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МАТЕМАТИКА

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On one inverse problem of reconstructing a subdiffusion process with degeneration from nonlocal data

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Dedicated to the memory of our Dear Teacher

Nakhushev Adam Maremovich

In recent years, the phenomena of anomalous diffusion has been observed in many fields, such as turbulence, seepage in porous media, pollution control. The demand for appropriate mathematical models is high from biomechanics to geophysics passing by acoustics. A most used approach to depicting a variety of complex anomalous diffusion phenomena is a nonlinear modeling that is generally mathematically challenging to analyze and computationally very expensive to simulate. In addition, the nonlinear models often require some parameters unavailable from experiments or field measurements. As alternative approaches, in recent decades, fractal and fractional derivatives have been found effective in modeling anomalous diffusion processes. The advantage of the fractal or the fractional derivative models over the standard integer-order derivative models is in that it can describe accurately the inherent abnormal-exponential or heavy tail decay processes.

Fractional powers in indicators also arise when describing fractal (multiscale, whole-like) media. In a fractal environment, unlike a continuous medium, a randomly wandering particle moves away from the launch site more slowly, since not all directions of motion become available for it. The slowing down of diffusion in the fractal media is so significant that the physical quantities begin to change more slowly than the first derivative and this effect can be taken into account only in an integral-differential equation containing the time derivative of fractional order:

$$D_t^{\alpha} u(x,t) = A_x u(x,t) + F.$$

In this paper, we consider an inverse problem close to that investigated in [1], [2]. Together with the solution, it is necessary to find the unknown source term of the equation. The equation contains a fractional derivative with respect to time and an involution with respect to the spatial variable. In contrast to [1], [2], we investigate the problem under nonlocal boundary conditions with respect to the spatial variable. The conditions for determination are initial and final states.

The second main difference in our problem is that the unknown function enters both in the right-hand side of the equation and in the conditions of the initial and final overdeterminations.

Let us consider a problem of modeling the thermal diffusion process which is close to that described in the paper of Cabada and Tojo [2], where an example that describes a concrete situation in physics is given. Consider a closed metal wire (length 2π) wrapped around a thin sheet of insulation material in the manner shown in Figure 1.

Assuming that the position $x = 0$ is the lowest of the wire, and the insulation goes up to the left at $-\pi$ and to the right up to π. Since the wire is closed, points $-\pi$ and π coincide.

The layer of insulation is assumed to be slightly permeable. Therefore, the temperature value from one side affects the diffusion process on the other side. For this reason, the standard heat equation is modified by adding an extra term $\varepsilon \frac{\partial^2 \Phi}{\partial x^2}(-x,t)$ to $\frac{\partial^2 \Phi}{\partial x^2}(x,t)$ (where $|\varepsilon| < 1$). Here $\Phi(x,t)$ is the temperature at point x of the wire at time t.

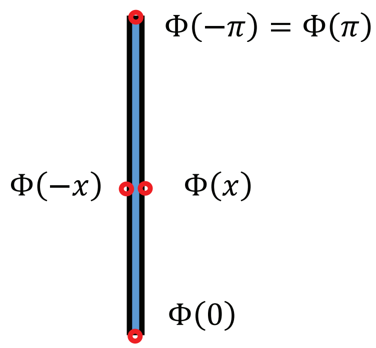

Рис. 1. The closed metal wire wrapped around a thin sheet of insulation material

We consider a process which is so slow that it is described by an evolution equation with a time fractional derivative:

$$t^{-\beta}D_t^{\alpha}\Phi\left(x,t\right)-\Phi_{xx}\left(x,t\right)+\varepsilon\Phi_{xx}\left(-x,t\right)=f\left(x\right),\quad \alpha+\beta>0, \tag{1}$$

in the domain $\Omega=\{(x,t): \ -\pi<x<\pi,\ 0<t<T\}$. Here $f(x)$ stands for an external source that does not change with time; $t=0$ is an initial time point and $t=T$ is a final one. The derivative D_t^{α} defined, for a differentiable function, as

$$\left(D_t^{\alpha}\varphi\right)(t)=I^{1-\alpha}\left[\frac{d}{dt}\varphi\left(t\right)\right],\ 0<\alpha<1,\quad t\in[0,T],$$

is the Caputo derivative built on the Riemann-Liouville fractional integral

$$I^{1-\alpha}[\varphi(t)]=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\varphi(s)}{(t-s)^{\alpha}}ds,\ 0<\alpha<1,\quad t\in[0,T].$$

Caputo derivative allows to impose initial conditions in a natural way.

