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ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ И ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ

DIFFERENTIAL EQUATIONS AND OPTIMAL CONTROL

УДК 517.95

НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА С НЕЛОКАЛЬНЫМ ГРАНИЧНЫМ УСЛОВИЕМ ДЛЯ НЕЛИНЕЙНОГО ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С ПАМЯТЬЮ

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Рассмотрено нелинейное параболическое уравнение с памятью $u_t = \Delta u + au^p \int_0^t u^q(x, \tau) d\tau - bu^m$ для $(x, t) \in \Omega \times (0, +\infty)$ с нелинейным нелокальным граничным условием $\frac{\partial u(x, t)}{\partial \nu} \Big|_{\partial \Omega \times (0, +\infty)} = \int_{\Omega} k(x, y, t) u^l(y, t) dy$ и начальными данными $u(x, 0) = u_0(x)$, $x \in \Omega$, где a, b, q, m, l – положительные постоянные; $p \geq 0$; Ω – ограниченная область в пространстве \mathbb{R}^n с гладкой границей $\partial \Omega$; ν – единичная внешняя нормаль к $\partial \Omega$. Неотрицательная непрерывная функция $k(x, y, t)$ определена при $x \in \partial \Omega$, $y \in \bar{\Omega}$, $t \geq 0$, неотрицательная функция $u_0(x) \in C^1(\bar{\Omega})$, при этом она удовлетворяет условию $\frac{\partial u_0(x)}{\partial \nu} = \int_{\Omega} k(x, y, 0) u_0^l(y) dy$ при $x \in \partial \Omega$. Рассмотрены классические решения.

Установлено существование локального максимального решения исходной задачи. Введены понятия верхнего и нижнего решений. Показано, что при выполнении определенных условий верхнее решение не меньше нижнего решения. Найдены условия положительности решений. Как следствие положительности решений и принципа сравнения решений доказана теорема единственности решения.

Ключевые слова: нелинейное параболическое уравнение; нелокальное граничное условие; существование решения; принцип сравнения.

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INITIAL BOUNDARY VALUE PROBLEM WITH NONLOCAL BOUNDARY CONDITION FOR A NONLINEAR PARABOLIC EQUATION WITH MEMORY

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We consider a nonlinear parabolic equation with memory $u_t = \Delta u + au^p \int_0^t u^q(x, \tau) d\tau - bu^m$ for $(x, t) \in \Omega \times (0, +\infty)$ under nonlinear nonlocal boundary condition $\frac{\partial u(x, t)}{\partial \nu} \Big|_{\partial\Omega \times (0, +\infty)} = \int_{\Omega} k(x, y, t) u^l(y, t) dy$ and initial data $u(x, 0) = u_0(x)$, $x \in \Omega$, where a, b, q, m, l are positive constants; $p \geq 0$; Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$; ν is unit outward normal on $\partial\Omega$. Nonnegative continuous function $k(x, y, t)$ is defined for $x \in \partial\Omega$, $y \in \bar{\Omega}$, $t \geq 0$, nonnegative function $u_0(x) \in C^1(\bar{\Omega})$, while it satisfies the condition $\frac{\partial u_0(x)}{\partial \nu} = \int_{\Omega} k(x, y, 0) u_0^l(y) dy$ for $x \in \partial\Omega$. In this paper we study classical solutions. We establish the existence of a local maximal solution of the original problem. We introduce definitions of a supersolution and a subsolution. It is shown that under some conditions a supersolution is not less than a subsolution. We find conditions for the positiveness of solutions. As a consequence of the positiveness of solutions and the comparison principle of solutions, we prove the uniqueness theorem.

Keywords: nonlinear parabolic equation; nonlocal boundary condition; existence of a solution; comparison principle.

Introduction

In this paper we consider the initial boundary value problem for the nonlinear parabolic equation

$$u_t = \Delta u + au^p \int_0^t u^q(x, \tau) d\tau - bu^m, \quad x \in \Omega, \quad t > 0, \quad (1)$$

with nonlinear nonlocal boundary condition

$$\frac{\partial u(x, t)}{\partial \nu} = \int_{\Omega} k(x, y, t) u^l(y, t) dy, \quad x \in \partial\Omega, \quad t > 0, \quad (2)$$

and initial datum

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (3)$$

where a, b, q, m, l are positive constants; $p \geq 0$; Ω is a bounded domain in \mathbb{R}^n ($n \geq 1$) with smooth boundary $\partial\Omega$; ν is unit outward normal on $\partial\Omega$.

