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О НЕКОТОРЫХ КЛАССАХ ПОДРЕШЕТОК РЕШЕТКИ ВСЕХ ПОДГРУПП

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В настоящей статье G всегда обозначает группу. Если K и H – подгруппы группы G, где K – нормальная подгруппа группы H, то фактор-группа группы H по K называется секцией группы G. Такая секция является нормальной, если K и H – нормальные подгруппы группы G, и тривиальной, если K и H равны. Назовем произвольное множество Σ нормальных секций группы G расслоением группы G, если оно содержит каждую тривиальную нормальную секцию группы G, и будем говорить, что расслоение Σ группы G является G-замкнутым, если Σ содержит каждую такую нормальную секцию группы G, которая G-изоморфна некоторой нормальной секции группы G, принадлежащей множеству Σ. Пусть теперь Σ – произвольное G-замкнутое расслоение группы G и пусть $\boldsymbol{L}$ – множество всех таких подгрупп A группы G, что фактор-группа группы V по W, где V – нормальное замыкание A в G, а W – нормальное ядро A в G, принадлежит Σ. Опишем условия на Σ, при которых множество $\boldsymbol{L}$ является подрешеткой решетки всех подгрупп группы G, а также обсудим некоторые применения этой подрешетки в теории обобщенных конечных T-групп.

Ключевые слова: группа; решетка подгрупп; модулярная решетка; формационное множество Фиттинга; формация Фиттинга.

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ON SOME CLASSES OF SUBLATTICES OF THE SUBGROUP LATTICE

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In this paper G always denotes a group. If K and H are subgroups of G, where K is a normal subgroup of H, then the factor group of H by K is called a section of G. Such a section is called normal, if K and H are normal subgroups of G, and trivial, if K and H are equal. We call any set Σ of normal sections of G a stratification of G, if Σ contains every trivial normal section of G, and we say that a stratification Σ of G is G-closed, if Σ contains every such a normal section of G, which is G-isomorphic to some normal section of G belonging Σ. Now let Σ be any G-closed stratification of G, and let $\boldsymbol{L}$ be the set of all subgroups A of G such that the factor group of V by W, where V is the normal closure of A in G and W is the normal core of A in G, belongs to Σ. In this paper we describe the conditions on Σ under which the set $\boldsymbol{L}$ is a sublattice of the lattice of all subgroups of G and we also discuss some applications of this sublattice in the theory of generalized finite T-groups.

Keywords: group; subgroup lattice; modular lattice; formation Fitting set; Fitting formation.

Introduction

In this paper G always denotes a group. Moreover, $\mathfrak{L}(G)$ denotes the set (the lattice) of all subgroups of G and $\mathfrak{L}_n(G)$ is the set (the lattice) of all normal subgroups of G. In this paper $\mathfrak{F}$ is a class of groups containing all identity groups, $\mathfrak{N}^*$ is the class of all finite quasinilpotent groups, $\mathfrak{N}$ is the class of all finite nilpotent groups and $\mathfrak{U}$ is the class of all finite supersoluble groups.

A class of groups $\mathfrak{F}$ is said to be a *Fitting formation* if the following conditions hold: (1) for every normal subgroup N of any group $G \in \mathfrak{F}$ both groups N and G/N belong to $\mathfrak{F}$; (2) $G \in \mathfrak{F}$ whenever G has normal subgroups A and B and either G/A, $G/B \in \mathfrak{F}$ and $A \cap B = 1$ or $G = AB$ and A, $B \in \mathfrak{F}$.

One of the organizing ideas of the group theory is the idea to study the group G depending on the presence in it a subgroup system $\mathfrak{L}$ having desired properties. Such an approach is the most effective in the case when $\mathfrak{L}$ forms a *sublattice* of $\mathfrak{L}(G)$, that is, $A \cap B \in \mathfrak{L}$ and $\langle A, B\rangle \in \mathfrak{L}$ for all A, $B \in \mathfrak{L}$. This circumstance makes the general problem of finding sublattices in $\mathfrak{L}(G)$ important and interesting.

One of the first results in this direction was obtained by Wielandt in his paper [1], where it was proved that the set $\mathfrak{L}_{sn}(G)$ of all subnormal subgroups of the group G having a composition series is a sublattice of $\mathfrak{L}(G)$. In the case when G is finite, an original generalization of the lattice $\mathfrak{L}_{sn}(G)$ was found by Kegel [2]. A subgroup A of G is called *$\mathfrak{F}$-subnormal in G in the sense of Kegel* [2] or *K-$\mathfrak{F}$-subnormal* in G [3, definition 6.1.4], if there is a subgroup chain $A = A_0 \leq A_1 \leq \ldots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/\left(A_{i-1}\right)_{A_i} \in \mathfrak{F}$ for all $i = 1, \ldots, t$. Kegel proved [2] that if the class $\mathfrak{F}$ is closed under extensions, epimorphic images and subgroups, then the set $\mathfrak{L}_{\mathfrak{F}sn}(G)$ of all K-$\mathfrak{F}$-subnormal subgroups of a finite group G is a sublattice of the lattice $\mathfrak{L}(G)$. For every set π of primes, we may choose the class $\mathfrak{F}$ of all π-groups. In this way we obtain infinitely many functors $\mathfrak{L}_{\mathfrak{F}sn}$ assigning to every finite group G a sublattice of $\mathfrak{L}(G)$ containing $\mathfrak{L}_{sn}(G)$. Subsequently, this result was generalized (also in the universe of all finite groups) on the basis of methods of the formation theory (see, in particular, [4; 5] and chapter 6 in [3]).

In this paper, we develop a new approach for finding sublattices in $\mathfrak{L}(G)$, where G is an arbitrary group, and we also discuss some applications of such sublattices.

The main concepts and results

If $K \trianglelefteq H \leq G$, then H/K is called a *section* of G; such a section is called: *normal* if H and K are normal subgroups of G; *trivial* if $H = K$; a *chief factor* of G provided $K < H$ and for any normal subgroup L of G with $K \leq L \leq H$ we have either $K = L$ or $L = H$. We write $H/K \simeq_G T/L$ provided the normal sections H/K and T/L of G are G-isomorphic; $Ch_G(H/K)$ denotes the set of all chief factors T/L of G with $K \leq L < T \leq H$; A^G is the normal closure of the subgroup A in G and $A_G = \bigcap_{x \in G} A^x$. If Δ is any set of chief factors of G (not necessary non-empty), then we write $\Sigma_G(\Delta)$ to denote the set of all normal sections H/K of G such that either $K = H$ or $K < H$ and the series $K < H$ can be refined to a chief series of G between K and H (of finite length) with $Ch_G(H/K) \subseteq \Delta$.

