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RECENT PROGRESS IN STUDYING THE BOUNDEDNESS OF CLASSICAL OPERATORS OF REAL ANALYSIS IN GENERAL MORREY-TYPE SPACES. II

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Abstract. The survey is aimed at providing detailed information about recent results in the problem of the boundedness in general Morrey-type spaces of various important operators of real analysis, namely of the maximal operator, fractional maximal operator, Riesz potential, singular integral operator, Hardy operator. The main focus is on the results which contain, for a certain range of the numerical parameters, necessary and sufficient conditions on the functional parameters characterizing general Morrey-type spaces, ensuring the boundedness of the aforementioned operators from one general Morrey-type space to another one. The major part of the survey is dedicated to the results obtained by the author jointly with his co-authores A. Gogatishvili, M.L. Goldman, D.K. Darbayeva, H.V. Guliyev, V.S. Guliyev, P. Jain, R. Mustafaev, E.D. Nursultanov, R. Oinarov, A. Serbetci, T.V. Tararykova. In Part I of the survey under discussion were the definition and basic properties of the local and global general Morrey-type spaces, embedding theorems, and the boundedness properties of the maximal operator. Part II of the survey contains discussion of boundedness properties of the fractional maximal operator, Riesz potential, singular integral operator, Hardy operator. All definitions and notation [1] in Part II are the same as in Part I.

7 Riesz potential

Let $f \in L_1^{loc}$. The Riesz potential I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n\,.$$

[1] Local Morrey-type spaces $LM_{p\theta,w(\cdot)}$, weak local Morrey-type spaces $WLM_{p\theta,w(\cdot)}$ $(0 < p,\theta \le \infty, w \in \Omega_\theta)$; global Morrey-type spaces $GM_{p\theta,w(\cdot)}$, weak global Morrey-type spaces $WGM_{p\theta,w(\cdot)}$ $(0 < p,\theta \le \infty, w \in \Omega_{p\theta})$.

Various applications of Morrey-type spaces are discussed in detail in the survey papers [36], [44], [46], [47]. Properties of the commutators of singular integrals in Morrey-type spaces are discussed in [32]. Interpolation theorems in Morrey-type spaces are studied in [9]. Complementary Morrey-type spaces are considered in [14], [3], [29].

Let $1 \le p_1 \le p_2 \le \infty$. The classical Hardy-Littlewood-Sobolev result states that I_α is bounded from L_{p_1} to L_{p_2} if and only if

$$1 < p_1 < p_2 < \infty \quad \text{and} \quad \alpha = n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big). \tag{7.1}$$

Also I_α is bounded from L_1 to WL_{p_2} if and only if

$$1 < p_2 < \infty \quad \text{and} \quad \alpha = n\Big(1 - \frac{1}{p_2}\Big). \tag{7.2}$$

The boundedness of I_α in Morrey spaces was investigated by S. Spanne, J. Peetre, and D. Adams. We start with the case $\alpha = n(\frac{1}{p_1} - \frac{1}{p_2})$. In [43] the next result is referred as Spanne's result.

Theorem 7.1. ([43])

Let conditions (7.1) *be satisfied. Then I_α is bounded from $M_{p_1}^\lambda$ to $M_{p_2}^\lambda$ for all* $0 \le \lambda < \frac{n}{p_2}$.

Let conditions (7.2) *be satisfied. Then I_α is bounded from M_1^λ to $WM_{p_2}^\lambda$ for all* $0 \le \lambda < \frac{n}{p_2}$.

If $\lambda = 0$ then the statement of this theorem reduces to the aforementioned result by Hardy-Littlewood-Sobolev.

The boundedness of I_α in Morrey spaces for $\alpha < n(\frac{1}{p_1} - \frac{1}{p_2})$ was investigated by D. Adams.

Theorem 7.2. ([1]) *Let $1 < p_1 < p_2 \le \infty$, $0 < \alpha < n$, $0 \le \lambda_1 < \frac{n}{p_1}$, $0 \le \lambda_2 < \frac{n}{p_2}$, and*

$$\lambda_1 p_1 = \lambda_2 p_2 \tag{7.3}$$

(hence $\lambda_2 < \lambda_1$ or $\lambda_1 = \lambda_2 = 0$).

If $p_1 > 1$ then the operator I_α is bounded from $M_{p_1}^{\lambda_1}$ to $M_{p_2}^{\lambda_2}$ if and only if

$$\alpha = \lambda_2 - \lambda_1 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right). \tag{7.4}$$

If $p_1 = 1$ then the operator I_α is bounded from $M_1^{\lambda_1}$ to $WM_{p_2}^{\lambda_2}$ if and only if condition (7.4) *is satisfied with $p_1 = 1$.*

If $\alpha = n(\frac{1}{p_1} - \frac{1}{p_2})$ condition (7.4) implies that $\lambda_1 = \lambda_2$ which by (7.3) can only happen only in the case $\lambda_1 = \lambda_2 = 0$ in which $M_{p_1}^0 = L_{p_1}$ and $M_{p_1}^0 = L_{p_1}$.

T. Mizuhara, E. Nakai, and V.S. Guliyev generalized Theorem 7.1 and obtained for the case $\alpha = n(\frac{1}{p_1} - \frac{1}{p_2})$ sufficient conditions for the boundedness of I_α from $GM_{p_1\infty,w_1(\cdot)}$ to $GM_{p_2\infty,w_2(\cdot)}$.

Theorem 7.3. ([30]) *Let $1 \le p_1 < p_2 < \infty$ and $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. Moreover, let functions $w_1 \in \Omega_{p\infty}, w_2 \in \Omega_{p\infty}$ satisfy the condition*

$$\big\| w_1^{-1}(r)\, r^{-\frac{n}{p_2}-1} \big\|_{L_1(t,\infty)} \lesssim w_2^{-1}(t)\, t^{-\frac{n}{p_2}} \tag{7.5}$$

uniformly in $t \in (0,\infty)$.

Then for $p_1 > 1$ I_α is bounded from $GM_{p_1\infty,w_1(\cdot)}$ to $GM_{p_2\infty,w_2(\cdot)}$ and for $p_1 = 1$ I_α is bounded from $GM_{1\infty,w_1(\cdot)}$ to $WGM_{p_2\infty,w_2(\cdot)}$.

In the [38], [40] this statement was proved under the following additional assumptions: it was assumed that $w_1 = w_2 = w$ and that w was a positive non-increasing function satisfying the pointwise doubling condition, namely that for some $c > 0$

$$c^{-1}w(r) \le w(t) \le cw(r)$$

for all t, $r > 0$ such that $0 < r \le t \le 2r$. . In [30] it was proved without these additional assumptions. (See also [33], [34], [31].)

Next the most general case will be considered. We start with necessary conditions on the numerical parameters.

Lemma 7.1. ([15], [16]) *Let $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 < \alpha < n$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$. Then the conditions*

$$p_1 < \infty \quad \text{and} \quad \alpha < \frac{n}{p_1}$$

are necessary for the boundedness of I_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

Lemma 7.2. ([15], [16]) *Let $1 \le p_1 < \infty$, $0 < p_2 \le \infty$, $0 < \alpha < \frac{n}{p_1}$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$. Moreover, let $w_1 \in L_{\theta_1}(0,\infty)$. Then the condition* [2]

$$\alpha \ge n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$$

is necessary for the boundedness of I_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

Remark 4. Without the assumption $w_1 \in L_{\theta_1}(0,\infty)$ this condition is not necessary. In particular in Theorem 7.2 $\alpha < n(\frac{1}{p_1} - \frac{1}{p_2})$ excluding the case $\lambda_1 = \lambda_2 = 0$.

The application of the known results about necessary and sufficient conditions for the boundedness of the operator I_α in weighted Lebesgue spaces and the relationship between general Morrey-type spaces and weighted Lebesgue spaces, described in Section 5 of Part I of the survey, immediately imply the following statement for the case of local Morrey-type spaces, including necessary and sufficient conditions for the boundedness of I_α from $LM_{p_1p_1,w_1(\cdot)}$ to $LM_{p_2p_2,w_2(\cdot)}$.