Throughout this paper we suppose that the functions $k(x, y, t)$ and $u_0(x)$ satisfy the following conditions:

$$k(x, y, t) \in C(\partial\Omega \times \bar{\Omega} \times [0, +\infty)), \quad k(x, y, t) \geq 0,$$

$$u_0(x) \in C^1(\bar{\Omega}), \quad u_0(x) \geq 0 \text{ in } \Omega, \quad \frac{\partial u_0(x)}{\partial \nu} = \int_{\Omega} k(x, y, 0) u_0^l(y) dy \text{ on } \partial\Omega.$$

Initial boundary value problems with nonlocal terms in parabolic equations or in boundary conditions have been considered in many papers (see, for example, [1–17] and the references therein). In particular, the initial boundary value problem (1)–(3) with $a = 0$ was considered for $b = b(x, t) \geq 0$ and $b = b(x, t) \leq 0$ in publications [18; 19] and [20; 21] respectively. Problem (1)–(3) with $p = 0$ and nonlocal boundary condition

$$u(x, t) = \int_{\Omega} k(x, y, t) u^l(y, t) dy, \quad x \in \partial\Omega, \quad t > 0, \quad (4)$$

was investigated in work [22].

The aim of this paper is to study problem (1)–(3) for any positive p, q, m and l . We prove existence of a local solution of problem (1)–(3). Comparison principle and the uniqueness of a solution are established. We show the nonuniqueness of solution of problem (1)–(3) with $u_0(x) \equiv 0$ also.

Local existence

In this section a local existence theorem for problem (1)–(3) will be proved. We begin with definitions of a supersolution, a subsolution and a maximal solution of problem (1)–(3). Let $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$, $\Gamma_T = S_T \cup \bar{\Omega} \times \{0\}$, $T > 0$.

Definition 1. We say that a nonnegative function $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a supersolution of problem (1)–(3) in Q_T if

$$u_t \geq \Delta u + au^p \int_0^t u^q(x, \tau) d\tau - bu^m, \quad (x, t) \in Q_T, \quad (5)$$

$$\frac{\partial u(x, t)}{\partial \nu} \geq \int_{\Omega} k(x, y, t) u^l(y, t) dy, \quad (x, t) \in S_T, \quad (6)$$

$$u(x, 0) \geq u_0(x), \quad x \in \Omega, \quad (7)$$

and a nonnegative function $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a subsolution of problem (1)–(3) in Q_T if $u \geq 0$ and it satisfies inequalities (5)–(7) in the reverse order. We say that $u(x, t)$ is a solution of problem (1)–(3) in Q_T if $u(x, t)$ is both a subsolution and a supersolution of problem (1)–(3) in Q_T .

Definition 2. We say that $u(x, t)$ is a maximal solution of problem (1)–(3) in Q_T if for any other solution $w(x, t)$ of problem (1)–(3) in Q_T the inequality $w(x, t) \leq u(x, t)$ is satisfied for $(x, t) \in Q_T \cup \Gamma_T$.

Let $\{\varepsilon_m\}$ be decreasing to zero a sequence such that $0 < \varepsilon_m < 1$ and $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$. For $\varepsilon = \varepsilon_m$, $m = 1, 2, \dots$, let $u_{0\varepsilon}(x)$ be the functions with the following properties:

$$\begin{aligned} u_{0\varepsilon}(x) &\in C^1(\bar{\Omega}), \quad u_{0\varepsilon}(x) \geq \varepsilon, \quad u_{0\varepsilon_i}(x) \geq u_{0\varepsilon_j}(x) \text{ for } \varepsilon_i \geq \varepsilon_j, \\ u_{0\varepsilon}(x) &\rightarrow u_0(x) \text{ as } \varepsilon \rightarrow 0 \text{ uniformly in } \bar{\Omega}, \\ \frac{\partial u_{0\varepsilon}(x)}{\partial \nu} &= \int_{\Omega} k(x, y, 0) u_{0\varepsilon}^l(y) dy, \quad x \in \partial\Omega. \end{aligned} \quad (8)$$

Let us consider the following auxiliary problem:

$$\begin{cases} u_t = \Delta u + au^p \int_0^t u^q(x, \tau) d\tau - bu^m + b\varepsilon^m, \quad (x, t) \in Q_T, \\ \frac{\partial u(x, t)}{\partial \nu} = \int_{\Omega} k(x, y, t) u^l(y, t) dy, \quad (x, t) \in S_T, \\ u(x, 0) = u_{0\varepsilon}(x), \quad x \in \Omega, \end{cases} \quad (9)$$

where $\varepsilon = \varepsilon_m$. The notion of a solution u_ε for problem (9) can be defined in a similar way as in the definition 1.

