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On a common generalization of symmetric rings and quasi duo rings

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Abstract. Let $J(R)$ denote the Jacobson radical of a ring R. We call a ring R as J-symmetric if for any $a, b, c \in R, abc = 0$ implies $bac \in J(R)$. It turns out that J-symmetric rings are a common generalization of left (right) quasi-duo rings and generalized weakly symmetric rings. Various properties of these rings are established and some results on exchange rings and the regularity of left SF-rings are generalized.

1. Introduction

All rings considered in this paper are associative ring with identity and R denotes a ring. The symbols $J(R)$, $N(R)$, $Z(R)$, $E(R)$ respectively stand for the Jacobson radical, the set of all nilpotent elements, the set of all central elements and the set of all idempotent elements of R. We also denote the set $\{a \in R : a^2 = 0\}$ by $N_2(R)$, the ring of $n \times n$ upper triangular matrix over R by $T_n(R)$ and the left (right) annihilator of any element $a \in R$ by $l(a)$ $(r(a))$. R is *abelian* if all its idempotents are central. R is *left quasi-duo* if every maximal left ideal of R is an ideal. As usual, a *reduced ring* is a ring without non zero nilpotent elements. R is *semiprimitive* if $J(R) = 0$. R is *semicommutative* if $l(a)$ is an ideal of R for any $a \in R$. It is well known that R is semicommutative if and only if for any $a \in R$, $r(a)$ is an ideal of R. R is *symmetric* if for any $a, b, c \in R, abc = 0$ implies $acb = 0$. R is *reversible* if $ab = 0$ implies

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$ba = 0$. It is clear that symmetric rings are reversible and reversible rings are semicommutative.

Various generalizations of symmetric rings have been done by many authors over the last several years. R is *weak symmetric* ([5]) if for any $a, b, c \in R$, $abc \in N(R)$ implies $acb \in N(R)$. R is *central symmetric* ([4]) if for any $a, b, c \in R$, $abc = 0$ implies $bac \in Z(R)$. R is *generalized weakly symmetric* (GWS) ([11]) if for any $a, b, c \in R$, $abc = 0$ implies $bac \in N(R)$. It follows that the class of GWS rings contains the class of weak symmetric rings. Again, it is known that central symmetric rings are GWS ([11]).

2. Main results

Definition 1. A ring R is *J-symmetric* if for any $a, b, c \in R$, $abc = 0$ implies $bac \in J(R)$.

Proposition 1. *Following conditions are equivalent for a ring R:*
1) *R is J-symmetric.*
2) *For any $a, b, c \in R$, $abc = 0$ implies $acb \in J(R)$.*

Proof. (1) $\Rightarrow$ (2). Let $a, b, c \in R$ such that $abc = 0$ but $acb \notin J(R)$. Then we get a maximal left ideal $M \subseteq R$ such that $acb \notin M$ so that $M + Racb = R$. Therefore $1 = x + yacb$ for some $x \in M$, $y \in R$. Now $(ya)bc = 0$. As R is J-symmetric, $byac \in J(R)$. Thus $(1 - x)^2 = yac(byac)b \in J(R) \subseteq M$. Then using $x \in M$ we get $1 \in M$, a contradiction.

(2) $\Rightarrow$ (1). If $a, b, c \in R$ such that $abc = 0$ and $bac \notin J(R)$, then there exists a maximal left ideal $M \subseteq R$ such that $M + Rbac = R$ which gives $1 = x + ybac$ for some $x \in M$, $y \in R$. Now $ab(cy) = 0$. Then by hypothesis, $acyb \in J(R)$. Therefore $(1 - x)^2 = yb(acyb)ac \in M$, whence $1 \in M$, a contradiction. Hence R is J-symmetric. □

Proposition 2. *Let R be a J-symmetric ring and $abc = 0$, then for each maximal left ideal M of $R, a \in M$ or $bc \in M$.*

Proof. If $a \notin M$, then $M + Ra = R$ which implies that $x + ya = 1$ for some $x \in M$, $y \in R$. Then using $abc = 0$ we get $(x - 1)bc = 0$. As R is J symmetric, $bc(x - 1) \in J(R) \subseteq M$ which leads to $bc \in M$. □

Corollary 1. *Let R be a J-symmetric ring, then $N_2(R) \subseteq J(R)$.*

Corollary 2. *Let R be a J symmetric ring, then for any $a, b, c \in R$, $abc = 0$ implies $cab \in J(R)$.*

The proof of the following proposition is trivial.