We call a set Σ of normal sections of G a *stratification* of G if Σ contains every trivial normal section of G and we say that a stratification Σ of G is *G-closed* provided $H/K \in \Sigma$ whenever H/K is a normal section of G with $H/K \simeq_G T/L \in \Sigma$.

Now let Σ be any stratification of G. Then write $\mathfrak{L}_\Sigma(G)$ to denote the set of all subgroups A of G with $A^G/A_G \in \Sigma$.

We will use $\Sigma_G(\mathfrak{F})$ to denote the set of normal sections H/K of G such that $H/K \in \mathfrak{F}$.

Definition. We say (by analogy with the definition of the *Fitting set* of a group [6, p. 537]) that a G-closed stratification Σ of G is a *formation Fitting set* of G if the following conditions hold:

(i) for every two normal sections H/K and T/K of G where $T/K \in \Sigma$ and $H \le T$, we have H/K, $T/H \in \Sigma$;

(ii) $H/(K \cap N) \in \Sigma$ for every two sections H/K, $H/N \in \Sigma$;

(iii) $HV/K \in \Sigma$ for every two sections H/K, $V/K \in \Sigma$.

The usefulness of this concept is primarily based on the following our three results.

Theorem 1. *If $\Sigma = \Sigma_G(\Delta)$ for some G-closed set Δ of chief factors of G or $\Sigma = \Sigma_G(\mathfrak{F})$ for some Fitting formation $\mathfrak{F}$, then Σ is a formation Fitting set of G.*

Theorem 2. *The set $\mathfrak{L}_\Sigma(G)$ forms a sublattice in $\mathfrak{L}(G)$ for each formation Fitting set Σ of G.*

Theorem 3. *The inclusion $\mathfrak{L}_n(G) \subseteq \mathfrak{L}_\Sigma(G)$ holds for every formation Fitting set Σ of G. Moreover, in the case when G satisfies the maximality condition the lattice $\mathfrak{L}_\Sigma(G)$ is distributive if and only if $\mathfrak{L}_\Sigma(G) = \mathfrak{L}_n(G)$ is distributive.*

From theorems 1 and 2 we get the following.

Corollary 1. *Let $\mathfrak{F}$ be either the class of all nilpotent groups, or the class of all soluble groups, or the class of all finite quasinilpotent groups. Then the set $\mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G)$ forms a sublattice in $\mathfrak{L}(G)$.*

We say that a chief factor H/K of G is *$\mathfrak{F}$-central* in G [7] if

$$(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}.$$

Let $D = M \rtimes A$ and $R = N \rtimes B$. Then the pairs (M, A) and (R, B) are said to be *equivalent* provided there are isomorphisms $f: M \to N$ and $g: A \to B$ such that $f(a^{-1}ma) = g(a^{-1})f(m)g(a)$ for all $m \in M$ and $a \in A$.

In fact, the following lemma is known (see, for example, lemma 3.27 in [7]) and it can be proved by the direct verification.

Lemma 1. *Let $D = M \rtimes A$ and $R = N \rtimes B$. If the pairs (M, A) and (R, B) are equivalent, then $D \simeq R$.*

Lemma 2. *Let N, M and $K < H \le G$ be normal subgroups of G, where H/K is a chief factor of G:*

(1) if $N \le K$, then $(H/K) \rtimes (G/C_G(H/K)) \simeq ((H/N)/(K/N)) \rtimes ((G/N)/C_{G/N}((H/N)/(K/N)))$;

(2) if T/L is a chief factor of G and H/K and T/L are G-isomorphic, then $C_G(H/K) = C_G(T/L)$ and $(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L))$;

(3) $(MN/N) \rtimes (G/C_G(MN/N)) \simeq (M/M \cap N) \rtimes (G/C_G(M/M \cap N))$.

P r o o f. (1) In view of the G-isomorphisms $H/K \simeq (H/N)/(K/N)$ and

$$G/C_G(H/K) \simeq (G/N)/(C_G(H/K)/N),$$

the pairs

$$(H/K,\ G/C_G(H/K)),\ ((H/N)/(K/N),\ (G/N)/C_{G/N}((H/N)/(K/N)))$$

are equivalent. Hence statement (1) is a corollary of lemma 1.

(2) A direct check shows that $C = C_{G/N}(H/K) = C_G(T/L)$ and that the pairs $(H/K,\ G/C)$ and $(T/L,\ G/C)$ are equivalent. Hence statement (2) is also a corollary of lemma 1.

(3) This follows from the G-isomorphism $MN/N \simeq M/M \cap N$ and part (2).

The lemma is proved.

In view of lemma 2, we get from theorems 1 and 2 the following fact.

Corollary 2. *Let Δ be the set of all $\mathfrak{F}$-central chief factors of G. Then the set $\mathfrak{L}_{\Sigma(\Delta)}(G)$ forms a sublattice in $\mathfrak{L}(G)$.*

Remark 1. (i) Let $\Sigma(G)$ be the set of all formation Fitting sets of G. It is clear that $\Sigma(G)$ is partially ordered with respect to set inclusion and the formation Fitting set $\{H/K \mid H, K \in \mathfrak{L}_n(G)\}$ is the greatest element in $\Sigma(G)$. Moreover, for every set $\{\Sigma_i \mid i \in I\}$ of formation Fitting sets of G the intersection $\bigcap_{i \in I} \Sigma_i$ is also a formation Fitting set of G and so $\bigcap_{i \in I} \Sigma_i$ is the greatest lower bound for $\{\Sigma_i \mid i \in I\}$ in $\Sigma(G)$. Therefore $\Sigma(G)$ is a complete lattice. The set $\{H/H \mid H \trianglelefteq G\}$ is the smallest element in $\Sigma(G)$.

