_{p,q_0}, (A_n)_n)$. We begin with a preparation.

Lemma 2.2. *Let $0 < p, q \leqslant \infty$ and $0 \leqslant \tau \leqslant 1/p$. Then we have*

$$c\, \|f - g_n\|_{B^{0,\tau}_{p,q}} \leqslant a_{2^{n-2}}(f, B^{0,\tau}_{p,q}) \leqslant \|f - g_{n-2}\|_{B^{0,\tau}_{p,q}}$$

with $c > 0$ independent of $f \in B^{0,\tau}_{p,q}$ and $n \geqslant 2$.

Proof. For given $f \in B^{0,\tau}_{p,q_0}$, $n \in \mathbb{N}_0$ and $\varepsilon > 0$ let $f_n \in A_{2^n}$ be an element such that

$$\|f - f_n\|_{B^{0,\tau}_{p,q_0}} - \varepsilon < a_{2^n}(f, B^{0,\tau}_{p,q_0}) .$$

By means of (7.1), (7.2) we conclude

$$f_n = \mathcal{F}^{-1}[\psi(2^{-n}\xi)\, \mathcal{F} f_n(\xi)] , \qquad n \in \mathbb{N}_0$$

and therefore

$$g_n - f_{n-2} = \mathcal{F}^{-1}\Big[\psi(2^{-n+2}\xi) \Big(\mathcal{F} f(\xi) - \mathcal{F} f_n(\xi)\Big)\Big] , \qquad n \geqslant 2 .$$

Hence

$$\begin{aligned} \|f - g_n\|_{B^{0,\tau}_{p,q_0}} &\lesssim \|f - f_{n-2}\|_{B^{0,\tau}_{p,q_0}} + \|g_n - f_{n-2}\|_{B^{0,\tau}_{p,q_0}} \\ &\lesssim \|f - f_{n-2}\|_{B^{0,\tau}_{p,q_0}} , \end{aligned}$$

where we used the same type of argument as in (2.9). This proves the claim. □

Recall, the spaces $\mathcal{B}^{s,\tau}_{p,q}$ have been defined in Definition 5 in the Appendix.

Lemma 2.3. *Let $s, s_0 \in \mathbb{R}$, $s > 0$, $0 < p \leqslant \infty$, $0 < q, q_0, q_1 \leqslant \infty$ and $0 \leqslant \tau \leqslant 1/p$. Then we have*

$$\mathcal{B}^{s_0+s,\tau}_{p,q} = \mathcal{A}^s_q(B^{s_0,\tau}_{p,q_0}, (A_n)_n)$$

in the sense of equivalent quasi-norms.

Proof. Step 1. Some preliminaries. We shall need a Fourier multiplier assertion for Morrey spaces. By $H^r_2 = F^r_{2,2}$ we denote the Bessel potential spaces. In case $0 < u \leqslant p < \infty$, $r > \frac{d}{2} + \frac{d}{\min(1,u)}$ and $f \in M^p_u \cap \mathcal{S}'$, and supp $\mathcal{F} f \subset \{\xi :\ |\xi| \leqslant D\}$ we have

$$\| \mathcal{F}^{-1}[M(\xi)\, \mathcal{F} f(\xi)] \|_{M^p_u} \leqslant c\, \|M(D\, \cdot)\|_{H^r_2} \| f \|_{M^p_u} , \tag{2.13}$$

where c is independent of f and $D \geqslant 1$, see [96]. For simplicity we shall deal with the case $s_0 = 0$ only. The general case follows by obvious modifications.

Step 2. We shall prove the continuous embedding

$$\mathcal{B}^{s,\tau}_{p,q} \hookrightarrow \mathcal{A}^s_q(B^{0,\tau}_{p,q_0}, (A_n)_n)\,.$$

In what follows we shall use the abbreviation $S_j f := \mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)]$, $j \in \mathbb{N}_0$, where $(\varphi_j)_j$ denotes a smooth dyadic decomposition of unity. Let $0 < q < \infty$. Then

$$\begin{aligned} \| f \|_{\mathcal{A}^s_q(B^{0,\tau}_{p,q_0},(A_n)_n)} & \lesssim \Big(\sum_{n=0}^\infty 2^{nsq}\, \| f - g_{2^n} \|^q_{B^{0,\tau}_{p,q_0}}\Big)^{1/q} \\ & = \Big(\sum_{n=0}^\infty 2^{nsq}\, \| \sum_{j=n}^\infty S_j f \|^q_{B^{0,\tau}_{p,q_0}}\Big)^{1/q}\,, \end{aligned}$$

see (2.6). The space $\ell_q(B^{0,\tau}_{p,q_0})$ with quasi-norm

$$\|(f_j)_j\|_{\ell_q(B^{0,\tau}_{p,q_0})} := \Big(\sum_{j=0}^\infty \|f_j\|^q_{B^{0,\tau}_{p,q_0}}\Big)^{1/q}$$

satisfies

$$\|(f_j)_j + (g_j)_j\|^\varepsilon_{\ell_q(B^{0,\tau}_{p,q_0})} \leqslant \|(f_j)_j\|^\varepsilon_{\ell_q(B^{0,\tau}_{p,q_0})} + \|(g_j)_j\|^\varepsilon_{\ell_q(B^{0,\tau}_{p,q_0})}$$

with $\varepsilon := \min(p, q_0, q)$. Hence

$$\begin{aligned} \| f \|^\varepsilon_{\mathcal{A}^s_q(B^{0,\tau}_{p,q_0},(A_n)_n)} & \lesssim \sum_{j=0}^\infty \Big(\sum_{n=0}^\infty 2^{nsq}\, \| S_{j+n} f \|^q_{B^{0,\tau}_{p,q_0}}\Big)^{\varepsilon/q} \\ & \lesssim \sum_{j=0}^\infty 2^{-js\varepsilon} \Big(\sum_{n=0}^\infty 2^{(n+j)sq}\, \| S_{j+n} f \|^q_{B^{0,\tau}_{p,q_0}}\Big)^{\varepsilon/q} \\ & \lesssim \| f \|^\varepsilon_{\mathcal{B}^{s,\tau}_{p,q_0}}\,, \end{aligned}$$

where we have used

$$\begin{aligned} \| S_{j+n} f \|_{B^{0,\tau}_{p,q_0}} & = \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left(\sum_{\max(j+n-1,0,j_Q)\leqslant \ell \leqslant j+n+1} \Big(\int_Q |S_\ell(S_{j+n} f)(x)|^p\, dx\Big)^{q_0/p} \right)^{1/q_0} \\ & \lesssim \max_{-1\leqslant \ell\leqslant 1} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big(\int_Q |S_{\ell+j+n}(S_{j+n} f)(x)|^p\, dx\Big)^{1/p} \end{aligned}$$

and

$$\mathrm{supp}\ \mathcal{F}(S_{j+n} f) \subset \{\xi:\ |\xi| \leqslant 2^{j+n+1}\}$$

since, in view of (2.13), we obtain

$$\begin{aligned} & \| S_{j+n} f \|_{B^{0,\tau}_{p,q_0}} \lesssim \| S_{j+n} f \|_{M^w_p} \\ & \qquad \times \max_{-1\leqslant \ell\leqslant 1} \max \Big(\| \varphi_1(2^{j+n+1}\, 2^{-j-n-\ell} x) \|_{H^r_2}, \| \varphi_0(2^{j+n+1}\, 2^{-j-n-\ell} x) \|_{H^r_2} \Big) \end{aligned}$$

where $\frac{1}{w} := \frac{1}{p} - \tau$. Formally this excludes $\tau = 1/p$, but in this case, i.e., for $w = \infty$, we know $M_p^\infty = L_\infty$. The inequality, we used, is, of course, also true in the L_∞-case.

Step 3. We shall prove the continuous embedding

$$\mathcal{A}_q^s(B_{p,q_0}^{0,\tau}, (A_n)_n) \hookrightarrow \mathcal{B}_{p,q}^{s,\tau} .$$

Obviously $S_j f = S_j(f - g_{j-2})$, $j \geqslant 2$. Hence, for any dyadic cube $Q \in \mathcal{Q}$, we have

$$\begin{aligned} \frac{1}{|Q|^\tau}\Big(\int_Q |S_j f(x)|^p\, dx\Big)^{1/p} &= \frac{1}{|Q|^\tau}\Big(\int_Q |S_j(f-g_{j-2})(x)|^p\, dx\Big)^{1/p} \\ &\leqslant \|f - g_j\|_{B_{p,\infty}^{0,\tau}} . \end{aligned}$$

But this proves

$$\begin{aligned} &\Big(\sum_{j=2}^\infty 2^{jsq} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\tau q}}\Big(\int_Q |S_j f(x)|^p\, dx\Big)^{q/p}\Big)^{1/q} \\ &\leqslant \Big(\sum_{j=2}^\infty 2^{jsq} \|f - g_{j-2}\|^q_{B_{p,\infty}^{0,\tau}}\Big)^{1/q} \lesssim \|f\|_{\mathcal{A}_q^s(B_{p,q_0}^{0,\tau},(A_n)_n)} \end{aligned}$$

Moreover,

$$\Big(\sum_{j=0}^1 2^{jsq} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\tau q}}\Big(\int_Q |S_j f(x)|^p\, dx\Big)^{q/p}\Big)^{1/q} \lesssim \|f\|_{B_{p,\infty}^{0,\tau}} .$$

Combining both inequalities we have proved the claim. □

2.3 Real interpolation of smoothness spaces related to Morrey spaces

After these preparations we can prove now the main result in this subsection.

Theorem 2.1. *Let $0 < \theta < 1$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $0 < p < \infty$, $0 \leqslant \tau \leqslant 1/p$ and $0 < q, q_0, q_1 \leqslant \infty$. We put $s := (1-\theta)\, s_0 + \theta\, s_1$. Then we have*

$$\mathcal{B}_{p,q}^{s,\tau} = (B_{p,q_0}^{s_0,\tau}, B_{p,q_1}^{s_1,\tau})_{\theta,q}$$

in the sense of equivalent quasi-norms.

Proof. Combining Lemma 2.1 and Lemma 2.3 we obtain

$$\mathcal{B}_{p,q}^{s,\tau} = \mathcal{A}_q^{s-s_0}(B_{p,q_0}^{s_0,\tau}, (A_n)_n) = (B_{p,q_0}^{s_0,\tau}, B_{p,q_1}^{s_1,\tau})_{\theta,q} .$$

This proves the claim. □

Remark 2. Theorem 2.1 is an extension of the well-known formula

$$B_{p,q}^s = (B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\theta,q}$$

see, e.g., [70, pp. 106/107], [98, 2.4] or [4, Theorem 6.4.5].

There is a number of consequences.

Corollary 2.1. *Let* $0 < \theta < 1$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $0 < p < \infty$, $0 \leqslant \tau \leqslant 1/p$ *and* $0 < q, q_0, q_1 \leqslant \infty$. *We put* $s := (1-\theta)\, s_0 + \theta\, s_1$. *Let* $A, \mathcal{A} \in \{B, F\}$. *Then we have*

$$\mathcal{B}^{s,\tau}_{p,q} = (A^{s_0,\tau}_{p,q_0}, \mathcal{A}^{s_1,\tau}_{p,q_1})_{\theta,q}$$

in the sense of equivalent quasi-norms.

Proof. The monotonicity of the real method and (7.5) yield

$$(B^{s_0,\tau}_{p,\min(p,q_0)}, B^{s_1,\tau}_{p,\min(p,q_1)})_{\theta,q} \hookrightarrow (F^{s_0,\tau}_{p,q_0}, F^{s_1,\tau}_{p,q_1})_{\theta,q} \hookrightarrow (B^{s_0,\tau}_{p,\infty}, B^{s_1,\tau}_{p,\infty})_{\theta,q}\,.$$

Now the independence in q_0, q_1 in Theorem 2.1 proves the claim in case $A = \mathcal{A} = F$. The needed modification in the mixed cases are obvious. □

Corollary 2.2. *Let* $0 < \theta < 1$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $0 < p \leqslant \infty$, $0 \leqslant \tau \leqslant 1/p$ *and* $0 < q, q_0, q_1 \leqslant \infty$. *We put* $s := (1-\theta)\, s_0 + \theta\, s_1$. *Let* $A, \mathcal{A} \in \{B, F\}$. *Then we have*

$$B^{s,\tau}_{p,\infty} = (A^{s_0,\tau}_{p,q_0}, \mathcal{A}^{s_1,\tau}_{p,q_1})_{\theta,\infty}$$

in the sense of equivalent quasi-norms.

Proof. Corollary 2.2 is an immediate consequence of Corollary 2.1 and the identity $\mathcal{B}^{s,\tau}_{p,\infty} = B^{s,\tau}_{p,\infty}$, see formula (7.8) in the Appendix. □

Finally, we can also deal with the real interpolation spaces with respect to Nikol'skii-Besov-Morrey and Lizorkin-Triebel-Morrey spaces.

Theorem 2.2. *Let* $0 < \theta < 1$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $0 < u \leqslant p \leqslant \infty$. *and* $0 < q, q_0, q_1 \leqslant \infty$. *We put* $s := (1-\theta)\, s_0 + \theta\, s_1$. *Let* $A, \mathcal{A} \in \{\mathcal{E}, \mathcal{N}\}$ *(*$p < \infty$ *if either* $A = \mathcal{E}$ *or* $\mathcal{A} = \mathcal{E}$*). Then we have*

$$\mathcal{N}^{s}_{p,q,u} = (A^{s_0}_{p,q_0,u}, \mathcal{A}^{s_1}_{p,q_1,u})_{\theta,q}$$

in the sense of equivalent quasi-norms.