Proposition 3. *The following conditions are equivalent for a ring R:*

1) *For any $a, b, c \in R$, $abc = 0$ implies $cab \in J(R)$.*
2) *For any $a, b, c \in R$, $abc = 0$ implies $bca \in J(R)$.*

Proposition 4. *If R is a ring such that for any $a, b, c \in R$, $abc = 0$ implies $cba \in J(R)$, then R is J symmetric.*

Proof. Let $a, b, c \in R$, $abc = 0$ but $bac \notin J(R)$. Then there exists a maximal left ideal $M \subseteq R$ such that $1 = x + ybac$ for some $x \in M$, $y \in R$. Now $ab(cy) = 0$. Then by hypothesis we get $cyba \in J(R)$. Hence $(1-x)^2 = yba(cyba)c \in M$ leading to $1 \in M$, a contradiction. Hence R is J-symmetric. □

Proposition 5. *If R is a left quasi-duo ring and $abc = 0$, then for each maximal left ideal M of R, $a \in M$ or $b \in M$ or $c \in M$.*

Proof. Let M be a maximal left ideal of R and $a \notin M$, then $M + Ra = R$ which implies that $x + ya = 1$ for some $x \in M$, $y \in R$ leading to $xbc = bc$. As R is left quasi-duo and $x \in M$, we get $bc \in M$. If $b \notin M$, then $M + Rb = R$ yielding $u + vb = 1$ for some $u \in M$, $v \in R$, whence $1 - vb \in M$ and so $(1 - vb)c \in M$. Therefore using $bc \in M$ we obtain $c \in M$. □

Proposition 6. *A left quasi-duo ring is J-symmetric.*

Proof. Let R be a left quasi duo ring and $abc = 0$ and M be a maximal left ideal of R. It follows from Proposition 5 that $a \in M$ or $b \in M$ or $c \in M$. As R is left quasi-duo, we get $bac \in M$. Therefore $bac \in J(R)$ which proves that R is J-symmetric. □

Proposition 7. *Central symmetric rings are J-symmetric.*

Proof. Let R be a central symmetric ring which is not J-symmetric. Then there exists $a, b, c \in R$ such that $abc = 0$ but $bac \notin J(R)$ so that there exists a maximal left ideal $M \subseteq R$ such that $1 = x + ybac$ for some $x \in M$, $y \in R$. Now for any $r_1, r_2 \in R$, $(ab)(cr_1)1 = 0$ and $(r_2a)bc = 0$. Hence $cr_1ab, br_2ac \in Z(R)$. Therefore

$$\begin{aligned}
(1-x)^4 &= (ybac)^4 = ybacyba(cybac)ybac = ybacyba(baccy)ybac \\
&= ybacybabacc(yybac) = ybacybabacc(bacyy) \\
&= y(b(acybab)ac)cbacyy = ycba(b(acybab)ac)cyy \\
&= ycbaba(c(yb)ab)accyy = ycbab(c(yb)ab)aaccyy \\
&= ycb(abc)ybabaaccyy = 0.
\end{aligned}$$

This leads to $1 \in M$, a contradiction. Hence R is J-symmetric. □

Proposition 8. *Generalized weakly symmetric rings are J-symmetric.*

Proof. Let R be a generalized weakly symmetric ring and $abc = 0$. If R is not J-symmetric, then there exists a maximal left ideal M of R such that $1 = x + ybac$ for some $x \in M$, $y \in R$. As R is generalized weakly symmetric and $abcy = 0$, $bacy \in N(R)$ so that $(bacy)^k = 0$ for some positive integer k. Therefore

$$(1-x)^{k+1} = (ybac)^{k+1} = y(bacy)^k bac = 0 \in M.$$

This together with $x \in M$ implies that $1 \in M$, a contradiction. Hence R is J-symmetric. □