(ii) Let $\mathfrak{X}$ be any set of normal sections of G. Then the set $\{\Sigma_i \mid i \in I\}$ of all formation Fitting sets of G containing $\mathfrak{X}$ is non-empty and the intersection $\bigcap_{i \in I} \Sigma_i$ is a formation Fitting set of G by part (i). We say that $\bigcap_{i \in I} \Sigma_i$ is the *formation Fitting set of G generated by* $\mathfrak{X}$ and denote it by $\mathrm{formfit}(\mathfrak{X})$. If $\mathfrak{X} = \{T/L\}$ is a singleton set, we write $\mathrm{formfit}(T/L)$ instead of $\mathrm{formfit}(\{T/L\})$ and say that $\mathrm{formfit}(T/L)$ is a *one-generated formation Fitting set of G*.

(iii) Let E and N be subgroups of G, where N is normal in G. Then for any stratification Σ of G we use $\Sigma N/N$ and $\Sigma \cap E$ to denote the stratification $\{(NH/N)/(NK/N) \mid H/K \in \Sigma\}$ of G/N and the stratification $\{(T \cap E)/(L \cap E) \mid T/L \in \Sigma\}$ of E, respectively. If Σ is a formation Fitting set of G, then $\Sigma N/N$ is a formation Fitting set of G/N (see proposition (iv) below).

From theorem 1 we get the following useful result.

Corollary 3. *Let $\mathfrak{X}$ be a set of normal sections of G and $T/L \in \Sigma = \mathrm{formfit}(\mathfrak{X})$. Then the following statements hold:*

(i) $T/L \in \mathfrak{F}$ for every Fitting formation $\mathfrak{F}$ containing $\mathfrak{X}$;

(ii) if $H/K \in Ch(T/L)$, then $H/K \simeq_G F/S$ for some $F/S \in Ch(V/W)$ and $V/W \in \mathfrak{X}$.

For any two sections H/K and T/L of G we write $H/K \leq T/L$ provided $K \leq L$ and $H \leq T$. Then the set of all sections of G is partially ordered with respect to $\leq$.

The proofs of theorems 2 and 3 are based on the following useful observation.

Proposition. *Let Σ be a formation Fitting set of G and let E and N be subgroups of G, where $N \trianglelefteq G$. Then:*

(i) $\langle \Sigma, \leq \rangle$ is a lattice in which HV/KW is the least upper bound and $(H \cap V)/(K \cap W)$ is the greatest lower bound of $\{H/K, V/W\}$ for any two its sections H/K, V/W;

(ii) if $T/L \in \Sigma$, then $\mathfrak{L}(T/L)$ is isomorphic to the interval $[T, L]$ in $\mathfrak{L}_\Sigma(G)$;

(iii) if $f : G \to G^*$ is an isomorphism, then $f(\Sigma) := \{T^f/L^f \mid T/L \in \Sigma\}$ is a formation Fitting set of G^*. Moreover, if Σ is hereditary, then $f(\Sigma)$ is hereditary;

(iv) $\Sigma N/N$ is a formation Fitting set of G/N and $\Sigma N/N = \{(H/N)/(K/N) \mid H/K \in \Sigma$ and $N \leq K\}$.

P r o o f. (i) Since $H/K \in \Sigma$ and $K(V \cap H)/K \leq H/K$, we have $K(V \cap H)/K \in \Sigma$. Hence from the G-isomorphism

$$(H \cap V)/(K \cap V) = (H \cap V)/(K \cap V \cap H) \simeq K(V \cap H)/K$$

we get that $(H \cap V)/(K \cap V) \in \Sigma$. Similarly, $(V \cap H)/(W \cap H) \in \Sigma$. But then we get that

$$(H \cap V)/((K \cap V) \cap (W \cap H)) = (H \cap V)/(K \cap W) \in \Sigma$$

since Σ is a formation Fitting set of G by hypothesis.

From the G-isomorphism

$$H(KW)/KW \simeq H/(H \cap KW) = H/K(H \cap W)$$

we get that $HKW/KW \in \Sigma$ since $(H \cap W)K/K \leq H/K$. Similarly, one can get that $VKW/KW \in \Sigma$. Moreover,

$$HV/KW = (HKW/KW)(VKW/KW)$$

and so $HV/KW \in \Sigma$. Hence statement (i) holds.

(ii) This statement follows from the fact that for every subgroup H of G with $L \leq H \leq T$ we have $L \leq H_G$ and $H^G \leq T$.

(iii) This assertion can be proved by direct checking.

(iv) First note that, in view of part (i), $V/W \in \Sigma$ always implies that $VN/WN \in \Sigma$, so every normal section of G/N in $\Sigma N/N$ is of the form $(V/N)/(W/N)$ for some $V/W \in \Sigma$.

(1) $\Sigma N/N$ *is* (G/N)*-closed.*

Indeed, if

$$(H/N)/(K/N) \simeq_{G/N} (V/N)/(W/N) \in \Sigma N/N,$$

then $H/K \simeq_G (V/W) \in \Sigma$. Hence $H/K \in \Sigma$, so $(H/N)/(K/N) \in \Sigma N/N$.

(2) *For every two normal sections* $(H/N)/(K/N)$ *and* $(T/N)/(K/N)$ *of* G/N, *where* $H/N \leq T/N$ *and* $(T/N)/(K/N) \in \Sigma N/N$ *both sections* $(H/N)/(K/N)$ *and* $(T/N)/(H/N)$ *belong to* $\Sigma N/N$. (This assertion is evident.)

(3) $(H/N)/((K/N) \cap (L/N)) \in \Sigma N/N$ *for every two normal sections* $(H/N)/(K/N)$, $(H/N)/(L/N) \in \Sigma N/N$.

From

$$(H/N)/(K/N),\ (H/N)/(L/N) \in \Sigma N/N$$

we get that H/K, $H/L \in \Sigma$ and so $H/(K \cap L) \in \Sigma$, which implies that

$$(H/N)/((K/N) \cap (L/N)) = (H/N)/((K \cap L)/N) \in \Sigma N/N.$$

(4) $(H/N)(V/N)/(K/N) \in \Sigma N/N$ *for every two normal sections* $(H/N)/(K/N)$, $(V/N)/(K/N) \in \Sigma N/N$.

From $(H/N)/(K/N)$, $(V/N)/(K/N) \in \Sigma N/N$ it follows that $HV/K \in \Sigma$, which implies that $(H/N)(V/N)/$ $/(K/N) \in \Sigma N/N$.