As additional information, we take

$$\Phi\left(x,0\right)=\phi\left(x\right),\quad \Phi(x,T)=\psi(x),\quad x\in[-\pi,\pi]. \tag{2}$$

Since the wire is closed, it is natural to assume that the temperatures at the tips of the wire are equal at all times:

$$\Phi\left(-\pi,t\right)=\Phi\left(\pi,t\right),\ t\in\left[0,T\right]. \tag{3}$$

If we consider a process in which the temperature at one end at every time point t is proportional to the (fractional) rate of change speed of the average value of the temperature throughout the wire, then,

$$\Phi\left(-\pi,t\right)=\gamma t^{-\beta}D_t^{\alpha}\int_{-\pi}^{\pi}\Phi\left(\xi,t\right)d\xi,\ t\in\left[0,T\right]; \tag{4}$$

here γ is the proportionality coefficient.

Thus the investigated process is reduced to the following inverse problem: *Find the source term $f(x)$ of the subdiffusion equation* (1), *and its solution $\Phi(x,t)$ subject to the initial and final conditions* (2), *the boundary condition* (3), *and condition* (4).

Let us mention the case when $\alpha=1$, $\beta=0$ was examined in [3], [4]. Note that such problems are considered in our paper [5].

1. Reduction of the problem Condition (4) is nonlocal. The integral along inner lines of the domain is present in this condition. Using the idea of Samarskii, we transform this condition. Taking into account equation (1) from (4), we get

$$\Phi\left(-\pi,t\right)=\gamma\int_{-\pi}^{\pi}\left\{\Phi_{\xi\xi}\left(\xi,t\right)-\varepsilon\Phi_{\xi\xi}\left(-\xi,t\right)+f\left(\xi\right)\right\}d\xi,\ t\in\left[0,T\right].$$

Hence

$$\Phi\left(-\pi,t\right)=\gamma(1-\varepsilon)[\Phi_x\left(\pi,t\right)-\Phi_x\left(-\pi,t\right)]+\gamma\int_{-\pi}^{\pi}f\left(\xi\right)d\xi,\ t\in\left[0,T\right].$$

As was shown in our work [5], for the existence of a solution, it is necessary to satisfy the equality

$$\int_{-\pi}^{\pi}f\left(\xi\right)d\xi=0. \tag{5}$$

In what follows, we will assume this equality to be fulfilled.

Let us set

$$u(x,t)=\Phi(x,t).$$

Then in terms of the new function $u(x,t)$, we get the following inverse problem: *In the domain $\Omega=\{(x,t):\ -\pi<x<\pi,\ 0<t<T\}$ find a right-hand side $f(x)$ of the time fractional evolution equation with involution*

$$t^{-\beta}D_t^{\alpha}u\left(x,t\right)-u_{xx}\left(x,t\right)+\varepsilon u_{xx}\left(-x,t\right)=f\left(x\right), \tag{6}$$

and its solution $u(x,t)$ that satisfies one initial condition

$$u\left(x,0\right)=\phi\left(x\right),\quad x\in[-\pi,\pi],\quad \alpha+\beta>0, \tag{7}$$

one final condition

$$u(x,T)=\psi(x),\quad x\in[-\pi,\pi], \tag{8}$$

and the boundary condition

$$\begin{cases} u_x\left(-\pi,t\right)-u_x\left(\pi,t\right)-au\left(\pi,t\right)=0, \\ u\left(-\pi,t\right)-u\left(\pi,t\right)=0, \end{cases}\quad t\in\left[0,T\right], \tag{9}$$

where $\phi(x)$ and $\psi\ (x)$ are given sufficiently smooth functions, $0<\alpha<1$, ε is a nonzero real number such that $|\varepsilon|<1$ and $a=\frac{1}{\gamma(\varepsilon-1)}$. Moreover, we assume that $f(x)$ satisfies condition (5).