Theorem 1. Problem (9) has a unique solution in Q_T for small values of $T > 0$.

Proof. Denote $K = \sup_{\partial\Omega \times Q_1} k(x, y, t)$ and introduce an auxiliary function $\psi(x)$ with the following properties:

$$\psi(x) \in C^2(\bar{\Omega}), \quad \inf_{\Omega} \psi(x) \geq \max\left(\sup_{\Omega} u_{0\varepsilon}(x), 1\right), \quad \inf_{\partial\Omega} \frac{\partial \psi(x)}{\partial \nu} \geq K \max(1, \exp(l-1)) \int_{\Omega} \psi^l(y) dy.$$

We put

$$w(x, t) = \exp(\alpha t) \psi(x),$$

where α will be defined below.

To prove the existence of a solution for problem (9) we introduce the set

$$B = \left\{ h(x, t) \in C(\bar{Q}_T) : \varepsilon \leq h(x, t) \leq w(x, t), \quad h(x, 0) = u_{0\varepsilon}(x) \right\}$$

and consider the problem

$$\begin{cases} u_t = \Delta u + a v^p \int_0^t v^q(x, \tau) d\tau - b u^m + b \varepsilon^m, & (x, t) \in Q_T, \\ \frac{\partial u(x, t)}{\partial \nu} = \int_{\Omega} k(x, y, t) v^l(y, t) dy, & (x, t) \in S_T, \\ u(x, 0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \quad (10)$$

where $v \in B$. It is obvious, B is a nonempty convex subset of $C(\bar{Q}_T)$. By classical theory [23] problem (10) has a solution $u \in C^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)$ for small values of T . Let us call $A(v) = u$, where $v \in B$, and u is a solution of problem (10). In order to show that A has a fixed point in B we verify that A is a continuous mapping from B into itself such that AB is relatively compact. Obviously, the function $u(x, t) = \varepsilon$ is a subsolution of problem (10). Let us show that $w(x, t)$ is a supersolution of problem (10) for suitable choice of $\alpha > 0$ and $T > 0$.

Indeed,

$$\begin{aligned} w_t - \Delta w - a v^p \int_0^t v^q(x, \tau) d\tau + b w^m - b \varepsilon^m &\geq w_t - \Delta w - a w^p \int_0^t w^q(x, \tau) d\tau + b w^m - b \varepsilon^m \geq \\ &\geq \exp(\alpha t) [\alpha \psi(x) - \Delta \psi(x)] - a \exp(p\alpha t) \frac{\exp(q\alpha t) - 1}{q\alpha} \psi^{p+q} + b (\exp(m\alpha t) \psi^m(x) - \varepsilon^m) \geq 0 \end{aligned}$$

for $(x, t) \in Q_T$ if

$$\alpha \geq \max \left\{ \frac{1}{q}, a \exp(1) \sup_{\Omega} \psi^{p+q-1}(x) + \sup_{\Omega} \frac{\Delta \psi(x)}{\psi(x)} \right\}, \quad T \leq \frac{1}{(p+q)\alpha}.$$

On the boundary S_T we have

$$\begin{aligned} \frac{\partial w(x, t)}{\partial \nu} - \int_{\Omega} k(x, y, t) v^l(y, t) dy &\geq \exp(\alpha t) K \max(1, \exp(l-1)) \int_{\Omega} \psi^l(y) dy - \\ &- K \exp(l\alpha t) \int_{\Omega} \psi^l(y) dy \geq 0 \end{aligned}$$

for $T \leq \frac{1}{\alpha}$. The inequality

$$w(x, 0) - u_{0\varepsilon}(x) \geq 0$$

holds for $x \in \Omega$. Then $w(x, t)$ is a supersolution of problem (10) and thanks to a comparison principle for problem (10) A maps B into itself.

Let $G(x, y; t - \tau)$ denote the Green's function for a heat equation with homogeneous Neumann boundary condition. The Green's function has the following properties (see, for example, [24]):

$$\begin{aligned} G(x, y; t - \tau) &\geq 0, \quad x, y \in \Omega, \quad 0 \leq \tau < t, \\ \int_{\Omega} G(x, y; t - \tau) dy &= 1, \quad x \in \Omega, \quad 0 \leq \tau < t. \end{aligned} \quad (11)$$