Hence statement (iv) holds.

The proposition is proved.

Before proceeding, consider some examples.

Example 1. (i) If $\mathfrak{X} = \{G/1\}$, then

$$\operatorname{formfit}(G/1) = \{H/K \mid H, K \trianglelefteq G\}$$

and so

$$\mathfrak{L}_{\operatorname{formfit}(G/1)}(G) = \mathfrak{L}(G).$$

(ii) If $\mathfrak{F}$ is the class of all identity groups, then $\mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G) = \mathfrak{L}_n(G)$.

(iii) Let $p > q > 2$ be primes, where q divides $p - 1$. Let Q be a non-abelian group of order q^3. Then Q has a unique minimal normal subgroup, so there exists a simple $\mathbb{F}_pQ$-module P which is faithful for Q. Then $|P| > p$. Let $G = (P \rtimes Q) \times (C_p \rtimes C_q)$, where $C_p \rtimes C_q$ is a non-abelian group of order pq. Let Δ is the set of all those chief factors of G on which G induces an abelian group of automorphisms. Then

$$\mathfrak{L}(P) \not\subseteq \mathfrak{L}_{\Sigma_G(\Delta)}(G) = \mathfrak{L}_n(G) \cup \{AC_q^x \mid A \trianglelefteq G,\ x \in G\}.$$

Therefore for every Fitting formation $\mathfrak{F}$ we have $\mathfrak{L}_{\Sigma_G(\Delta)}(G) \neq \mathfrak{L}_{\Sigma_G(\mathfrak{F})}$ since otherwise $P \in \mathfrak{F}$ and so

$$\mathfrak{L}(P) \subseteq \mathfrak{L}_{\Sigma_G(\mathfrak{F})} = \mathfrak{L}_{\Sigma_G(\Delta)}(G).$$

(iv) Let A be a non-abelian simple group and $\mathfrak{F}$ the class of all groups B such that either $B = 1$ or B is the direct product of isomorphic copies of A. Let $G = A_0 \wr A = K \rtimes A$, where $A_0 \simeq A$ and $K = A_1 \times \cdots \times A_{|A|}$ is the base group of the regular wreath product G. Then K is the unique minimal normal subgroup of G by [6, chapter A, proposition 18.5]. Moreover,

$$\Sigma := \Sigma_G(\mathfrak{F}) = \{G/K,\ K/1,\ G/G,\ K/K,\ 1/1\}$$

is clearly a formation Fitting set of G, so $\mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G)$ is a sublattice of $\mathfrak{L}(G)$. We show that $\mathfrak{L}_{\Sigma_G(\mathfrak{F})} \neq \mathfrak{L}_{\Sigma_G(\Delta)}(G)$ for every G-closed set Δ of chief factors of G. Indeed, assume that $\mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G) = \mathfrak{L}_{\Sigma_G(\Delta)}(G)$. Then for all subgroups $L \leq K$ and $K \leq R \leq G$ we have $L^G/L_G = K/1$ and $R^G/R_G = G/K$, so $L, R \in \mathfrak{L}_{\Sigma(\Delta)}(G)$. Therefore $R/1$, $G/K \in \Delta$ and hence $G/1 \in \Sigma_G(\Delta)$. Thus $\mathfrak{L}_{\Sigma_G(\Delta)}(G) = \mathfrak{L}(G)$ and so $A \in \mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G)$. But then $G/1 = A^G/A_G \in \mathfrak{F}$, which means that G is the direct product of isomorphic copies of A. This contradiction shows that

$$\mathfrak{L}_{\Sigma_G(\mathfrak{F})} \neq \mathfrak{L}_{\Sigma_G(\Delta)}(G)$$

for every G-closed set Δ of chief factors of G.

(v) The class of groups $\mathfrak{F}$ is called a *saturated* if $\mathfrak{F}$ contains every finite group G with $G/\Phi(G) \in \mathfrak{F}$.

Now let A be a maximal subgroup of a finite group G and let $\mathfrak{F}$ be a saturated Fitting formation. Let Δ be the set of all $\mathfrak{F}$-central chief factors of G. Then $G/A_G = A^G/A_G \in \mathfrak{F}$ if and only if $A^G/A_G \in \Sigma_G(\Delta)$ (see lemma 5 below). Therefore $A \in \mathfrak{L}_{\Sigma_G(\mathfrak{F})}(G)$ if and only if $A \in \mathfrak{L}_{\Sigma_G(\Delta)}(G)$.

In conclusion of this section note that some special versions of theorems 2 and 3 were proved in the papers [8–10]. In particular, in the paper [9], the following results were proved.

Corollary 4 (see theorem 1.4(ii) in [9]). *Let G be a finite group and* $\Sigma = \Sigma(\Delta)$*, where* Δ *is the set of all central chief factors of G. Then the lattice* $\mathfrak{L}_\Sigma(G)$ *is distributive if and only if* $\mathfrak{L}_\Sigma(G) = \mathfrak{L}_n(G)$ *is distributive.*

Corollary 5 (see theorem 1.2 in [9]). *Let G be a finite group and either* $\Sigma = \Sigma(\Delta)$*, where* Δ *is the set of all* $\mathfrak{F}$*-central chief factors of G for some class of groups containing all identity groups* $\mathfrak{F}$*, or* $\Sigma = \Sigma_G(\mathfrak{F})$ *for some Fitting formation* $\mathfrak{F}$*, then* $\mathfrak{L}_\Sigma(G)$ *is a sublattice in* $\mathfrak{L}(G)$.

Some further applications

A group is called *primary* if it is a finite p-group for some prime p. If $\sigma = \{\sigma_i \mid i \in I\}$ is any partition of the set of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$, then we say, following [11], that the group G is: σ-*primary* if it is a finite σ_i-group for some i; σ-*soluble* if G is finite and every its chief factor is σ-primary; σ-*nilpotent* or σ-*decomposable* [12] if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$. Observe that a finite group is primary (respectively soluble, nilpotent) if and only if it is σ-primary (respectively σ-soluble, σ-nilpotent), where $\sigma = \{\{2\}, \{3\}, \ldots\}$.