Corollary 3. *Weak symmetric rings are J-symmetric.*

Remark 1. For a field $\mathbb{F}$ and $n > 1, R = T_n(\mathbb{F})$ is weak symmetric ([5], Proposition 2.3) and hence GWS and J-symmetric. As R is not abelian, R is neither central symmetric nor semicommutative. Also, it is worth mentioning here that an abelian ring need not be J-symmetric.

Take

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ a - d \equiv b \equiv c \equiv 0 \pmod 2 \right\}$$

Then $E(R) = \{0, I\}$ where I is the identity matrix over $\mathbb{Z}$. Therefore R is abelian. Consider $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $A^2 = 0$ but $A \notin J(R)$ as for $K = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, $I - KA$ is not a unit in R. Therefore $N_2(R) \nsubseteq J(R)$, hence R is not J-symmetric.

A ring R is *directly finite* if for any $a, b \in R$, $ab = 1$ implies $ba = 1$.

Proposition 9. *Every J-symmetric ring is directly finite.*

Proof. Let $a, b \in R$ such that $ab = 1$. Take $e = ba$, then $e^2 = e$. If $c = b(1-e)$, then $c^2 = 0$ so that by Corollary 1, $c \in J(R)$ which implies that $ac \in J(R)$ and hence $1 - ac = 1 - ab(1-e) = e$ is invertible which leads to $e = ba = 1$. □

Recall that a ring R is *left min-abel* if $(1-e)Re = 0$ for any $e \in E(R)$ satisfying Re is a minimal left ideal of R.

Lemma 1. *For any $e \in E(R)$, $J(eRe) = eJ(R)e$*

Theorem 1. *Let R be a J-symmetric ring. Then*
1. *If $e \in E(R)$ such that $ReR = R$, then $e = 1$.*
2. *If $e \in E(R)$ and M be a maximal left ideal of R, then either $e \in M$ or $(1-e) \in M$.*
3. *$Ra + R(ae-1) = R$ for any $a \in R$ and $e \in E(R)$.*
4. *R is left min-abel.*
5. *For any $e \in E(R)$, eRe is J-symmetric.*

Proof. (1) Since R is J-symmetric and $Re(1-e) = 0$, $eR(1-e) \subseteq J(R)$. By hypothesis, $ReR = R$ which implies that $R(1-e) = ReR(1-e) \subseteq J(R)$, whence $1-e \in J(R)$ so that $e = 1$.

(2) Follows from Proposition 2 as $e(1-e) = 0$.

(3) Assume $Ra + R(ae-1) \neq R$ for some $a \in R$ and $e \in E(R)$, then there exists a maximal left ideal M of R such that $Ra + R(ae-1) \subseteq M$. If $e \in M$, then $ae \in M$, hence $1 = -(ae-1) + ae \in M$, a contradiction. If $e \notin M$, then $1-e \in M$ implying $a - ae = a(1-e) \in M$. As $ae - 1 \in M$, this leads to $1 \in M$, a contradiction. Hence $Ra + R(ae-1) = R$ for each $a \in R$ and $e \in E(R)$.

(4) Let $e \in E(R)$ and Re be a minimal left ideal and $(1-e)Re \neq 0$. Then $R(1-e)Re = Re$. Now $e \in eRe = eR(1-e)Re \subseteq J(R)$ which is a contradiction. Therefore $(1-e)Re = 0$ and R is left min-abel.

(5) Let $e \in E(R)$ and $eae, ebe, ece \in eRe$ with $(eae)(ebe)(ece) = 0$. By hypothesis, $(ebe)(eae)(ece) \in J(R)$ and so $e(ebe)(eae)(ece)e = (ebe)(eae)(ece) \in eJ(R)e = J(eRe)$ by Lemma 1. □

Converse of (5) of Theorem 1 need not be true. The following example shows this fact.