It is well known that $u(x, t)$ is a solution of problem (10) in Q_T if and only if for $(x, t) \in Q_T$

$$\begin{aligned} u(x, t) &= \int_{\Omega} G(x, y; t) u_{0\varepsilon}(y) dy + \\ &+ \int_0^t \int_{\Omega} G(x, y; t - \tau) \left(a v^p(y, \tau) \int_0^{\tau} v^q(y, \sigma) d\sigma + b (\varepsilon^m - u^m(y, \tau)) \right) dy d\tau + \\ &+ \int_0^t \int_{\partial \Omega} G(x, \xi; t - \tau) \int_{\Omega} k(\xi, y, \tau) v^l(y, \tau) dy dS_{\xi} d\tau. \end{aligned} \quad (12)$$

We claim that A is continuous. In fact let v_k be a sequence in B converging to $v \in B$ in $C(\bar{Q}_T)$. Denote $u_k = A v_k$. Then by (11) and (12) we see that

$$\begin{aligned}
 |u - u_k| &= \left| \int_0^t \int_{\Omega} G(x, y; t - \tau) \left\{ a(v^p(y, \tau) - v_k^p(y, \tau)) \int_0^{\tau} v^q(y, \sigma) d\sigma + \right. \right. \\
 &+ a v_k^p(y, \tau) \int_0^{\tau} (v^q(y, \sigma) - v_k^q(y, \sigma)) d\sigma - b(u^m(y, \tau) - u_k^m(y, \tau)) \left. \right\} dy d\tau + \\
 &+ \left. \int_0^t \int_{\partial\Omega} G(x, \xi; t - \tau) \int_{\Omega} k(\xi, y, \tau) (v^l(y, \tau) - v_k^l(y, \tau)) dy dS_{\xi} d\tau \right| \leq \\
 &\leq aT^2 \sup_{Q_T} |v^p - v_k^p| \sup_{Q_T} w^q + aT^2 \sup_{Q_T} |v^q - v_k^q| \sup_{Q_T} w^p + \\
 &+ \theta T \sup_{Q_T} |u - u_k| + KT |\Omega| \sup_{Q_T} |v^l - v_k^l|,
 \end{aligned}$$

where $\theta = mb \max \left(\varepsilon^{m-1}, \sup_{Q_T} w^{m-1}(x, t) \right)$; $T \leq \min \left\{ 1, \frac{1}{2\theta} \right\}$. Now we can conclude that u_k converges to u in $C(\bar{Q}_T)$ as $k \rightarrow \infty$.

The equicontinuity of AB follows from equation (12) and the properties of the Green's function (see, for example, [25]). The Ascoli – Arzelà theorem guarantees the relative compactness of AB . Thus we are able to apply the Schauder – Tychonoff fixed point theorem and conclude that A has a fixed point in B if T is small. Now if u_{ε} is a fixed point of A , $u_{\varepsilon} \in C^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)$ and it is a solution of problem (9) in Q_T . Uniqueness of the solution follows from a comparison principle for problem (9) which can be proved in a similar way as in the next section. Theorem 1 is proved.

Now, let $\varepsilon_2 > \varepsilon_1$. Then it is easy to see that $u_{\varepsilon_2}(x, t)$ is a supersolution of problem (9) with $\varepsilon = \varepsilon_1$. Applying to problem (9) a comparison principle we have $u_{\varepsilon_1}(x, t) \leq u_{\varepsilon_2}(x, t)$. Using the last inequality and the continuation principle of solutions we deduce that the existence time of u_{ε} does not decrease as $\varepsilon \rightarrow 0$. Taking $\varepsilon \rightarrow 0$, we get $u_M(x, t) = \lim_{\varepsilon \rightarrow 0} u_{\varepsilon}(x, t) \geq 0$ and $u_M(x, t)$ exists in Q_T for some $T > 0$. We know that $u_{\varepsilon}(x, t)$ is a solution of problem (9) in Q_T if and only if for $(x, t) \in Q_T$

$$\begin{aligned}
 u_{\varepsilon}(x, t) &= \int_{\Omega} G(x, y; t) u_{0\varepsilon}(y) dy + \\
 &+ \int_0^t \int_{\Omega} G(x, y; t - \tau) \left(a u_{\varepsilon}^p(y, \tau) \int_0^{\tau} u_{\varepsilon}^q(y, \sigma) d\sigma + b(\varepsilon^m - u_{\varepsilon}^m(y, \tau)) \right) dy d\tau + \\
 &+ \int_0^t \int_{\partial\Omega} G(x, \xi; t - \tau) \int_{\Omega} k(\xi, y, \tau) u_{\varepsilon}^l(y, \tau) dy dS_{\xi} d\tau.
 \end{aligned} \tag{13}$$