In this section we discuss some applications of the lattice $\mathfrak{L}_\Sigma(G)$ in the theory of finite groups. And we start with one application of the lattices $\mathfrak{L}_{\Sigma_G(\mathfrak{N}_\sigma)}(G)$ and $\mathfrak{L}_{\Sigma_G(\Delta)}(G)$, where $\mathfrak{N}_\sigma$ is the class of all σ-nilpotent groups and Δ is the set of all σ-central, that is, $\mathfrak{N}_\sigma$-central chief factors of G, in the theory of generalized T-groups.

Lattice characterizations of finite σ-soluble *PσT*-groups. We say, following [11], that the subgroup A of G is σ-*subnormal in G* if it is $\mathfrak{N}_\sigma$-*subnormal in G* in the sense of Kegel. Note that a subgroup A of G is subnormal in G if and only if A is σ-subnormal in G, where $\sigma = \{\{2\}, \{3\}, \ldots\}$.

A subgroup A of a finite group G is said to be: *quasinormal* (respectively *S-quasinormal* or *S-permutable* [13]) in G if A permutes with all subgroups (respectively with all Sylow subgroups) H of G, that is, $AH = HA$; σ-*permutable* in G [11] if A permutes with all Hall σ_i-subgroups of G for all i.

Recall that a finite group G is said to be a *T-group* (respectively *PT-group*, *PST-group*) if every subnormal subgroup of G is normal (respectively permutable, S-permutable) in G; G is said to be a *PσT-group* if every σ-subnormal subgroup of G is σ-permutable in G.

The description of *PST*-groups, that are groups, in which every subnormal subgroup is S-permutable, was first obtained by Agrawal [14], for the soluble case, and by Robinson in [15], for the general case. In the further publications, authors (see, for example, the recent papers [16–25]) have found out and described many other interesting characterizations of soluble *PST*-groups. Some characterizations of *PσT*-groups were obtained in the papers [11; 26]. Theorem 2.4 allows to prove the following result in this line research.

Theorem 4. *Suppose that G is a finite* σ*-soluble group. Then G is a PσT-group if and only if* $\mathfrak{L}_{\Sigma_G(\mathfrak{N}_\sigma)}(G) = \mathfrak{L}_{\Sigma_G(\Delta)}(G)$*, where* Δ *is the set of all* σ*-central chief factors of G.*

The proof of theorem 4 consists of many steps and it uses theorems 1 and 2 and also the following lemmas.

Lemma 3. *Let* $\mathfrak{F}$ *be a class of groups, N be a normal subgroup of G and* Σ *be a formation Fitting set of G.*

(1) If $\Sigma = \Sigma_G(\Delta)$*, where* Δ *is the set of all* $\mathfrak{F}$*-central chief factors of G, then* $\Sigma N/N = \Sigma_{G/N}(\Delta^*)$*, where* Δ^* *is the set of all* $\mathfrak{F}$*-central chief factors of G/N.*

(2) $\Sigma_G(\mathfrak{F})N/N = \Sigma_{G/N}(\mathfrak{F})$.

P r o o f. (1) This follows from proposition (iv) and the fact that a chief factor $(H/N)/(K/N)$ is $\mathfrak{F}$-central in G/N if and only if the chief factor H/K is $\mathfrak{F}$-central in G (see lemma 2(1)).

(2) This follows from proposition (iv).

The lemma is proved.

Lemma 4. *Let* Σ *be a formation Fitting set of* G *and let* $A \in \mathfrak{L}_\Sigma(G)$ *and* $N \le H \le G$, *where* $N \trianglelefteq G$:

(1) $AN/N \in \mathfrak{L}_{\Sigma N/N}(G/N)$;

(2) if $H/N \in \mathfrak{L}_{\Sigma N/N}(G/N)$, *then* $H \in \mathfrak{L}_\Sigma(G)$;

(3) $A \cap E \in \mathfrak{L}_{\mathrm{formfit}(\Sigma \cap E)}(E)$ *for every subgroup* E *of* G.

P r o o f. (1) Since $A \in \mathfrak{L}_\Sigma(G)$, $A^G/A_G \in \Sigma$ and so

$$(A^G N/N)/(A_G N/N) \in \Sigma N/N.$$

On the other hand, we have that

$$(AN/N)^{G/N} = (AN)^G/N = A^G N/N,$$

where $A_G N/N \le (AN/N)_{G/N}$. Hence

$$(AN/N)^{G/N}/(AN/N)_{G/N} \in \Sigma N/N$$

since $\Sigma N/N$ is a formation Fitting set of G/N by proposition (iv), so $AN/N \in \mathfrak{L}_{\Sigma N/N}(G/N)$.

(2) Since $H/N \in \mathfrak{L}_{\Sigma N/N}(G/N)$, we have

$$(H^G/N)/(H_G/N) = (H/N)^{G/N}/(H/N)_{G/N} \in \Sigma N/N$$

and so $H^G/H_G \in \Sigma$ by proposition (i). Hence $H \in \mathfrak{L}_\Sigma(G)$.

(3) Let $\Sigma_0 = \mathrm{formfit}(\Sigma \cap E)$. It is clear that

$$(A^G \cap E)/(A_G \cap E) \in \Sigma \cap E \subseteq \Sigma_0.$$

On the other hand, we have

$$A_G \cap E \le (A \cap E)_E \le A \cap E \le (A \cap E)^E \le A^G \cap E$$

and so $(A \cap E)^E/(A \cap E)_E \in \Sigma_0$ since Σ_0 is a formation Fitting set of E. Hence $A \cap E \in \mathfrak{L}_{\Sigma_0}(E)$.

The lemma is proved.

Lemma 5. *Let* $\mathfrak{F}$ *be a saturated formation and* G *be a finite group:*

(1) if $G \in \mathfrak{F}$, *then every chief factor of* G *is* $\mathfrak{F}$*-central in* G;

(2) if G *has a normal subgroup* N *with* $G/N \in \mathfrak{F}$ *such that every chief factor of* G *below* N *is* $\mathfrak{F}$*-central in* G, *then* $G \in \mathfrak{F}$.

P r o o f. (1) This part directly follows from the Barnes – Kegel result [6, chapter IV, proposition 1.5].