In the physical sense, the second of conditions (9) means the equality of the distribution density at the ends of the interval. And the first of conditions (9) means the proportionality of the difference of fluxes across opposite boundaries to the density value at the boundary. We note that in [1] the Dirichlet boundary conditions $u\left(-\pi,t\right)=u\left(\pi,t\right)=0$ were used instead of condition (9).

Let us mention that the well-posedness of direct and inverse problems for parabolic equations with involution is considered in [6]-[8], and the solvability of various inverse problems for parabolic equations was studied in papers of Anikonov and Belov, Bubnov Prilepko and Kostin, Monakhov, Kozhanov, Kaliev, Sabitov and many others, see [9] and [10]. In [1], good references on related issues are cited. We note [5]-[31] from recent papers close to the theme of our article. In these papers different variants of direct and inverse initial-boundary value problems for evolution equations are considered, including problems with nonlocal boundary conditions and problems for equations with fractional derivatives.

Problem (6)-(9) for $a = 0$ was considered in [30], and for $a = \beta = 0$ in [31].

Let us finally mention that we will use the Fourier method to solve our problem. Here, we use a spectral problem for ordinary differential operators with involution. Similar spectral problems are considered in [32]-[43].

Definition. *A regular solution* of the inverse problem (6)-(9), is a pair of functions $(u(x,t), f(x))$, $u(x,t) \in C^{2,1}_{x,t}(\overline{\Omega})$, $f(x) \in C[-\pi, \pi]$ satisfying Eq. (6) and conditions (7)-(9).

Definition. *A generalised solution* of the inverse problem (6)-(9), is a pair of functions $(u(x,t), f(x))$, $u(x,t) \in W^{2,1}_2(\Omega) \cap C\left(\overline{\Omega}\right)$, $f(x) \in L_2(-\pi, \pi)$ that satisfy Eq. (6) and conditions (7)-(9) almost everywhere.

2. Spectral problem A similar spectral problem was considered in [4].

The use of the Fourier method for solving problem (6)–(9) leads to a spectral problem for the operator $\mathcal{L}$ given by the differential expression

$$\mathcal{L}X(x) \equiv -X''(x) + \varepsilon X''(-x) = \lambda X(x), \; -\pi < x < \pi, \tag{10}$$

and the boundary conditions

$$\begin{cases} X'(-\pi) - X'(\pi) - aX(\pi) = 0, \\ X(-\pi) - X(\pi) = 0, \end{cases} \tag{11}$$

where λ is the spectral parameter.

Spectral problems for Eq. (10) were first considered, apparently, in [34]. There was considered cases of Dirichlet and Neumann boundary conditions, and cases of conditions in the form (11) for $a = 0$. Here we consider the case $a > 0$.

We search a solution of Eq. (10) in the form:

$$X(x) = A\sin(\mu_1 x) + B\cos(\mu_2 x), \quad \mu_1 = \sqrt{\frac{\lambda}{1+\varepsilon}}, \quad \mu_2 = \sqrt{\frac{\lambda}{1-\varepsilon}},$$

where A and B are arbitrary complex numbers. The boundary conditions (11) lead to equations

$$\sin(\mu_1 \pi) = 0, \quad \tan(\mu_2 \pi) = \frac{a}{2\mu_2}.$$

Therefore, the spectral problem (10)-(11) has two series of eigenvalues:

$$\lambda_{k,1} = (1+\varepsilon) k^2, \; k \in \mathbb{N};$$

$$\lambda_{k,2} = (1-\varepsilon)(k+\delta_k)^2, \; \delta_k = \frac{a}{k+1} O(1) > 0, \; k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\},$$