Proof. Step 1. To begin with we consider the special case $A = \mathcal{A} = \mathcal{N}$. We shall work with the identity

$$\mathcal{B}^{s,\frac{1}{p}-\frac{1}{u}}_{p,q} = \mathcal{N}^{s}_{u,q,p}\,, \qquad 0 < p \leqslant u \leqslant \infty \qquad \text{(see (1.1))}.$$

Then the claim consists in proving

$$\mathcal{B}^{s,\frac{1}{p}-\frac{1}{u}}_{p,q} = \left(\mathcal{B}^{s_0,\frac{1}{p}-\frac{1}{u}}_{p,q_0}, \mathcal{B}^{s_1,\frac{1}{p}-\frac{1}{u}}_{p,q_1}\right)_{\theta,q}. \tag{2.14}$$

There are at least two possibilities to proves this claim. One could use the method of retractions and coretraction, e.g., as in [48] or [59]. However, we shall continue to use properties of approximation spaces, in particular Proposition 2.2. Let $s_2 < s_0 < s_1$ and let $\tau := \frac{1}{p} - \frac{1}{u}$. We are going to employ Proposition 2.2 with $X = B^{s_2,\tau}_{p,\infty}$ and

$$\mathcal{B}^{s_i,\tau}_{p,q_i} = \mathcal{A}^{s_i-s_2}_{q_i}(B^{s_2,\tau}_{p,\infty}, (A_n)_n)\,, \qquad i = 1, 2\,,$$

see Lemma 2.3. It follows

$$A_q^{s-s_2}(B_{p,\infty}^{s_2,\tau},(A_n)_n) = \Big(\mathcal{B}_{p,q_0}^{s_0,\frac{1}{p}-\frac{1}{u}}, \mathcal{B}_{p,q_1}^{s_1,\frac{1}{p}-\frac{1}{u}}\Big)_{\theta,q}$$

since $s-s_2 = (1-\theta)\,(s_0-s_2)+\theta\,(s_1-s_2)$. Using once again Lemma 2.3 we have done.

Step 2. The elementary embeddings

$$\mathcal{N}_{p,\min(q,u),u}^{s} \hookrightarrow \mathcal{E}_{p,q,u}^{s} \hookrightarrow \mathcal{N}_{p,\infty,u}^{s}\,,$$

see (1.2), allow to argue as in proof of Corollary 2.1. □

Remark 3. Kozono and Yamazaki [48] already proved

$$N_{p,q,u}^{s} = (N_{p,q_0,u}^{s_0}, N_{p,q_1,u}^{s_1})_{\theta,q} \tag{2.15}$$

if $0<\theta<1$, $s_0,s_1\in\mathbb{R}$, $s_0<s_1$, $1<u\leqslant p<\infty$, $1\leqslant q,q_0,q_1\leqslant\infty$ and $s := (1-\theta)\,s_0+\theta\,s_1$. They used the method of retraction and coretraction. They also considered a counterpart for the homogeneous spaces $\dot{\mathcal{N}}_{p,q,u}^{s}$.

Corollary 2.3. *Let $0<\theta<1$, $s_1\in\mathbb{R}$, $0<s_1$, $1\leqslant u\leqslant p\leqslant\infty$. and $0<q,q_1\leqslant\infty$. We put $s:=\theta\,s_1$. Let $\mathcal{A}\in\{\mathcal{E},\mathcal{N}\}$ ($p<\infty$ if $\mathcal{A}=\mathcal{E}$). Then we have*

$$\mathcal{N}_{p,q,u}^{s} = (\mathcal{M}_u^p, \mathcal{A}_{p,q_1,u}^{s_1})_{\theta,q}$$

in the sense of equivalent quasi-norms.

Proof. This follows from the elementary embedding in (1.3) and Theorem 2.2. □

Corollary 2.4. *Let $0<\theta<1$, $s_1\in\mathbb{R}$, $0<s_1$, $1\leqslant p\leqslant u\leqslant\infty$. and $0<q,q_1\leqslant\infty$. We put $s:=\theta\,s_1$. Then we have*

$$\mathcal{N}_{u,q,p}^{s} = (\mathcal{M}_u^p, B_{p,q_1}^{s_1,\frac{1}{p}-\frac{1}{u}})_{\theta,q} = (\mathcal{M}_u^p, \mathcal{B}_{p,q_1}^{s_1,\frac{1}{p}-\frac{1}{u}})_{\theta,q}\,,$$

in the sense of equivalent quasi-norms.

Proof. We employ Theorem 2.2, the formulas (2.14), (1.3) and the monotonicity of the real method. □

2.4 Interpolation of Nikol'skii-Besov-Morrey spaces with fixed p and fixed s

There are some other interesting results on real interpolation in this field. First we mention the following.

Proposition 2.3. *Let $0<\theta<1$, $s,s_0,s_1\in\mathbb{R}$, and $0<p,q,q_0,q_1\leqslant\infty$.*

(i) *Let $1\leqslant u\leqslant p\leqslant\infty$. Then*

$$\mathcal{N}_{p,q,u}^{s} = (\mathcal{N}_{p,q_0,u}^{s}, \mathcal{N}_{p,q_1,u}^{s})_{\theta,q} \qquad \text{if} \qquad \frac{1}{q} := \frac{1-\theta}{q_0}+\frac{\theta}{q_1} \tag{2.16}$$

holds in the sense of equivalent quasi-norms.

(ii) *Let* $0 < u_0 \leqslant u_1 \leqslant p \leqslant \infty$. *We require*

$$\frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1}\,.$$

Then

$$(\mathcal{M}^p_{u_0}, \mathcal{M}^p_{u_1})_{\theta,u} \hookrightarrow \mathcal{M}^p_u \tag{2.17}$$

follows.

(iii) *Let* $1 \leqslant u \leqslant p \leqslant \infty$. *We put* $s := (1-\theta)\, s_0 + \theta\, s_1$,

$$\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \qquad \text{and} \qquad \frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1}\,.$$

Then, if $u = q$, *we have*

$$(\mathcal{N}^{s_0}_{p,q_0,u_0}, \mathcal{N}^{s_1}_{p,q_1,u_1})_{\theta,q} \hookrightarrow \mathcal{N}^{s}_{p,q,u} \tag{2.18}$$

follows.

Proof. Step 1. We deal with (i). The standard method of proof by means of retraction and coretraction (cf. e.g. [97, 1.2.4, 2.4.1, 2.4.2] or [4, 6.4]) can be applied here. We proceed as in [90]. Let $\{\varphi_j\}_j$ denote the smooth dyadic decomposition of unity. Next we define an associated system $(\varrho)_k$ by

$$\begin{aligned} \varrho_0 &:= \varphi_0 + \varphi_1 \\ \varrho_k &:= \varphi_{k-1} + \varphi_k + \varphi_{k+1}\,, \qquad k = 1, 2, \ldots\,. \end{aligned}$$

For $f \in \mathcal{S}'$ we put

$$Sf(x) := \Big\{ \mathcal{F}^{-1}[\varphi_j\, \mathcal{F} f](x) \Big\}_{j=0}^{\infty}\,.$$

For a sequence $(g_k)_k \subset \mathcal{S}'$ we define (formally)

$$R\,\{g_k\}_k := \sum_{k=0}^{\infty} \mathcal{F}^{-1}[\varrho_k\, \mathcal{F} g_k](x) \qquad (\text{convergence in } \mathcal{S}')\,.$$

Furthermore, we need the spaces $\ell_q(\mathcal{M}^p_u)$ quasi-normed by

$$\|\,(f_j)_j\,\|_{\ell_q(\mathcal{M}^p_u)} := \Big\| (\|\, f_j\,\|_{\mathcal{M}^p_u}) \Big\|_{\ell_q}\,.$$

It is obvious that the restriction of S is a linear and bounded operator

$$S\,:\,\mathcal{N}^0_{p,q,u} \mapsto \ell_q(\mathcal{M}^p_u)\,.$$

Moreover, it is clear by definition of the ϱ_k that (at least formally)

$$R(Sf) = f \qquad \text{for all} \quad f \in \mathcal{N}^0_{p,q,u}$$

holds. We claim that

$$R\,:\,\ell_q(\mathcal{M}^p_u) \mapsto \mathcal{N}^0_{p,q,u}$$

is bounded. By using the properties of supp φ_j and the triangle inequality we have

$$\| R\{g_k\}_k |\mathcal{N}^0_{p,q,u}\| \lesssim \max_{-2\leqslant k\leqslant 2} \Big(\sum_{j=0}^\infty \| \mathcal{F}^{-1}[\varphi_j\, \varphi_{j+k}\, \mathcal{F} g_j] \|^q_{\mathcal{M}^p_u} \Big)^{1/q} ,$$

where $\varphi_{-1} \equiv \varphi_{-2} \equiv 0$. The right-hand side can be estimated by using the convolution inequality

$$\begin{aligned} \| \mathcal{F}^{-1}[\varphi_j\, \varphi_{j+k}\, \mathcal{F} g_j](x) \|_{\mathcal{M}^p_u} & \lesssim \| \mathcal{F}^{-1}\varphi_j \|_{L_1} \| \mathcal{F}^{-1}\varphi_{j+k} \|_{L_1} \| g_j \|_{\mathcal{M}^p_u} \\ & \lesssim \max \Big(\| \mathcal{F}^{-1}\varphi_0 \|_{L_1}, \| \mathcal{F}^{-1}\varphi_1 \|_{L_1} \Big)^2 \| g_j \|_{\mathcal{M}^p_u} \end{aligned}$$

with constants independent of j,k and $(g_j)_j$. Now, in case $s=0$, the formula (2.16) follows from the well-known formula (cf. [97, Theorem 1.18.6/2], [4, Theorem 5.6.1] or [24, Theorem 6.7.5])

$$\Big(\ell_{q_0}(X), \ell_{q_1}(X) \Big)_{\Theta,q} = \ell_q(X)\, , \tag{2.19}$$

where $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and X is a complete quasi-normed space. The general case $s \neq 0$ is a consequence of the lift property of the spaces under consideration, see Corollary 2 in [93].

Step 2. Proof of (ii). Let B be a ball with radius 1. We shall use the formula

$$(L_{u_0}(B), L_{u_1}(B))_{\theta,u} = L_u(B)\, , \qquad \frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1}\, ,$$

see [4, Theorem 5.2.1]. As usual, Peetre's K-functional $K(t,f,X_0,X_1)$ of the interpolation couple (X_0,X_1) is defined as

$$K(t,f,X_0,X_1) := \inf_{\substack{f=f_0+f_1\\ f_i\in X_i,\, i=1,2}} \| f_0 \|_{X_0} + t\, \| f_1 \|_{X_1}\, , \qquad t>0\, .$$

This implies

$$\begin{aligned} \Big(\int_B |f(x)|^u\, du \Big)^{1/u} & \lesssim \Big(\int_0^\infty [t^{-\theta}\, K(t,f,L_{u_0}(B), L_{u_1}(B))]^u \frac{dt}{t} \Big)^{1/u} \\ & \lesssim \Big(\int_0^\infty [t^{-\theta}\, K(t,f,\mathcal{M}^p_{u_0}, \mathcal{M}^p_{u_1})]^u \frac{dt}{t} \Big)^{1/u} \end{aligned}$$

with constants independent of f and B. Now, let B be a ball with radius $\lambda > 0$. Then a change of coordinates and the previous inequality yield

$$\begin{aligned} \Big(\int_B |f(x)|^u\, dx \Big)^{1/u} & = \lambda^{d/u} \Big(\int_{\widetilde{B}} |f(\lambda y)|^u\, dy \Big)^{1/u} \\ & \lesssim \lambda^{d/u} \Big(\int_0^\infty [t^{-\theta}\, K(t,f(\lambda\,\cdot\,),\mathcal{M}^p_{u_0}, \mathcal{M}^p_{u_1})]^u \frac{dt}{t} \Big)^{1/u} , \end{aligned}$$

where $\widetilde{B}$ is a ball with volume 1. Let $g_t \in \mathcal{M}^p_{u_0}$ and $h_t \in \mathcal{M}^p_{u_1}$ be a pair of functions such that $f = g_t + h_t$ on $\mathbb{R}$. Using the scaling property of the Morrey spaces $\mathcal{M}^p_{u_i}$, i=1,2, i.e., the equality

$$\| f(\lambda\,\cdot\,) \|_{\mathcal{M}^p_u} = \lambda^{-d/p}\, \| f(\,\cdot\,) \|_{\mathcal{M}^p_u}\, , \qquad \lambda > 0\, , \quad f \in \mathcal{M}^p_u\, ,$$

we obtain

$$\begin{aligned} K(t, f(\lambda\,\cdot\,), \mathcal{M}^p_{u_0}, \mathcal{M}^p_{u_1}) & \leqslant \| \, g_t(\lambda\,\cdot\,) \, \|_{\mathcal{M}^p_{u_0}} + t \, \| \, h_t(\lambda\,\cdot\,) \, \|_{\mathcal{M}^p_{u_1}} \\ & \leqslant \lambda^{-d/p} \, \| \, g_t \, \|_{\mathcal{M}^p_{u_0}} + t \, \| \, h_t \, \|_{\mathcal{M}^p_{u_1}} \, . \end{aligned}$$

Inserting this in the above inequality we get

$$\begin{aligned} \Big(\int_B |f(x)|^u \, dx \Big)^{1/u} & \lesssim \lambda^{\frac{d}{u} - \frac{d}{p}} \Big(\int_0^\infty [t^{-\theta} \, (\| \, g_t \, \|_{\mathcal{M}^p_{u_0}} + t \, \| \, h_t \, \|_{\mathcal{M}^p_{u_1}})]^u \frac{dt}{t} \Big)^{1/u} \\ & \lesssim |B|^{-\frac{1}{p} + \frac{1}{u}} \, \| \, f \, \|_{(\mathcal{M}^p_{u_0}, \mathcal{M}^p_{u_1})_{\theta,u}} \end{aligned}$$

by choosing g_t and h_t in an appropriate way.