Theorem 7.4. *Let $0 < \alpha < n$, $1 < p_1 \le p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$. If $p_1 \ge \theta_1$, $p_2 \le \theta_2$ and*

$$\sup_{x\in\mathbb{R}^n, r>0} \|\widehat{W}_2\|_{L_{p_2}(B(x,r))} \, \| \, |x-y|^{\alpha-n}\widehat{W}_1(y)^{-1}\|_{L_{p_1'}({}^{\complement}B(x,r))} < \infty \tag{7.6}$$

and

$$\sup_{x\in\mathbb{R}^n, r>0} \|\widehat{W}_1^{-1}\|_{L_{p_1'}(B(x,r))} \, \| \, |x-y|^{\alpha-n}\widehat{W}_2(y)\|_{L_{p_2}({}^{\complement}B(x,r))} < \infty, \tag{7.7}$$

where $p_1' = \frac{p_1}{p_1-1}$ and

$$\widehat{W}_1(x) = \|w_1\|_{L_{\theta_1}(|x|,\infty)}, \qquad \widehat{W}_2(x) = \|w_2\|_{L_{\theta_2}(|x|,\infty)}, \tag{7.8}$$

[2] For $a \in \mathbb{R}$, a_+ is the positive part of a ($a_+ = \max\{0, a\}$).

for all $x \in \mathbb{R}^n$, *then the operator* I_α *is bounded from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$ *and from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$. *(In the latter case it is assumed that* $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$.*)*

If $p_1 \le \theta_1$ *and* $p_2 \ge \theta_2$, *then conditions* (7.6)–(7.7) *are necessary for the boundedness of* I_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

In particular, if $\theta_1 = p_1$ *and* $\theta_2 = p_2$, *then conditions* (7.6)–(7.7) *are necessary and sufficient for the boundedness of* I_α *from* $LM_{p_1p_1,w_1(\cdot)}$ *to* $LM_{p_2p_2,w_2(\cdot)}$.

The following theorem contains necessary and sufficient for the boundedness of I_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ without the assumptions $p_1 = \theta_1$ and $p_2 = \theta_2$.

Theorem 7.5. ([15], [16], [11])

1. *If* $1 \le p_1 < \infty$, $0 < p_2 \le \infty$, $0 < \alpha < \frac{n}{p_1}$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$t^{\alpha-\frac{n}{p_1}+\min\{n-\alpha,\frac{n}{p_2}\}} \left\| w_2(r) \frac{r^{\frac{n}{p_2}}}{(t+r)^{\min\{n-\alpha,\frac{n}{p_2}\}}} \right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)},$$

uniformly in $t \in (0,\infty)$ *is necessary for the boundedness of* I_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

2. *If condition* (7.1) *or the condition*

$$1 \le p_1 < \infty,\ 0 < p_2 < \infty \ \text{ and } \ n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ < \alpha < \frac{n}{p_1} \tag{7.9}$$

is satisfied, $0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$\left\| w_2(r) \frac{r^{\frac{n}{p_2}}}{(t+r)^{\frac{n}{p_1}-\alpha}} \right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)}.$$

uniformly in $t \in (0,\infty)$ *is sufficient for the boundedness of* I_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$ *and from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$. *(In the latter case it is assumed that* $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$.*)*

3. *In particular, if condition* (7.1) *is satisfied,* $0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$, *then the condition*

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)}, \tag{7.10}$$

uniformly in $t \in (0,\infty)$ *is necessary and sufficient for the boundedness of* I_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

4. *Let*

$$1 \le p_1 < p_2 < \infty,\ \alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right), \tag{1'}$$

$0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$, *then condition* (7.10) *is necessary and sufficient for the boundedness of* I_α *from* $LM_{p_1\theta_1,w_2(\cdot)}$ *to* $WLM_{p_2\theta_2,w_2(\cdot)}$.

Remark 5. In [15], [16] this statement is proved under the additional assumptions: either $\theta_1 \le 1$ or, if w_2 satisfies certain regularity conditions, $\theta_1 \le p_1$. In [11] it is proved without additional assumptions on θ_1 by using a different method.

The next theorem contains sufficient conditions on w_1, w_2 ensuring the boundedness of I_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ for all values of the parameters satisfying (7.1) or (7.9), which are close to necessary ones and are necessary ones if $p_1 = 1$.

Theorem 7.6. ([11]) *Let condition* (7.1) *or* (7.9) *be satisfied. Moreover, let* $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$.

1. *The operator* I_α *is bounded from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$ *if, and in the case* $p_1 = 1$ *only if,*

(a) *if* $1 < \theta_1 \le \theta_2 < \infty$, *then*

$$B_1^1 := \sup_{t>0} \left(\int_t^\infty w_2^{\theta_2}(r) r^{\theta_2\left(\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty w_1^{\theta_1}(r) dr \right)^{-\frac{1}{\theta_1}} < \infty, \tag{7.11}$$

and

$$B_2^1 := \sup_{t>0} \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r) r^{\theta_1'\left(\alpha - \frac{n}{p_1}\right)}}{\left(\int_r^\infty w_1^{\theta_1}(\rho) d\rho\right)^{\theta_1'}} dr \right)^{\frac{1}{\theta_1'}} < \infty;$$

(b) *if* $0 < \theta_1 \le 1$, $0 < \theta_1 \le \theta_2 < \infty$, *then* $B_1^1 < \infty$ *and*

$$B_2^2 := \sup_{t>0} t^{\alpha - \frac{n}{p_1}} \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty w_1^{\theta_1}(r) dr \right)^{-\frac{1}{\theta_1}} < \infty; \tag{7.12}$$

(c) *if* $1 < \theta_1 < \infty$, $0 < \theta_2 < \theta_1 < \infty$, $\theta_2 \ne 1$, *then*

$$B_1^3 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2^{\theta_2}(r) r^{\theta_2\left(\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} dr}{\int_t^\infty w_1^{\theta_1}(r) dr} \right)^{\frac{\theta_2}{\theta_1 - \theta_2}} w_2^{\theta_2}(t) t^{\theta_2\left(\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} dt \right)^{\frac{\theta_1 - \theta_2}{\theta_1 \theta_2}} < \infty,$$

and

$$B_2^3 := \left(\int_0^\infty \left[\left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r) r^{\theta_1'\left(\alpha - \frac{n}{p_1}\right)}}{\left(\int_r^\infty w_1^{\theta_1}(\rho) d\rho\right)^{\theta_1'}} dr \right)^{\frac{\theta_2 - 1}{\theta_2}} \right]^{\frac{\theta_1 \theta_2}{\theta_1 - \theta_2}} \times \right.$$

$$\left. \times \frac{w_1^{\theta_1}(t) t^{\theta_1'\left(\alpha - \frac{n}{p_1}\right)}}{\left(\int_t^\infty w_1^{\theta_1}(\rho) d\rho\right)^{\theta_1'}} dt \right)^{\frac{\theta_1 - \theta_2}{\theta_1 \theta_2}} < \infty;$$

(d) *if* $1 = \theta_2 < \theta_1 < \infty$, *then*

$$B_1^4 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2(r) r^{\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} dr}{\int_t^\infty w_1^{\theta_1}(r) dr} \right)^{\frac{1}{\theta_1 - 1}} w_2(t) t^{\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} dt \right)^{\frac{\theta_1 - 1}{\theta_1}} < \infty,$$

and

$$B_2^4 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2(r) r^{\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} dr + t^{\alpha - \frac{n}{p_1}} \int_0^t w_2(r) r^{\frac{n}{p_2}} dr}{\int_t^\infty w_1^{\theta_1}(r) dr} \right)^{\theta_1' - 1} \times \right.$$

$$\left. \times t^{\alpha - \frac{n}{p_1}} \left(\int_0^t w_2(r) r^{\frac{n}{p_2}} dr \right) \frac{dt}{t} \right)^{\theta_1'} < \infty\,;$$

(e) *if* $0 < \theta_2 < \theta_1 = 1$, *then*

$$B_1^5 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2^{\theta_2}(r) r^{\theta_2\left(\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} dr}{\int_t^\infty w_1(r) dr} \right)^{\frac{\theta_2}{1-\theta_2}} w_2^{\theta_2}(t) t^{\theta_2\left(\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} dt \right)^{\frac{1-\theta_2}{\theta_2}} < \infty,$$

and

$$B_2^5 := \left(\int_0^\infty \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{\theta_2}{1-\theta_2}} \left(\inf_{t<s<\infty} s^{\frac{n}{p_1} - \alpha} \int_s^\infty w_1(\rho) d\rho \right)^{\frac{\theta_2}{\theta_2 - 1}} \times \right.$$

$$\left. \times w_2^{\theta_2}(t) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{1-\theta_2}{\theta_2}} < \infty\,;$$

(f) *if* $0 < \theta_2 < \theta_1 < 1$, *then* $B_1^3 < \infty$ *and*

$$B_2^6 := \left(\int_0^\infty \sup_{t \le s < \infty} \frac{s^{\left(\alpha - \frac{n}{p_1}\right)\frac{\theta_1\theta_2}{\theta_1 - \theta_2}}}{\left(\int_s^\infty w_1^{\theta_1}(\rho) d\rho\right)^{\frac{\theta_2}{\theta_1 - \theta_2}}} \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{\theta_2}{\theta_1 - \theta_2}} \times \right.$$

$$\left. \times w_2^{\theta_2}(t) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{\theta_1 - \theta_2}{\theta_1\theta_2}} < \infty\,;$$

(g) *if* $0 < \theta_1 \le 1$, $\theta_2 = \infty$, *then*

$$B^7 := \operatorname*{ess\,sup}_{0<t\le s<\infty} \frac{w_2(t) t^{\frac{n}{p_2}}}{s^{\frac{n}{p_1} - \alpha} \left(\int_s^\infty w_1^{\theta_1}(r) dr\right)^{\frac{1}{\theta_1}}} < \infty\,;$$