with corresponding normalized eigenfunctions given by

$$X_{k,1}(x) = \frac{1}{\sqrt{\pi}} \sin(kx), \quad k \in \mathbb{N}; \quad X_{k,2}(x) = \nu_k \cos((k+\delta_k)x), \; k \in \mathbb{N}_0; \tag{12}$$

here ν_k is the normalization coefficient:

$$\nu_k^{-2} = \| \cos((k+\delta_k)x) \|^2 = \pi + \frac{a^2}{(k+\delta_k)\left[a^2 + (k+\delta_k)^2 \pi^2\right]}.$$

It is easy to see that system (12) is simultaneously a system of eigenfunctions for the Sturm-Liouville operator

$$\mathcal{L}_1 X(x) \equiv -X''(x) = \lambda X(x), \ -\pi < x < \pi,$$

with the self-adjoint boundary conditions (11) corresponding to the eigenvalues

$$\hat{\lambda}_{k,1} = k^2, \ k \in \mathbb{N}; \ \hat{\lambda}_{k,2} = (k+\delta_k)^2, \ k \in \mathbb{N}_0.$$

Consequently, system (12) forms an orthonormal basis of $L_2(-\pi,\pi)$.

3. Uniqueness of the solution Let the pair of functions $(u(x,t), f(x))$ be a solution of the inverse problem (6)-(9). Let us set

$$u_{k,i}(t) = \int_{-\pi}^{\pi} u(x,t) X_{k,i}(x)\,dx, \quad f_{k,i} = \int_{-\pi}^{\pi} f(x) X_{k,i}(x)\,dx, \ i = 1,2. \tag{13}$$

We apply the operator $t^{-\beta} D^\alpha$ to $u_{k,i}(t)$. Then, using Eq. (6) and integrating by parts, we obtain the problem

$$t^{-\beta} D^\alpha u_{k,i}(t) + \lambda_{k,i} u_{k,i}(t) = f_{k,i}, \quad 0 < t < T, \quad i = 1,2; \tag{14}$$

$$u_{k,i}(0) = \phi_{k,i}, \quad i = 1,2; \tag{15}$$

$$u_{k,i}(T) = \psi_{k,i}, \quad i = 1,2, \tag{16}$$

where

$$\phi_{k,i} = \int_{-\pi}^{\pi} \phi(x) X_{k,i}(x)\,dx, \ \ \psi_{k,i} = \int_{-\pi}^{\pi} \psi(x) X_{k,i}(x)\,dx,$$

It is easy to see that the function $\tilde{u}_{k,1}(t) = (\lambda_{k,i})^{-1} f_{k,i}$ is a partial solution of the inhomogeneous equation (14). Using the general solution of the homogeneous equation (14), which is constructed in ([44], p. 233) for $\alpha + \beta > 0$, we get

$$u_{k,i}(t) = \frac{f_{k,i}}{\lambda_{k,i}} + C_{k,i} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\lambda_{k,i}\, t^{\alpha+\beta}\right), \quad 0 < t < T, \quad i = 1,2,$$

where $E_{\alpha+\beta,1,1-\alpha}$ is the generalized Mittag-Leffler function ([43], p. 48):

$$E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(z) = \sum_{k=0}^{\infty} c_k z^k; \ \ c_0 = 1, \ c_k = \prod_{j=0}^{k-1} \frac{\Gamma(j(\alpha+\beta)+\beta+1)}{\Gamma(j(\alpha+\beta)+\alpha+\beta+1)}, \ k \in \mathbb{N}$$

and the constants $C_{k,i}$ and $f_{k,i}$ are unknown.