Step 3. Proof of (iii). We argue as in Step 1. This time we shall work with the spaces $\ell^s_q(\mathcal{M}^p_u)$ quasi-normed by

$$\| \, (f_j)_j \, \|_{\ell^s_q(\mathcal{M}^p_u)} := \Big\| (\| \, 2^{js} \, f_j \, \|_{\mathcal{M}^p_u}) \Big\|_{\ell_q} \, .$$

It follows that the operators S and R have the properties:

- $S \in \mathcal{L}(\mathcal{N}^{s_i}_{p,q_i,u_i}, \ell^{s_i}_{q_i}(\mathcal{M}^p_{u_i}))$, $i = 1, 2$;
- $R \in \mathcal{L}(\ell^{s_i}_{q_i}(\mathcal{M}^p_{u_i}), \mathcal{N}^{s_i}_{p,q_i,u_i})$, $i = 1, 2$;
- $R(Sf) = f$ for all $f \in \mathcal{N}^{s_i}_{p,q_i,u_i}$, $i = 1, 2$.

We conclude that $(\mathcal{N}^{s_0}_{p,q_0,u_0}, \mathcal{N}^{s_1}_{p,q_1,u_1})_{\theta,q}$ is a retract of $(\ell^{s_0}_{q_0}(\mathcal{M}^p_{u_0}), \ell^{s_1}_{q_1}(\mathcal{M}^p_{u_1})_{\theta,q}$. From

$$\Big(\ell^{s_0}_{q_0}(X_0), \ell^{s_1}_{q_1}(X_1) \Big)_{\theta,q} = \ell^s_q((X_0, X_1)_{\theta,q}) \, ,$$

where

$$s = (1-\theta) \, s_0 + \theta \, s_1 \, , \qquad s_0, s_1 > 0 \, , \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and X_0, X_1 are complete quasi-normed spaces, see [4, Theorem 5.6.2] or [24, Theorem 6.7.5], and (2.17) we derive

$$\ell^s_u((\mathcal{M}^p_{u_0}, \mathcal{M}^p_{u_1})_{\theta,u}) \hookrightarrow \ell^s_u(\mathcal{M}^p_u) \, , \qquad \frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1} \, .$$

The proof is complete. □

Remark 4. (i) Under some technical conditions Ruiz and Vega [74] and Blasco, Ruiz and Vega [7] constructed counterexamples showing

$$\mathcal{M}^p_u \neq (\mathcal{M}^p_{u_0}, \mathcal{M}^p_{u_1})_{\theta,q} \qquad \text{and} \qquad \mathcal{M}^p_u \neq [\mathcal{M}^p_{u_0}, \mathcal{M}^p_{u_1}]_\theta \, . \tag{2.20}$$

A more clear picture has been obtained quite recently by Lemarié-Rieusset [52]. He proved

$$(\mathcal{M}^{p_0}_{u_0}, \mathcal{M}^{p_1}_{u_1})_{\theta,u} \hookrightarrow \mathcal{M}^p_u$$

if $1 < u_0 \leqslant p_0 < \infty$, $1 < u_1 \leqslant p_1 < \infty$

$$\frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1} \qquad \text{and} \qquad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

(compare with Proposition 2.3). Under the same conditions he was able to show

$$\mathcal{M}^p_u \hookrightarrow (\mathcal{M}^{p_0}_{u_0}, \mathcal{M}^{p_1}_{u_1})_{\theta,\infty} \qquad \Longleftrightarrow \qquad \frac{u_0}{p_0} = \frac{u_1}{p_1}\,.$$

For his results concerning the complex method we refer to the next subsection.

(ii) Mazzucato [59] proved Proposition 2.3 in the context of the homogeneous spaces $\dot{\mathcal{N}}^s_{p,q,u}$ and restricted to Banach spaces. However, the proof given here is essentially the same. Concerning part (i) we also refer to Sawano and Tanaka [87].

2.5 Complex interpolation of Nikol'skii-Besov-Morrey and Lizorkin-Triebel-Morrey spaces

There are also some results on complex interpolation. For the basics of complex interpolation we refer to [4] and [97].

Proposition 2.4. *Let $0 < \theta < 1$, $s \in \mathbb{R}$, $1 < u \leqslant p < \infty$. and $1 < q_0, q_1 \leqslant \infty$.*

(i) *Let $1 < u_0 \leqslant p_0 < \infty$, $1 < u_1 \leqslant p_1 < \infty$, and*

$$\frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1} \qquad \text{and} \qquad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,.$$

Then

$$[\mathcal{M}^{p_0}_{u_0}, \mathcal{M}^{p_1}_{u_1}]_\theta = \mathcal{M}^p_u$$

if, and only if, $u_0\, p_1 = u_1\, p_0$.

(ii) *Furthermore,*

$$\mathcal{N}^s_{p,q,u} = [\mathcal{N}^s_{p,q_0,u}, \mathcal{N}^s_{p,q_1,u}]_\theta \qquad \text{if} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{2.21}$$

holds in the sense of equivalent norms.

(iii) *We also have*

$$\mathcal{E}^s_{p,q,u} = [\mathcal{E}^s_{p,q_0,u}, \mathcal{E}^s_{p,q_1,u}]_\theta \qquad \text{if} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{2.22}$$

in the sense of equivalent norms.

Remark 5. (i) Part (i) has been proved by Lemarié-Rieusset [52]. The remaining parts (ii) and (iii) can be found in Sawano and Tanaka [87].

(ii) Some more formulas with respect to real and complex interpolation, but dealing either with the homogeneous spaces $\dot{\mathcal{N}}^s_{p,q,u}$ or with the scale $N^s_{p,q,u}$, may be found in the papers of Kozono and Yamazaki [48], [49].

3 Gagliardo-Nirenberg type inequalities

The classical Gagliardo-Nirenberg inequalities represent a very useful tool in connection with pde. For that reason there is some interest also in counterparts in various non-classical situations. Here we concentrate in consequences of Gagliardo-Nirenberg type inequalities for embeddings.

First we deal with Lizorkin-Triebel type spaces. We follow Brezis and Mironescu [8] (but traced there to Oru [68]).

Proposition 3.1. *Let* $0 < q, q_0, q_1 \leqslant \infty$, $-\infty < s_0 < s_1 < \infty$, $\tau_0, \tau_1 \geqslant 0$ *and* $0 < \theta < 1$.

(i) *Let* $0 < p_0, p_1 < \infty$. *We put*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,, \qquad \tau := (1-\theta)\,\tau_0 + \theta\,\tau_1 \qquad \text{and} \qquad s := (1-\theta)\,s_0 + \theta\,s_1\,. \tag{3.1}$$

Then there exists a positive constant c such that

$$\|\, f\,|F^{s,\tau}_{p,q}\| \leqslant c\,\|\, f\,|F^{s_0,\tau_0}_{p_0,q_0}\|^{1-\theta}\|\, f\,|F^{s_1,\tau_1}_{p_1,q_1}\|^{\theta} \tag{3.2}$$

holds for all $f \in \mathcal{S}'(\mathbb{R})$.

(ii) *Let* $0 < p_0 < \infty$. *We put*

$$\frac{1}{p} := \frac{1-\theta}{p_0}\,, \qquad \tau := (1-\theta)\,\tau_0 + \theta\,\tau_1 \qquad \text{and} \qquad s := (1-\theta)\,s_0 + \theta\,s_1\,.$$

Then there exists a positive constant c such that

$$\|\, f\,|F^{s,\tau}_{p,q}\| \leqslant c\,\|\, f\,|F^{s_0,\tau_0}_{p_0,q_0}\|^{1-\theta}\|\, f\,|B^{s_1,\tau_1}_{\infty,\infty}\|^{\theta} \tag{3.3}$$

holds for all $f \in \mathcal{S}'$.

Proof. The proof is based on the following elementary inequality due to Oru, cf. [68] and [8]. For any sequence $(a_j)_j$ of complex numbers it holds

$$\|\,(2^{sj}\,a_j)_j\,|\ell_q\| \leqslant c\,\|\,(2^{s_0 j}\,a_j)_j\,|\ell_\infty\|^{1-\theta}\ \|\,(2^{s_1 j}\,a_j)_j\,|\ell_\infty\|^{\theta}\,,$$

where $s = (1-\theta)\,s_0 + \theta\,s_1$ and $0 < q \leqslant \infty$. Here $c = c(s_0, s_1, \theta, q)$, but does not depend on $(a_j)_j$. This can be applied with $a_j = |\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|$. Then Hölder's inequality with respect to p yields

$$\begin{aligned}
\|f\|_{F^{s,\tau}_{p,q}} \;&\leqslant\; c \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\tau}} \left\{\int_Q \sup_{j\geqslant\max(j_Q,0)} 2^{js_0}|\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|^{p_0}\,dx\right\}^{(1-\theta)/p_0}\\
&\qquad\times\left\{\int_Q \sup_{j\geqslant\max(j_Q,0)} 2^{js_1}|\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|^{p_1}\,dx\right\}^{\theta/p_1}\\
&\leqslant\; c \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{(1-\theta)\tau_0}} \left\{\int_Q \sup_{j\geqslant\max(j_Q,0)} 2^{js_0}|\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|^{p_0}\,dx\right\}^{(1-\theta)/p_0}
\end{aligned}$$

$$\times \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\theta\tau_1}} \left\{ \int_Q \sup_{j\geqslant \max(j_Q,0)} 2^{js_1} |\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)|^{p_1}\, dx \right\}^{\theta/p_1} .$$

Hence

$$\| f \,|F^{s,\tau}_{p,q}\| \leqslant c \, \| f \,|F^{s_0,\tau_0}_{p_0,\infty}\|^{1-\theta} \| f \,|F^{s_1,\tau_1}_{p_1,\infty}\|^{\theta} ,$$

as long as $p_1 < \infty$. In the case $p_1 = \infty$ an obvious modification leads to

$$\begin{aligned} \| f \,|F^{s,\tau}_{p,q}\| & \leqslant c \, \| f \,|F^{s_0,\tau_0}_{p_0,\infty}\|^{1-\theta} \\ & \quad \times \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\theta\tau_1}} \sup_{x\in Q} \left(\sup_{j\geqslant \max(j_Q,0)} |\, 2^{s_1 j}\, \mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x)\,| \right)^{\theta} . \end{aligned}$$

The monotonicity with respect to the third parameter complements the proof. □

Remark 6. The surprising fact is the independence of the microscopic parameters q, q_0, q_1. The proof given above is due to Brezis and Mironescu [8] for $\tau = 0$, but see also [90].

Proposition 3.2. *Let $0 < q, q_0, q_1 \leqslant \infty$, $-\infty < s_0 < s_1 < \infty$, $\tau_0, \tau_1 \geqslant 0$, $0 < \theta < 1$ and $0 < p_0, p_1 \leqslant \infty$. With p, τ and s as in (3.1) and with*

$$\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} ,$$

we have the following: there exists a positive constant c such that

$$\| f \,|B^{s,\tau}_{p,q}\| \leqslant c \, \| f \,|B^{s_0,\tau_0}_{p_0,q_0}\|^{1-\theta} \| f \,|B^{s_1,\tau_1}_{p_1,q_1}\|^{\theta} \tag{3.4}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^d)$.

Proof. This time we only apply Hölder's inequality. □

Remark 7. (i) In contrast to Lizorkin-Triebel type spaces this time the parameter q is coupled with q_0 and q_1.

(ii) With $\tau = 0$ Proposition 3.2 is a classical result of Peetre [70]. Necessary and sufficient conditions for the validity of Gagliardo-Nirenberg type inequalities are known in the framework of homogeneous Nikol'skii-Besov-Lizorkin-Triebel spaces, we refer to the recent paper Hajaiej, Molinet, Ozawa and Wang [37].

4 Embeddings

Embeddings are a classical part of the theory of function spaces, we refer in particular to the monographs [5], [6], [67], [70], [98], [99] and [100]. By switching from Lebesgue spaces to Morrey spaces we change the local and the global behaviour. Here we are more interested in the second phenomenon.

4.1 Some Sobolev type embeddings with fixed τ

Concerning Sobolev type embeddings with fixed τ there are several results parallel to the classical case $\tau = 0$, we refer to [108, Corollary 2.2].