(h) *if* $1 < \theta_1 < \infty$, $\theta_2 = \infty$, *then*

$$B^8 := \operatorname*{ess\,sup}_{t>0} w_2(t) t^{\frac{n}{p_2}} \left(\int_t^\infty \frac{r^{\theta_1'\left(\alpha - \frac{n}{p_1}\right)}}{\left(\int_r^\infty w_1^{\theta_1}(s) ds\right)^{\theta_1' - 1}} \frac{dr}{r} \right)^{\frac{1}{\theta_1'}} < \infty\,;$$

(i) *if* $\theta_1 = \infty$, $0 < \theta_2 < \infty$, *then*

$$B^{10} := \left(\int_0^\infty \left(t^{\frac{n}{p_1} - \alpha} \int_t^\infty \frac{s^{\alpha - \frac{n}{p_1} - 1} ds}{\operatorname{ess\,sup}_{s<y<\infty} w_1(y)} \right)^{\theta_2} \times \right.$$

$$\left. \times w_2^{\theta_2}(t) t^{\theta_2\left(\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} dt \right)^{\frac{1}{\theta_2}} < \infty\,;$$

(j) *if* $\theta_1 = \theta_2 = \infty$, *then*

$$B^9 := \operatorname*{ess\,sup}_{t>0} w_2(t) t^{\frac{n}{p_2}} \int_t^\infty \frac{s^{\alpha - \frac{n}{p_1} - 1}}{\operatorname{ess\,sup}_{s<y<\infty} w_1(y)} ds < \infty \,. \tag{7.13}$$

Moreover, in case (a)

$$\|I_\alpha\|_{LM_{p_1\theta_1,w_1(\cdot)} \to LM_{p_2\theta_2,w_2(\cdot)}} \lesssim B_1^1 + B_2^1$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and in* $w_2 \in \Omega_{\theta_2}$, *where the sign* $\lesssim$ *should be replaced by* $\approx$ *if* $p_1 = 1$, *and similar inequalities and equivalencies hold in cases* (b)-(j).

2. *If* $p_1 = 1$, $0 < p_2 < \infty$ *and* $n\left(1 - \frac{1}{p_2}\right)_+ < \alpha < n$ *or* $1 < p_2 < \infty$ *and* $\alpha = n\left(1 - \frac{1}{p_2}\right)$, *then* I_α *is bounded from* $LM_{1\theta_1,w_1(\cdot)}$ *to* $WLM_{p_2\theta_2,w_2(\cdot)}$ *if and only if conditions* (a)-(j) *are satisfied.*

Moreover, in case (a)

$$\|I_\alpha\|_{LM_{1\theta_1,w_1(\cdot)} \to WLM_{p_2\theta_2,w_2(\cdot)}} \approx B_1^1 + B_2^1$$

uniformly in $w_1 \in \Omega_{\theta_1}$ *and in* $w_2 \in \Omega_{\theta_2}$, *and similar equivalencies hold in cases* (b)-(j).

Remark 6. Note that two conditions (7.11) and (7.12) are equivalent to one condition (7.5).

Remark 7. Statement (j) of Theorem 7.6 is stronger than that of Theorem 7.5: first of all it holds for a wider range of the parameters, but even for the same range of the parameters as in Theorem 7.3, i. e. for $1 \le p_1 < p_2 < \infty$ and $\alpha = n(\frac{1}{p_1} - \frac{1}{p_2})$, condition (7.13) is weaker than condition (7.5). It is obvious that if condition (7.5) holds, then condition (7.13) holds too. Moreover for non-increasing continuous functions w_1 conditions (7.5) and (7.13) coincide. However, in general, condition (7.13) does not imply condition (7.5). For example, the functions

$$w_1(r) = \chi_{(1,\infty)}(r) r^{-\beta}, \quad w_2(t) = \frac{1}{t^\beta + 1}, \quad 0 < \beta < \frac{n}{p_1} - \alpha$$

satisfy condition (7.13) but do not satisfy condition (7.5).

Remark 8. Note that under the assumptions on the parameters of the second part of Theorem 7.6

$$\|I_\alpha\|_{LM_{1\theta_1,w_1(\cdot)} \to LM_{p_2\theta_2,w_2(\cdot)}} \approx \|I_\alpha\|_{LM_{1\theta_1,w_1(\cdot)} \to WLM_{p_2\theta_2,w_2(\cdot)}} \,.$$

Corollary 7.1. *If*

$$1 < p_1 < p_2 < \infty,\ 0 < \theta_2 \le \infty,\ \alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right), \quad \text{and } w_2 \in \Omega_{\theta_2},$$

or

$$1 \le p_1 < \infty,\ 0 < p_2 < \infty,\ \theta_2 = \infty,\ n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ < \alpha < \frac{n}{p_1}, \quad \text{and } w_2 \in \Omega_\infty,$$

then the condition

$$w_2(r)r^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \in L_{\theta_2}(0,\infty) \qquad (7.14)$$

is necessary and sufficient for the boundedness of I_α *from* L_{p_1} *to* $LM_{p_2\theta_2,w_2(\cdot)}$ *and from* L_{p_1} *to* $GM_{p_2\theta_2,w_2(\cdot)}$. *(In the case of the spaces* $GM_{p_2\theta_2,w_2(\cdot)}$ *it is assumed that* $w_2 \in \Omega_{p_2\theta_2}$.*)*

Further information on the properties of the Riesz potential can be found in survey papers [41], [35].

8 Fractional maximal operator

Let $f \in L_1^{loc}$. The fractional maximal operator M_α is defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy,$$

where $0 \le \alpha < n$. If $\alpha = 0$, then $M \equiv M_0$ is the maximal operator.

Note that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x), \qquad (8.1)$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$, hence the boundedness of the Riesz potential also implies the boundedness of the fractional maximal operator M_α.

Therefore Theorems 7.1, 7.2 and 7.3 are also valid for the fractional maximal operator. Moreover, they are valid for a wider range of the parameter p_2, namely for $p_1 \le p_2 \le \infty$, which, in the limiting cases $p_1 = p_2$ and $p_2 = \infty$, follows by theorems for the maximal operator formulated in Section 6 of Part I.

There are minor distinctions in necessary conditions on the parameters compared with the case of the Riesz potential.

Lemma 8.1. *Let* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 \le \alpha < n$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$. *Then the condition*

$$\alpha \le \frac{n}{p_1}$$

is necessary for the boundedness of M_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

Lemma 8.2. *Let* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 \le \alpha \le \frac{n}{p_1}, \alpha < n$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$. *Moreover, let* $w_1 \in L_{\theta_1}(0,\infty)$. *Then the condition*

$$\alpha \ge n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$$

is necessary for the boundedness of M_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

Remark 9. If $w_1 \notin L_{\theta_1}(0,\infty)$ then this condition is not necessary. See Remark 4.

An analogue of Theorem 7.4 takes a different form. The known results on the boundedness of the fractional maximal operator in general weighted Lebesgue spaces (see [45], [26], [25], [28]) and the relationship between general Morrey-type spaces and weighted Lebesgue spaces, described in Section 5 of Part I of the survey, imply the following statement.

Theorem 8.1. *Let $0 \le \alpha < n$, $1 < p_1 \le p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$.*
If $\theta_1 \le p_1$ and $p_2 \le \theta_2$ and

$$\sup_{R>0} R^{\alpha-n} \left\| t^{\frac{n-1}{p_1'}} \widehat{W}_1(t)^{-1} \right\|_{L_{p_1'}(0,R)} \left\| t^{\frac{n-1}{p_2}} \widehat{W}_2(t) \right\|_{L_{p_2}(0,R)} < \infty. \tag{8.2}$$

or equivalently

$$\left\| M_\alpha \left(\chi_B W_1^{\frac{p_1}{1-p_1}} \right) \right\|_{L_{p_2,W_2}(B)} \lesssim \left\| W_1^{\frac{1}{1-p_1}} \right\|_{L_{p_1}(B)}, \tag{8.3}$$

uniformly in balls $B \subset \mathbb{R}^n$, where $\widehat{W}_1, \widehat{W}_2$ are the same as in Theorem 7.4 (formula (7.8)) and

$$W_1(t) = \|w_1\|_{L_{\theta_1}(t,\infty)}, \qquad W_2(t) = \|w_2\|_{L_{\theta_2}(t,\infty)}$$

for all $t > 0$, then M_α is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ and from $GM_{p_1\theta_1,w_1(\cdot)}$ to $GM_{p_2\theta_2,w_2(\cdot)}$. (In the latter case it is assumed that $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2,\theta_2}$).

If $p_1 \le \theta_1$ and $p_2 \ge \theta_2$, then condition (8.2), or equivalently (8.3), is necessary for the boundedness of M_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

In particular, if $\theta_1 = p_1$ and $\theta_2 = p_2$, then condition (8.2), or equivalently (8.3), is necessary and sufficient for the boundedness of M_α from $LM_{p_1p_1,w_1(\cdot)}$ to $LM_{p_2p_2,w_2(\cdot)}$.

