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F-supplemented modules

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ABSTRACT. Let R be a ring, let M be a left R-module, and let U, V, F be submodules of M with F proper. We call V an F-supplement of U in M if V is minimal in the set $F \subseteq X \subseteq M$ such that $U + X = M$, or equivalently, $F \subseteq V$, $U + V = M$ and $U \cap V$ is F-small in V. If every submodule of M has an F-supplement, then we call M an F-supplemented module. In this paper, we introduce and investigate F-supplement submodules and (amply) F-supplemented modules. We give some properties of these modules, and characterize finitely generated (amply) F-supplemented modules in terms of their certain submodules.

Introduction

All rings considered in this paper will be associative with an identity element. Unless otherwise stated, R denotes an arbitrary ring and all modules will be *left* unitary R-modules. Let M be a module. By $X \subseteq M$, we mean X is a submodule of M, and $X \subsetneqq M$ means X is a proper submodule of M. As usual, $\mathrm{Rad}(M)$ denotes the radical of M. Throughout the paper, unless otherwise stated, F will be a proper submodule, and we follow the terminology and notation as in [2].

A submodule $K \subseteq M$ is called *small* in M, denoted by $K \ll M$, if, for every submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$. The notion of a supplement submodule was introduced in [3] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective covers. For submodules U and V of a module M,

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V is said to be a *supplement* of U in M or U is said to *have a supplement* V in M if $U+V=M$ and $U\cap V\ll V$, and M is called *supplemented* if every submodule of M has a supplement in M. See [4, §41] and [2] for results and definitions related to supplements, supplemented modules and small submodules.

Recently, several authors have studied different generalizations of small submodules (see, for example, [6], [5], [1]). In [1], F-small submodules are defined and studied as a generalization of small submodules. Let F be a proper submodule of a module M. A submodule $K\subseteq M$ is called *F-small in M*, denoted $K\ll_F M$, if, for every submodule $L\subseteq M$ containing F, the equality $K+L=M$ implies $L=M$. Motivated by the relation between supplement submodules and small submodules, we introduce the notion of an F-supplement submodule. We call a submodule $V\subseteq M$ an *F-supplement* of $U\subseteq M$ in M if V is minimal in the set $\{L\subseteq M\mid U+L=M \text{ and } F\subseteq L\}$, or equivalently, $F\subseteq V$, $U+V=M$ and $U\cap V\ll_F V$ (Proposition 1). We say $U\subseteq M$ has *ample F-supplements* if, for every $V\subseteq M$ with $U+V=M$, there is an F-supplement V' of U with $V'\subseteq V$. If every submodule of M has an (ample) F-supplement, then M is called an (*amply*) *F-supplemented* module. Like small submodules, of course, for $F=0$, supplement submodules and F-supplement submodules coincide in a module. Also, any supplement submodule containing F is always F-supplement, and the converse is true when $F\ll M$ (Remark 1). So, for instance, for a finitely generated module M, if we take $F=\mathrm{Rad}(M)$, then any submodule of M containing F is F-supplement if and only if it is supplement in M.

In Section 1, we investigate F-small submodules and F-supplement submodules, and we give some properties of F-supplement submodules which are adapted from supplement submodules. For instance, for submodules $K\subseteq N\subseteq M$, we show that if N is an F-supplement in M, then K is an F-supplement in M if and only if K is an F-supplement in N (Theorem 1). Also, we prove that if N is an F-supplement in M, then N/K is an $(F+K)/K$-supplement in M/K, and the converse is true if, in addition, K is an F-supplement in M (Propositions 4 and 5). Moreover, we show that if V is an F-supplement in M, then $\mathrm{Rad}_F(V)=V\cap\mathrm{Rad}_F(M)$, where $\mathrm{Rad}_F(M)$ is the intersection of all maximal submodules of M containing F (Proposition 2).

In Section 2, we introduce and study F-supplemented modules. We show that every finite (direct) sum of F-supplemented modules is F-supplemented (Corollary 4), and that a module M is F-supplemented if and only if M/F is supplemented (i.e. 0-supplemented) (Theorem 2). Also, we prove that if M is F-supplemented, then $M/\operatorname{Rad}_F(M)$ is semisimple (Proposition 8). Finally, we characterize finitely generated F-supplemented modules (Corollary 7). Namely, we prove the equivalence of the following statements: (1) M is F-supplemented, (2) every maximal submodule of M containing F has an F-supplement, (3) M is a sum of F-hollow submodules (i.e. modules in which every proper submodule containing F is F-small), (4) M is an irredundant sum of F-local submodules (i.e. modules L with $\operatorname{Rad}_F(L)$ is the largest submodule of L containing F).