Proposition 4.1. *Let $0 \leqslant \tau < \infty$, $0 < r, q \leqslant \infty$ and $-\infty < s_1 < s_0 < \infty$.*

(i) *If $0 < p_0 < p_1 \leqslant \infty$ such that $s_0 - n/p_0 = s_1 - n/p_1$, then*

$$B^{s_0,\tau}_{p_0,q} \hookrightarrow B^{s_1,\tau}_{p_1,q} .$$

(ii) *If $0 < p_0 < p_1 < \infty$ such that $s_0 - n/p_0 = s_1 - n/p_1$, then*

$$F^{s_0,\tau}_{p_0,r} \hookrightarrow F^{s_1,\tau}_{p_1,q} .$$

Remark 8. The proof given in [108] is partly based on a related result in the framework of homogeneous spaces, see Yang and Yuan [106].

4.2 Sobolev type embeddings with a change of τ

Next we turn to the first case of different τ. We choose $\tau = 0$ and $p = \infty$ in the target space.

Proposition 4.2. *Let $s \in \mathbb{R}$, $0 \leqslant \tau < \infty$ and $0 < p, q \leqslant \infty$. Then*

$$A^{s,\tau}_{p,q} \hookrightarrow B^{s+d\tau-d/p}_{\infty,\infty} , \tag{4.1}$$

in particular,

$$A^{s,\tau}_{p,q} \hookrightarrow \mathcal{Z}^{s+d\tau-d/p} \qquad \text{if} \qquad s + d\tau - d/p > 0 . \tag{4.2}$$

Proof. Elementary embeddings, see Lemma 7.2 in the Appendix, combined with Corollary 2.2 and Proposition 2.6 in [108], yield Proposition 4.2. □

Remark 9. (i) The embedding of $B^{s,\tau}_{p,\infty,\mathrm{unif}} \hookrightarrow \mathcal{Z}^{s+d\tau-d/p}$, $1 \leqslant p \leqslant \infty$, $0 < s < 1$, $0 \leqslant \tau \leqslant 1/p$, has been proved by Ross [73]. Here one has to use the characterization by differences of $B^{s,\tau}_{p,\infty}$, see [108, Theorem 4.7] or [93, Theorem 7], for getting an appropriate description of $B^{s,\tau}_{p,\infty,\mathrm{unif}}$.

(ii) Let Q be some fixed dyadic cube in $\mathbb{R}^d$. For embeddings of $B^{s,\frac{1}{u}-\frac{1}{p}}_{u,\infty}(Q) = \mathcal{N}^s_{p,\infty,u}(Q)$ (defined by using the classical definition of Nikol'skii-Besov spaces by differences on $\mathbb{R}^d$ as the starting point, but replacing the $L_p(\mathbb{R}^d)$-norm by the $\mathcal{M}^p_u(Q)$-norm) into Lebesgue and Lorentz spaces we refer to Ross [73] and Adams and Lewis [1].

In view of the classical theory of embeddings of Besov spaces one would expect that

$$B^{s,\tau}_{p,q} \hookrightarrow B^{s+d\tau-d/p}_{\infty,q} , \qquad 0 < q < \infty ,$$

holds as well. But this is not true in general.

Lemma 4.1. *Let $0 < q_0, q_1 < \infty$.*

(i) *Let $p = \infty$ and suppose*

$$0 < \tau < \begin{cases} \frac{s}{d} & \text{if} \quad s > 0; \\ \frac{1-|s|}{d} & \text{if} \quad -1 < s \leqslant 0\,. \end{cases} \tag{4.3}$$

Then

$$B^{s,\tau}_{\infty,q_0} \not\subset B^{s+d\tau}_{\infty,q_1}\,. \tag{4.4}$$

(ii) *Let $0 < p < \infty$ and $0 < \tau < 1/p$. Then*

$$B^{s,\tau}_{p,q_0} \not\subset B^{s+d\tau-d/p}_{\infty,q_1}\,. \tag{4.5}$$

(iii) Let $0 < p < \infty$ and $\tau = 1/p$. Then

$$B^{s,1/p}_{p,q_0} \not\subset B^{s}_{\infty,q_1}\,. \tag{4.6}$$

Proof. Step 1. Proof of (i). We employ the wavelet characterization of the Besov spaces B^s_{∞,q_1}. We shall use standard notation, but see [93, Subsection 3.8] for all details. Choosing the univariate generators $\widetilde{\varphi}$ and $\widetilde{\psi}$ of the wavelet system appropriate (depending on s) we have

$$\|\, f\,\|_{B^s_{\infty,q}} \asymp \sup_{k\in\mathbb{Z}} |\langle f, \varphi_{0,k}\rangle| + \left\{ \sum_{j=0}^{\infty} \left[2^{j(s+\frac{d}{2})} \sup_{i=1,\ldots,2^d-1}\ \sup_{k\in\mathbb{Z}} |\langle f, \psi_{i,j,k}\rangle| \right]^q \right\}^{1/q},$$

see, e.g., [101, Theorem 1.20]. Our test function is given by

$$f(x) := \sum_{j=1}^{\infty} c_j\, 2^{-j(s+d\tau+\frac{d}{2})}\, \psi_{1,j,k^j} \tag{4.7}$$

where $k^j := (2^j, 0, \ldots 0)$, $j \in \mathbb{N}$. It follows

$$\|\, f\,\|_{B^{s+d\tau}_{\infty,q_1}} \asymp \Big(\sum_{j=1}^{\infty} |c_j|^{q_1} \Big)^{1/q_1}.$$

Under the restrictions given in (4.3) the spaces $B^{s,\tau}_{\infty,q}$ allow a characterization in terms of atoms, see [108, Theorem 3.3]. The function ψ_{1,j,k^j} is an inhomogeneous smooth atom for $B^{s,\tau}_{\infty,q}$ supported near Q_{j,k^j} as long as $\widetilde{\varphi}$, $\widetilde{\psi}$ are smooth enough and $\widetilde{\psi}$ satisfies an appropriate moment condition. This implies in view of [108, Theorem 3.3]

$$\|\, f\,\|_{B^{s,\tau}_{\infty,q_0}} \lesssim \sup_{j\in\mathbb{N}} |c_j| + \sup_{M\in\mathbb{N}} 2^{-Md\tau} \Big(\sum_{j=1}^{M} |2^{-jd\tau}\, c_j|^{q_0} \Big)^{1/q_0}.$$

Choosing $c_j := j^{-1/q_1}$ the claim follows.

Step 2. Proof of (ii) and (iii). Since $0 \leqslant \tau \leqslant 1/p$ the spaces $B^{s,\tau}_{p,q_0}$ allow a characterization in terms of atoms for all s, see [108, Theorem 3.3]. For convenience of the reader we argue for $s > \sigma_p$ since in this case we may employ Theorem 4.1 in [108], see also Theorem 4 in [93]. Recall,

$$\|f\|_{B^{s,\tau}_{p,q}}^{\blacktriangle} := \sup_{\{Q\in\mathcal{Q}:\,|Q|\geqslant 1\}} \frac{1}{|Q|^\tau}\left(\sum_{k\in\mathcal{J}_Q} |\langle f,\varphi_{0,k}\rangle|^p\right)^{\frac{1}{p}} + \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau}\left\{\sum_{j=\max(j_Q,0)}^{\infty} 2^{j(s+d/2)q}\sum_{i=1}^{2^d-1}\left[\sum_{k\in\mathcal{I}_{Q,j}} 2^{-jd}|\langle f,\psi_{i,j,k}\rangle|^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

is an equivalent norm on $B^{s,\tau}_{p,q}$ under the given restrictions. By considering small cubes ($|Q|<1$) and large cubes ($|Q|\geqslant 1$) separately, we obtain for f as in (4.7)

$$\|\, f\,\|_{B^{s,\tau}_{p,q_0}} \lesssim \sup_{j\in\mathbb{N}} |2^{-\frac{jd}{p}}\, c_j| + \sup_{M\in\mathbb{N}} 2^{-Md\tau}\left(\sum_{j=1}^{M} |2^{-j(d\tau+\frac{d}{p})}\, c_j|^{q_0}\right)^{1/q_0}. \tag{4.8}$$

Choosing again $c_j := j^{-1/q_1}$ the claim follows. For $s \leqslant \sigma_p$ we have to argue by using atoms and [108, Theorem 3.3]. □

Remark 10. Let $0<p<\infty$. Recall $B^{s,1/p}_{p,p} = F^{s,1/p}_{p,p} = F^{s}_{\infty,p}$, see [29]. It is well-known that $B^s_{\infty,p} \hookrightarrow F^s_{\infty,p}$ and the embedding is strict.

Gagliardo-Nirenberg type inequalities are a powerful tool in investigating embeddings. We present some examples for this claim. To begin with we consider all spaces lying on the line connecting $(s_0, 1/p_0)$ and $(s_0 - d/p_0, 1/\infty)$ in the $(s,(1/p))$-plane, cf. Figure 1. At the same time we allow a change of the Morrey parameter τ.

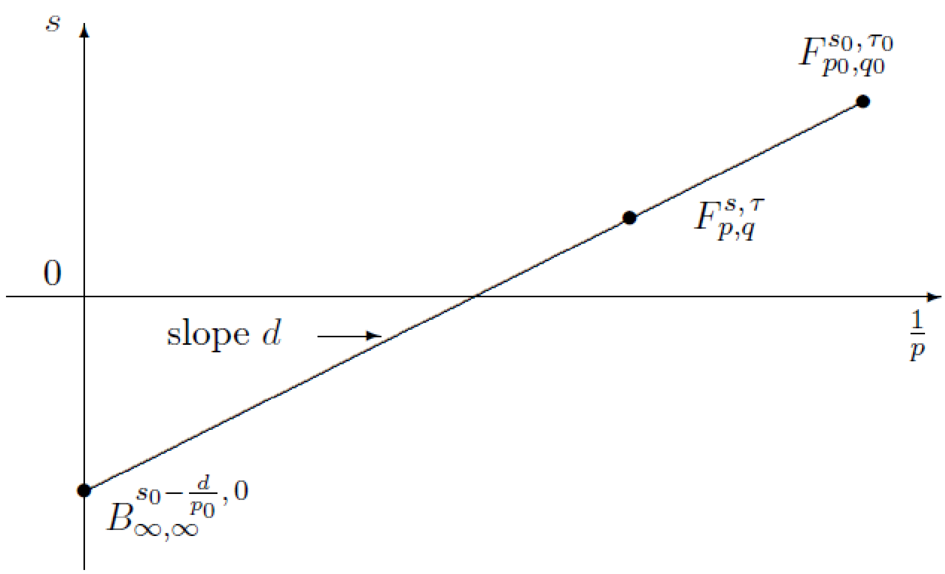

Fig. 1

These restrictions correspond to $F^{s,\tau}_{p,q}$ with

$$s_0 - \frac{d}{p_0} = s - \frac{d}{p} \qquad \text{and} \qquad p_0 < p < \infty\,.$$

Applying Proposition 3.1(ii) and the embedding in (4.1) with $1/p = (1-\theta)/p_0$, choosing $\tau_1 = 0$, and using the abbreviations from (3.1), we find

$$\begin{aligned} \| f \,|F^{s,\tau}_{p,q}\| &\leqslant c \,\| f \,|F^{s_0,\tau_0}_{p_0,q_0}\|^{1-\theta} \| f \,|B^{s_0-d/p_0,\tau_1}_{\infty,\infty}\|^{\theta} \\ &\leqslant c \,\| f \,|F^{s_0,\tau_0}_{p_0,q_0}\|^{1-\theta} \| f \,|F^{s,\tau}_{p,q}\|^{\theta} \end{aligned}$$

without any influence of the microscopic parameters q and q_0, except possibly on the constant c. We summarize this in the following.

Theorem 4.1. *Suppose $s_0 \in \mathbb{R}$, $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1 \leqslant \infty$ and $\tau_0 \geqslant 0$. Further, let*

$$\tau := \frac{p_0\,\tau_0}{p_1} \qquad \text{and} \qquad s := s_0 + d\left(\tau_0 - \frac{1}{p_0}\right) - d\left(\tau - \frac{1}{p_1}\right). \tag{4.9}$$

Then we have the continuous embedding $F^{s_0,\tau_0}_{p_0,q_0} \hookrightarrow F^{s,\tau}_{p_1,q_1}$.

Proof. We put $\theta := 1 - p_0/p_1$. Then $\tau = (1-\theta)\,\tau_0 + \theta\,\tau_1$ with $\tau_1 = 0$. In addition we obtain the relation $s = (1-\theta)\,s_0 + \theta\,(s_0 - d/p_0 + d\tau_0)$. Now we use (4.1) as indicated above. This yields the claim. □

Remark 11. (i) By means of the monotonicity of the scale $F^{s,\tau}_{p,q}$ with respect to s we obtain $F^{s_0,\tau_0}_{p_0,q_0} \hookrightarrow F^{s,\tau}_{p_1,q_1}$ also in case $s \leqslant s_0 + d\left(\tau_0 - \frac{1}{p_0}\right) - d\left(\tau - \frac{1}{p}\right)$ and τ as in (4.9).

(ii) Observe, that the case $\tau = \tau_0 = \tau_1 > 0$, excluded by (4.9), has been treated in Proposition 4.1. For the classical case itself we refer to Jawerth [42] and Triebel [98, 2.7.1].

For convenience of the reader (and for later use) we reformulate Theorem 4.1 for $\tau < 1/p$ in terms of the scale $\mathcal{E}^s_{p,q,u}$.