The following theorem contains necessary and sufficient for the boundedness of M_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ without the assumptions $p_1 = \theta_1$ and $p_2 = \theta_2$.

Theorem 8.2. ([12], [13], [10]) 1. *If*

$$1 < p_1 \le p_2 < \infty, \ \alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \tag{1''}$$

$0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$, then condition (7.10) is necessary and sufficient for the boundedness of M_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

2. *If*

$$1 \le p_1 \le p_2 < \infty, \ \alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \tag{1'''}$$

$0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$, then condition (7.10) is necessary and sufficient for the boundedness of M_α from $LM_{p_1\theta_1,w_2(\cdot)}$ to $WLM_{p_2\theta_2,w_2(\cdot)}$.

Remark 10. In [12], [13] this statement is proved under the additional assumption $\theta_1 \le p_1$, in [10] without this assumption by using a different method.

The next theorem contains sufficient conditions on w_1, w_2 ensuring the boundedness of M_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ for all values of the parameters satisfying (7.1) or (7.9), which are close to necessary ones and are necessary ones if $p_1 = 1$.

Theorem 8.3. *Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \le \alpha < \frac{n}{p_1}$ if $p_1 > 1$, and $n\left(1 - \frac{1}{p_2}\right)_+ < \alpha < n$ if $p_1 = 1$. Let also $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

Then the operator M_α is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if, and in the case $p_1 = 1$ only if,

(i) *if* $\theta_1 \le \theta_2$ *and* $\theta_1 < \infty$, *then*

$$\sup_{t>0} \left\| w_2(r) \frac{r^{\frac{n}{p_2}}}{(t+r)^{\frac{n}{p_1}-\alpha}} \right\|_{L_{\theta_2}(0,\infty)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty;$$

(ii) *if* $\theta_2 < \theta_1 < \infty$, *then*

$$\left\| w_2(t) t^{\alpha - n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|w_2(r) r^{\alpha - n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|^{\frac{\theta_2}{\theta_1-\theta_2}}_{L_{\theta_2}(t,\infty)} \|w_1\|^{-\frac{\theta_1}{\theta_1-\theta_2}}_{L_{\theta_1}(t,\infty)} \right\|_{L_{\theta_2}(0,\infty)} < \infty$$

and

$$\left\| w_2(t) t^{\frac{n}{p_2}} \|w_2(r) r^{\frac{n}{p_2}}\|^{\frac{\theta_2}{\theta_1-\theta_2}}_{L_{\theta_2}(0,t)} \sup_{r>t} \left(r^{\alpha-\frac{n}{p_1}} \|w_1\|^{-1}_{L_{\theta_1}(r,\infty)} \right)^{\frac{\theta_1}{\theta_1-\theta_2}} \right\|_{L_{\theta_2}(0,\infty)} < \infty;$$

(iii) *if* $\theta_1 = \infty$, *then*

$$\left\| w_2(t) t^{\frac{n}{p_2}} \sup_{r>t} \left(r^{\alpha-\frac{n}{p_1}} \|w_1\|^{-1}_{L_\infty(r,\infty)} \right) \right\|_{L_{\theta_2}(0,\infty)} < \infty.$$

Corollary 8.1. *Let* $1 < p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $w_2 \in \Omega_{\theta_2}$, *and*

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} < \infty \tag{8.4}$$

for all $t > 0$. *Moreover, if* $\theta_2 = \infty$ *and* $\theta_1 < \infty$ *it is also assumed that*

$$\lim_{t\to\infty} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p_2}} \right\|_{L_\infty(0,\infty)} = 0\,. \tag{8.5}$$

Then

1) M_α *is bounded from* $LM_{p_1\theta_1,w_1^*}$ *to* $LM_{p_2\theta_2,w_2}$, *where* w_1^* *is a non-increasing continuous function on* $(0,\infty)$ *defined by*

$$\|w_1^*\|_{L_{\theta_1}(t,\infty)} = \left\| w_2(r) \left(\frac{r}{t+r}\right)^{n/p_2} \right\|_{L_{\theta_2}(0,\infty)}, \qquad t \in (0,\infty). \tag{8.6}$$

2) *If* $w_1 \in \Omega_{\theta_1}$ *and* M_α *is bounded from* $LM_{p_1\theta_1,w_1}$ *to* $LM_{p_2\theta_2,w_2}$, *then*

$$LM_{p_1\theta_1,w_1} \subset LM_{p_1\theta_1,w_1^*}.$$

(*Hence* $LM_{p_1\theta_1,w_1^*}$ *is the maximal among spaces* $LM_{p_1\theta_1,w_1}$ *for which* M_α *is bounded from* $LM_{p_1\theta_1,w_1}$ *to* $LM_{p_2\theta_2,w_2}$.)

Note that equality (8.6), under the assumptions (8.4) and (if $\theta_2 = \infty$ and $\theta_1 < \infty$) (8.5), defines a non-increasing continuous function w_1^* uniquely. In particular, if $\theta_1 = \infty$, then

$$w_1^*(t) = \left\| w_2(r) \left(\frac{r}{t+r} \right)^{n/p_2} \right\|_{L_{\theta_2}(0,\infty)}, \qquad t \in (0,\infty).$$

We also note that Corollary 8.1 holds for all $1 \le p_1 \le p_2 < \infty$ if the space $LM_{p_2\theta_2,w_2(\cdot)}$ is replaced by the space $LWM_{p_2\theta_2,w_2(\cdot)}$. So $LM_{p_1\theta_1,w_1^*(\cdot)}$ is the maximal among spaces $LM_{p_1\theta_1,w_1(\cdot)}$ for which M_α is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LWM_{p_2\theta_2,w_2(\cdot)}$.

Corollary 8.2. *If*

$$1 < p_1 \le p_2 < \infty, \; 0 < \theta_2 \le \infty, \; \alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right), \quad \text{and } w_2 \in \Omega_{\theta_2},$$

or

$$1 \le p_1 < \infty, \; 0 < p_2 < \infty, \; \theta_2 = \infty \; n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \le \alpha \le \frac{n}{p_1}, \quad \text{and } w_2 \in \Omega_\infty,$$

then condition (7.14) *is necessary and sufficient for the boundedness of M_α from L_{p_1} to $LM_{p_2\theta_2,w_2(\cdot)}$ and from L_{p_1} to $GM_{p_2\theta_2,w_2(\cdot)}$. (In the case of the spaces $GM_{p_2\theta_2,w_2(\cdot)}$ we assume that $w_2 \in \Omega_{p_2\theta_2}$.)*

9 Anisotropic fractional maximal operator

Let $d = (d_1, \dots, d_n)$, $d_i \ge 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv (t^{d_1}x_1, \dots, t^{d_n}x_n)$. By [5, 27], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([5, 6, 27]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$. If $d = \mathbf{1} \equiv (1, \dots, 1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = B(x,r)$. Note that in the standard parabolic case $d = (1, \dots, 1, 2)$

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $0 \le \alpha < |d|$ and $f \in L_1^{loc}$. The anisotropic fractional maximal function $M_\alpha^d f$ is defined by

$$M_\alpha^d f(x) = \sup_{t>0} |\mathcal{E}_d(x,t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}_d(x,t)} |f(y)| dy \, .$$

If $\alpha = 0$, then $M^d \equiv M_0^d$ is the anisotropic maximal operator. If $d = \mathbf{1}$, then $M_\alpha \equiv M_\alpha^{\mathbf{1}}$ is the fractional maximal operator and $M \equiv M_0^{\mathbf{1}}$ is the Hardy-Littlewood maximal operator.

In order to investigate the boundedness properties of the anisotropic fractional maximal function M_α^d it is natural to consider anisotropic local and global Morrey-type spaces.

Definition 6. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0, \infty)$. We denote by $LM_{p\theta,w(\cdot),d}$, $GM_{p\theta,w(\cdot),d}$, the anisotropic local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions f measurable on $\mathbb{R}^n$ with finite quasi-norms

$$\|f\|_{LM_{p\theta,w(\cdot),d}} \equiv \|f\|_{LM_{p\theta,w(\cdot),d}(\mathbb{R}^n)} = \left\| w(r) \|f\|_{L_p(\mathcal{E}_d(0,r))} \right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p\theta,w(\cdot),d}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{LM_{p\theta,w(\cdot),d}}$$

respectively.

Note that $GM_{p\theta,w,\mathbf{1}} = GM_{p\theta,w}$, $LM_{p\theta,w,\mathbf{1}} = LM_{p\theta,w}$ and

$$\|f\|_{LM_{p\infty,1,d}} = \|f\|_{GM_{p\infty,1,d}} = \|f\|_{L_p}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p},d} \equiv \mathcal{M}_{p,\lambda,d}$, $0 \le \lambda \le |d|$.