In Section 3, we define and investigate amply F-supplemented modules. We prove that M is amply F-supplemented if and only if every submodule $U \subseteq M$ is of the form $U = X + Y$, where X is F-supplemented and $Y \ll_F M$; and if M is finitely generated, these are equivalent to the statement that every maximal submodule of M containing F has ample F-supplements in M (Theorem 4).

1. *F*-small and *F*-supplement submodules

In this section, we give some useful properties of F-small submodules and some results on F-supplement submodules.

Clearly, small submodules and F-small submodules are the same for $F = 0$. Moreover, small submodules are always F-small, but the converse is not true in general.

Example 1. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. Taking a submodule $F = 4\mathbb{Z}$ of M, we see that $8\mathbb{Z}$ is F-small in M since $8\mathbb{Z} \subseteq F$. However, $8\mathbb{Z}$ is not small in M since, for example, $8\mathbb{Z} + 3\mathbb{Z} = M$. In fact, 0 is the only small submodule of M.

We collect some known properties of F-small submodules which will be useful in the sequel in the following lemma (see [1]).

Lemma 1. *Let M be a module and let K, L be submodules of M.*

1) *If $f : M \to N$ is a homomorphism of modules, then $K \ll_F M$ implies $f(K) \ll_{f(F)} N$. In particular, if $K \ll_F M \subseteq N$, then $K \ll_F N$.*
2) *If $K \subseteq N \subseteq M$, then $N \ll_F M$ if and only if $K \ll_F M$ and $N/K \ll_{(F+K)/K} M/K$.*
3) *$K + L \ll_F M$ if and only if $K \ll_F M$ and $L \ll_F M$.*

Let M be a module and let U, V be submodules of M. Recall that V is said to be an F-*supplement* of U in M if V is minimal in the set $F \subseteq L \subseteq M$ with $U + L = M$.

Proposition 1. *Let M be a module and let U, V be submodules of M. Then V is an F-*supplement *of U in M if and only if $F \subseteq V$, $U + V = M$ and $U \cap V \ll_F V$.*

Proof. ($\Rightarrow$) Assume that $U \cap V + X = V$ with $F \subseteq X \subseteq M$. Then $M = U + V = U + X$, and so $X = V$ by the minimality of V. Thus $U \cap V \ll_F V$.

($\Leftarrow$) Assume that $U + Y = M$ for a submodule Y of M with $F \subseteq Y \subseteq V$. Then we have $V = M \cap V = (U + Y) \cap V = U \cap V + Y$. Since $U \cap V \ll_F V$ and $F \subseteq Y$, it follows that $V = Y$. This minimality of V shows that V is an F-supplement of U in M. □

As a generalization of the radical of a module, $\mathrm{Rad}_F(M)$ is defined in [1, Definition 3.1] to be the intersection of all maximal submodules of M that contain F. If there is no such maximal submodule of M, then $\mathrm{Rad}_F(M) = M$. Also, it was proved that $\mathrm{Rad}_F(M)$ is equal to the sum of all F-small submodules of M (see [1, Theorem 3.3]).

Lemma 2. *Let M be a module and let $x \in M$. Then $x \in \mathrm{Rad}_F(M)$ if and only if $Rx \ll_F M$.*

Proof. ($\Rightarrow$) Let $Rx + U = M$ with $F \subseteq U \subseteq M$. Assume that $U \neq M$. Then, by Zorn's Lemma, there is a submodule $L \subseteq M$ maximal with respect to $U \subseteq L$ and $x \notin L$. Since $L + Rx = M$, it follows that L is a maximal submodule of M. So, $\mathrm{Rad}_F(M) \subseteq L$ since $F \subseteq L$, and this implies that $x \in L$. This contradiction shows that $U = M$. Hence $Rx \ll_F M$.

($\Leftarrow$) Since $\mathrm{Rad}_F(M)$ is the sum of all F-small submodules of M and $Rx \ll_F M$, we have $Rx \subseteq \mathrm{Rad}_F(M)$. Hence $x \in \mathrm{Rad}_F(M)$. □

Proposition 2. *Let M be a module and let V be an F-supplement of U in M. If $K \ll_F M$, then $K \cap V \ll_F V$ and so $\mathrm{Rad}_F(V) = V \cap \mathrm{Rad}_F(M)$.*

Proof. First, assume that $(K \cap V) + X = V$ with $F \subseteq X \subseteq V$. Then $M = U + V = U + (K \cap V) + X = K + (U + X)$. Since $F \subseteq U + X$ and $K \ll_F M$, we have $U + X = M$. Thus $X = V$ by the minimality of V. Hence $K \cap V \ll_F V$. Next, it is clear that $\mathrm{Rad}_F(V) \subseteq V \cap \mathrm{Rad}_F(M)$. Now, let $x \in V \cap \mathrm{Rad}_F(M)$. Then $x \in \mathrm{Rad}_F(M)$ and so $Rx \ll_F M$ by Lemma 2. Since $Rx \subseteq V$, it follows by the first part that $Rx = V \cap Rx \ll_F V$. Thus $x \in \mathrm{Rad}_F(V)$. □