Example 1. Take $R = M_2(\mathbb{F})$, where $\mathbb{F}$ is a field and consider the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to check that $eRe = \begin{pmatrix} \mathbb{F} & 0 \\ 0 & 0 \end{pmatrix}$ is J-symmetric but R is not.

Proposition 10. *If R is a J-symmetric ring and idempotents can be lifted modulo $J(R)$, then $R/J(R)$ is abelian.*

Proof. Let $\overline{R} = R/J(R)$ and $\overline{a} \in E(\overline{R})$. As idempotents can be lifted modulo $J(R)$, there exists $e \in E(R)$ such that $\overline{e} = \overline{a}$. For any $\overline{x} \in \overline{R}$, write $h = xe - exe$. Then $h^2 = 0$ and hence by Corollary 1, $h \in J(R)$. Therefore $\overline{xe} = \overline{exe}$, that is $\overline{xa} = \overline{axa}$. Similarly $\overline{ax} = \overline{axa}$. Hence $\overline{R}$ is abelian. □

Proposition 11. *If $R/J(R)$ is symmetric, then R is J-symmetric.*

Proof. Let $a, b, c \in R$ such that abc=0. Then $\overline{a}\overline{b}\overline{c} = \overline{0}$. As $R/J(R)$ is symmetric, $\overline{b}\overline{a}\overline{c} = \overline{0}$ which yields $bac \in J(R)$. Therefore R is J-symmetric. □

Proposition 12. *Direct product of arbitrary family of J-symmetric rings is J-symmetric.*

Proof. For any arbitrary family of rings $\{R_i : i \in I\}$, we know that $J\left(\prod_{i\in I} R_i\right) = \prod_{i\in I}(J(R_i))$. Hence the result easily follows. □

Corollary 4. *A ring R is J-symmetric if eR and $(1-e)R$ are J-symmetric for any central idempotent e.*

Example 2. A homomorphic image of a J-symmetric ring need not be J-symmetric

Consider $\mathbb{Z}_2(y)$, the rational functions field of polynomial ring $\mathbb{Z}_2[y]$ and $R = \mathbb{Z}_2(y)[x]$ be the ring of polynomials in x over $\mathbb{Z}_2(y)$ subject to the relation $xy + yx = 1$. Now by ([4], Example 2.11), R is central symmetric and therefore J-symmetric. Let $L = x^2R$, which is a maximal ideal of R. Consider $\overline{R} = R/L$. Now $(\overline{x})^2 = \overline{0}$. So $0 \neq \overline{x} \in N_2(\overline{R})$. But $\overline{R}$ being a simple ring, we have $J(\overline{R}) = 0$. Thus we have $N_2(\overline{R}) \nsubseteq J(\overline{R})$, hence $\overline{R}$, a homomorphic image of R is not J-symmetric.

The next two propositions gives the condition on an ideal of a ring which forces the ring to be J-symmetric.

Proposition 13. *Let I be a nil ideal of a ring R such that R/I is J-symmetric. Then R is J-symmetric.*

Proof. Let $a, b, c \in R$ such that $abc = 0$. Then $\overline{abc} = \overline{0}$ in R/I. Since R/I is J-symmetric, $\overline{bac} \in J(R/I)$. Then for any $r \in R$, there exists $t \in R$ such that $1 - t(1 - rbac) \in I \subseteq J(R)$ since I is nil. It follows that $(1 - rbac)$ is left invertible and hence $bac \in J(R)$. □

Proposition 14. *Let I be an ideal of a J-symmetric ring S and let R be a subring of S containing I. Then R/I is J-symmetric implies R is J-symmertic.*