Lemma 9.1. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0, \infty)$.*

1. *If for all $t > 0$*
$$\|w(r)\|_{L_\theta(t,\infty)} = \infty,$$
then $LM_{p\theta,w(\cdot),d} = GM_{p\theta,w(\cdot),d} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

2. *If for all $t > 0$*
$$\|w(r) r^{|d|/p}\|_{L_\theta(0,t)} = \infty,$$
then for all functions $f \in LM_{p\theta,w(\cdot),d}$, continuous at 0, $f(0) = 0$, and for $0 < p < \infty$ $GM_{p\theta,w(\cdot),d} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Definition 7. Let $0 < p, \theta \le \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

Moreover, we denote by $\Omega_{p\theta,d}$ the set of all functions w which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty, \quad \text{and} \quad \|w(r) r^{|d|/p}\|_{L_\theta(0,t)} < \infty.$$

Keeping in mind Lemma 9.1, when considering the spaces $LM_{p\theta,w,d}$ we always assume that $w \in \Omega_\theta$, and when considering the spaces $GM_{p\theta,w,d}$ we always assume that $w \in \Omega_{p\theta,d}$.

Lemma 9.2. *Let $1 < p_1 \le \infty$, $0 < p_2 \le \infty$, $0 \le \alpha < |d|$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$. Then the condition*

$$\alpha \le \frac{|d|}{p_1}$$

is necessary for the boundedness of M_α^d from $LM_{p_1\theta_1,w_1(\cdot),d}$ to $LM_{p_2\theta_2,w_2(\cdot),d}$.

For the isotropic case $d = \mathbf{1}$ Lemma 9.2 reduces to Lemma 8.1.

Theorem 9.1. *Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $|d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \le \alpha < \frac{|d|}{p_1}$ if $p_1 > 1$, and $|d|\left(1 - \frac{1}{p_2}\right)_+ < \alpha < |d|$ if $p_1 = 1$. Let also $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

Then the operator M_α^d is bounded from $LM_{p_1\theta_1,w_1(\cdot),d}$ to $LM_{p_2\theta_2,w_2(\cdot),d}$ if, and in the case $p_1 = 1$ only if,

(i) *if $\theta_1 \le \theta_2$ and $\theta_1 < \infty$, then*

$$\sup_{t>0}\left(t^{\alpha-\frac{|d|}{p_1}}\|w_2(r)r^{\frac{|d|}{p_2}}\|_{L_{\theta_2}(0,t)} + \|w_2(r)r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|_{L_{\theta_2}(t,\infty)}\right)\|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty;$$

(ii) *if $\theta_2 < \theta_1 < \infty$, then*

$$\left\|w_2(t)t^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|w_2(r)r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|^{\frac{\theta_2}{\theta_1-\theta_2}}_{L_{\theta_2}(t,\infty)}\|w_1\|^{-\frac{\theta_1}{\theta_1-\theta_2}}_{L_{\theta_1}(t,\infty)}\right\|_{L_{\theta_2}(0,\infty)} < \infty$$

and

$$\left\|w_2(t)t^{\frac{|d|}{p_2}}\|w_2(r)r^{\frac{|d|}{p_2}}\|^{\frac{\theta_2}{\theta_1-\theta_2}}_{L_{\theta_2}(0,t)}\ \overline{S}\big(r^{\alpha-\frac{|d|}{p_1}}\|w_1\|^{-1}_{L_{\theta_1}(r,\infty)}\big)(t)^{\frac{\theta_1}{\theta_1-\theta_2}}\right\|_{L_{\theta_2}(0,\infty)} < \infty;$$

(iii) *if $\theta_1 = \infty$, then*

$$\left\|w_2(t)t^{\frac{|d|}{p_2}}\,\overline{S}\left(r^{\alpha-\frac{|d|}{p_1}}\|w_1\|^{-1}_{L_\infty(r,\infty)}\right)(t)\right\|_{L_{\theta_2}(0,\infty)} < \infty.$$

Theorem 9.1 contains necessary and sufficient conditions if $p_1 = 1$. If $p_1 > 1$ it contains sufficient conditions. However for $\theta_1 \le \theta_2$ and the limiting case $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ Theorem 9.1 together with the appropriate necessity condition imply the following necessary and sufficient conditions.

Theorem 9.2. 1. *Let*

$$1 < p_1 \le p_2 < \infty,\ \alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right),$$

$0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$, then the condition

$$\left\|w_2(r)\left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}}\right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{9.1}$$

uniformly in $t \in (0,\infty)$ *is necessary and sufficient for the boundedness of* M_α^d *from* $LM_{p_1\theta_1,w_1(\cdot),d}$ *to* $LM_{p_2\theta_2,w_2(\cdot),d}$.

2. *Let*

$$1 \le p_1 \le p_2 < \infty,\ \alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right),$$

$0 < \theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, *and* $w_2 \in \Omega_{\theta_2}$, *then condition* (9.1) *is necessary and sufficient for the boundedness of* M_α^d *from* $LM_{p_1\theta_1,w_1(\cdot),d}$ *to* $WLM_{p_2\theta_2,w_2(\cdot),d}$.

Corollary 9.1. *Let* $1 < p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, *and* $w_2 \in \Omega_{\theta_2}$.

Then the statement of Corollary 8.1 *holds under the assumption that* n *is replaced by* $|d|$ *and* $LM_{p_1\theta_1,w_1(\cdot)}$, $LM_{p_1\theta_1,w_1^*(\cdot)}$, $LM_{p_2\theta_2,w_2(\cdot)}$ *are replaced by* $LM_{p_1\theta_1,w_1(\cdot),d}$, $LM_{p_1\theta_1,w_1^*(\cdot),d}$, $LM_{p_2\theta_2,w_2(\cdot),d}$ *respectively.*

The same refers to the comments related to Corillary 8.1.

Remark 11. The assumption made at the beginning of this section $d_i \ge 1$, $i = 1,\dots,n$, is not essential. One may assume that $d_i > 0$, $i = 1,\dots,n$. However, under this assumption the function $\rho(x-y)$, $x,y \in \mathbb{R}^n$, is in general a quasi-distance, which does note cause any problem. The results for arbitrary $d_i > 0$, $i = 1,\dots,n$ can be derived from the results for the case $d_i \ge 1$, $i = 1,\dots,n$ by using the following equality: for any $\nu > 0$

$$\|M_\alpha^d f\|_{LM_{p_1\theta_1,w_1(\rho),\nu d} \to LM_{p_2\theta_2,w_2(\rho),\nu d}} = \|M_{\nu\alpha}^{\nu d} f\|_{LM_{p_1\theta_1,w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}},\nu d} \to LM_{p_2\theta_2,w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}},\nu d}}.$$

(See [4], Section 7.)

10 Singular integrals

Let T be a Calderon-Zygmund operator, i.e. a linear operator taking C_0^∞ into L_1^{loc}, bounded on L_2 and represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy \qquad \text{a.e.} \quad \text{on } \ \mathbb{R}^n \setminus \text{supp} f$$

for every function $f \in L^\infty(\mathbb{R}^n)$ with compact support. Here $K(x,y)$ is a continuous function away from the diagonal and satisfies the standard estimates: for some $c_1 > 0$ and $0 < \varepsilon \le 1$

$$|K(x,y)| \le c|x-y|^{-n},$$

for all $x, y \in \mathbb{R}^n, x \ne y$ and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')|$$

$$\le c_1 \left(\frac{|x-x'|}{|x-y|}\right)^\varepsilon |x-y|^{-n}$$

whenever $2|x-x'| \le |x-y|$ for some constants $c > 0$, $\varepsilon \in]0,1]$. This class of operators was introduced by R. Coifman and I. Meyers [22].

The classical results for Calderon-Zygmund operators state that if $1 < p < \infty$ then T is bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$, and if $p = 1$ then T is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ (see, for example, [48], [22]).

J. Peetre [42] studied the boundedness of singular integral operators in Morrey spaces, and his results imply the following statement for Calderon-Zygmund operators T.

Theorem 10.1. *Let $1 < p < \infty$, $0 \le \lambda < \frac{n}{p}$. Then Calderon-Zygmund operators T are bounded from M_p^λ to M_p^λ.*

If $\lambda = 0$, the statement of Theorem 10.1 reduces to the aforementioned result for L_p.

In [17], [18] the class of *genuine* Calderon-Zygmund operators was introduced: an operator T belongs to this class if it is a Calderon-Zygmund operator and for $n \ge 2$ there exists $c_1, c_2 > 0$ $n \ge 2$ and a rotation $\mathcal{R}$ such that

$$K(x,y) \ge \frac{c_1}{|x-y|^n}$$

for all $x \in \mathbb{R}^n$ and $y \in C_x$ where

$$C_x = x + \mathcal{R}(C)$$

and

$$C = \left\{ y = (\overline{y}, y_n) \in \mathbb{R}^n : y_n > c_2 \left|\overline{y}\right|, \ \overline{y} \in \mathbb{R}^{n-1} \right\}.$$

If $n = 1$ then it is assumed that there exists $c_1 > 0$ such that

$$K(x,y) \ge \frac{c_1}{|x-y|}$$

for all $x \in \mathbb{R}$ and for all $y > x$ or for all $x \in \mathbb{R}$ and for all $y < x$.