(2) In fact, in view of part (1) and the Jordan – Hölder's theorem for the chief series, it is enough to show that if every chief factor of G is $\mathfrak{F}$-central in G, then $G \in \mathfrak{F}$. Assume that this is false and let G be a counterexample of minimal order. Then G has a unique minimal normal subgroup, R say, and $R \nleq \Phi(G)$. Moreover, R is abelian since otherwise we have $G \simeq G/C_G(R) = G/1 \in \mathfrak{F}$. Hence $R = C_G(R)$ by [6, chapter A, theorem 15.6] and for some maximal subgroup M of G we have $G = R \rtimes M$. Therefore the map

$$f : G \to R \rtimes (G/C_G(R)) = R \rtimes (G/R)$$

with $f(rm) = (r, mR)$ for all $r \in R$ and $m \in M$ is isomorphism, so $G \in \mathfrak{F}$ since the factor $R/1$ is $\mathfrak{F}$-central in G by hypothesis.

The lemma is proved.

Recall that the σ-*nilpotent residual* $G^{\mathfrak{N}_\sigma}$ of a finite groups G is the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

Lemma 6 (see theorem A in [26]). *Let* $D = G^{\mathfrak{N}_\sigma}$ *be the* σ*-nilpotent residual of a finite group* G. *If* G *is* σ*-soluble* $P\sigma T$*-group, then the following conditions hold:*

(1) $G = D \rtimes M$, *where* D *is an abelian Hall subgroup of* G *of odd order,* M *is* σ*-nilpotent and every element of* G *induces a power automorphism in* D;

(2) $O_{\sigma_i}(D)$ *has a normal complement in a Hall* σ_i*-subgroup of* G *for all* i.

Conversely, if conditions (1) and (2) hold for some subgroups D *and* M *of* G, *then* G *is a* $P\sigma T$*-group.*

Lemma 7. *Let* N *be a normal subgroup of a finite group* G *such that every chief factor of* G *below* N *is* G*-central in* G. *Then* N *is* σ*-nilpotent, and if* N *is a* σ_i*-group, then* $O^{\sigma_i}(G) \le C_G(N)$.

P r o o f. Let $1 = Z_0 < Z_1 < \ldots < Z_t = N$ be a chief series of G below N and $C_i = C_G\left(Z_i/Z_{i-1}\right)$. First we show that N is σ-nilpotent. By hypothesis, Z_1 and G/G_1 are σ_j-groups for some j. Now let H/K be any chief factor of N such that $H \le Z_1$. From the isomorphism $C_1N/N \simeq N/\left(C_1 \cap N\right)$ it follows that H/K and $N/C_N\left(H/K\right)$ are σ_j-groups. Therefore every chief factor of N below Z_1 is N_σ-central in N. On the other hand, N/Z_1 is σ-nilpotent by induction and so N is σ-nilpotent by lemma 5, condition (2).

Finally, assume that N is a σ_i-group and let $C = C_1 \cap \ldots \cap C_t$. Then G/C is a σ_i-group. On the other hand, $C/C_G\left(N\right) \simeq A \le Aut\left(N\right)$ stabilizes the series $1 = Z_0 < Z_1 < \ldots < Z_t = N$, so $C/C_G\left(N\right)$ is a $\pi\left(N\right)$-group by [6, chapter A, corollary 12.4]. Hence $C/C_G\left(N\right)$ is a σ_i-group, so $O^{\sigma_i}\left(G\right) \le C_G\left(N\right)$. The lemma is proved.

Now consider some applications of theorem 4.

Recall that $Z_\sigma\left(G\right)$ denotes the σ-*hypercentre* of G [11], that is, the largest normal subgroup of G such that every chief factor of G below $Z_\sigma\left(G\right)$ is σ-central in G. We say, following [13, p. 20], that a subgroup H of a finite group G is σ-*hypercentrally embedded in G* if $H/H_G \le Z_\sigma\left(G/H_G\right)$ and *hypercentrally embedded in G* if $H/H_G \le Z_\infty\left(G/H_G\right)$.

Corollary 6 (see theorem 4.1 in [11]). *Let G be a finite* σ*-soluble group. If every* σ*-subnormal subgroup of G is* σ*-hypercentrally embedded in G, then G is a P*σ*T-group.*

In the case where $\sigma = \left\{\{2\}, \{3\}, \ldots\right\}$ we get from theorem 3.1 the following known characterization of finite soluble *PST*-groups.

Corollary 7 (see theorem 1.3 in [10]). *Suppose that G is a finite soluble group. Then G is a PST-group if and only if* $\mathfrak{L}_{\Sigma_G(\mathfrak{N})}\left(G\right) = \mathfrak{L}_{\Sigma(\Delta)}\left(G\right)$*, where* Δ *is the set of all central chief factors H/K of G, that is,* $C_G\left(H/K\right) = G$.

Corollary 8 (see theorem 2.4.4 in [13]). *Let G be a finite group. G is a soluble PST-group if and only if every subnormal subgroup H of G is hypercentrally embedded in G (that is* $H/H_G \le Z_\infty\left(G/H_G\right)$*).*

Groups with Σ-normal and Σ-abnormal subgroups. Let Σ be a formation Fitting set of G. Then we say that a subgroup A of G is: (i) Σ-*normal* in G if $A \in \mathfrak{L}_\Sigma\left(G\right)$; (ii) Σ-*abnormal* in G provided $H \notin \mathfrak{L}_{\mathrm{formfit}(\Sigma \cap E)}\left(E\right)$ for all subgroups $H < E$ of G, where $A \le H$.

Example 2. (i) A subgroup A of G is normal in G if and only if it is Σ-normal in G, where $\Sigma = \left\{H/H \mid H \trianglelefteq G\right\}$.

(ii) A subgroup A of G is called *abnormal* in G if $g \in \left\langle A, A^g\right\rangle$ for all $g \in G$. If G is a soluble finite group, then A is abnormal in G if and only if A is Σ-abnormal in G, where $\Sigma = \Sigma_G\left(\mathfrak{N}\right)$, by [12, chapter IV, theorem 1.7.1].

(iii) Let Δ be the set of all $\mathfrak{F}$-central chief factors of G and $\Sigma = \Sigma_G\left(\Delta\right)$. If G is finite, then a subgroup A of G is called: (a) $\mathfrak{F}$-*normal* in G [8] if $A^G/A_G \in \Sigma$, (b) $\mathfrak{F}$-*abnormal* in G [8] if H is not $\mathfrak{F}$-normal in E for every two subgroups $H < E$ of G such that $A \le H$. Therefore a subgroup A of G is $\mathfrak{F}$-normal ($\mathfrak{F}$-abnormal) in G if and only if it is Σ-normal (respectively Σ-abnormal) in G, where $\Sigma = \Sigma_G\left(\Delta\right)$.