Proof. Let $a, b, c \in R$ such that $abc = 0$ in $R \subseteq S$. Since S is J-symmetric, $bac \in J(S)$. Then for any $r \in R \subseteq S$, there exists $s \in S$ such that $s(1 - rbac) = 1$. Now $\overline{abc} = \overline{0}$ in R/I. Since R/I is J-symmetric, $\overline{bac} \in J(R/I)$. Therefore there exists $t \in R$ such that $(1 - (1 - rbac)t) \in I$. This yields $s - s(1 - rbac)t \in I$ and so $s - t \in I \subseteq R$. This implies $s \in R$ and hence $(1 - rbac)$ is left invertible in R so that $bac \in J(R)$. □

Proposition 15. *Subdirect product of arbitrary family of J-symmetric rings is J-symmetric.*

Proof. Let R be a subdirect product of a family of J-symmetric rings $\{R_i\}_{i\in I}$. Then for each $i \in I$, we have epimorphism $\phi_i : R \to R_i$ and hence $\prod_{i\in I} R/\operatorname{Ker}(\phi_i) \simeq \prod_{i\in I} R_i$ is J-symmetric. The map

$$\Phi : R \longrightarrow \prod_{i\in I} R/\operatorname{Ker}(\phi_i), \qquad \Phi(r) = (r + \operatorname{Ker}(\phi_i))_{i\in I}$$

is a monomorphism. Then $R \cong \operatorname{Im}(\Phi)$. Also $\operatorname{Im}(\Phi)/\Phi(\operatorname{Ker}(\phi_i)) \simeq R/\operatorname{Ker}(\phi_i)$ is J-symmetric. Now $\Phi(\operatorname{Ker}(\phi_i)) \subseteq \operatorname{Im}(\Phi) \subseteq \prod_{i\in I} R/\operatorname{Ker}(\phi_i)$. Hence by Proposition 14, $\operatorname{Im}(\Phi) \cong R$ is J-symmetric. □

Theorem 2. *The following conditions are equivalent for a ring R:*

(1) *R is J-symmetric.*

(2) *$T_n(R)$ is J-symmetric for any $n \geqslant 2$.*

(3) *$R[x]/(x^n)$ is J-symmetric for any $n \geqslant 2$.*

(4) $S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \dots & a_{1n} \\ 0 & a & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix} : a, a_{ij} \in R, i < j \leqslant n \right\}$ *is J-symmetric for any $n \geqslant 2$.*

Proof. Let

$$I = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots\ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} : a_{ij} \in R,\ i < j \leqslant n \right\}.$$

Then I is a nil ideal of $T_n(R)$ as well as $S_n(R)$.

(2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1) are trivial.

(1) $\Rightarrow$ (2). $T_n(R)/I$ is isomorphic to direct product of n-copies of R. Hence by Proposition 12 and Proposition 13, $T_n(R)$ is J-symmetric.

(1) $\Rightarrow$ (3). Since $S_n(R)/I \simeq R$, it follows that $S_n(R)$ is also J-symmetric.

(1) $\Rightarrow$ (4). $R[x]/(x^n) \simeq V_n(R)$ where

$$V_n(R) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 \dots & a_{n-1} & a_n \\ 0 & a_0 & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots \\ 0 & 0 & 0 & 0\dots & a_0 \end{pmatrix} : a_i \in R, i = 0, 1, 2, \cdots n \right\}.$$

As $K = I \cap V_n(R)$ is a nil ideal of $V_n(R)$ and $V_n(R)/K \simeq R$, $V_n(R)$ is J-symmetric. □

If R is J-symmetric then $M_n(R)$ need not be J-symmetric. The following example shows this fact:

Example 3. Let $\mathbb{F}$ be a field and consider $R = M_2(\mathbb{F})$. Now $J(M_2(\mathbb{F})) = M_2(J(\mathbb{F})) = 0$. If $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then $ABC = 0$, but $BAC \neq 0$.