Corollary 4.1. *Suppose $s_0 \in \mathbb{R}$, $0 < u_0 < u_1 < \infty$, $0 < q_0, q_1 \leqslant \infty$ and*

$$0 < \frac{u_0}{p_0} = \frac{u_1}{p_1} \leqslant 1\,. \tag{4.10}$$

Further, let

$$s_1 := s_0 - d\left(\frac{1}{p_0} - \frac{1}{p_1}\right). \tag{4.11}$$

Then we have the continuous embedding $\mathcal{E}^{s_0}_{p_0,q_0,u_0} \hookrightarrow \mathcal{E}^{s_1}_{p_1,q_1,u_1}$.

Remark 12. (i) Observe, that this result can be supplemented by the monotonicity of the scale with respect to u, i.e. $\mathcal{E}^{s_0,}_{p_0,q_0,u_0} \hookrightarrow \mathcal{E}^{s_0}_{p_0,q_0,u_1}$ if $0 < u_1 \leqslant u_0 \leqslant p_0$. This is a direct consequence of

$$\mathcal{M}^{p_0}_{u_0} \hookrightarrow \mathcal{M}^{p_0}_{u_1}\,, \qquad 0 < u_1 \leqslant u_0 \leqslant p \leqslant \infty\,,$$

which can be proved by applying Hölder's inequality, see [48].

(ii) The homogeneous counterpart of Corollary 4.1 has been proved in Sawano, Sugano, Tanaka [83].

Now we turn to the B-case.

Theorem 4.2. *Suppose $s_0 \in \mathbb{R}$, $0 < p_0 < p_1 \leqslant \infty$, $0 < q_0 \leqslant \infty$ and $\tau_0 \geqslant 0$. Further, let τ and s be as in (4.9). In addition we put $q_1 := q_0\, p_1/p_0$. Then we have the continuous embedding $B^{s_0,\tau_0}_{p_0,q_0} \hookrightarrow B^{s,\tau}_{p_1,q_1}$.*

Proof. As above we put $\theta := 1 - p_0/p_1$. Then $\tau = (1-\theta)\,\tau_0 + \theta\,\tau_1$ with $\tau_1 = 0$. In addition we obtain the relations $s = (1-\theta)\, s_0 + \theta\,(s_0 - d/p_0 + d\,\tau_0)$ and $q_1 = q_0/(1-\theta)$. The embedding (4.1) yields

$$\begin{aligned} \|\, f\,|B^{s,\tau}_{p_1,q_1}\| &\lesssim \|\, f\,|B^{s_0,\tau_0}_{p_0,q_0}\|^{1-\theta} \|\, f\,|B^{s_0-d/p_0,\tau_1}_{\infty,\infty}\|^{\theta} \\ &\lesssim \|\, f\,|B^{s_0,\tau_0}_{p_0,q_0}\|^{1-\theta} \|\, f\,|B^{s,\tau}_{p_1,q_1}\|^{\theta}\,, \end{aligned}$$

which completes the proof. □

Remark 13. (i) We do not know whether q_1 is optimal in this context.

(ii) As in the F-case, the monotonicity of the scale $B^{s,\tau}_{p,q}$ with respect to s yields $B^{s_0,\tau_0}_{p_0,q_0} \hookrightarrow B^{s,\tau}_{p_1,q_1}$ also in case $s < s_0 + d\left(\tau_0 - \frac{1}{p_0}\right) - d\left(\tau - \frac{1}{p}\right)$ and τ as in (4.9). This time q_0 and q_1 can be chosen independent from each other.

(iii) For $\tau = \tau_0 = \tau_1 > 0$, excluded by (4.9), we refer to Proposition 4.1. For the classical case $\tau = \tau_0 = \tau_1 = 0$ we refer to Nikol'skii [67], see also Peetre [70], Jawerth [42] and Triebel [98, 2.7.1].

4.3 Embeddings for the scale $\mathcal{N}^s_{p,q,u}$

In a certain sense the theory of embeddings for the scale $\mathcal{N}^s_{p,q,u}$ is more satisfactory than for the scale $B^{s,\tau}_{p,q}$. In this case a final result is known, see Haroske and Skrzypczak [38].

Theorem 4.3. *Suppose $s_0, s_1 \in \mathbb{R}$, $0 < u_0 \leqslant p_0 < \infty$, $0 < u_1 \leqslant p_1 < \infty$, and $0 < q_0,\, q_1 \leqslant \infty$. Then we have the continuous embedding $\mathcal{N}^{s_0}_{p_0,q_0,u_0} \hookrightarrow \mathcal{N}^{s_1}_{p_1,q_1,u_1}$ if, and only if, $p_0 \leqslant p_1$,*

$$\frac{u_1}{p_1} \leqslant \frac{u_0}{p_0} \qquad \textit{and either} \qquad s_1 < s_0 - d\left(\frac{1}{p_0} - \frac{1}{p_1}\right) \tag{4.12}$$

and no conditions with respect to q_0 and q_1 or

$$s_1 = s_0 - d\left(\frac{1}{p_0} - \frac{1}{p_1}\right) \qquad \textit{and} \qquad q_0 \leqslant q_1\,. \tag{4.13}$$

Remark 14. Sufficiency of

$$\frac{u_1}{p_1} = \frac{u_0}{p_0}, \qquad s_1 = s_0 - d\left(\frac{1}{p_0} - \frac{1}{p_1}\right), \qquad q_0 = q_1\,,$$

in the homogeneous context, i.e., for the scale $\dot{\mathcal{N}}^s_{p,q,u}$, has been known before, see Kozono and Yamazaki [48] and Sawano, Sugano, Tanaka [83].

Also the particular case $p = \infty$ can be treated now.

Lemma 4.2. *Let $s \in \mathbb{R}$, $0 < u \leqslant p \leqslant \infty$ and $0 < q \leqslant \infty$. Then*

$$\mathcal{N}^s_{p,q,u} \hookrightarrow B^{s-d/p}_{\infty,q}\,.$$

Proof. We argue by using Peetre's maximal function, defined as

$$f_j^*(x) := \sup_{z\in\mathbb{R}^d} \frac{\mathcal{F}^{-1}[\varphi_j(\xi)\,\mathcal{F}f(\xi)](x-z)}{(1+2^j|z|)^a}\,, \qquad x \in \mathbb{R}^d\,. \tag{4.14}$$

Here $f \in \mathcal{S}'$, $j \in \mathbb{N}_0$ and $a > 0$ will be chosen later on. Obviously, if $|x-y| < \sqrt{d}\,2^{-j}$, we find

$$f_j^*(x) \leqslant f_j^*(y) \sup_{z\in\mathbb{R}^d} \frac{(1+2^j|z-y-x|)^a}{(1+2^j|z|)^a} \leqslant (1+\sqrt{d})^a\, f_j^*(y)\,. \tag{4.15}$$

Without loss of generality we consider the case $q < \infty$. For given x let $Q_j(x)$ denote the unique dyadic cube of volume 2^{-jd} which contains x. Then, applying (4.15), we find

$$\begin{aligned}
\sum_{j=0}^\infty \quad 2^{j(s-d/p)q} \|\, \mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)]\,\|_{L_\infty} \;&\leqslant\; \sum_{j=0}^\infty 2^{jsq} \|\, f_j^* \|_{L_\infty}^q \\
\lesssim\;& \sum_{j=0}^\infty 2^{j(s-d/p)q} \Big[\sup_{x\in\mathbb{R}^d} |Q_j(x)|^{-1/u} \Big(\int_{Q_j(x)} f_j^*(y)^u\, dy\Big)^{1/u}\Big]^q \\
=\;& \sum_{j=0}^\infty 2^{jsq} \Big[\sup_{x\in\mathbb{R}^d} |Q_j(x)|^{\frac{1}{p}-\frac{1}{u}} \Big(\int_{Q_j(x)} f_j^*(y)^u\, dy\Big)^{1/u}\Big]^q \\
\lesssim\;& \sum_{j=0}^\infty 2^{jsq} \Big[\sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{u}} \Big(\int_{Q} f_j^*(y)^u\, dy\Big)^{1/u}\Big]^q \\
=\;& \sum_{j=0}^\infty 2^{jsq} \|\, f_j^* \,\|^q_{\mathcal{M}^p_u}\,.
\end{aligned} \tag{4.16}$$

To continue we shall employ the inequality

$$f_j^*(x) \leqslant c_1\, (M|\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)]|^r)^{1/r}(x)\,,$$

where c_1 does not depend on f, j and x, proved in [98, 1.6.2/(2)]. Moreover, r is at our disposal. The boundedness of the Hardy-Littlewood maximal function in Morrey spaces, see [18], yields

$$\begin{aligned}
\|\, f_j^* \,\|_{\mathcal{M}^p_u} \;&\leqslant\; c_2 \Big(\|\, (M|\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)]|^r)\,\|_{\mathcal{M}^{p/r}_{u/r}}\Big)^{1/r} \\
&\leqslant\; c_3 \Big(\|\, |\mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)]|^r \,\|_{\mathcal{M}^{p/r}_{u/r}}\Big)^{1/r} \\
&=\; c_3 \,\|\, \mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)]\,\|_{\mathcal{M}^p_u}
\end{aligned}$$

if $0 < r < u \leqslant p$. Hence,

$$\|\, f \,\|_{B^{s-d/p}_{\infty,q}} \leqslant c_4 \,\|\, f \,\|_{\mathcal{N}^s_{p,q,u}}$$

with c_4 independent of f. □

Remark 15. (i) Lemma 4.2 has to be compared with Lemma 4.1. In case $p = \infty$ we have equality of the spaces, i.e., $\mathcal{N}^s_{\infty,q,u} = B^s_{\infty,q}$, a consequence of $\mathcal{M}^\infty_u = L_\infty$.

(ii) Again Sawano, Sugano, Tanaka [83] have proved the homogeneous counterpart $\dot{\mathcal{N}}^s_{p,q,u} \hookrightarrow \dot{B}^{s-d/p}_{\infty,q}$.

(iii) Also Netrusov [64] has investigated embeddings of some Besov-Morrey spaces $BM^s_{p,q,a,\varkappa}(G)$ defined on domains G. In case of $G = \mathbb{R}^d$ the spaces he studied are subspaces of $\mathcal{N}^s_{p,q,u,\mathrm{unif}}$, more exactly,

$$BM^s_{p,q,a,\varkappa}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{\frac{p}{1-a},q,u,\mathrm{unif}}\,, \qquad 0 \leqslant a \leqslant 1\,, \quad \varkappa = (\varkappa_0, \ldots, \varkappa_0)\,, \quad \varkappa_0 > 0\,.$$

The spaces $BM^s_{p,q,a,\varkappa}(G)$ are built on modified Morrey spaces, see (4.18) below.

4.4 Embeddings into C_{ub}

Of particular interest are embeddings into C_{ub}, the collection of all complex-valued, uniformly continuous and bounded functions defined on $\mathbb{R}^d$. Proposition 4.2 implies $A^{s,\tau}_{p,q} \hookrightarrow C_{ub}$ if $s + d\tau - d/p > 0$, see also [83]. Now we partly improve this embedding.

Theorem 4.4. *(i) Let $0 < u \leqslant p \leqslant \infty$. Then*

$$\mathcal{N}^{\frac{d}{p}}_{p,1,u} \hookrightarrow N^{\frac{d}{p}}_{p,1,u} \hookrightarrow C_{ub}\,.$$

(ii) Let $0 < p < \infty$, $0 < q \leqslant \infty$, $0 < \tau < 1/p$ and $s + d\tau - \frac{d}{p} = 0$. Then

$$B^{s,\tau}_{p,q} \not\subset C_{ub} \qquad \text{and} \qquad F^{s,\tau}_{p,q} \not\subset C_{ub}\,.$$

(iii) Let $0 < p < \infty$ and $0 < q \leqslant \infty$. Then

$$F^{0,1/p}_{p,q} \not\subset C_{ub} \qquad \text{and} \qquad B^{0,1/p}_{p,q} \not\subset C_{ub}\,.$$

(iv) Let $0 < q \leqslant \infty$, $0 < \tau < \frac{1}{2d}$ and $s + d\tau = 0$. Then

$$B^{s,\tau}_{\infty,q} \not\subset C_{ub}\,.$$

Proof. Step 1. For a proof of part (i) we refer to Kozono and Yamazaki [48] or simply to the proof of Lemma 4.2.

Step 2. Proof of (ii). This time our test function is given by

$$f(x) := \sum_{j=1}^{\infty} c_j\, 2^{-j(s+d(\tau-\frac{1}{p})+\frac{d}{2})}\, \psi_{1,j,k^j} = \sum_{j=1}^{\infty} c_j\, 2^{-\frac{jd}{2}}\, \psi_{1,j,k^j}\,, \qquad x \in \mathbb{R}^d\,, \tag{4.17}$$

where $k^j := (2^j, 0, \ldots 0)$, $j \in \mathbb{N}$. By obvious modifications in the estimate (4.8) we obtain

$$\|\, f\,\|_{B^{s,\tau}_{p,q}} \lesssim \sup_{j\in\mathbb{N}} |c_j| + \sup_{M\in\mathbb{N}} 2^{-Md\tau} \Big(\sum_{j=1}^{M} |2^{-jd\tau}\, c_j|^q\Big)^{1/q}\,.$$

With $c_j := 1$ for all $j \in \mathbb{N}$ the right-hand side is bounded. On the other hand the function

$$f(x) = \sum_{j=1}^{\infty} \psi_1(2^j x - k^j)$$

is obviously continuous but not uniformly continuous. This proves the claim for $B^{s,\tau}_{p,q}$. By means of the elementary embeddings, see Lemma 7.2 in the Appendix, also the claim for $F^{s,\tau}_{p,q}$ follows.