Corollary 1. *Let M be a module and let V be an F-supplement of U in M. If $\mathrm{Rad}_F(M) \ll_F M$, then U is contained in a maximal submodule of M containing F.*

Proof. By Proposition 2, we have $\mathrm{Rad}_F(V) = V \cap \mathrm{Rad}_F(M) \ll_F V$. Therefore, $\mathrm{Rad}_F(V) \neq V$, and so V has a maximal submodule V' that contains F. So, $M/(U+V') = (U+V)/(U+V') \cong V/V'$, and $U+V'$ is the desired maximal submodule of M. □

Corollary 2. *Let M be a module, and let V be an F-supplement of U in M. If U is a maximal submodule of M containing F, then $U \cap V = \mathrm{Rad}_F(V)$ is the unique maximal submodule of V that contains F.*

Proof. Since U is a maximal submodule of M containing F, we have $\mathrm{Rad}_F(M) \subseteq U$. So, it follows by Proposition 2 that $\mathrm{Rad}_F(V) = V \cap \mathrm{Rad}_F(M) \subseteq U \cap V$. Conversely, since $U \cap V \ll_F V$ as V is an F-supplement of U, we get $U \cap V \subseteq \mathrm{Rad}_F(V)$. □

For modules which don't have maximal submodules containing F (for instance, for the $\mathbb{Z}$-module $\mathbb{Q}$), we have the following result.

Proposition 3. *Let M be a module. Then $\mathrm{Rad}_F(M) = M$ if and only if every finitely generated submodule of M is F-small in M.*

Proof. (⇒) Let N be any finitely generated submodule of M. Then $N = Rm_1 + Rm_2 + \cdots + Rm_k$ for some $m_i \in M$. Since, by assumption, $m_i \in \mathrm{Rad}_F(M)$, it follows that $Rm_i \ll_F M$ for all $i = 1, 2, \ldots k$ by Lemma 2. Therefore $N \ll_F M$ by Lemma 1-(3).

(⇐) Clearly, $\mathrm{Rad}_F(M) \subseteq M$. To show that $M \subseteq \mathrm{Rad}_F(M)$, let $x \in M$. Then $Rx \ll_F M$ by assumption, and so $Rx \subseteq \mathrm{Rad}_F(M)$. Thus $x \in \mathrm{Rad}_F(M)$ as desired. □

Corollary 3. *Let M be a module. If $\mathrm{Rad}_F(M) = M$, then every finitely generated submodule of M has an F-supplement in M.*

Proof. Since every finitely generated submodule U of M is F-small in M by Proposition 3, then M is an F-supplement of U in M. □

The following result shows F-supplements in submodules.

Theorem 1. *Let $K \subseteq N \subseteq M$ be submodules.*

1) *If K is an F-supplement in M, then K is an F-supplement in N.*
2) *If N is an F-supplement in M, then*
   (a) *K is an F-supplement in M if and only if K is an F-supplement in N;*
   (b) *$K \ll_F M$ if and only if $K \ll_F N$.*

Proof. 1) Since K is an F-supplement in M, there is a submodule $P \subseteq M$ such that $K + P = M$ and $K \cap P \ll_F K$. By modular law, we have $N = K + N \cap P$. Moreover, $K \cap N \cap P = K \cap P \ll_F K$. Thus K is an F-supplement of $N \cap P$ in N.

2) Suppose that N is F-supplement in M. Then there is a submodule $L \subseteq M$ such that $N + L = M$ and $N \cap L \ll_F N$.

(a) ($\Rightarrow$) It follows by (1).

($\Leftarrow$) Since K is an F-supplement in N, there is a submodule $T \subseteq N$ such that $K+T = N$ and $K\cap T \ll_F K$. So, $K+(T+L) = N+L = M$. Assume that $K' + (T + L) = M$ for a submodule $F \subseteq K' \subseteq K$. Since $F \subseteq K' + T$ and N is F-supplement of L in M, it follows that $K' + T = N$ by the minimality of N. Now by the minimality of K, we conclude that $K' = K$. This means that K is F-supplement of $T + L$ in M.

(b) ($\Rightarrow$) Assume that $K + X = N$ for a submodule $F \subseteq X \subseteq N$. Then $K + X + L = N + L = M$, and so $X + L = M$ since $F \subseteq X + L$ and $K \ll_F M$ by assumption. Thus by modular law, we have $N = X + N \cap L$. Since $N \cap L \ll_F N$, it follows that $X = N$. Hence $K \ll_F N$.

($\Leftarrow$) It is always true by Lemma 1-(1). □

Supplement submodules and F-supplement submodules coincide in a module under some extra conditions over F.