R is *(von Neumann) regular* if for any $a \in R$, there exists some $b \in R$ such that $a = aba$. R is *strongly regular* if for any $a \in R$, there exists some $b \in R$ such that $a = a^2b$. It is known that R is strongly regular if and only if R is reduced regular. R is *left SF-ring* if its simple left modules are flat. In 1975, Ramamurthy initiated the study of SF-rings in [10]. It is known that regular rings are left SF-rings. However, till date, it is unknown whether left SF-rings are regular. The regularity of left SF-rings satisfying certain additional conditions have been proved by various authors over the last four deacades (see, [6], [9], [10], [11], [14]). The strong regularity of left (right) quasi-duo left SF-rings, central symmetric left SF rings are proved respectively in [6], [11]. These results are generalized as follows:

Theorem 3. *A J-symmetric left SF-ring is strongly regular.*

Proof. $R/J(R)$ is a left SF-ring by ([6], Proposition 3.2). Let $b^2 \in J(R)$ such that $b \notin J(R)$. We claim that $Rr(b) + J(R) \neq R$. If this is not true, then $1 = c + \sum r_i t_i$, where $c \in J(R)$, $r_i \in R$, $t_i \in r(b)$. This yields $b = cb + \sum r_i t_i b$. Now for each i, $(t_i b)^2 = t_i(bt_i)b = 0$ and hence by Corollary 1, $t_i b \in J(R)$. Therefore $\sum r_i t_i b \in J(R)$ yielding $b \in J(R)$, a contradiction to $b \notin J(R)$. Therefore $Rr(b) + J(R) \neq R$ and so there exists a maximal left ideal M of R containing $Rr(b) + J(R)$. Since R is a left SF-ring and $b^2 \in J(R) \subseteq M$, by ([6], Lemma 3.14), there exists some $d \in M$ such that $b^2 = b^2 d$, that is $b - bd \in r(b) \subseteq M$, whence $b \in M$. Hence, again there exists some $e \in M$ such that $b = be$. Then $1 - e \in r(b) \subseteq M$, so that $1 \in M$, contradicting $M \neq R$. Therefore $R/J(R)$ is reduced. Hence by ([6], Remark 3.13), $R/J(R)$ is strongly regular. This implies that R is left quasi-duo. Therefore by ([6], Theorem 4.10), R is strongly regular. □

R is *clean* if every element of R can be written as a sum of an idempotent and a unit. R is *exchange* if for any $a \in R$, there exists $e \in E(R)$ such that $e \in Ra$ and $(1-e) \in R(1-a)$. In [7], Nicholson proved that every clean ring is exchange. Exchange rings need not be clean but under certain additional conditions exchange rings turns out to be clean (see [1], [2], [3], [7], [11], [12]). It is known that left (right) quasi-duo exchange rings are clean ([12]). Also GWS exchange rings are clean ([11]). These results are extended to J-symmetric rings as follows:

Theorem 4. *Let R be a J-symmetric exchange ring. Then R is clean.*

Proof. Let $x \in R$. By hypothesis, there exists $e \in E(R)$ such that $e \in Rx$ and $(1-e) \in R(1-x)$. It is easy to see that $e = yx$ and $1-e = z(1-x)$ for some $y,\ z \in R$ such that $y = ey$ and $z = (1-e)z$. Therefore $(ze)^2 = 0 = [y(1-e)]^2$ and so by Corollary 1, ze, $y(1-e) \in J(R)$. Now $1-ze-y(1-e) = (e-zx+z)-ze-y(1-e) = yx-zx+z-ze-y+ye = (y-z)(x-(1-e))$. As ze, $y(1-e) \in J(R)$, $1-ze-y(1-e)$ is a unit so that that $x-(1-e)$ is left invertible. Since a J- symmetric ring is directly finite, it follows that $x-(1-e)$ is a unit and hence x is clean which implies that R is clean □

R has *stable range one* if for any a, $b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a unit. It is known that left (right) quasi-duo exchange rings have stable range one. In [11], Wei proved that GWS exchange rings have stable range one. Observing that a J-symmetric ring R satisfies $eR(1-e) \subseteq J(R)$ for any $e \in E(R)$ and using ([8], Theorem 5.4(1)), we get the following theorem which is a generalization of these existing results.

Theorem 5. *A J-symmetric exchange ring have stable range one.*

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