Step 3. Proof of (iii). In case of $B^{0,1/p}_{p,q}$ we argue as in Step 2. Again Lemma 7.2 can be used to derive the conclusion for $F^{0,1/p}_{p,q}$.

Step 4. Proof of (iv). We employ the test function (4.7). Also this function is not uniformly continuous. □

Remark 16. (i) As a matter of fact, Nikol'skii-Besov-Lizorkin-Triebel spaces, i.e., the spaces $B^s_{p,q}$ and $F^s_{p,q}$, can not distinguish between boundedness, continuity and uniform continuity. All counterexamples, studied in Lemma 4.1 and Theorem 4.4, are continuous but not uniformly continuous. It would be of certain interest to study under which conditions on s, p, q, τ unboundedness or discontinuity become possible. Also of certain interest would be the questions whether the above assertions extend to the local situation. Observe in this connection, that the counterexamples, defined in (4.7) and (4.17), do not have compact support.

(ii) Recall $F^{0,1/p}_{p,q} = F^0_{\infty,q}$, $0 < q < \infty$, see [29]. It is well-known that $F^0_{\infty,q}$ contains unbounded functions, see [94].

(iii) Let us comment on the special case of Sobolev-Morrey spaces by taking $s = m \in \mathbb{N}$, $q = 2$ and $1 < u \leqslant p < \infty$. Then part (ii) of Theorem 4.4 implies

$$F^{m,\frac{1}{u}-\frac{1}{p}}_{u,2} = W^m(\mathcal{M}^p_u) \not\subset C_{ub}\,, \qquad m = \frac{d}{p}\,.$$

This supplements earlier results of Dchumakaeva [22]. In [22] it is also proved

$$W^m(\mathcal{M}^p_u)((0,1)^d) \hookrightarrow C_{ub}((0,1)^d)\,, \qquad m > \frac{d}{p}\,,$$

which is covered by Proposition 4.2.

(iv) Besov, Il'in and Nikol'skii [6, Section 27] have studied a slightly modified version of Sobolev-Morrey spaces and related embeddings. Instead of $\| f \|_{\mathcal{M}^p_u}$ they worked with

$$\sup_B \min(|B|, 1)^{\frac{1}{p}-\frac{1}{u}} \Big(\int_B |f(x)|^u \, dx \Big)^{1/u}, \tag{4.18}$$

where the supremum is taken with respect to all balls in $\mathbb{R}^d$. In fact, these authors have treated a much more general nonisotropic situation on domains with some regularity of the boundary.

5 Different approaches to smoothness spaces related to Morrey spaces

The aim of this section consists in mentioning three further approaches to smoothness spaces related to Morrey spaces. First of all we direct the attention of the reader to a very recent variant of Morrey type spaces oriented on wavelet analysis and due to Triebel [103]. Secondly, in recent years there is an increasing interest in developing general frameworks of Nikol'skii-Besov-Lizorkin-Triebel spaces. Here we would like to mention on the one hand the approach of Hedberg and Netrusov [39] and on the other side to approach of Liang, Sawano, Ullrich, Yang, Yuan [54].

5.1 Triebel's local spaces

We are using the notation from Subsection 3.8 in [93], in particular the wavelet system from formula (50) there.

Let $0 < p \leqslant \infty$ and $-d/p \leqslant r < \infty$. Then we require that the generators $\widetilde{\varphi}, \widetilde{\psi}$ of the wavelet system Ψ belong to $C^u(\mathbb{R})$ for some u satisfying

$$u > \begin{cases} \max(s, s+r, \sigma_p - s) & \text{if} \quad A = B\,, \\ \max(s, s+r, \sigma_{p,q} - s) & \text{if} \quad A = F\,. \end{cases} \tag{5.1}$$

For a given dyadic cube Q we denote by $2\,Q$ the cube, having the same center, sides parallel to the original cube and side-length twice larger. In the spirit of the wavelet characterizations of $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$, see Theorems 4 and 5 in [93] or Theorem 4.1 in [108], Triebel introduces the following scales of spaces.

Definition 1. Let $0 < p, q \leqslant \infty$, $s \in \mathbb{R}$ and $-d/p \leqslant r < \infty$. Let Ψ denote the wavelet system in [93, formula (50)] satisfying (5.1). Then, with $A \in \{B, F\}$,

$$\mathcal{L}^r A^s_{p,q} := \Big\{ f \in \mathcal{S}' : \quad \| \, f \, \|_{\mathcal{L}^r A^s_{p,q}, \Psi} < \infty \Big\}\,,$$

where

$$\begin{aligned} & \| \, f \, \|_{\mathcal{L}^r A^s_{p,q}, \Psi} \\ := \; & \sup_{J \in \mathbb{N}_0} \sup_{M \in \mathbb{Z}} 2^{J(\frac{d}{p}+r)} \Big\| \sum_{\substack{k \in \mathbb{Z}^d \\ Q_{0,k} \subset 2\,Q_{J,M}}} \langle f, \varphi_{0,k} \rangle \, \psi_{0,k} + \sum_{i=1}^{2^d-1} \sum_{\substack{j \in \mathbb{N}_0,\, k \in \mathbb{Z}^d \\ Q_{j,k} \subset 2\,Q_{J,M}}} \langle f, \psi_{i,j,k} \rangle \, \psi_{i,j,k} \Big\|_{A^s_{p,q}} \end{aligned}$$

($p < \infty$ if $A = F$).

Remark 17. Of course,

$$\sum_{\substack{k \in \mathbb{Z}^d \\ Q_{0,k} \subset 2\,Q_{J,M}}} \langle f, \varphi_{0,k} \rangle \, \psi_{0,k} + \sum_{i=1}^{2^d-1} \sum_{\substack{j \in \mathbb{N}_0,\, k \in \mathbb{Z}^d \\ Q_{j,k} \subset 2\,Q_{J,M}}} \langle f, \psi_{i,j,k} \rangle \, \psi_{i,j,k}$$

is the part of the wavelet expansion of f which lives near $Q_{J,M}$. Since the supremum with respect to J runs through $\mathbb{N}_0$, only dyadic cubes with side-length $\leqslant 1$ come into the competition. Comparing with Theorems 4, 5 in [93] one would expect relations of $\mathcal{L}^r A^s_{p,q}$ to the spaces $A^{s,\tau}_{p,q,\mathrm{unif}}$, $\tau = \frac{r}{d} + \frac{1}{p}$.

Triebel [103] announced some results.

Theorem 5.1. *Let $0 < p,q \leqslant \infty$, $s,\sigma \in \mathbb{R}$ and $-d/p \leqslant r < \infty$.*

(i) *Then the operator I_σ, defined by*

$$I_\sigma f := \mathcal{F}^{-1}[(1+|\cdot|^2)^{\sigma/2}\mathcal{F}f]\,, \qquad f \in \mathcal{S}',$$

maps $\mathcal{L}^r B^s_{p,q}$ isomorphically onto $\mathcal{L}^r B^{s-\sigma}_{p,q}$.

(ii) *If $r > 0$, then $\mathcal{L}^r B^s_{p,q} = B^{s+r}_{\infty,\infty}$ in the sense of equivalent quasi-norms.*

Remark 18. (i) We refer to Proposition 9 and Corollary 2 in [93] for comparison with the spaces $A^{s,\tau}_{p,q}$.

(ii) Also several results on embeddings are stated in [103]. We omit details. As the main open problems with respect to the scales $\mathcal{L}^r A^s_{p,q}$ Triebel is mentioning the following:

(a) Find necessary and sufficient conditions for the validity of embeddings

$$\mathcal{L}^{r_0} A^{s_0}_{p_0,q_0} \hookrightarrow \mathcal{L}^{r_1} A^{s_1}_{p_1,q_1}\,.$$

(b) Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We define $\mathcal{L}^r A^s_{p,q}(\Omega)$ by restrictions, i.e., $\mathcal{L}^r A^s_{p,q}(\Omega)$ is the set of all $f \in D'(\Omega)$ such that there exists a tempered distribution $g \in \mathcal{L}^r A^s_{p,q}$ and $g_{|\Omega} = f$, equipped with the quotient quasi-norm. Then find necessary and sufficient conditions for the compactness of the embedding

$$\mathcal{L}^{r_0} A^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow \mathcal{L}^{r_1} A^{s_1}_{p_1,q_1}(\Omega)\,.$$

(c) Find the asymptotic behaviour of the entropy numbers of the compact embeddings in part (b).

5.2 The approach of Hedberg and Netrusov

Hedberg and Netrusov [39] have used the following general concept.

Let E be a quasi-Banach space of sequences of Lebesgue measurable functions on $\mathbb{R}^d$, which has the lattice property with respect to the natural ordering.

Definition 2. Let $\varepsilon_+, \varepsilon_- \in \mathbb{R}$, $0 < r < \infty$ and $\alpha \geqslant 0$. We say that the space E (as above) belongs to the class $S(\varepsilon_+, \varepsilon_-, r, \alpha)$ if the following conditions are satisfied:

(a) The linear shift operators S_+ and S_-, defined by

$$\begin{aligned} S_+\Big(\{f_i\}_{i=0}^\infty\Big) &:= \{f_{i+1}\}_{i=0}^\infty\,, \\ S_-\Big(\{f_i\}_{i=0}^\infty\Big) &:= \{f_{i-1}\}_{i=0}^\infty\,, \qquad f_{-1} \equiv 0\,, \end{aligned}$$

are continuous on E, and their quasi-norms satisfy for all $j \in \mathbb{N}$ the inequalities

$$\| (S_+)^j \|_{\mathcal{L}(E)} \leqslant c_1\, 2^{-j\varepsilon_+} \qquad \text{and} \qquad \| (S_-)^j \|_{\mathcal{L}(E)} \leqslant c_2\, 2^{-j\varepsilon_-}$$

(with constants c_1, c_2 independent of j).
(b) The vector-valued maximal operator $\mathcal{M}_{r,\alpha}$, defined by

$$\mathcal{M}_{r,\alpha}\Big(\{f_i\}_{i=0}^\infty\Big) := \{M_{r,\alpha} f_i\}_{i=0}^\infty\,,$$

with

$$M_{r,\alpha} f(x) := \sup_{a>0}\, \Big(\frac{1}{a^d}\int_{|y|<a} |f(x+y)|^r\, (1+|y|)^{-r\alpha}\, dy\Big)^{1/r}\,,$$

is bounded on E.

We define two different types of spaces of sequences of Lebesgue measurable functions. Let $0 < q \leqslant \infty$ and $s \in \mathbb{R}$. For a given quasi-Banach space E_0 of Lebesgue measurable functions we denote by $\ell_q^s(E_0)$ the collection of all sequences $\{f_j\}_{j=0}^\infty$ of those functions such that

$$\|\{f_j\}_{j=0}^\infty\|_{\ell_q^s(E_0)} := \left(\sum_{j=0}^\infty 2^{jsq}\, \|f_j\|_{E_0}^q\right)^{1/q} < \infty\,.$$

Furthermore, by $E_0(\ell_q^s)$ we denote the collection of all sequences $\{f_j\}_{j=0}^\infty$ of Lebesgue-integrable functions such that

$$\|\{f_j\}_{j=0}^\infty\|_{E_0(\ell_q^s)} := \left\| \left(\sum_{j=0}^\infty 2^{jsq}\, |f_j|^q\right)^{1/q} \right\|_{E_0} < \infty\,.$$

Of course, both, $\ell_q^s(E_0)$ and $E_0(\ell_q^s)$ are quasi-Banach spaces. Now we are specializing $E_0 := \mathcal{M}_u^p$.

Lemma 5.1. *Let $0 < u \leqslant p \leqslant \infty$, $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.*
(i) *Then $\ell_q^s(\mathcal{M}_u^p)$ belongs to the class $S(s,s,r,0)$ if $0 < r < u$.*
(ii) *Let $p < \infty$. Then $\mathcal{M}_u^p(\ell_q^s)$ belongs to the class $S(s,s,r,0)$ if $0 < r < \min(u,q)$.*

Proof. Only the boundedness of the maximal operators $M_{r,0}$ requires a comment. For the scalar case we refer to Chiarenza and Frasca [18]. The vector-valued situation has been treated by Tang and Xu [96]. □

On spaces $E \in S(s,s,r,\alpha)$ Hedberg and Netrusov have built smoothness spaces called $Y(E)$.