The Hilbert transform in which case $K(x,y) = \frac{1}{x-y}$ and an operator of the form

$$K(x,y) = \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n}$$

where Ω is a continuous function on the unit sphere homogeneous of order zero whose modulus of continuity satisfies the Dini condition and such that $\Omega \not\equiv 0$ and $\int_{S^{n-1}} \Omega(\eta) d\eta = 0$, are examples of genuine Calderon-Zygmund operators.

Theorem 10.2. ([17], [18]) *Let $1 < p < \infty$, $0 < \theta_1$, $\theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$.*

1. *If T is a genuine Calderon-Zygmund operator, then the condition*

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{n/p} \right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{10.1}$$

uniformly in $t \in (0,\infty)$ is necessary for the boundedness of T from $LM_{p\theta_1,w_1(\cdot)}$ to $LM_{p\theta_2,w_2(\cdot)}$.

2. *If T is a Calderon-Zygmund operator, $\theta_1 \le \theta_2$ and $\theta_1 \le 1$, then condition* (10.1) *is sufficient for the boundedness of T from $LM_{p\theta_1,w_1(\cdot)}$ to $LM_{p\theta_2,w_2(\cdot)}$ and from $GM_{p\theta_1,w_1(\cdot)}$ to $GM_{p\theta_2,w_2(\cdot)}$. (In the latter case we assume that $w_1 \in \Omega_{p\theta_1}$ and $w_2 \in \Omega_{p\theta_2}$.)*

3. *In particular, if T is a genuine Calderon-Zygmund operator, $\theta_1 \le \theta_2$ and $\theta_1 \le 1$, then condition* (10.1) *is necessary and sufficient for the boundedness of T from $LM_{p\theta_1,w_1(\cdot)}$ to $LM_{p\theta_2,w_2(\cdot)}$.*

4. *If T is a genuine Calderon-Zygmund operator, $1 \le p < \infty$, $\theta_1 \le \theta_2$ and $\theta_1 \le 1$, then condition* (10.1) *is necessary and sufficient for the boundedness of T from $LM_{p\theta_1,w_1(\cdot)}$ to $WLM_{p\theta_2,w_2(\cdot)}$.*

Remark 12. If w_2 has certain regularity, namely if

$$\left\| w_2(r) r^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,t)} \lesssim w_2(t) t^{\frac{n}{p}+\frac{1}{\theta_2}}$$

uniformly in $t \in (0,\infty)$, then the assumption $\theta_1 \le 1$ in Theorem 10.2 can be replaced by $\theta_1 \le p$.

Remark 13. Recall that for $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$ condition (10.1) is necessary and sufficient for the boundedness of the maximal operator M from $LM_{p\theta_1,w_1(\cdot)}$ to $LM_{p\theta_2,w_2(\cdot)}$, and for $1 \le p < \infty$, $0 < \theta_1, \theta_2 \le \infty$ it is necessary and sufficient for the boundedness of M from $LM_{p\theta_1,w_1(\cdot)}$ to $WLM_{p\theta_2,w_2(\cdot)}$ (Section 7 in Part I of the survey).

11 Hardy operator

We consider, for $-\infty < \alpha < \infty$, the Hardy operator $H_\alpha \equiv H_{n,\alpha}$ defined for $f \in L_1^{loc}(\mathbb{R}^n)$ by

$$(H_\alpha f)(x) = \frac{1}{|B(0,|x|)|^{1-\frac{\alpha}{n}}} \int_{B(0,|x|)} f(y)dy, \qquad x \in \mathbb{R}^n .$$

This operator has certain relationship with the fractional maximal operator M_α defined for $0 \le \alpha < n$.

One can easily verify that

$$(M_\alpha f)(x) = \sup_{z \in \mathbb{R}^n} (H_\alpha(|f(\cdot + x)|))(z), \qquad x \in \mathbb{R}^n ,$$

and

$$(H_\alpha(|f|))(x) \le 2^{n-\alpha}(M_\alpha(f))(x), \quad x \in \mathbb{R}^n . \tag{11.1}$$

However the latter estimate is rather rough. It may easily happen that $(M_\alpha f)(x) = +\infty$ for all $x \in \mathbb{R}^n$ whilst $(H_\alpha(|f|)(x) < +\infty$ for all $x \in \mathbb{R}^n$. (For example, this happens if $f(x) = 0$ for $|x| \le 1$ and $f(x) = |x|^\beta$ for $|x| > 1$ where $\beta > -\alpha$.) The reason for that is that, for a fixed $x \in \mathbb{R}^n$, the definition of $(M_\alpha f)(x)$ takes into account the values of $f(y)$ for all $y \in \mathbb{R}^n$ while the definition of $(H_\alpha f)(x)$ takes into account the values of $f(y)$ only for $y \in B(0,|x|)$.

Let, for $1 \le p_1, p_2 \le \infty$ and for functions u_1, u_2 of one variable measurable on $(0,\infty)$, for [3] $p_1 \le p_2$

$$I(u_1,u_2) = \Big\| \big\| u_2(\tau)\tau^{\alpha-n+\frac{n-1}{p_2}} \big\|_{L_{p_2}(t,\infty)} \big\| u_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}} \big\|_{L_{p_1'}(0,t)} \Big\|_{L_\infty(0,\infty)}$$

and for $p_2 < p_1$

$$I(u_1,u_2) = \Big\| \big\| u_2(\tau)\tau^{\alpha-n+\frac{n-1}{p_2}} \big\|_{L_{p_2}(t,\infty)} \Lambda(t) \Big\|_{L_s(0,\infty)},$$

where [4]

$$\Lambda(t) = \big\| u_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}} \big\|_{L_{p_1'}(0,t)}^{\frac{p_1'}{p_2'}} u_1(t)^{-\frac{p_1'}{s}} t^{\frac{n-1}{s}}$$

and s is defined by

$$\frac{1}{s} = \frac{1}{p_2} - \frac{1}{p_1}.$$

Direct application of the results of [49], [51], [37], where necessary and sufficient conditions ensuring the boundedness of the Hardy operator from one Lebesgue space to another one were obtained and the relationship between general Morrey-type spaces and weighted Lebesgue spaces, described in Section 5 of Part I of the survey, imply the following statement for the case of local Morrey-type spaces.

Theorem 11.1. *Let $1 \le p_1, p_2 \le \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, and $w_2 \in \Omega_{\theta_2}$.*

If $p_1 \ge \theta_1$ and $p_2 \le \theta_2$, then the condition

$$I\Big(\|w_1\|_{L_{p_1}(t,\infty)}, \|w_2\|_{L_{p_2}(t,\infty)} \Big) < \infty \tag{11.2}$$

is sufficient for the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

If $p_1 \le \theta_1$ and $p_2 \ge \theta_2$, then this condition is necessary for the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

In particular, if $\theta_1 = p_1$ and $\theta_2 = p_2$, then this condition is necessary and sufficient for the boundedness of H_α from $LM_{p_1p_1,w_1(\cdot)}$ to $LM_{p_2p_2,w_2(\cdot)}$.

Under certain regularity assumptions on w_1 or w_2 necessary and sufficient conditions ensuring the boundedness of the Hardy operator from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ can be simplified. (See [19] for details.)

Corollary 11.1. *If $p_1 \ge \theta_1$, $p_2 \le \theta_2$, $\alpha < \frac{n}{p_2'}$ for $p_2 < \infty$, and $\alpha \le n$ for $p_2 = \infty$, then the condition*

$$\Big\| t^{\alpha-n(\frac{1}{p_1}-\frac{1}{p_2})-\frac{1}{s}} \|w_1\|_{L_{\theta_1}(t,\infty)}^{-1} \|w_2\|_{L_{\theta_2}(t,\infty)} \Big\|_{L_s(0,\infty)} < \infty, \tag{11.3}$$

[3] If $p_1 = 1$, then the factor $\big\| u_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}} \big\|_{L_{p_1'}(0,t)}$ should be replaced by $u_1(t)^{-1}$ and if $p_2 = \infty$, then the factor $\big\| u_2(\tau)\tau^{\alpha-n+\frac{n-1}{p_2}} \big\|_{L_{p_2}(t,\infty)}$ should be replaced by $u_2(t)t^{\alpha-n}$.