To find these constants, we use conditions (15) and (16). From (15), we obtain a unique solution of the Cauchy problem (14)-(15) in the form

$$u_{k,i}(t) = \left[1 - E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\lambda_{k,i}\, t^{\alpha+\beta}\right)\right] \frac{f_{k,i}}{\lambda_{k,i}} + (\phi_{k,i})\, E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\lambda_{k,i}\, t^{\alpha+\beta}\right). \tag{17}$$

Since $\lambda_{k,i} > 0$, then by virtue of the well-known asymptotics [44]:

$$\left| E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(z) \right| \le \frac{M}{1+|z|}, \ \arg(z) = \pi, \ |z| \to \infty, \ M = Const > 0, \tag{18}$$

$T \gg 1$, the estimate

$$1-E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\,T^{\alpha+\beta}\right) \geq m^* > 0, \tag{19}$$

holds; the constant m^* does not depend on values of the indices k, i.

Therefore, using condition (16), we get

$$f_{k,i} = \lambda_{k,i}\frac{\psi_{k,i} - \phi_{k,i}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\,T^{\alpha+\beta}\right)}{1 - E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\,T^{\alpha+\beta}\right)}. \tag{20}$$

Lemma. *If* (19) *holds for all values of the indices* k, i, *then the solution* $(u(x,t), f(x))$ *of the inverse problem* (6)-(9) *is unique.*

Proof. Suppose that there are two solutions $(u_1(x,t), f_1(x))$ and $(u_2(x,t), f_2(x))$ of the inverse problem (6)-(9). Set

$$u\left(x,t\right) = u_1\left(x,t\right) - u_2\left(x,t\right), \quad f\left(x\right) = f_1\left(x\right) - f_2\left(x\right).$$

Then the functions $u(x,t)$ and $f(x)$ satisfy Eq. (6), the boundary conditions (9) and the homogeneous conditions (7) and (8):

$$u\left(x,0\right) = u(x,T) = 0, \quad x \in [-\pi, \pi].$$

Therefore, by using (13) from (20), we find $f_{k,i} = 0$.

Whereupon, from (17) and (20), we find

$$u_{k,i}\left(t\right) \equiv \int_{-\pi}^{\pi} u(x,t)X_{k,i}\left(x\right)dx = 0, \;\; f_{k,i} \equiv \int_{-\pi}^{\pi} f(x)X_{k,i}\left(x\right)dx = 0$$

for all values of the indices $k \in \mathbb{N}$ for $i = 1$ and $k \in \mathbb{N}_0$ for $i = 2$. Furthermore, by the completeness of system (12) in $L_2\left(-\pi,\pi\right)$, we obtain

$$u\left(x,t\right) \equiv 0, \;\; f\left(x\right) \equiv 0 \;\; \text{for all} \;\; (x,t) \in \overline{\Omega}.$$

The uniqueness of the solution of the inverse problem (6)-(9) is verified.

4. Construction of a formal solution As the eigenfunctions system (12) forms an orthonormal basis in $L_2\left(-\pi,\pi\right)$, the unknown functions $u(x,t)$ and $f\,(x)$ can be formally represented as

$$u\left(x,t\right) = \sum_{k=1}^{\infty} u_{k,1}\left(t\right)X_{k,1}\left(x\right) + \sum_{k=0}^{\infty} u_{k,2}\left(t\right)X_{k,2}\left(x\right), \tag{21}$$

$$f\left(x\right) = \sum_{k=1}^{\infty} f_{k,1}X_{k,1}\left(x\right) + \sum_{k=0}^{\infty} f_{k,2}X_{k,2}\left(x\right), \tag{22}$$

where $u_{k,1}\left(t\right)$ and $u_{k,2}\left(t\right)$ are unknown functions; $f_{k,1}$ and $f_{k,2}$ are unknown constants.

Substituting (21) and (22) into equation (6), we obtain the inverse problems (14)-(16). If the constants $\sigma_{k,i}$ are assumed to be given, then the solutions of these inverse problems exist, are unique and are represented by formulas (17) and (20). Substituting (17) and (20) into series (21) and (22), we obtain a formal solution of the inverse problem (6)-(9).