(iv) Let G be finite. If A is σ-hypercentrally embedded in G, that is, $A/A_G \le Z_\sigma\left(G/A_G\right)$, then $A^G/A_G \le Z_\sigma\left(G/A_G\right)$. In particular, if A is hypercentrally embedded in G, then $A^G/A_G \le Z_\infty\left(G/A_G\right)$. Therefore A is σ-hypercentrally (hypercentrally) embedded in G if and only if it is Σ-abnormal in G, where $\Sigma = \Sigma_G\left(\Delta\right)$ and Δ is the set of all σ-central (respectively central) chief factors of G.

Recall that a finite group G is a *DM-group* [8] if $G = D \rtimes M$ and the following conditions hold: (1) $D = G' \ne 1$ is abelian; (2) $M = \langle x\rangle$ is a cyclic abnormal Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$; (3) $M_G = \left\langle x^p\right\rangle = Z\left(G\right)$; (4) x induces a fixed-point-free power automorphism on D.

In the paper [27], Fattahi defined B-groups to be a finite groups in which every subgroup is either normal or abnormal and he showed that a non-nilpotent finite group G is a B-group if and only if G is a DM-group. As a generalization of this result, Ebert and Bauman classified the group in which every subgroup is either subnormal or abnormal [28]. In further, the results in [27] have been developed in many other directions (see, for example, the recent papers [8; 29–33]).

We say that G is a Σ*NA-group* if every subgroup of G is either Σ-normal or Σ-abnormal in G for some formation Fitting set Σ of G.

The results in [8; 27–33] and also many other known results of this type are the motivation for the following question.

Question 1. Let Σ be a formation Fitting set of a finite group G. What we can say about the structure of G in the case when at least one of the following conditions holds: *(i) every subgroup of G is Σ-normal in G; (ii) G is a ΣNA-group, where $\Sigma = \Sigma_G(\Delta)$ for some G-closed set Δ of chief factors of G or $\Sigma = \Sigma_G(\mathfrak{F})$ for some hereditary (in the sense of Mal'cev* [34]*) Fitting formation $\mathfrak{F}$?*

Note that the answer to question 1 for some special Σ is known. Let, for example, $\Sigma = \{H/H \mid H \trianglelefteq G\}$. Then: (i) every subgroup of G is Σ-normal in G if and only if G is a Dedekind group; (ii) G is a ΣNA-group if and only if G is a P-group by example 2(i) and 2(ii) since every P-group is clearly soluble.

Now let Δ be the set of all $\mathfrak{F}$-central chief factors of a finite group G and $\Sigma = \Sigma_G(\Delta)$, where $\mathfrak{F}$ is a hereditary saturated formation containing all nilpotent groups. Then G is a ΣNA-group if and only if every subgroup of G is either $\mathfrak{F}$-normal or $\mathfrak{F}$-abnormal in G by example 2(iii). Such a class of finite groups is also known.

Theorem 5 (see theorem 1.4 in [8]). *Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups. If every subgroup of a finite group G is either $\mathfrak{F}$-normal or $\mathfrak{F}$-abnormal in G, then G is of either of the following types:*

(I) $G \in \mathfrak{F}$;

(II) $G = D \rtimes M$ is a DM-group, where $D = G^{\mathfrak{F}}$, and M is an $\mathfrak{F}$-abnormal subgroup of G with $M_G = Z_{\mathfrak{F}}(G)$.

Conversely, in a group G of type (I) or (II) every subgroup is either $\mathfrak{F}$-normal or $\mathfrak{F}$-abnormal.

In this theorem $Z_{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-*hypercentre of* G, that is the product of all normal subgroups N of G such that either $N = 1$ or $N \neq 1$ and every chief factor of G below N is $\mathfrak{F}$-central in G.

Finite groups G with modular lattices $\mathfrak{L}_{\Sigma}(G)$ and $\mathfrak{L}_{sn}(G)$. A subgroup A of G is called: *subnormal* in G if there exists a subgroup series $A = A_0 \trianglelefteq A_1 \trianglelefteq \cdots \trianglelefteq A_{t-1} \trianglelefteq A_t = G$ (*); *composition* in G if every factor A_i/A_{i-1} of the series (*) is a simple group. Note that a subgroup A of a finite group G is subnormal in G if and only if it is composition in G.

Now let Σ be a formation Fitting set of G. We say a subgroup A of G is Σ-*subnormal* in G if there exists a subgroup series $A = A_0 \trianglelefteq A_1 \trianglelefteq \cdots \trianglelefteq A_{t-1} \trianglelefteq A_t = G$ of G such that A_{i-1} is Σ_i-normal in A_i, where $\Sigma_i = \mathrm{formfit}(\Sigma \cap A_i)$, for all $i = 1, \ldots, t$.

By classical Wielandt's result [35, theorem 1.1.5], the set $\mathfrak{L}_{sn}(G)$ of all composition subgroups of G forms a sublattice of $\mathfrak{L}(G)$.

Question 2. Let G be finite. For which conditions on the formation Fitting set Σ of G the set of all Σ-subnormal subgroups of G forms a sublattice of $\mathfrak{L}(G)$?

In some special cases the answer to question 2 is known. Indeed, $\mathfrak{L}_n(G) = \mathfrak{L}_{\Sigma}(G)$, where $\Sigma = \{H/H \mid H \trianglelefteq G\}$, is modular. In the paper [9] the following result in this direction was obtained.

Theorem 6 (see theorem 1.4 in [9]). *Let G be finite and $\Sigma = \Sigma_G(\Delta)$, where Δ is the set of all central chief factors of G. Then the lattice $\mathfrak{L}_{\Sigma}(G)$ is modular if and only if every two subgroups $A, B \in \mathfrak{L}_{\Sigma}(G)$ are permutable, that is $AB = BA$.*

Zappa, in his paper [36], described conditions under which the lattice $\mathfrak{L}_{sn}(G)$, where G is finite, is modular.