Remark 1. Clearly, supplement submodules and 0-supplement submodules coincide in a module, and any supplement submodule containing F is always F-supplement. However an F-supplement submodule need not be supplement in general (since F-smallness need not imply smallness, see Example 1). But, for example, if $F \ll M$ and V is an F-supplement of U in M, then we have $F \ll V$ (by Theorem 1-(2b)), and so $U \cap V \ll_F V$ implies that $U \cap V \ll V$ (by [1, Proposition 2.3]). This means that V is a supplement of U in M.

Proposition 4. *Let M be a module. If N is an F-supplement of U in M, then for $K \subseteq U$, $(N + K)/K$ is an $(F + K)/K$-supplement of U/K in M/K.*

Proof. Since N is an F-supplement of U in M, we have $U + N = M$ and $U\cap N \ll_F N$. So, we have $(U/K)+(N+K)/K = (U+N+K)/K = M/K$, and by modular law, $(U/K) \cap (N + K)/K = (U \cap N + K)/K$. Since $U \cap N \ll_F N$, it follows by Lemma 1-(1) that $(U\cap N+K)/K \ll_{(F+K)/K}$

$(N+K)/K$ (by considering the epimorphism $f : N \to (N+K)/K$). Thus $(N+K)/K$ is an $(F+K)/K$-supplement of U/K in M/K. □

The converse of the previous statement is true under a special condition.

Proposition 5. *Let $K \subseteq N \subseteq M$ be submodules. If K is an F-supplement in M, and N/K is an $(F+K)/K$-supplement in M/K, then N is an F-supplement in M.*

Proof. First of all, since $F \subseteq K$, we see by assumption that N/K is a supplement in M/K. Now, let K be an F-supplement of a submodule K' in M. Then $K + K' = M$ and $K \cap K' \ll_F K$. Moreover, let N/K be a supplement of N'/K in M/K where $K \subseteq N'$. Then we have $N/K + N'/K = M/K$ and $N/K \cap N'/K = (N \cap N')/K \ll N/K$. Since $M = N \cap N' + K'$ as $K \subseteq N \cap N'$, and $N + N' = M$ it follows that $M = N + K' \cap N'$ by [2, 1.24]. Since $N = K + (N \cap K')$ by modular law, we get $N \cap (K' \cap N')/(K \cap K') \ll N/(K \cap K')$ by [2, 2.3-(1)]. So, clearly, we have $N \cap (K' \cap N')/(K \cap K') \ll_{F+(K\cap K')/(K\cap K')} N/(K \cap K')$. Thus, the fact that $K \cap K' \ll_F N$, which follows from $K \cap K' \ll_F K$, implies that $N \cap (K' \cap N') \ll_F N$ by Lemma 1-(2). This means that N is F-supplement of $K' \cap N'$ in M. □

2. *F*-supplemented modules

In this section, we define the concept of F-supplemented modules and we give a characterization for finitely generated F-supplemented modules.

A module M is called F-*supplemented* if every submodule of M has an F-supplement in M.

Proposition 6. *Let M be a module. Assume that U and M_1 are submodules of M, where M_1 is F-supplemented. If $M_1 + U$ has an F-supplement in M, then U also has an F-supplement in M.*

Proof. Let X be an F-supplement of $M_1 + U$ in M. Since M_1 is F-supplemented, the submodule $(X+U) \cap M_1$ has an F-supplement in M_1, say Y. We claim that $X+Y$ is an F-supplement of U in M. First, we have $M = X + M_1 + U = X + (Y + (X+U) \cap M_1) + U = (X+Y) + U$. Next, since $Y \subseteq M_1$, we have $Y \cap (X+U) = (Y \cap M_1) \cap (X+U) = Y \cap [(X+U) \cap M_1] \ll_F Y$, and since $Y + U \subseteq M_1 + U$, it follows that $X \cap (Y+U) \ll_F X$. Therefore, the inclusion $(X+Y) \cap U \subseteq X \cap (Y+U) + Y \cap (X+U)$ implies that $(X+Y) \cap U \ll_F X + Y$ by Lemma 1, as claimed. □

Remark 2. Since the zero submodule 0 is small in any module M, we observe that M is always a supplement of 0 in M, and vice versa. Likewise, since any submodule of M contained in F is F-small in M, we see that F is F-small in M. So, M is an F-supplement of F in M by Proposition 1, because $F + M = M$ and $F \cap M = F \ll_F M$. In fact, F is also an F-supplement of M in M since $F \ll_F F$.

Proposition 7. *If M_1 and M_2 are F-supplemented modules, then $M = M_1 + M_2$ is also an F-supplemented module.*

Proof. Let $U \subseteq M$. Since $M_1 + (M_2 + U)$ has trivially an F-supplement F in M, it follows, by Proposition 6, that there is an F-supplement for $M_2 + U$, and so there is an F-supplement for U in M. □

Corollary 4. *Every finite (direct) sum of F-supplemented modules is F-supplemented.*

Now, we investigate factor modules of F-supplemented modules.