Indeed, from (5) and Eq. (1) we have

$$0 = \int_{-\pi}^{\pi} f\left(\xi\right)d\xi = \int_{-\pi}^{\pi} t^{-\beta}D_t^{\alpha}\Phi\left(\xi,t\right)d\xi - \int_{-\pi}^{\pi}\Big\{\Phi_{\xi\xi}\left(\xi,t\right) + \varepsilon\Phi_{\xi\xi}\left(-\xi,t\right)\Big\}d\xi.$$

For the first integral, we apply condition (4), and calculate the second integral. Then we obtain

$$0 = (1-\varepsilon)\left[\Phi_x(-\pi,t) - \Phi_x(\pi,t) + \frac{1}{\gamma(1-\varepsilon)}\Phi(-\pi,t)\right].$$

This means that the boundary conditions (4) and (9) coincide. Hence, problems (6)-(9) and (1)-(4) also coincide.

Thus, in what follows we shall consider problem (1)-(3) with the boundary condition

$$\Phi_x(-\pi,t) - \Phi_x(\pi,t) - a\,\Phi(-\pi,t) = 0. \tag{23}$$

Thus, in what follows we will consider the inverse problem (1)-(3), (23).

Similarly, as before, the formal solution of this problem can be constructed in the form of series

$$\Phi(x,t) = \sum_{k=1}^{\infty}\Phi_{k,1}(t)\,X_{k,1}(x) + \sum_{k=0}^{\infty}\Phi_{k,2}(t)\,X_{k,2}(x), \tag{24}$$

$$f(x) = \sum_{k=1}^{\infty} f_{k,1}X_{k,1}(x) + \sum_{k=0}^{\infty} f_{k,2}X_{k,2}(x), \tag{25}$$

where

$$\Phi_{k,i}(t) = \left(\phi_{k,i} - \frac{f_{k,i}}{\lambda_{k,i}}\right)E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\,t^{\alpha+\beta}\right) + \frac{f_{k,i}}{\lambda_{k,i}}, \tag{26}$$

$$f_{k,i} = \lambda_{k,i}\frac{\psi_{k,i} - \phi_{k,i}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\,T^{\alpha+\beta}\right)}{1 - E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\,T^{\alpha+\beta}\right)}. \tag{27}$$

In order to complete our study, it is necessary, to justify the smoothness of the resulting formal solutions and the convergence of all appearing series.

5. Main results Here we present the existence and uniqueness results for our inverse problem.

Theorem. *Let $a>0$, $\alpha+\beta>0$ and T be large enough that condition* (19) *holds for all values of the indices k,i.*
(A) Let $\phi(x)$, $\psi(x) \in W_2^2(-\pi,\pi)$ and satisfy the boundary conditions (11). *Then, for a real number ε such that $|\varepsilon|<1$, the inverse problem* (1)-(3), (23) *has a unique generalized solution, which is stable in norm:*

$$\left\|t^{-\beta}D_t^{\alpha}\Phi\right\|^2_{L_2(\Omega)} + \|\Phi_{xx}\|^2_{L_2(\Omega)} + \|f\|^2_{L_2(-\pi,\pi)} \le C\left\{\|\phi\|^2_{W_2^2(-\pi,\pi)} + \|\psi\|^2_{W_2^2(-\pi,\pi)}\right\},$$

where the constant C does not depend on $\phi(x)$, $\psi(x)$.
(B) Let $\phi(x), \psi(x) \in C^4[-\pi,\pi]$ and the functions $\phi(x), \psi(x)$ and $\phi''(x)$, $\psi''(x)$ satisfy the boundary conditions (11), *then, for a real number ε such that $|\varepsilon|<1$, the inverse problem* (1)-(3), (23) *has a unique regular solution.*