Theorem 7 (see theorem 9.2.3 in [35]). *The following properties of the finite group G are equivalent:*

(a) the lattice $\mathfrak{L}_{sn}(G)$ is modular;

(b) if $T \trianglelefteq S$, where S is subnormal in G and S/T is a p-group, p a prime, then $\mathfrak{L}(S/T)$ is modular;

(c) if $T \trianglelefteq S$, where S is subnormal in G and $|S/T| = p^3$, p a prime, then $\mathfrak{L}(S/T)$ is modular.

A new characterization of finite groups with modular lattice of the subnormal subgroups was given in the paper [9].

Theorem 8 (see theorem 1.3 in [9]). *Let G be a finite group. Then the lattice $\mathfrak{L}_{sn}(G)$ is modular if and only if for every two subnormal subgroups $L \leq T$ of G, where $L \in \mathfrak{L}_{\Sigma}(T)$ and $\Sigma = \Sigma_T(\mathfrak{N}^*)$, L permutes with every subnormal subgroup M of T.*

Finite groups factorized by Σ-normal subgroups. It is well-known that the product $G = AB$ of two normal finite supersoluble groups A and B is not supersoluble in general. Nevertheless, such a product is supersoluble if the indices $|G : A|$ and $|G : B|$ are coprime [37, chapter 4, theorem 3.4]. Moreover, by Doerk's result [38], the finite group G is supersoluble if it has four supersoluble subgroups A_1, A_2, A_3, A_4 whose indices $|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$ are pairwise coprime. In this paper, we prove the following result in this line research.

Theorem 9. *Suppose that G is finite and let* Δ *is the set of all cyclic chief factors of G and* $\Sigma = \Sigma_G(\Delta)$. *Then G is supersoluble if and only if G has three* Σ*-normal supersoluble subgroups* A_1, A_2, A_3 *whose indices* $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ *are pair coprime.*

Lemma 8 (see lemma 4.5 in [6, chapter IV]). *Let G be a finite group in* $\mathfrak{F}$, *where* $\mathfrak{F}$ *is a saturated Fitting formation and let* $p \in \pi(G)$. *If* $X = G/O_{p',p}(G)$ *and R is an irreducible* $\mathbb{F}_p X$*-module, then* $R \rtimes X \in \mathfrak{F}$.

Proof of theorem 9. We need only to show that the sufficiency of the condition of the theorem holds. Assume that this is false and let G be a counterexample of minimal order. Then $G \neq A_i \neq 1$ for all i and G is soluble by Wielandt's theorem [6, chapter I, theorem 3.4]. Moreover, from $(|G : A_i|, |G : A_j|) = 1$ for $i \neq j$ it follows that $G = A_1A_2 = A_1A_3 = A_2A_3$.

Let R be a minimal normal subgroup of G. Then R is a p-group for some prime p. Note also that $\Sigma R/R = \Sigma_{G/R}(\Delta^*)$, where Δ^* is the set of all cyclic chief factors of G/R by lemma 3(1). On the other hand, the subgroup A_iR/R belongs the lattice $\mathfrak{L}_{\Sigma R/R}(G)$ by lemma 4(1), so $A_iR/R \in \mathfrak{L}_{\Sigma_{G/R}(\Delta^*)}(G/R)$. Note also that $A_iR/R \simeq A_i/(A_i \cap R)$ is supersoluble. Therefore the hypothesis hods for G/R. Hence G/R is supersoluble, so R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$. Thus $R = C_G(R) = O_p(G)$ for some prime p by [6, chapter A, theorem 15.6]. Let G_p be a Sylow p-subgroup of G.

From the hypothesis it follows that for some $i \neq j$ and some $x, y \in G$ we have $R \leq G_p^x \leq A_i$ and $R \leq G_p^y \leq A_j$. Since $R = C_G(R)$, $F(A_i) = O_p(A_i)$. On the other hand, A_i is supersoluble and so $A_i/F(A_i) = A_i/O_p(A_i)$ is abelian. Hence $A_i \leq N_G(G_p^x)$. It follows that $A_i^{x^{-1}} \leq N_G(G_p)$. Similarly, $A_j^{y^{-1}} \leq N_G(G_p)$. Then

$$G = A_iA_j = A_i^{x^{-1}}A_j^{y^{-1}} \leq N_G(G_p)$$

and so

$$R = O_p(G) = G_p = O_p(A_i) = O_p(A_j).$$

Now we show that $R \leq A_k$, where $j \neq k \neq i$. Assume that $R \nleq A_k$. Then $(A_k)_G = 1$ and $A_k^G \neq 1$ since $A_k \neq 1$. Hence $R \leq A_k^G$, which implies that $R/1$ is cyclic and so G is supersoluble. This contradiction shows that $R \leq A_3$, so $R = G_p = O_p(A_k) = F(A_k)$.

Therefore A_1R/R, A_2R/R, A_3R/R are abelian subgroup of G/R whose indices

$$|G/R : A_1R/R|, |G/R : A_2R/R|, |G/R : A_3R/R|$$

are pair coprime, so G/R is nilpotent by Kegel's theorem [39]. Moreover, for every Sylow subgroup Q/R of G/R we have that $Q/R \leq A_i/R$ or $Q/R \leq A_j/R$. Hence for some subgroups $A/R \trianglelefteq A_i/R$ and $B/R \trianglelefteq A_j/R$ we have $G/R = (A/R) \times (B/R)$. It is clear that the subgroups A and B are supersoluble and so the group $A \times B$ is supersoluble. It is clear also that $O_{p',p}(A) = R = O_{p',p}(B)$. Hence

$$X = (A \times B)/O_{p',p}(A \times B) \simeq (A/R) \times (B/R) \simeq G/R.$$

But then G is supersoluble by lemma 8. This contradiction completes the proof of the result.

A subgroup M of G is called *modular* in G if M is a modular element (in the sense of Kurosh [35, p. 43]) of the lattice $\mathfrak{L}(G)$. It is known that [35, theorem 5.2.3] for every modular subgroup A of G all chief factors of G between A_G and A^G are cyclic. Therefore we get from theorem 9 the following result.

Corollary 9. *If G is finite and G has three modular supersoluble subgroups* A_1, A_2, A_3 *whose indices* $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ *are pair coprime, then G is supersoluble.*

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