Proposition 8. *If M is an F-supplemented module, then $M/\operatorname{Rad}_F(M)$ is semisimple (and so supplemented).*

Proof. Let $X/\operatorname{Rad}_F(M) \subseteq M/\operatorname{Rad}_F(M)$. Since M is F-supplemented, there is an F-supplement Y of X in M, that is, $X + Y = M$ and $X \cap Y \ll_F Y$. So, we have $(X/\operatorname{Rad}_F(M)) + (Y + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) = M/\operatorname{Rad}_F(M)$, and $(X/\operatorname{Rad}_F(M)) \cap (Y + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) = X \cap (Y + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) = (X \cap Y + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) = 0$, because $X \cap Y \ll_F Y$ implies that $X \cap Y \subseteq \operatorname{Rad}_F(M)$. Thus $X/\operatorname{Rad}_F(M)$ is a direct summand of $M/\operatorname{Rad}_F(M)$. Hence $M/\operatorname{Rad}_F(M)$ is semisimple. □

Proposition 9. *Let M be a module and let $K \subseteq M$ be a submodule. If M is F-supplemented, then M/K is $(F + K)/K$-supplemented.*

Proof. Take any submodule N/K of M/K where $K \subseteq N \subseteq M$. Since M is F-supplemented, N has an F-supplement in M, say V. Thus $(V+K)/K$ is an $(F + K)/K$-supplement of N/K in M/K by Proposition 4. Hence M/K is $(F + K)/K$-supplemented. □

Theorem 2. *Let M be a module. Then M is F-supplemented if and only if M/F is supplemented.*

Proof. ($\Rightarrow$) Suppose that M is F-supplemented. Then, by Proposition 9, we see that M/F is 0-supplemented, that is, supplemented.

($\Leftarrow$) Take any submodule $U \subseteq M$. Since M/F is supplemented, $(U + F)/F$ has a supplement in M/F, say V/F. Then $[(U + F)/F] + V/F = M/F$ which implies that $U + V = M$, and $[U \cap V) + F]/F = [(U + F)/F] \cap V/F \ll V/F$ from which it follows that $U \cap V \ll_F V$ by [1, Proposition 2.9]. Thus V is F-supplement of U in M. Hence M is F-supplemented. □

Taking $F = 0$, we obtain the following corollary.

Corollary 5. *A module M is* 0*-supplemented if and only if it is supplemented.*

A nonzero module M is called *hollow* if every proper submodule is small in M; and *local* if it has a largest submodule (namely $\mathrm{Rad}(M)$). It is clear that local modules are hollow, and that any finitely generated module is hollow if and only if it is local (see, for example, [2, p. 15]).

Let M be a nonzero module and let F be a proper submodule of M. We call M an F-*hollow* module if every proper submodule containing F is F-small in M; and F-*local* if $\mathrm{Rad}_F(M)$ is the largest submodule of M containing F (i.e. a proper submodule which contains all other proper submodules containing F). In this case, $\mathrm{Rad}_F(M) \ll_F M$.

Remark 3. Since M is hollow if and only if every proper submodule is F-small in M (see [1, Proposition 2.21]), it follows easily that a hollow module is F-hollow, and the converse is true when $F = 0$. Moreover, local modules are always F-local (because, in this case, $F \subseteq \mathrm{Rad}(M)$ and so $\mathrm{Rad}(M) = \mathrm{Rad}_F(M)$), and the converse is also true for $F = 0$.

In general, F-hollow (respectively F-local) modules need not be hollow (respectively local).

Example 2. Let M be the $\mathbb{Z}$-module $\mathbb{Z}$, and let $F = 4\mathbb{Z}$. Then we have $\mathrm{Rad}_F(M) = 2\mathbb{Z}$, and so M is F-local which implies that M is F-hollow. But, since $\mathrm{Rad}(M) = 0$, M is not local, and so it is not hollow as a cyclic module.

Now we give some results which are needed to characterize finitely generated F-supplemented modules.

Proposition 10. *A nonzero module M is F-local if and only if it is F-hollow and* $\mathrm{Rad}_F(M) \neq M$.

Proof. ($\Rightarrow$) Let K be a proper submodule of M containing F. Since M is F-local, $K \subseteq \mathrm{Rad}_F(M)$. Now since $\mathrm{Rad}_F(M) \ll_F M$, it follows that $K \ll_F M$ by Lemma 1-(2), and that $\mathrm{Rad}_F(M) \neq M$.