[4] If $p_2 = 1$, then the factor $\big\| u_1(\tau)^{-1}\tau^{\frac{n-1}{p_1'}} \big\|_{L_{p_1'}(0,t)}^{\frac{p_1'}{p_2'}}$ should be omitted.

is sufficient for the boundedness of H_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

If $p_1 \le \theta_1$ *and* $p_2 \ge \theta_2$, *then for any* $\mu > 1$ *both conditions*

$$\left\| t^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2}) - \frac{1}{s}} \, \|w_1\|^{-1}_{L_{\theta_1}(\frac{t}{\mu},\infty)} \, \|w_2\|_{L_{\theta_2}(t,\infty)} \right\|_{L_s(0,\infty)} < \infty \,,$$

and

$$\left\| t^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2}) - \frac{1}{s}} \, \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \, \|w_2\|_{L_{\theta_2}(\mu t,\infty)} \right\|_{L_s(0,\infty)} < \infty$$

are necessary for the boundedness of H_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

In particular, if $\theta_1 = p_1$, $\theta_2 = p_2$, $\alpha < \frac{n}{p_2'}$ *for* $p_2 < \infty$, $\alpha \le n$ *for* $p_2 = \infty$ *and, for some* $\mu > 1$, *one of the conditions*

$$\|w_1\|_{L_{p_1}(t,\infty)} \lesssim \|w_1\|_{L_{p_1}(\mu t,\infty)} \quad \text{or} \quad \|w_1\|_{L_{p_2}(t,\infty)} \lesssim \|w_2\|_{L_{p_2}(\mu t,\infty)}$$

uniformly in $t \in (0,\infty)$ *is satisfied, then condition* (11.3) *is necessary and sufficient for the boundedness of* H_α *from* $LM_{p_1p_1,w_1(\cdot)}$ *to* $LM_{p_2p_2,w_2(\cdot)}$.

Lemma 11.1. *Let* $\alpha \in \mathbb{R}, 1 \le p_1 \le \infty, 0 < p_2, \theta_1, \theta_2 \le \infty$.

If $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the condition: for all* $t > 0$

$$\|w_2(r) r^{\alpha - \frac{n}{p_2'}}\|_{L_{\theta_2}(t,\infty)} < \infty \tag{11.4}$$

is necessary for the boundedness of H_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

If $w_1 \in \Omega_{p_1\theta_1}$ *and* $w_2 \in \Omega_{p_2\theta_2}$, *then this condition is also necessary for the boundedness of* H_α *from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$.

Remark 14. For $w_2 \in \Omega_{\theta_2}$ condition (11.4) implies that $\|w_2\|_{L_{\theta_2}(t,\infty)} < \infty$ not only for some $t > 0$ (which is the meaning of the condition $w_2 \in \Omega_{\theta_2}$) but also for all $t > 0$.

Lemma 11.2. *Let* $\alpha \in \mathbb{R}, 1 \le p_1 \le \infty, 0 < p_2, \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{p_1\theta_1}$, *and* $w_2 \in \Omega_{p_2\theta_2}$.

Then the condition

$$\alpha \le \frac{n}{p_1}$$

is necessary for the boundedness of H_α *from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$.

Moreover, if in addition $\|w_2(r) r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty$, *then the condition*

$$\alpha < \frac{n}{p_1}$$

is necessary for the boundedness of H_α *from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$.

In [19] the investigation of the boundedness of H_α in local and global Morrey-type spaces was carried out under the following assumptions on the parameters:

$$\alpha \ge n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \quad \text{if} \quad 1 < p_1 \le p_2 \le \infty \quad \text{or} \quad p_1 = 1 \text{ and } p_2 = \infty \tag{11.5}$$

and

$$\alpha > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \quad \text{if} \quad p_1 = 1 \le p_2 < \infty \quad \text{or} \quad 0 < p_2 < p_1 \le \infty \,. \tag{11.6}$$

Theorem 11.2. *Let* $1 \le p_1 \le \infty$, $0 < p_2, \theta_1, \theta_2 \le \infty$, *and conditions* (11.5), (11.6) *be satisfied.*

1. *Assume that* $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ *and condition* (11.4) *is satisfied. Then for* $\theta_1 \le \theta_2$ *the condition* [5]

$$\Big\| \|w_2(t) t^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})}\|_{L_{\theta_2}(t,\infty)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} \Big\|_{L_\infty(0,\infty)} < \infty \tag{11.7}$$

and for $\theta_2 < \theta_1 < \infty$ *the condition*

$$\Big\| \|w_2(t) t^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})}\|_{L_{\theta_2}(t,\infty)} \|w_1\|^{-\frac{\theta_1}{\theta_2}}_{L_{\theta_1}(t,\infty)} w_1(t)^{\frac{\theta_1}{\sigma}} \Big\|_{L_\sigma(0,\infty)} < \infty\,, \tag{11.8}$$

where σ *is defined by*

$$\frac{1}{\sigma} = \frac{1}{\theta_2} - \frac{1}{\theta_1}\,,$$

are sufficient for the boundedness of H_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

2. *Assume that* $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$, *condition* (11.4) *is satisfied, the function* $w_2(r) r^{\frac{n}{p_2}}$ *is almost increasing,* [6] $\alpha \le \frac{n}{p_1}$, *and* $\alpha < \frac{n}{p_1}$ *if* $\|w_2(r) r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty$. *Then conditions* (11.7) *and* (11.8) *are sufficient for the boundedness of* H_α *also from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$.

Remark 15. In contrast to the operators M_α and I_α, the operator H_α does not possess property

$$(H_\alpha(f(\cdot + h)))(x) = (H_\alpha f)(x+h), \quad x, h \in \mathbb{R}^n.$$

This is the reason why in the second part of this theorem there are additional assumptions on w_2 which allow passing from the case of local Morrey-type spaces to the case of global Morrey-type spaces.

Theorem 11.3. *Let* $1 \le p_1 \le \infty$, $0 < p_2 \le \infty, 0 < \theta_1 \le \theta_2 \le \infty$, *and conditions* (11.5), (11.6) *be satisfied.*

1. *Assume that* $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ *and condition* (11.4) *is satisfied. If for* $p_1 = 1$, *for some* $\gamma > 1$,

$$\|w_2(r) r^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})}\|_{L_{\theta_2}(t,\infty)} \lesssim \|w_2(r) r^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})}\|_{L_{\theta_2(\gamma t,\infty)}} \tag{11.9}$$

uniformly in $t \in (0,\infty)$ *or for* $p_1 > 1$, *for some* $\varepsilon > 0, \gamma > 1$,

$$\|w_2(r) r^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2})}\|_{L_{\theta_2}(t,\infty)} \lesssim t^\varepsilon \|w_2(r) r^{\alpha - n(\frac{1}{p_1} - \frac{1}{p_2}) - \varepsilon}\|_{L_{\theta_2(\gamma t,\infty)}} \tag{11.10}$$

uniformly in $t \in (0,\infty)$, *then condition* (11.7) *is necessary and sufficient for the boundedness of* H_α *from* $LM_{p_1\theta_1,w_1(\cdot)}$ *to* $LM_{p_2\theta_2,w_2(\cdot)}$.

2. *Assume that* $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$, *condition* (11.4) *is satisfied, the function* $w_2(r) r^{\frac{n}{p_2}}$ *is almost increasing,* $\alpha \le \frac{n}{p_1}$, *and* $\alpha < \frac{n}{p_1}$ *if* $\|w_2(r) r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,\infty)} = \infty$. *If, in addition to* (11.9) *and* (11.10)

$$t^{-\frac{n}{p_1}} \|w_1(r) r^{\frac{n}{p_1}}\|_{L_{\theta_1(0,t)}} \lesssim \|w_1(r)\|_{L_{\theta_1}(t,\infty)} \tag{11.11}$$

uniformly in $t \in (0,\infty)$, *then condition* (11.7) *is also necessary and sufficient for the boundedness of* H_α *from* $GM_{p_1\theta_1,w_1(\cdot)}$ *to* $GM_{p_2\theta_2,w_2(\cdot)}$.

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[5] If $\alpha = n(\frac{1}{p_1} - \frac{1}{p_2})$, then it coincides with condition (11.3).

[6] i. e., for some $c \ge 1$, $w(r) r^{\frac{n}{p_2}} \le c w(\varrho) \varrho^{\frac{n}{p_2}}$ for all $0 < r < \varrho < \infty$.

Remark 16. Let us compare the necessary and sufficient conditions ensuring the boundedness of the operators H_α, M_α, and I_α in general local Morrey-type spaces.

This can be done if

$$1 < p_1 < p_2 < \infty, \quad 0 < \theta_1 \le \theta_2 \le \infty, \quad \alpha = n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big),$$

$w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ and conditions (11.4), (11.10) are satisfied, when the necessary and sufficient conditions for all three operators H_α, M_α, and I_α are known.