Passing to the limit as $\varepsilon \rightarrow 0$ in equation (13), we obtain by dominated convergence theorem

$$\begin{aligned}
 u_M(x, t) &= \int_{\Omega} G(x, y; t) u_0(y) dy + \\
 &+ \int_0^t \int_{\Omega} G(x, y; t - \tau) \left(a u_M^p(y, \tau) \int_0^{\tau} u_M^q(y, \sigma) d\sigma - b u_M^m(y, \tau) \right) dy d\tau + \\
 &+ \int_0^t \int_{\partial\Omega} G(x, \xi; t - \tau) \int_{\Omega} k(\xi, y, \tau) u_M^l(y, \tau) dy dS_{\xi} d\tau
 \end{aligned}$$

for $(x, t) \in Q_T$. Therefore, $u_M(x, t)$ is a solution of problem (1)–(3). Let $u(x, t)$ be any other solution of problem (1)–(3). Then by comparison principle from the next section $u_{\varepsilon}(x, t) \geq u(x, t)$. Taking $\varepsilon \rightarrow 0$, we conclude $u_M(x, t) \geq u(x, t)$. Now we proved the following local existence theorem.

Theorem 2. *Problem (1)–(3) has a maximal solution in Q_T for small values of T .*

Comparison principle

Theorem 3. Let $\bar{u}(x, t)$ and $\underline{u}(x, t)$ be a supersolution and a subsolution of problem (1)–(3) in Q_T respectively. Suppose that $\underline{u}(x, t) > 0$ or $\bar{u}(x, t) > 0$ in $Q_T \cup \Gamma_T$ if either $\min(q, l) < 1$ or $0 < p < 1$. Then $\bar{u}(x, t) \geq \underline{u}(x, t)$ in $Q_T \cup \Gamma_T$.

Proof. Suppose that $\min(p, q, l) \geq 1$. Let $T_0 \in (0, T)$ and $u_{0\varepsilon}(x)$ have the same properties as in (8) but only $u_{0\varepsilon}(x) \rightarrow \underline{u}(x, 0)$ as $\varepsilon \rightarrow 0$ uniformly in $\bar{\Omega}$. We can construct a solution $u_M(x, t)$ of problem (1)–(3) with $u_0(x) = \underline{u}(x, 0)$ in the following way: $u_M(x, t) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t)$, where $u_\varepsilon(x, t)$ is a solution of problem (9). To establish the theorem we will show that

$$\underline{u}(x, t) \leq u_M(x, t) \leq \bar{u}(x, t), \quad (x, t) \in \bar{Q}_{T_0}. \quad (14)$$

We prove the second inequality in relations (14) only since the proof of the first one is similar. Let $\varphi(x, \tau) \in C^{2,1}(\bar{Q}_{T_0})$ be a nonnegative function such that

$$\frac{\partial \varphi(x, t)}{\partial \nu} = 0$$

for $(x, t) \in S_{T_0}$. If we multiply the first equation in problem (9) by $\varphi(x, t)$ and then integrate over Q_t for $t \in (0, T_0)$, we obtain

$$\begin{aligned} & \int_0^t \int_{\Omega} u_{\varepsilon\tau}(x, \tau) \varphi(x, \tau) dx d\tau = \\ & = \int_0^t \int_{\Omega} \left(\Delta u_\varepsilon(x, \tau) + a u_\varepsilon^p(x, \tau) \int_0^\tau u_\varepsilon^q(x, \sigma) d\sigma + b(\varepsilon^m - u_\varepsilon^m(x, \tau)) \right) \varphi(x, \tau) dx d\tau. \end{aligned}$$

Integrating by parts and using Green's identity, we have

$$\begin{aligned} & \int_{\Omega} u_\varepsilon(x, t) \varphi(x, t) dx \leq \int_{\Omega} u_\varepsilon(x, 0) \varphi(x, 0) dx + \\ & + \int_0^t \int_{\Omega} (u_\varepsilon(x, \tau) \varphi_\tau(x, \tau) + u_\varepsilon(x, \tau) \Delta \varphi(x, \tau)) dx d\tau + \\ & + \int_0^t \int_{\Omega} \left(a u_\varepsilon^p(x, \tau) \int_0^\tau u_\varepsilon^q(x, \sigma) d\sigma + b(\varepsilon^m - u_\varepsilon^m(x, \tau)) \right) \varphi(x, \tau) dx d\tau + \\ & + \int_0^t \int_{\partial\Omega} \varphi(x, \tau) \int_{\Omega} k(x, y, \tau) u_\varepsilon'(y, \tau) dy dS_x d\tau. \end{aligned} \quad (15)$$