($\Leftarrow$) First, we claim that $\overline{M} = M/\operatorname{Rad}_F(M)$ is simple. By assumption, $\overline{M} \neq 0$. Now, if $N/\operatorname{Rad}_F(M) \subsetneqq \overline{M}$, then we have $F \subseteq N \subsetneqq M$. Since M is F-hollow, then $N \ll_F M$, and so $N \subseteq \operatorname{Rad}_F(M)$. Hence $N = \operatorname{Rad}_F(M)$. Next, if $F \subseteq X \subsetneqq M$, then $(X + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) \subsetneqq \overline{M}$. So, $X + \operatorname{Rad}_F(M) = \operatorname{Rad}_F(M)$ since $\overline{M}$ is simple, which implies that $X \subseteq \operatorname{Rad}_F(M)$. Hence $\operatorname{Rad}_F(M)$ is the largest submodule of M that contains F, that is, M is F-local. □

Proposition 11. *Let M be a module. If $M/\operatorname{Rad}_F(M)$ is semisimple and $\operatorname{Rad}_F(M) \ll_F M$, then every proper submodule of M containing F is contained in a maximal submodule.*

Proof. Consider the natural epimorphism $\sigma : M \to M/\operatorname{Rad}_F(M) = \overline{M}$. Let $F \subseteq U \subsetneqq M$. Since $\operatorname{Rad}_F(M) \ll_F M$, we have $\sigma(U) \neq \overline{M}$. Thus $\sigma(U)$ is contained in a maximal submodule $\overline{N}$ of $\overline{M}$ (as $\overline{M}$ is semisimple). Hence U is contained in the maximal submodule $\sigma^{-1}(\overline{N})$ of M. □

Corollary 6. *If M is F-supplemented and $\operatorname{Rad}_F(M) \ll_F M$, then every proper submodule of M containing F is contained in a maximal submodule.*

Proof. It follows from Propositions 8 and 11. □

Proposition 12. *Every F-hollow module M is F-supplemented.*

Proof. Let U be any submodule of M. Since F is always an F-supplement of M in M by Remark 2, we may assume that $U \neq M$. There are two cases to consider. First, if $F \subseteq U$, then $U \ll_F M$ since M is F-hollow by assumption. So, M is an F-supplement of U in M, because $U + M = M$ and $U \cap M = U \ll_F M$. Next, assume that $F \nsubseteq U$. If $U + F = M$, then F is an F-supplement of U in M since $U \cap F \subseteq F \ll_F F$. Otherwise, $U + F \neq M$. Therefore, $U + F \ll_F M$ by assumption, from which we get $U \ll_F M$. So, M is an F-supplement of U in M as in the first case. Thus, in each case, U has an F-supplement in M. Hence M is F-supplemented. □

Proposition 13. *Let M be a module. Every F-supplement of a maximal submodule of M containing F is F-local.*

Proof. Let U be a maximal submodule of M containing F. Assume that V is an F-supplement of U in M. Then by Corollary 2, we have $\operatorname{Rad}_F(V) = V \cap U$ is the unique maximal submodule of V containing F. In fact, $\operatorname{Rad}_F(V)$ is the largest proper submodule of V containing F, because, for any submodule $F \subseteq X \subseteq V$ with $X \nsubseteq \operatorname{Rad}_F(V)$, we have $X + \operatorname{Rad}_F(V) = V$ since $\operatorname{Rad}_F(V)$ is maximal in V, and so $X = V$ since $\operatorname{Rad}_F(V) \ll_F V$ as V is an F-supplement of U. Hence V is F-local. □

Let $\{M_i\}_I$ be a family of modules for some index set I. The sum $M = \sum_I M_i$ is called *irredundant* if, for every $i_0 \in I$, $\sum_{i \neq i_0} M_i \neq M$ holds.

Theorem 3. *For a module M, the following are equivalent.*

1) *M is a sum of F-hollow submodules and $\operatorname{Rad}_F(M) \ll_F M$.*
2) *Every proper submodule of M containing F is contained in a maximal submodule, and*
   (a) *every maximal submodule containing F has an F-supplement in M, or*
   (b) *every submodule K of M, with M/K is finitely generated, has an F-supplement in M.*
3) *M is an irredundant sum of F-local modules and $\operatorname{Rad}_F(M) \ll_F M$.*