Proof. The generalized Mittag-Leffler function's estimates (18) and (19) are known. Therefore, from representations (17) и (20), we get estimates

$$|f_{k,i}| \le C_1\,|\lambda_{k,i}|\left\{|\phi_{k,i}| + |\psi_{k,i}|\right\}, \tag{28}$$

$$|\Phi_{k,i}(t)| \le C_1\left\{|\phi_{k,i}| + |\psi_{k,i}|\right\}, \tag{29}$$

where the constant C_1 does not depend on the indices k, i and on the functions $\phi(x), \psi(x)$. Since the eigenfunctions system (12) forms an orthonormal basis in $L_2(-\pi,\pi)$, then by virtue of the Parseval equality, it is easy to obtain estimates

$$\|f\|^2_{L_2(-\pi,\pi)} \le C\left\{\|\phi''\|^2_{L_2(-\pi,\pi)} + \|\psi''\|^2_{L_2(-\pi,\pi)}\right\}, \tag{30}$$

$$\|\Phi_{xx}\|^2_{L_2(\Omega)} \le C\left\{\|\phi''\|^2_{L_2(-\pi,\pi)} + \|\psi''\|^2_{L_2(-\pi,\pi)}\right\}. \tag{31}$$

In deriving these inequalities, we use the fact that the functions $\phi(x), \psi(x)$ satisfy the boundary conditions (11). Now we can easily obtain an estimate for $t^{-\beta} D_t^{\alpha}\Phi(x,t)$ from Eq. (6). This together with (30) and (31) gives the necessary estimate for the solution.

From the obtained estimates it also follows that in the constructed formal solution of the inverse problem all the series converge, they can be term-by-term differentiated, and the series obtained during differentiation also converge in the metrics of L_2.

From (21) and (29), by using Holder's inequality, it is easy to justify the inequality

$$\max_{(x,t)\in\overline{\Omega}} |\Phi(x,t)|^2 \le C\left\{\|\phi''\|^2_{L_2(-\pi,\pi)} + \|\psi''\|^2_{L_2(-\pi,\pi)}\right\},$$

which justifies the continuity of $\Phi(x,t)$ in the closed domain $\overline{\Omega}$.

From the representation of the solution in the form of series (21), (22) and inequalities (28), (29), it is easy to justify estimate

$$|\Phi_{xx}(x,t)| + |\Phi_{tt}(x,t)| + |f(x)| \le C\sum_{k=1}^{\infty} |\lambda_{k,i}|^2 \left\{ |\phi_{k,i}| + |\psi_{k,i}| \right\}. \tag{32}$$

Let $\phi(x), \psi(x) \in C^4[-\pi,\pi]$ and the functions $\phi(x), \psi(x)$ and $\phi''(x)$, $\psi''(x)$ satisfy the boundary conditions (11), then the series in the right-hand side of (32) converges. Therefore, in such case, the formal solution gives the regular solution of the inverse problem (6)-(9). The Theorem is completely proved.

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ABSTRACT

In this article we consider an inverse problem for one-dimensional degenerate fractional heat equation with involution and with periodic boundary conditions with respect to a spatial variable. This problem simulates the process of heat propagation in a thin closed wire wrapped around a weakly permeable insulation. The inverse problem consists in the restoration (simultaneously with the solution) of an unknown right-hand side of the equation, which depends only on the spatial variable. The conditions for redefinition are initial and final states. Existence and uniqueness results for the given problem are obtained via the method of separation of variables.

Keywords: inverse problem, heat equation, equation with involution, subdiffusion process, equation with degeneration, periodic boundary conditions, method of separation of variables.

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АННОТАЦИЯ

В этой статье рассматривается одна обратная задача для одномерного вырождающегося уравнения дробной теплопроводности с инволюцией и с периодическими граничными условиями относительно пространственной переменной. Эта проблема имитирует процесс распространения тепла в тонкой замкнутой проволоке, обернутой вокруг слабо проницаемой изоляции. Обратная задача состоит в восстановлении (одновременно с решением) уравнения неизвестная правая часть уравнения, зависящая только от пространственная переменная. Условиями переопределения являются начальное и конечное состояния. Результаты существования и единственности для данной задачи получены методом разделения переменных.

Ключевые слова: обратная задача, уравнение теплопроводности, уравнение с инволюцией, субдиффузионный процесс, уравнение с вырождением, периодические граничные условия, метод разделения переменных.

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