On the other hand, \bar{u} satisfies (15) with reversed inequality and with $\varepsilon = 0$. Set $w(x, t) = u_\varepsilon(x, t) - \bar{u}(x, t)$. Then $w(x, t)$ satisfies

$$\begin{aligned} & \int_{\Omega} w(x, t) \varphi(x, t) dx \leq \int_{\Omega} w(x, 0) \varphi(x, 0) dx + \varepsilon^m b \int_0^t \int_{\Omega} \varphi(x, \tau) dx d\tau + \\ & + \int_0^t \int_{\Omega} w(x, \tau) (\varphi_\tau(x, \tau) + \Delta \varphi(x, \tau) - m b \theta_1^{m-1}(x, \tau)) \varphi(x, \tau) dx d\tau + \\ & + \int_0^t \int_{\Omega} \left(a \bar{u}^p(x, \tau) \varphi(x, \tau) \int_0^\tau q \theta_2^{q-1}(x, \sigma) w(x, \sigma) d\sigma \right) dx d\tau + \\ & + \int_0^t \int_{\Omega} \left(a p \theta_3^{p-1}(x, \tau) w(x, \tau) \varphi(x, \tau) \int_0^\tau u_\varepsilon^q(x, \sigma) d\sigma \right) dx d\tau + \\ & + \int_0^t \int_{\partial\Omega} \varphi(x, \tau) \int_{\Omega} k(x, y, \tau) l \theta_4^{l-1}(y, \tau) w(y, \tau) dy dS_x d\tau, \end{aligned} \quad (16)$$

where $\theta_i(x, \tau)$, $i = 1, 2, 3, 4$, are some continuous functions between $u_\varepsilon(x, t)$ and $\bar{u}(x, t)$. Note here that by hypotheses for $k(x, y, \tau)$, $u_\varepsilon(x, t)$ and $\bar{u}(x, t)$, we have

$$\begin{aligned} 0 \leq \bar{u}(x, t) \leq M, \quad \varepsilon \leq u_\varepsilon(x, t) \leq M, \quad (x, t) \in \bar{Q}_{T_0}, \\ 0 \leq k(x, y, t) \leq M, \quad (x, y, t) \in \partial\Omega \times \bar{\Omega} \times [0, T_0], \end{aligned} \quad (17)$$

where M is some positive constant. Then it is easy to see from (17) that $\theta_1^{m-1}(x, \tau)$, $\theta_2^{q-1}(x, \tau)$, $\theta_3^{p-1}(x, \tau)$ and $\theta_4^{l-1}(x, \tau)$ are positive and bounded functions in \bar{Q}_{T_0} and, moreover, $\theta_1^{m-1}(x, \tau) \leq \max\{\varepsilon^{m-1}, M^{m-1}\}$, $\theta_2^{q-1}(x, \tau) \leq M^{q-1}$, $\theta_3^{p-1}(x, \tau) \leq M^{p-1}$, $\theta_4^{l-1}(x, \tau) \leq M^{l-1}$. Define a sequence $\{a_n\}$ in the following way: $a_n(x, t) \in C^\infty(\bar{Q}_{T_0})$, $a_n(x, t) \geq 0$ and $a_n(x, t) \rightarrow mb\theta_1^{m-1}(x, t)$ as $n \rightarrow \infty$ in $L^1(\bar{Q}_{T_0})$. Now, we consider a backward problem given by

$$\begin{cases} \varphi_\tau + \Delta\varphi - a_n\varphi = 0, & (x, \tau) \in Q_t, \\ \frac{\partial\varphi(x, \tau)}{\partial\nu} = 0, & (x, \tau) \in S_t, \\ \varphi(x, t) = \psi(x), & x \in \Omega, \end{cases} \quad (18)$$

where $\psi(x) \in C_0^\infty(\Omega)$ and $0 \leq \psi(x) \leq 1$. Denote a solution of problem (18) as $\varphi_n(x, \tau)$. Then by the standard theory for linear parabolic equations (see, for example, [25]), we find that $\varphi_n(x, \tau) \in C^{2,1}(\bar{Q}_t)$, $0 \leq \varphi_n(x, \tau) \leq 1$ in \bar{Q}_t . Putting $\varphi = \varphi_n$ in inequality (16) and passing to the limit as $n \rightarrow \infty$, we infer

$$\begin{aligned} \int_{\Omega} w(x, t)\psi(x)dx \leq \int_{\Omega} w_+(x, 0)dx + \varepsilon^m b T_0 |\Omega| + \\ + \left\{ a(p+q)M^{p+q-1}T_0 + l|\partial\Omega|M^l \right\} \int_0^t \int_{\Omega} w_+(x, \tau)dx d\tau, \end{aligned} \quad (19)$$

where $w_+ = \max(w, 0)$; $|\partial\Omega|$ and $|\Omega|$ are the Lebesgue measures of $\partial\Omega$ in \mathbb{R}^{n-1} and Ω in \mathbb{R}^n respectively. Since inequality (19) holds for every $\psi(x)$, we can choose a sequence $\{\psi_n(x)\}$ converging in $L^1(\Omega)$ to

$$\psi(x) = \begin{cases} 1, & \text{if } w(x, t) > 0, \\ 0, & \text{if } w(x, t) \leq 0. \end{cases}$$