Proof. (1) $\Leftrightarrow$ (3) Let $M = \sum_I L_i$ with F-hollow submodules L_i of M for some index set I. Then $M/\operatorname{Rad}_F(M) = \sum_I (L_i + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M)$. Since $(L_i + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) \cong L_i/(L_i \cap \operatorname{Rad}_F(M))$, we claim that these factors are simple or zero. If $L_i \cap \operatorname{Rad}_F(M) \neq L_i$ and $X/(L_i \cap \operatorname{Rad}_F(M))$ is any proper submodule of $L_i/(L_i \cap \operatorname{Rad}_F(M))$, then we have $F \subseteq X \subsetneq L_i$ as $F \subseteq L_i \cap \operatorname{Rad}_F(M)$. But, then $X \ll_F L_i$ as L_i is F-hollow, and so $X \subseteq \operatorname{Rad}_F(L_i) \subseteq L_i \cap \operatorname{Rad}_F(M)$. This implies that $X = L_i \cap \operatorname{Rad}_F(M)$, and $L_i/(L_i \cap \operatorname{Rad}_F(M))$ is simple as claimed. Therefore, we obtain that $M/\operatorname{Rad}_F(M) = \oplus_J (L_i + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M)$ for some subset $J \subseteq I$. Since $\operatorname{Rad}_F(M) \ll_F M$, it follows that $M = \sum_J L_i$ with F-local modules L_i by Proposition 10 (since $\operatorname{Rad}_F(L_i) \neq L_i$).

Since $(b) \Rightarrow (a)$ is clear, it suffices to prove the following implications:

(3) $\Rightarrow$ (2)(b) Clearly, $M/\operatorname{Rad}_F(M)$ is semisimple (see (1) $\Leftrightarrow$ (3)). Since $\operatorname{Rad}_F(M) \ll_F M$, it follows by Proposition 11 that every proper submodule of M containing F is contained in a maximal submodule.

Now, assume that K is a submodule of M with M/K is finitely generated. By assumption, there are finitely many F-local (and so F-hollow) submodules $L_1, L_2, \ldots, L_n$ with $M = K + L_1 + L_2 + \cdots + L_n$. Then by Proposition 12 and Corollary 4, it follows that $L_1 + L_2 + \cdots + L_n$ is F-supplemented. Moreover, since M has trivially an F-supplement F in M, by Proposition 6, K also has an F-supplement in M.

(2)(a) $\Rightarrow$ (1) Let $H = \sum_I L_i$ with F-hollow submodules L_i of M for some index set I. Observe that $F \subseteq L_i$ for each $i \in I$. We show that $H = M$. Suppose to the contrary that $H \neq M$. Since $F \subseteq H$, it follows by assumption that H is contained in a maximal submodule N of M. By assumption, N has an F-supplement in M, say L. Since $F \subseteq H \subseteq N$, by Proposition 13, we obtain that L is F-local, and so it is F-hollow. Thus we get $L \subseteq H \subseteq N$ by the choice of H, which implies that $N = M$ as $N + L = M$. This contradiction shows that $H = M$. □

The following result gives a characterization for finitely generated F-supplemented modules.

Corollary 7. *For a finitely generated module M, the following statements are equivalent.*

1) *M is F-supplemented.*
2) *Every maximal submodule containing F has an F-supplement in M.*
3) *M is a sum of F-hollow submodules.*
4) *M is an irredundant sum of F-local submodules.*

Proof. Since M is finitely generated, first, we have $\mathrm{Rad}_F(M) \ll_F M$. Indeed, assume that $\mathrm{Rad}_F(M) + X = M$ with $F \subseteq X$, and that $X \neq M$. Then X is contained in a maximal submodule N of M. So, we have $\mathrm{Rad}_F(M) + N = M$. Since $F \subseteq N$, it follows that $\mathrm{Rad}_F(M) \subseteq N$, and so $N = M$. It is a contradiction. Next, for every submodule K of M, we have M/K is finitely generated. Thus the proof follows immediately by Theorem 3. □

3. Amply F-supplemented modules

In this section, we introduce and characterize amply F-supplemented modules.

Let M be a module. We say a submodule $U \subseteq M$ has *ample* (or *enough*) *F-supplements* in M if, for every $V \subseteq M$ with $U + V = M$, there is an F-supplement V' of U with $V' \subseteq V$. If every submodule of M has ample F-supplements in M, then we call M *amply F-supplemented.*

Since for each submodule $U \subseteq M$, we have $U + M = M$, it follows that every amply F-supplemented module is F-supplemented.

Proposition 14. *Let M be an amply F-supplemented module. Then*

1) *Every F-supplement submodule of M is amply F-supplemented.*
2) *Every direct summand of M containing F is amply F-supplemented.*

Proof. 1) Let $V \subseteq M$ be an F-supplement of $U \subseteq M$. For $X \subseteq V$, assume that $V = X + Y$. Then $M = U + V = (U + X) + Y$, and so there is an F-supplement Y' of $U + X$ in M with $Y' \subseteq Y$ by assumption. We claim that Y' is an F-supplement of X in V. Since $X \cap Y' \subseteq (U + X) \cap Y' \ll_F Y'$, we have $X \cap Y' \ll_F Y'$ by Lemma 1-(2). Now, since $F \subseteq X + Y'$, $M = U + X + Y'$ implies that $V = X + Y'$ by the minimality of V.