Definition 3. Let $E \in S(s,s,r,\alpha)$ for some $\alpha \geqslant 0$. The space $Y(E)$ consists of all distributions $f \in \mathcal{S}'$, which have a representation $f = \sum_{i=0}^\infty f_i$, converging in $\mathcal{S}'$, satisfying $\|\{f_i\}_{i=0}^\infty\|_E < \infty$ and

$$\operatorname{supp} \mathcal{F} f_0 \subset B(0,2)\,, \qquad \operatorname{supp} \mathcal{F} f_i \subset B(0,2^{i+1}) \setminus B(0,2^{i-1})\,, \quad i \in \mathbb{N}\,.$$

We put

$$\| f \|_{Y(E)} := \inf \Big\{ \|\{f_i\}_{i=0}^\infty\|_E : \quad (f_i)_i \quad \text{is as above}\Big\}\,.$$

Corollary 5.1. *Let $0 < u \leqslant p \leqslant \infty$, $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.*

(i) *We have $Y(\ell_q^s(\mathcal{M}_u^p)) = \mathcal{N}_{p,q,u}^s$ in the sense of equivalent quasi-norms.*

(ii) *Let $p < \infty$. Then we have $Y(\mathcal{M}_u^p(\ell_q^s)) = \mathcal{E}_{p,q,u}^s$ in the sense of equivalent quasi-norms.*

Proof. Let $(\varphi_i)_{i=0}^\infty$ be a smooth dyadic decomposition of unity as in (7.2). For $f \in \mathcal{N}_{p,q,u}^s(\mathcal{E}_{p,q,u}^s)$ we may choose

$$f_i := \mathcal{F}^{-1}[\varphi_i(\xi)\,\mathcal{F}f(\xi)]\,, \qquad i \in \mathbb{N}_0\,.$$

This implies $\mathcal{N}_{p,q,u}^s \hookrightarrow Y(\ell_q^s(\mathcal{M}_u^p))$ $(\mathcal{E}_{p,q,u}^s \hookrightarrow Y(\mathcal{M}_u^p(\ell_q^s)))$. The converse direction follows by the same type of arguments as used in proving the independence of the smooth dyadic decomposition of unity, see, e.g., Kozono and Yamazaki [48], Tang and Xu [96], [108, Lemma 4.1]. □

Hedberg and Netrusov [39] have been able to characterize the spaces $Y(E)$ in terms of differences, by means of local polynomial approximation and by atoms. These characterizations carry over to the classes $\mathcal{N}_{p,q,u}^s$ and $\mathcal{E}_{p,q,u}^s$, respectively, by applying Corollary 5.1. We omit details and refer to Subsections 3.8.4 and 3.9 in [93], see also [108, Section 4.5].

5.3 A further recent approach to generalized Nikol'skii-Besov-Lizorkin-Triebel spaces

We will be brief here. Recently Liang, Sawano, Ullrich, Yang, Yuan [54] modified the axiomatic background of the Hedberg-Netrusov approach. Roughly speaking, these authors replaced the boundedness of the vector-valued maximal operator $\mathcal{M}_{r,\alpha}$ by the boundedness of the Peetre maximal function, see (4.14). As in [39] a number of interesting properties of the associated scales of Nikol'skii-Besov-Lizorkin-Triebel spaces are treated, in particular, characterizations by atoms, molecules and wavelets, as well as characterizations by differences and oscillations (local polynomial approximation). Within the examples also generalized Morrey spaces are treated.

6 Open problems

We finish this survey with a number of open problems. These problems are ordered into two groups. The first group is dealing with further properties of the scales $A_{p,q}^{s,\tau}$ which could be of a certain interest. The second one deals with generalizations.

6.1 Open problems I - further properties

- **Relations within our scales**.

  (a) Under which conditions on the parameters s, u, p, q we have the coincidence

$$E_{u,q,p}^s = F_{p,q,\mathrm{unif}}^{s,\frac{1}{p}-\frac{1}{u}} \qquad \text{(see Proposition 7(iii) in [93])}\,. \tag{6.1}$$

(b) Under which conditions on the parameters s, u, p, q we have the coincidence

$$N^s_{u,q,p} = \mathcal{B}^{s,\frac{1}{p}-\frac{1}{u}}_{p,q,\mathrm{unif}} \qquad \text{(see Proposition 7(i) in [93])}\,. \tag{6.2}$$

- **Boundedness, continuity and uniform continuity**. Find necessary and sufficient conditions such that either $A^{s,\tau}_{p,q} \hookrightarrow L_\infty$ or $A^{s,\tau}_{p,q} \hookrightarrow C_b$, where C_b is the collection of all complex-valued, continuous and bounded functions on $\mathbb{R}^d$, see Remark 16.

- **Interpolation**. There are many open questions with respect to real and complex interpolation, e.g., is the formula

$$\mathcal{E}^s_{p,q,u} = [\mathcal{E}^{s_0}_{p,q_0,u_0}, \mathcal{E}^{s_1}_{p,q_1,u_1}]_\theta \qquad \text{if} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$\frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1}\,, \qquad s := (1-\theta)\, s_0 + \theta\, s_1\,,$$

correct, see Proposition 2.4. With $s = s_0 = s_1$ it partly follows from Proposition 2.4(i). For some more results in this direction we refer to Kozono and Yamazaki [48, 49].

- **Regularity of characteristic functions.** For a better understanding of the smoothness conditions one should study the question under which conditions on the boundary the characteristic function $\mathcal{X}_\Omega$ of the open, bounded and simply connected set Ω belongs to $A^{s,\tau}_{p,q}$. For some results in this direction we refer to [75] and [91, 92]. Applications can be found in Clop, Faraco, Ruiz [19] and Faraco, Rogers [26].

- **Haar basis**. Find necessary and sufficient conditions on s, p, q, τ such that the Haar wavelet can be used to characterize $A^{s,\tau}_{p,q}$. For $B^s_{p,q}$ and $F^s_{p,q}$ this problem has a certain history. In case of Nikol'skii-Besov spaces one knows the final answer, in case of Lizorkin-Triebel spaces this question is partly open. We refer to Triebel [102] and the references given there for more details.

- **Embeddings**. Find necessary and sufficient conditions on $s_0, s_1, p_0, p_1, q_0, q_1, \tau_0, \tau_1$ such that the embedding $A^{s_0,\tau_0}_{p_0,q_0} \hookrightarrow A^{s_1,\tau_1}_{p_1,q_1}$, $A \in \{B, F\}$, takes place.

- **Jawerth-Franke embeddings.** Are there counterparts of the Jawerth-Franke embeddings, see Jawerth [41] and Franke [28] ? This means, find necessary and sufficient conditions on $s_0, s_1, p_0, p_1, q_0, q_1, \tau_0, \tau_1$ such that either $B^{s_0,\tau_0}_{p_0,q_0} \hookrightarrow F^{s_1,\tau_1}_{p_1,q_1}$ or $F^{s_0,\tau_0}_{p_0,q_0} \hookrightarrow B^{s_1,\tau_1}_{p_1,q_1}$ holds. For necessary conditions in case $\tau_0 = \tau_1 = 0$ we refer to [94].

- **Spaces on domains**. In view of possible applications of these scales it is important to develop the theory of the scales $A^{s,\tau}_{p,q}(\Omega)$ for reasonable domains Ω. Here it would be convenient to construct linear and bounded extension operators, if possible, independent on s, p, q and τ. For $\tau = 0$ we refer to Rychkov [76] and the references given there.

- **Compact embeddings**. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Find necessary and sufficient conditions on $s_0, s_1, p_0, p_1, q_0, q_1, \tau_0, \tau_1$ such that the embedding $A^{s_0,\tau_0}_{p_0,q_0} \hookrightarrow A^{s_1,\tau_1}_{p_1,q_1}$, $A \in \{B, F\}$, is compact. For the case $\tau_0 = \tau_1 = 0$ we refer to [25].

- **Pointwise multipliers**. Recall, if $0 < s < 1$, then

$$f \in M(B^s_{1,1}) \qquad \Longleftrightarrow \qquad f \in L_\infty \cap B^{s,\tau}_{1,1,\mathrm{unif}}\,, \tau = 1 - s/d\,,$$

see Corollary 8 in [93]. There is another interesting result which connects pointwise multipliers, (homogeneous) Besov spaces and Morrey spaces. Lemarié-Rieusset [51] proved the following very interesting result. Suppose $0 < s \leqslant 3/2$. For a locally square integrable function f, defined on $\mathbb{R}^3$, there is the following equivalence:

$$f \in M(\dot{B}^s_{2,1}(\mathbb{R}^3), L_2(\mathbb{R}^3)) \qquad \Longleftrightarrow \qquad f \in \mathcal{M}^{3/s}_2(\mathbb{R}^3)\,.$$

Hence, the problem consists in clarifying the role of Morrey type spaces as spaces of pointwise multipliers for pairs of Nikol'skii-Besov and Lizorkin-Triebel spaces. For more results in this direction see also Lemarié-Rieusset [52].

- **Duality**. Duality seems to be a difficult task in the framework of Morrey spaces. Zorko [110] has found spaces (block spaces) such that their dual is a Morrey space, see also Kalita [43] and Adams, Xiao [2]. Concerning preduals of Morrey spaces we refer to Blasco, Ruiz, Vega [7] and to Sawano, Tanaka [88]. Some open problems with this respect can be found in Izuki, Sawano and Tanaka [40]. Dual spaces of the closure of C_0^∞ in $A^{s,\tau}_{p,p}$, $s \in \mathbb{R}$, $1 < p < \infty$, $0 \leqslant \tau \leqslant 1/p$, are determined in [108, Theorem 7.12]. For dual spaces of the closure of $\mathcal{S}_\infty$ in $\dot{F}^{s,\tau}_{p,p}$, $s \in \mathbb{R}$, $1 < p < \infty$, $0 \leqslant \tau \leqslant 1/p$, where

$$\mathcal{S}_\infty := \Big\{\varphi \in \mathcal{S} : \int_{\mathbb{R}^d} \varphi(x)\, x^\gamma\, dx = 0 \quad \text{for all} \quad \gamma \in \mathbb{N}_0^d\Big\}\,,$$

we refer to Yang, Yuan [107]. For the homogeneous spaces $\dot{A}^{s,\tau}_{p,q}$, $A \in \{F, B\}$, we refer to Yang, Yuan [105, 106, 107] and [108, Chapt. 8].

6.2 Open problems II - generalizations

We already indicated some possible modifications and/or generalizations. Here are some more.

(a) Replace the Morrey space $\mathcal{M}^p_u$ by generalized Morrey spaces $\mathcal{M}^\varphi_u$, see, e.g., Spanne [95], Peetre [69] or Nakai [63]. Introduce the spaces $A^{s,\varphi}_{p,q}$ just by replacing the factor $|Q|^\tau$ by $\varphi(|Q|)$ in Definition 4. Try to develop the theory as above and ask for applications of the fine tuning with φ in case of embeddings. For the particular example of embeddings of generalized Sobolev-Morrey spaces into C we refer to Dchumakaeva [22]. The boundedness of the Hardy-Littlewood maximal function has been proved by Mizuhara [57] and Nakai [62, 63] (scalar case) and by Sawano [80] (vector-valued case), always under certain restrictions on φ. We also refer to Sawano, Sugano and Tanaka [84] in this context. Consequently, parts of the theory are covered by the general approaches of Hedberg and Netrusov [39], see Subsection 5.2, and Liang, Sawano, Ullrich, Yang, Yuan [54].

There is also a number of papers dealing with the more general variant, where φ not only depends on the size of Q but also on its position. We refer, e.g., to Mizuhara [57], Nakai [62], Guliyev [34], Guliyev, Aliyev, Karaman [35], Akbulut,Guliyev and Mustafayev [3], Guliyev, Aliyev, Karaman and Shukurov [36].

(b) Local Morrey spaces. For $\lambda > 0$ and $0 < p, q \leqslant \infty$ the local Morrey space $LM^{\lambda}_{p,q}$ is the collection of all $f \in L^{\ell oc}_p$ such that

$$\| f \|_{LM^{\lambda}_{p,q}} := \Big(\int_0^\infty \big[r^{\lambda} \| f \|_{L_p(B(0,r))} \big]^q \, \frac{dr}{r} \Big)^{1/q} < \infty \, .$$

The relation to the original Morrey spaces follow from

$$\| f \|_{\mathcal{M}^{\lambda}_p} = \sup_{x \in \mathbb{R}^d} \| f(\,\cdot\, + x) \|_{LM^{\lambda}_{p,\infty}} \, .$$

The remarkable property of this new scale consists in the interpolation formula

$$(LM^{\lambda_0}_{p,q_0}, LM^{\lambda_1}_{p,q_1})_{\theta,q} = LM^{\lambda}_{p,q} \, , \qquad \lambda := (1-\theta)\,\lambda_0 + \theta\,\lambda_1 \, ,$$

valid, if $0 < p, q_0, q_1, q \leqslant \infty$, $0 < \theta < 1$, $\lambda_0, \lambda_1 > 0$, $\lambda_0 \neq \lambda_1$. For all these facts we refer to Burenkov and Nursultanov [16] and Burenkov, Darbayeva, Nursultanov [11].

The boundedness of the Hardy-Littlewood maximal function in this context has been proved by Burenkov and H. V. Guliyev [12]. There is a number of papers dealing with those types of spaces, we refer to Burenkov, H. V. Guliyev, V. S. Guliyev [13] - [15], Mizuhara [57], Nakai [62], and survey papers by Burenkov [9], [10].

(c) Refine the scales $A^{s,\tau}_{p,q}$ with respect to s and study the consequences in case of limiting embeddings. Function spaces of generalized smoothness have been introduced and considered by several authors, in particular since the middle of the seventies up to the end of the eighties, with different starting points and in different contexts. We refer in particular to the works of Goldman [30, 31, 32, 33] and Kalyabin [44, 45] (with Ul'yanov (1968) and Dzhafarov (1965) as certain forerunners). The surveys of Kalyabin, Lizorkin [46], Kudryavtsev, Nikol'skii [50, Chapter 5] and Leopold, Farkas [27] cover the literature to a certain extend. But let us mention also the contributions of Netrusov [65, 66] and Cobos, Fernandez [20] in this context.