Passing to the limit as $n \rightarrow \infty$ in inequality (19), we obtain

$$\begin{aligned} \int_{\Omega} w_+(x, t)dx \leq \int_{\Omega} w_+(x, 0)dx + \varepsilon^m b T_0 |\Omega| + \\ + \left\{ a(p+q)M^{p+q-1}T_0 + l|\partial\Omega|M^l \right\} \int_0^t \int_{\Omega} w_+(x, \tau)dx d\tau, \quad t \in (0, T_0]. \end{aligned}$$

Applying now Gronwall's inequality, we have

$$\int_{\Omega} w_+(x, t)dx \leq \left(\int_{\Omega} w_+(x, 0)dx + \varepsilon^m b T_0 |\Omega| \right) \exp \left[\left\{ a(p+q)M^{p+q-1}T_0 + l|\partial\Omega|M^l \right\} t \right]$$

for $t \in (0, T_0]$. Passing to the limit as $\varepsilon \rightarrow 0$, the conclusion of the theorem follows for $\min(p, q, l) \geq 1$. For the case $p = 0$, $\min(q, l) \geq 1$ we prove the theorem in the same way. If $\min(q, l) < 1$ or $0 < p < 1$ we can consider $w(x, t) = \underline{u}(x, t) - \bar{u}(x, t)$ and prove the theorem in a similar way using the positiveness of a subsolution or a supersolution. Theorem 3 is proved.

Remark. For similar problem (1), (3), (4) with $p = 0$ the authors of work [22] suppose in the comparison principle that $\underline{u}(x, t) > 0$ or $\bar{u}(x, t) > 0$ in $Q_T \cup \Gamma_T$ if $\min(q, m, l) < 1$.

Lemma 1. Let $u(x, t)$ be a solution of problem (1)–(3) in Q_T . Let $u_0(x) \equiv 0$ in Ω and $m \geq 1$. Then $u(x, t) > 0$ in $Q_T \cup S_T$. If $u_0(x) > 0$ in $\bar{\Omega}$ and $p < m < 1$ then $u(x, t) > 0$ in $Q_T \cup \Gamma_T$.

Proof. Let $u_0(x) \equiv 0$ in Ω and $m \geq 1$. We denote

$$M = \sup_{Q_{T_0}} u(x, t),$$

where M is some positive constant; $T_0 \in (0, T)$. Now we put $h(x, t) = u(x, t) \exp(\lambda t)$ with $\lambda \geq bM^{m-1}$. Then in Q_{T_0} we have

$$h_t - \Delta h = \exp(\lambda t)(\lambda u + u_t - \Delta u) \geq u \exp(\lambda t)(\lambda - bu^{m-1}) \geq 0.$$

Since $h(x, 0) = u_0(x) \geq 0$, $x \in \Omega$, and $u_0(x) \equiv 0$ in Ω , by the strong maximum principle $h(x, t) > 0$ in Q_{T_0} . Hence, $u(x, t) > 0$ in Q_{T_0} . Let $h(x_0, t_0) = 0$ in some point $(x_0, t_0) \in S_T$. Then according to theorem 3.6 of work [26] it yields $\frac{\partial h(x_0, t_0)}{\partial \nu} < 0$, which contradicts boundary condition (2).

Let $u_0(x) > 0$ in $\bar{\Omega}$ and $p < m < 1$. Then there exist $\tau \in (0, T)$ and $\varepsilon > 0$ such that

$$u(x, t) \geq \varepsilon \text{ in } \bar{Q}_\tau$$

and, moreover, $u(x, t) \equiv \varepsilon_1 = \min \left(\varepsilon, \left[\frac{a\tau\varepsilon^q}{b} \right]^{\frac{1}{m-p}} \right)$ is the subsolution of problem (1)–(3) in $Q_{T_0} \setminus \bar{Q}_\tau$ with initial

function $u(x, \tau)$ for $t = \tau$ instead of initial datum (3). Putting $\underline{u}(x, t) \equiv \varepsilon_1$ and $\bar{u}(x, t) \equiv u(x, t)$ and arguing as in the proof of theorem 3, we get

$$u(x, t) \geq \varepsilon_1 \text{ in } \bar{Q}_{T_0} \text{ for any } T_0 \in (0, T).$$

Lemma 1 is proved.