2) Since any direct summand of M containing F is an F-supplement in M in the obvious way, it is amply F-supplemented by (1). □

Proposition 15. *Let M be a module with $M = U_1+U_2$. If the submodules U_1, U_2 have ample F-supplements in M, then so does $U_1 \cap U_2$.*

Proof. Let $V \subseteq M$ with $U_1 \cap U_2 + V = M$. Then by modular law, we have $U_1 \cap U_2 + U_2 \cap V = U_2$, and so $U_1 + U_2 \cap V = M$. So, by assumption, there is an F-supplement V_2' of U_1 with $V_2' \subseteq U_2 \cap V$. Similarly, there is also an F-supplement V_1' of U_2 with $V_1' \subseteq U_1 \cap V$. Thus, for $V_1' + V_2' \subseteq V$, we obtain that $U_1 \cap U_2 + (V_1' + V_2') = M$, and $(V_1' + V_2') \cap (U_1 \cap U_2) = (V_1' \cap U_2) + (V_2' \cap U_1) \ll_F V_1' + V_2'$ by Lemma 1. Hence $V_1' + V_2'$ is the desired F-supplement of $U_1 \cap U_2$ in M. □

Proposition 16. *Let M be a module, and $U, K \subseteq M$. If $K \ll_F M$ and $U + K$ has ample F-supplements, then U has also ample F-supplements.*

Proof. Let $V \subseteq M$ with $U+V = M$. Then $M = (U+K)+V$, and so there is an F-supplement $V' \subseteq V$ of $U + K$ by assumption. Since $K \ll_F M$ and $F \subseteq V' + U$, the equality $M = V' + U + K$ implies that $V' + U = M$. Moreover, $V' \cap U \subseteq V' \cap (U + K) \ll_F V'$ implies that $V' \cap U \ll_F V'$ by Lemma 1-(2). Hence V' is an F-supplement of U in M. □

Now we give a characterization for amply F-supplemented modules.

Theorem 4. *For a module M, the following statements are equivalent.*

1) *M is amply F-supplemented.*
2) *Every submodule $U \subseteq M$ is of the form $U = X + Y$, where X is F-supplemented and $Y \ll_F M$.*
3) *For every submodule $U \subseteq M$, there is an F-supplemented submodule $X \subseteq U$ such that $U/X \ll M/X$.*

   If M is finitely generated, then $(1) - (3)$ are equivalent to:
4) *Every maximal submodule containing F has ample F-supplements in M.*

Proof. (1) ⇒ (2) Clearly, M is F-supplemented. So, let V be an F-supplement of U in M. Then $U + V = M$, and by assumption there is an F-supplement X of V in M with $X \subseteq U$. Therefore, $U = U \cap M = U \cap (X + V) = X + U \cap V$, where $U \cap V \ll_F M$ since $U \cap V \ll_F V \subseteq M$, and X is (amply) F-supplemented by Proposition 14-(1).

(2) ⇒ (3) Let $U = X + Y$, where X is F-supplemented and $Y \ll_F M$. Then $U/X = (Y + X)/X \ll_{(F+X)/X} M/X$ by Lemma 1-(1), that is, $U/X \ll M/X$ since $F \subseteq X$ implies that $(F + X)/X = 0$

(3) ⇒ (1) Let $U \subseteq M$ with $U + V = M$. By assumption, there is an F-supplemented submodule X of V in M with $V/X \ll M/X$. So, the equality $(U + X)/X + V/X = M/X$ implies that $U + X = M$. Now, the submodule $U \cap X \subseteq X$ has an F-supplement in X, say V'. Therefore, we get $M = U + (U \cap X) + V' = U + V'$, and $U \cap V' = (U \cap X) \cap V' \ll_F V'$. Thus, V' is an F-supplement of U in M with $V' \subseteq V$. Hence M is amply F-supplemented.

(1) ⇒ (4) Clear.

(4) ⇒ (1) Now suppose that M is finitely generated, and that all maximal submodules of M containing F have ample F-supplements (so F-supplements). Then M is F-supplemented by Corollary 7, and $M/\operatorname{Rad}_F(M)$ is semisimple by Proposition 8. Therefore, for any submodule $U \subseteq M$, we have $M/(U+\operatorname{Rad}_F(M))$ is semisimple. Thus, $\operatorname{Rad}(M/(U+\operatorname{Rad}_F(M))) = 0$, and so $U+\operatorname{Rad}_F(M) = \cap_{i=1}^{k} N_i$, where N_i's are maximal submodules of M containing $U + \operatorname{Rad}_F(M)$ (and so containing F). From assumption and Proposition 15, we obtain that $U + \operatorname{Rad}_F(M)$ has ample F-supplements. Since $\operatorname{Rad}_F(M) \ll_F M$, Proposition 16 implies that U also has ample F-supplements. □

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