Under these assumptions by Theorem 11.3 $H_{n(\frac{1}{p_1}-\frac{1}{p_2})}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if

$$\sup_{t>0} \|w_2\|_{L_{\theta_2}(t,\infty)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty\,,$$

by Theorem 8.2 $M_{n(\frac{1}{p_1}-\frac{1}{p_2})}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if

$$\sup_{t>0} \big(t^{-\frac{n}{p_2}} \|w_2(r) r^{\frac{n}{p_2}}\|_{L_{\theta_1}(0,t)} + \|w_2\|_{L_{\theta_2}(t,\infty)}\big) \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty\,,$$

and by Theorem 7.5 this condition is also necessary and sufficient for the boundedness of $I_{n(\frac{1}{p_1}-\frac{1}{p_2})}$ from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

Moreover, if

$$p_1 = 1, \quad 0 < p_2 < \infty, \quad 0 < \theta_1 \le \theta_2 < \infty, \quad n\Big(1 - \frac{1}{p_2}\Big)_+ < \alpha < n\,,$$

$w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$ and conditions (11.4), (11.9) are satisfied, then by Theorem 11.3 H_α is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if

$$\sup_{t>0} \|w_2(r) r^{\alpha - n(1-\frac{1}{p_2})}\|_{L_{\theta_2}(t,\infty)} \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty$$

and by Theorem 8.3 M_α is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if

$$\sup_{t>0} \big(t^{\alpha-n} \|w_2(r) r^{\frac{n}{p_2}}\|_{L_{\theta_2}(t,\infty)} + \|w_2(r) r^{\alpha - n(1-\frac{1}{p_2})}\|_{L_{\theta_2}(t,\infty)}\big) \|w_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty\,. \tag{11.12}$$

If $0 < \theta_1 \le 1$, then condition (11.12) is also necessary and sufficient for the boundedness of I_α from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$. If $\theta_1 > 1$, then I_α is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if apart from condition (11.12) also

$$\sup_{t>0} \|w_2(r) r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,t)} \bigg\| \frac{w_1^{\theta_1 - 1}(r) r^{\alpha-n}}{\|w_1\|^{\theta_1}_{L_{\theta_1}(r,\infty)}} \bigg\|_{L_{\theta_1'}(t,\infty)} < \infty\,.$$

(See Theorem 7.6.)

Clearly the conditions for the boundedness of H_α are in general weaker than for M_α and the conditions for the boundedness of M_α are in general weaker than for I_α which conforms with inequalities (8.1) and (11.1), though sometimes they coincide.

In [21] further necessary and sufficient conditions are obtained ensuring the boundedness of H_α from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ for the case $\theta_1 = p_1$. Recall that $LM_{p_1p_1,w_1(\cdot)} = L_{p_1,u_1(\cdot)}$ where

$$u_1(x) = \|w_1\|_{L_{p_1}(|x|,\infty)},$$

so in this case the problem under consideration is a problem of boundedness of the operator H_α from a weighted space $L_{p_1,u_1(\cdot)}$ with a radially symmetric non-negative non-increasing weight u_1 to a local Morrey-type space $LM_{p_2\theta_2,w_2(\cdot)}$.

In [21] this problem is considered for a more general multi-dimensional Hardy operator $H_{\varphi(\cdot)}$ defined for all functions $f \in L_1^{loc}$ by

$$(H_{\varphi(\cdot)}f)(x) = \varphi(|x|) \int\limits_{B(0,|x|)} f(y)dy, \ \ x \in \mathbb{R}^n,$$

where φ is a fixed non-negative measurable function on $(0,\infty)$ which is not equivalent to 0, and for radially symmetric non-negative weights u_1, not necessarily non-increasing. Clearly $H_{|B(0,|x|)|^{\frac{\alpha}{n}-1}} \equiv H_\alpha$.

Lemma 11.3. *Let $1 \le p \le \infty$, $0 < p_2, \theta \le \infty$, $w \in \Omega_\theta$, $u(x) = v(|x|)$, $x \in \mathbb{R}^n$, where v is a non-negative measurable function on $(0,\infty)$, and $c_1 > 0$.*

The inequality

$$\|H_{\varphi(\cdot)}f\|_{LM_{p_2\theta,w(\cdot)}} \le c_1 \|f\|_{L_{p_1,u(\cdot)}}$$

for all functions $f \in L_{p_1,u(\cdot)}$ is equivalent to the inequality

$$\left(\int\limits_0^\infty w^\theta(r) \left(\int\limits_0^r (H_{\tilde\varphi} g)^{p_2} dt \right)^{\frac{\theta}{p_2}} dr \right)^{\frac{1}{\theta}} \le c_2 \|g\|_{L_{p_1,\tilde u}(0,\infty)}$$

for all non-negative functions $g \in L_{p_1,\tilde u(\cdot)}(0,\infty)$, where

$$(H_{\tilde\varphi(\cdot)}g(t) = \tilde\varphi(t) \int\limits_0^t g(s)ds,$$

$$\tilde\varphi(t) = \varphi(t) t^{\frac{n-1}{p_2}}, \ \ \tilde u(t) = v(t) t^{-\frac{n-1}{p_1'}}, \ \ c_2 = c_1 \sigma_n^{-(\frac{1}{p_1'}+\frac{1}{p_2})},$$

and $\sigma_n = n v_n$ is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$.

Theorem 11.4. *Let $1 < p_1 \le p_2 \le \theta < \infty$ or $1 < p_1 \le \theta < p_2 < \infty$, and let u, w be as in Lemma 11.3. Then the operator $H_{\varphi(\cdot)}$ is bounded from $L_{p_1,u(\cdot)}$ to $LM_{p_2\theta,w(\cdot)}$ if and only if*

$$B_1 = \sup_{\beta>0} \left(\int\limits_\beta^\infty w^\theta(r) \left(\int\limits_\beta^r \tilde\varphi^{p_2} ds \right)^{\frac{\theta}{p_2}} dr \right)^{\frac{1}{\theta}} \left(\int\limits_0^\beta \tilde u^{-p_1'} dr \right)^{\frac{1}{p_1'}} < \infty \,. \tag{11.13}$$

Moreover,

$$\sigma_n^{\frac{1}{p_1'}+\frac{1}{p_2}} B_1 \le \|H_{\varphi(\cdot)}\|_{L_{p_1,u(\cdot)} \to LM_{p_2\theta,w(\cdot)}} \le 4 \sigma_n^{\frac{1}{p_1'}+\frac{1}{p_2}} B_1.$$

Remark 17. Since the functions w and φ are not equivalent to 0 on $(0,\infty)$ it follows from (11.13) that $\tilde{u}^{-p_1'} \in L_1(0,\beta)$ for all $\beta > 0$.

Theorem 11.5. *Let $0 < p_2 < p_1 \le \theta < \infty$, $p_1 > 1$, and $\tilde{u}^{-p_1'} \in L_1(0,\beta)$ for all $\beta > 0$ or $p_2 > 1$. Then the operator $H_{\varphi(\cdot)}$ is bounded from $L_{p_1,u(\cdot)}$ to $LM_{p_2\theta,w(\cdot)}$ if and only if* $\max\{B_1,B_2\} < \infty$, *where*

$$B_2 = \sup_{\beta>0} \Big(\int_\beta^\infty w^\theta dr\Big)^{\frac{1}{\theta}} \Big(\int_0^\beta \Big(\int_t^\beta \widetilde{\varphi}^{p_2} dr\Big)^{\frac{p_2}{p_1-p_2}} \widetilde{\varphi}^{p_2}(t) \Big(\int_0^t \widetilde{u}^{-p_1'} dr\Big)^{\frac{p_2(p_1-1)}{p_1-p_2}} dt\Big)^{\frac{p_1-p_2}{p_1p_2}}.$$

Moreover,

$$\|H_{\varphi(\cdot)}\|_{L_{p_1,u(\cdot)}\to LM_{p_2\theta,w(\cdot)}} \approx \max\{B_1,B_2\}$$

uniformly in u and w.

Theorem 11.6. *Let $0 < p_2 < \theta < p_1 < \infty$, $\theta > 1$, and $\tilde{u}^{-p_1'} \in L_1(0,\beta)$ for all $\beta > 0$ or $p_2 > 1$. Then the operator $H_{\varphi(\cdot)}$ is bounded from $L_{p_1,u(\cdot)}$ to $LM_{p_2\theta,w(\cdot)}$ if and only if* $\max\{B_3,B_4\} < \infty$, *where*

$$B_3 = \Big(\int_0^\infty \Big(\int_\beta^\infty w^\theta(r) \Big(\int_\beta^r \widetilde{\varphi}(s) ds\Big)^{\frac{\theta}{p_2}} dr\Big)^{\frac{p_1}{p_1-\theta}} \Big(\int_0^\beta \widetilde{u}^{-p_1'} dt\Big)^{\frac{p_1(\theta-1)}{p_1-\theta}} \widetilde{u}^{-p_1'}(\beta) d\beta\Big)^{\frac{p_1-\theta}{p_1\theta}},$$

$$B_4 = \Big(\int_0^\infty \Big(\int_\beta^\infty w^\theta(r) dr\Big)^{\frac{\theta}{p_1-\theta}} \Lambda(\beta) w^\theta(\beta) d\beta\Big)^{\frac{p_1-\theta}{p_1\theta}},$$

and

$$\Lambda(\beta) = \Big(\int_0^\beta \Big(\int_t^\beta \widetilde{\varphi}(s) ds\Big)^{\frac{p_2}{p_1-p_2}} \Big(\int_0^t \widetilde{u}^{-p_1'} d\tau\Big)^{\frac{p_2(p_1-1)}{p_1-p_2}} dt\Big)^{\frac{\theta(p_1-p_2)}{p_2(p_1-\theta)}}.$$

Moreover,

$$\|H_{\varphi(\cdot)}\|_{L_{p_1,u(\cdot)}\to LM_{p_2\theta,w(\cdot)}} \approx \max\{B_3,B_4\},$$

uniformly in u and w.

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