As a simple consequence of theorem 3 and lemma 1, we get the following uniqueness result for problem (1)–(3).

Theorem 4. Let problem (1)–(3) have a positive in $Q_T \cup \Gamma_T$ solution or a solution in Q_T either with non-negative initial data in Ω for $\min(p, q, l) \geq 1$ or with positive initial data in $\bar{\Omega}$ under the conditions $m \geq 1$ or $p < m < 1$. Then a solution of problem (1)–(3) is unique in Q_T .

Now we will prove the nonuniqueness of solution of problem (1)–(3) with $u_0(x) \equiv 0$ for $l < \min(1, m)$ or $p + q < \min(1, m)$. We note that problem (1)–(3) with $u_0(x) \equiv 0$ has solution $u(x, t) \equiv 0$.

Theorem 5. Let $u_0(x) \equiv 0$ and either $l < \min(1, m)$ and

$$k(x, y_0, t_0) > 0 \text{ for any } x \in \partial\Omega \text{ and some } y_0 \in \partial\Omega \text{ and } t_0 \in [0, T) \tag{20}$$

or $p + q < \min(1, m)$. Then a maximal solution of problem (1)–(3) $u_M(x, t) \not\equiv 0$ in Q_T .

Proof. As shown in theorem 2 a maximal solution $u_M(x, t) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t)$, where $u_\varepsilon(x, t)$ is some positive in \bar{Q}_T supersolution of problem (1)–(3). To prove the theorem we construct a subsolution $\underline{u}(x, t) \equiv 0$ of problem (1)–(3) with $u_0(x) \equiv 0$. By theorem 3 we have $u_\varepsilon(x, t) \geq \underline{u}(x, t)$ and therefore maximal solution $u_M(x, t) \not\equiv 0$.

At first let $l < \min(1, m)$ and inequality (20) hold. To construct a subsolution we use the change of variables in a neighbourhood of $\partial\Omega$ as in work [27]. Let \bar{x} be a point on $\partial\Omega$. We denote by $\bar{n}(\bar{x})$ the inner unit normal to $\partial\Omega$ at the point \bar{x} . Since $\partial\Omega$ is smooth it is well-known that there exists $\delta > 0$ such that the mapping $\psi : \partial\Omega \times [0, \delta] \rightarrow \mathbb{R}^n$ given by $\psi(\bar{x}, s) = \bar{x} + s\bar{n}(\bar{x})$ defines new coordinates (\bar{x}, s) in a neighbourhood of $\partial\Omega$ in $\bar{\Omega}$.

Under the assumptions of the theorem, there exists \bar{t} such that $k(x, y, t) > 0$ for $t_0 \leq t \leq t_0 + \bar{t}$, $x \in \partial\Omega$ and $y \in V(y_0)$, where $V(y_0)$ is some neighbourhood of y_0 in $\bar{\Omega}$.

Let $\frac{1}{1-l} < \alpha \leq \frac{1}{1-m}$ for $m < 1$ and $\alpha > \frac{1}{1-l}$ for $m \geq 1$, $2 < \beta < \frac{2}{1-m}$ for $m < 1$ and $\beta > 2$ for $m \geq 1$. Assume that $A > 0$, $0 < \xi_0 \leq 1$ and $0 < T_0 < \min(T - t_0, \bar{t}, \delta^2)$. For points in $\partial\Omega \times [0, \delta] \times (t_0, t_0 + T_0]$ of coordinates (\bar{x}, s, t) define

$$\underline{u}(\bar{x}, s, t) = A(t - t_0)^\alpha \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^\beta$$

and extend \underline{u} as zero to the whole of \bar{Q}_τ with $\tau = t_0 + T_0$. Arguing as in work [18] we prove that \underline{u} is the subsolution of problem (1)–(3) with $u_0(x) \equiv 0$ in Q_τ .

Now we suppose that $p + q < \min(1, m)$. Then it is easy to check that $\underline{u}(x, t) = t^\gamma$ is the subsolution of problem (1)–(3) with $u_0(x) \equiv 0$ in Q_τ for small values of τ if

$$\gamma > \max \left(\frac{2}{1 - (p + q)}, \frac{1}{m - (p + q)} \right).$$

Theorem 5 is proved.

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