(d) **Weights**. Izuki, Sawano and Tanaka [40] investigated weighted variants of the spaces $\mathcal{E}^s_{p,q,u}$ and $\mathcal{N}^s_{p,q,u}$. The class of weights, used by these authors, is $A^{\ell oc}_\infty$. Based on a weighted local-Morrey maximal inequality (here the authors are oriented on the earlier work of Rychkov [77]) they prove characterizations via atoms. For the boundedness of the Hardy-Littlewood maximal function on weighted Morrey spaces we refer to Komori and Shirai [47] and to [40]. It is of certain interest to study embeddings of weighted spaces into unweighted spaces (e.g., to obtain compactness of embeddings also for spaces defined on $\mathbb{R}^d$). First attempts in this direction have been undertaken by Haroske and Skrzypczak [38] with respect to the weighted scale associated to $\mathcal{N}^s_{p,q,u}$ (Muckenhoupt weights).

(e) **Non-doubling measures**. In a series of papers Sawano and Tanaka developed the theory of the scales $\mathcal{E}^s_{p,q,u}$ and $\mathcal{N}^s_{p,q,u}$ associated to Morrey spaces defined by using a general non-doubling measure, see [78, 80, 85, 86, 81, 87].

(f) **Non-isotropic smoothness**. Of course, there is also some motivation to consider anisotropic versions of $A^{s,\tau}_{p,q}$. First steps in this directions have been done in Besov, Il'in, Nikol'skii [6, Section 27] and Netrusov [64]. Function spaces of dominating mixed smoothness based on Morrey spaces have been considered in Dchumakaeva [21] and Najavov [60, 61].

Note added in proof. In the meanwhile Triebel has written a book [104] about the spaces $\mathcal{L}^r A^s_{p,q}$, see Subsection 5.1. In addition, the conjecture in Remark 17 has been proved to be true, i.e.,

$$\mathcal{L}^{d(\tau-1/p)} A^s_{p,q} = A^{s,\tau}_{p,q}$$

in the sense of equivalent quasi-norms, we refer to Yuan, S. and Yang [109].

7 Appendix

Here we recall some basic notions from Part I of this survey.

A cube Q such that

$$Q = Q_{j,k} := \{x \in \mathbb{R}^d : \; 2^{-j} k_\ell \leqslant x_\ell < 2^{-j}(k_\ell + 1) \, , \; \ell = 1, \ldots , d\} \, ,$$

for some $j \in \mathbb{Z}$ and some $k \in \mathbb{Z}^d$ is called *dyadic*. The collection of all dyadic cubes will be denoted by $\mathcal{Q}$. For a given cube Q the number $\ell(Q)$ is its side-length. To each dyadic cube we associate one more number, namely

$$j_Q := -\log_2 \ell(Q) \, , \qquad Q \in \mathcal{Q} \, .$$

Let $\psi \in C_0^\infty$ be a function such that

$$\psi(x) := 1 \quad \text{if} \quad |x| \leqslant 1 \qquad \text{and} \qquad \psi(x) := 0 \quad \text{if} \quad |x| \geqslant \frac{3}{2} \, . \tag{7.1}$$

Then, with $\varphi_0 := \psi$,

$$\varphi(x) := \varphi_0(x/2) - \varphi_0(x) \qquad \text{and} \qquad \varphi_j(x) := \varphi(2^{-j+1}x) \, , \quad j \in \mathbb{N} \, , \tag{7.2}$$

we have

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d .$$

We call the system $(\varphi_j)_{j=0}^{\infty}$ an (inhomogeneous) smooth dyadic decomposition of unity. By means of such a smooth dyadic decomposition of unity we define the spaces we are interested in.

Definition 4. Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (7.1), (7.2). Let τ, $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.

(i) Let $0 < p < \infty$. Then the inhomogeneous Triebel-Lizorkin type space $F^{s,\tau}_{p,q}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{F^{s,\tau}_{p,q}} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \int_Q \left[\sum_{j=\max(j_Q,0)}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^q \right]^{p/q} dx \right\}^{1/p} < \infty .$$

(ii) Let $0 < p \leqslant \infty$. Then the inhomogeneous Nikol'skii-Besov type space $B^{s,\tau}_{p,q}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{B^{s,\tau}_{p,q}} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{j=\max(j_Q,0)}^{\infty} \left[\int_Q (2^{js} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|)^p \, dx \right]^{q/p} \right\}^{1/q} < \infty . \tag{7.3}$$

Many times the scale $F^{s,\tau}_{p,q}$ behave as the scale $B^{s,\tau}_{p,q}$. In those situations, to avoid unattractive repetitions, we shall use the notation $A^{s,\tau}_{p,q}$ with $A \in \{F, B\}$. For convenience of the reader we also recall the following two lemmas, see [108, Lemma 2.1, Proposition 2.3] and [108, Proposition 2.1].

Lemma 7.1. (i) *The classes $A^{s,\tau}_{p,q}$ are quasi-Banach spaces, i. e., complete quasi-normed spaces. With $\varepsilon := \min\{1, p, q\}$ it holds*

$$\|f+g\|^\varepsilon_{A^{s,\tau}_{p,q}} \leqslant \|f\|^\varepsilon_{A^{s,\tau}_{p,q}} + \|g\|^\varepsilon_{A^{s,\tau}_{p,q}}$$

for all f, $g \in A^{s,\tau}_{p,q}$.

(ii) *We always have*

$$\mathcal{S} \hookrightarrow A^{s,\tau}_{p,q} \hookrightarrow \mathcal{S}' .$$

(iii) One can replace the set $\mathcal{Q}$ by the set of all cubes with sides parallel to the axes in Definition 4 obtaining an equivalent quasi-norm on that way. With the same argument one can replace the set of all such cubes by the set of all balls.

Lemma 7.2. *With $q_0 \leqslant q_1$ we have*

$$A^{s,\tau}_{p,q_0} \hookrightarrow A^{s,\tau}_{p,q_1} . \tag{7.4}$$

Furthermore, we have

$$B^{s,\tau}_{p,\min(p,q)} \hookrightarrow F^{s,\tau}_{p,q} \hookrightarrow B^{s,\tau}_{p,\max(p,q)} \tag{7.5}$$

and

$$A^{s,\tau}_{p,q} \hookrightarrow B^{s,\tau}_{p,\infty} \qquad A \in \{B, F\}\,. \tag{7.6}$$

By replacing in the definition of $B^{s,\tau}_{p,q}$ the sum $\sum_{j=\max(j_Q,0)}^{\infty}$ by $\sum_{j=0}^{\infty}$ and interchanging summation and taking the supremum we obtain a further scale of spaces.

Definition 5. Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (7.1), (7.2). Let τ, $s \in \mathbb{R}$ and $0 < q, p \leqslant \infty$. Then the space $\mathcal{B}^{s,\tau}_{p,q}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathcal{B}^{s,\tau}_{p,q}} := \left\{ \sum_{j=0}^{\infty} \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \left[\int_Q (2^{js}|\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|)^p \, dx \right]^{q/p} \right\}^{1/q} < \infty\,. \tag{7.7}$$

Remark 19. In [93] we proved the embedding $\mathcal{B}^{s,\tau}_{p,\infty} \hookrightarrow B^{s,\tau}_{p,q}$, valid for all admissible values of the parameters, as well as the identity

$$\mathcal{B}^{s,\tau}_{p,\infty} = B^{s,\tau}_{p,\infty}, \qquad 0 < p \leqslant \infty, \quad 0 \leqslant \tau \leqslant 1/p\,, \tag{7.8}$$

in the sense of equivalent quasi-norms.

Some more scales will be defined by using the Morrey norm directly.

Definition 6. Let $0 < u \leqslant p \leqslant \infty$. The space $\mathcal{M}^p_u$ is defined to be the set of all u-locally Lebesgue-integrable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{\mathcal{M}^p_u} := \sup_B |B|^{1/p-1/u} \Big(\int_B |f(x)|^u \, dx \Big)^{1/u} < \infty\,,$$

where the supremum is taken over all balls B in $\mathbb{R}^d$.

Now we turn to the definition of Lizorkin-Triebel-Morrey spaces.

Definition 7. Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (7.1), (7.2). Let $s \in \mathbb{R}$, $0 < u \leqslant p < \infty$ and $0 < q \leqslant \infty$. Then $\mathcal{E}^s_{p,q,u}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathcal{E}^s_{p,q,u}} := \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^q \Big)^{1/q} \Big\|_{\mathcal{M}^p_u} < \infty\,. \tag{7.9}$$

Finally, we define Nikol'skii-Besov-Morrey spaces.

Definition 8. Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (7.1), (7.2). Let $s \in \mathbb{R}$, $0 < u \leqslant p \leqslant \infty$ and $0 < q \leqslant \infty$.
Then $\mathcal{N}^s_{p,q,u}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathcal{N}^s_{p,q,u}} := \Big(\sum_{j=0}^{\infty} 2^{jsq} \, \|\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)]\,\|^q_{\mathcal{M}^p_u} \Big)^{1/q} < \infty\,. \tag{7.10}$$

In the framework of Morrey spaces it happens many times that the supremum is taken only with respect to balls of volume less than 1. Here are some scales related to such a procedure.

Definition 9. Let ψ be as in (7.1). Let E be a quasi-Banach space of distributions in $\mathcal{S}'$. Then E_{unif} is the collection of all distributions $f \in \mathcal{S}'$ such that

$$\|f\|_{E_{\mathrm{unif}}} := \sup_{\lambda \in \mathbb{Z}} \|f\, \psi(\,\cdot\, - \lambda)\|_E < \infty\,.$$

Remark 20. For us the cases $E = F^{s,\tau}_{p,q}$ and $E = B^{s,\tau}_{p,q}$ are of interest.

Definition 10. Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (7.1), (7.2). Let $s \in \mathbb{R}$, $0 < u \leqslant p \leqslant \infty$ and $0 < q \leqslant \infty$. Then $N^s_{p,q,u}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{N^s_{p,q,u}} := \Big\{ \sum_{j=0}^{\infty} 2^{jsq} \Big[\sup_{|B|\leqslant 1} |B|^{\frac{1}{p}-\frac{1}{u}} \Big(\int_B |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^u dx \Big)^{1/u} \Big]^q \Big\}^{1/q} < \infty\,. \tag{7.11}$$

Here the supremum is taken with respect to all balls in $\mathbb{R}^d$ with volume $\leqslant 1$.

Definition 11. Let $(\varphi_j)_j$ be a smooth dyadic decomposition of unity as defined in (7.1), (7.2). Let $s \in \mathbb{R}$, $0 < u \leqslant p < \infty$ and $0 < q \leqslant \infty$. Then $E^s_{p,q,u}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{E^s_{p,q,u}} := \sup_{|B|\leqslant 1} |B|^{\frac{1}{p}-\frac{1}{u}} \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)](x)|^q \Big)^{1/q} \Big\|_{L_u(B)} < \infty\,. \tag{7.12}$$

Also here the supremum is taken with respect to all balls in $\mathbb{R}^d$ with volume $\leqslant 1$.

Notation

As usual, $\mathbb{N}$ denotes the set all natural numbers, $\mathbb{N}_0$ the set all natural numbers and 0, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. $\mathbb{C}$ denotes the complex numbers and $\mathbb{R}^d$ the Euclidean d-space. All functions are assumed to be complex-valued, i.e., we consider functions $f : \mathbb{R}^d \to \mathbb{C}$. In general the classes of functions (distributions) are defined on $\mathbb{R}^d$. So we will drop it in notation. Let $\mathcal{S}$ denote the Schwartz space of all rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^d$. By $\mathcal{S}'$ we denote the collection of all complex-valued tempered distributions on $\mathbb{R}^d$, i.e., the topological dual of $\mathcal{S}$, equipped with the strong topology. The symbol $\mathcal{F}$ refers to the Fourier transform, $\mathcal{F}^{-1}$ to its inverse transformation, both defined on $\mathcal{S}'$. All function spaces, which we consider in this paper, are subspaces of $\mathcal{S}'$, i.e. spaces of equivalence classes with respect to almost everywhere equality. However, if such an equivalence class contains a continuous representative, then usually we work with this representative and call also the equivalence class a continuous function.

If E and F are two quasi-Banach spaces, then the symbol $E \hookrightarrow F$ indicates that the embedding is continuous. By C_0^∞ we denote the set of all test functions, i.e., the set of all compactly supported and infinitely differentiable functions. If E is a quasi-Banach function space on $\mathbb{R}^d$ we denote by $E^{\ell oc}$ the collection of all functions f having the property that the products $\varphi f \in E$ for all $\varphi \in C_0^\infty$. The symbol $\mathcal{L}(E,F)$ denotes the set of all linear and bounded operators $T: \ E \to F$. In case $E = F$ we simply write $\mathcal{L}(E)$.

As usual, the symbol c denotes a positive constant which depends only on the fixed parameters d, s, τ, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbol "$\lesssim$" instead of "$\leqslant$". The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \leqslant c\,B$. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. Many times we shall need the following abbreviations:

$$\sigma_p := d \max\Big(0, \frac{1}{p} - 1\Big) \qquad \text{and} \qquad \sigma_{p,q} := d \max\Big(0, \frac{1}{p} - 1, \frac{1}{q} - 1\Big). \tag{7.13}$$

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