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# Baer semisimple modules and Baer rings

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Abstract. We consider Baer rings and Baer semisimple $R$-modules which are generalizations of semisimple modules. Several characterization theorems of Baer semisimple modules are obtained. In particular, we prove that a ring $R$ is a Baer ring if and only if $R$ itself, regarded as a regular $R$-module, is Baer semisimple.

Throughout this paper, $R$ is an associative ring with identity 1 and all R-modules are unital. Denote the set of idempotents of $R$ by $E(R)$. Let $M$ be a left $R$-module and a right $S$-module. Also, let $X$ be a subset of $M$, $R$ or $S$, respectively. Then we denote the left [resp. right] annihilator of $X$ by $ann_\ell(X)$ [resp. $ann_r(X)$]. We also write $ann_\ell(\{m\})$ [resp. $ann_r(\{m\})$] by $ann_\ell(m)$ [resp. $ann_r(m)$].

We call a ring $R$ a *Baer ring* if the left annihilator of any subset of $R$ is generated by an idempotent. The properties of Baer rings and its generalizations have been studied by many authors, for example, see ([3], [4], [11] and [13]). We observe that Baer rings can be generalized into other forms, for example, rpp rings, etc. The rpp-rings and their generalizations have been extensively studied in the literature after Hattori (see, [2]-[15]). Recently, the authors have introduced the concept of right perpetual ideals and consequently, reduced pp rings are characterized by using right perpetual submodules (see [8]).

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Recall that a right ideal $I$ of $R$ is a *right perpetual ideal* of $R$ if for every $x \in I$ and $y \in R$, $ann_\ell(x) \subseteq ann_\ell(y)$ implies that $y \in I$ (see [8]). Clearly, for any $X \subseteq R$, there exists the smallest right perpetual ideal of $R$ containing $X$. We usually call this smallest right perpetual ideal of $R$ containing $X$ the *right perpetual ideal* generated by $X$ and is denoted by $R^*(X)$. If $X = \{a\}$, then we write $R^*(X) = R^*(a)$.

The following results are known.

**Lemma 1.** [8] *The following statements hold in a ring $R$:*

(1) *If $e \in E(R)$, then $R^*(e) = eR$.*

(2) *For all $X \subseteq R$, $ann_r(X)$ is a right perpetual ideal of $R$.*

(3) *A ring $R$ is lpp if and only if for any $a \in R$, $R^*(a)$ is generated by an idempotent.*

Let $M$ be a right $R$-module. Denote the ring of $R$-endomorphisms of $M$ by $End(M_R)$. If $End(M_R)$ is regarded as a set of left operations, in notation, $End_\ell(M_R)$, then $M$ can be regarded as a left $End_\ell(M_R)$- right $R$-module. Inspiring by the definition of right perpetual ideals, we now define the perpetual submodules.

**Definition 1.** *Let $M$ be a right $R$-module. Then, we call a (right $R$-)submodule $N$ of $M$ a perpetual submodule of $M$ if for all $x \in N$ and $y \in M$, $ann_\ell(x) \subseteq ann_\ell(y)$ implies $y \in N$.*

It is clear that $M$ and $(0)$ are both trivial perpetual submodules of $M$. Also, the intersection of perpetual submodules of $M$ is still a perpetual submodule of $M$ and hence, there exists the smallest perpetual submodule of $M$ containing $X$ for $X \subseteq M$. Denote the smallest perpetual submodule of $M$ containing $X$ by $SM^*(X)$. On the other hand, if $R$ is regarded as a regular right $R$-module $R_R$, then the left $End(R_R)$-right $R$-module $R$ becomes a regular bimodule ${}_{End(R_R)}R_R$. Thus in this case, every perpetual submodule of $R$ is a right perpetual ideal of $R$ (same as in rings).

The following lemma can be easily proved.

**Lemma 2.** *Let $M$ be a right $R$-module and $X \subseteq End_\ell(M_R)$. Then*

(1) *$ann_r(X)$ is a perpetual submodule of $M$.*

(2) *If $\varphi^2 = \varphi \in End_\ell(M_R)$, then $\varphi M$ is a perpetual submodule of $M$.*

The proof of the following lemma is straightforward.

**Lemma 3.** *Let $M$ be a right $R$-module and $x \in M$. Then $SM^*(x) = ann_r(ann_\ell(x))$.*

The following result lemma is crucial in this paper but the proof can be found in [1].

**Lemma 4.** [1] *A R-submodule K of the right R-module M is a direct summand of M if and only if $K = eM$ for some idempotent $e \in End_\ell(M_R)$.*

Now, we formulate the following definition.

**Definition 2.** *Let M be a right R-module. Then*

(1) *M is called a* ***Baer simple R-module*** *if $M \neq 0$, and M contains no perpetual submodules of M other than M itself and (0).*

(2) *M is called a* ***Baer semismiple R-module*** *if every perpetual submodule of M is a direct summand of M.*

Evidently, a Baer simple $R$-module is itself Baer semisimple and the usual semisimple $R$-module is also Baer semisimple. Indeed, if $M$ is a semisimple $R$-module, then every $R$-submodule $N$ of $M$ is a direct summand of $M$. By Lemma 4, $N = eM$, for some $e^2 = e \in End_\ell(M_R)$. This implies that every $R$-submodule of $M$ is a perpetual submodule of $M$. Thus $M$ is Baer semisimple.

**Proposition 1.** *Let M be a Baer semisimple R-module and N a perpetual submodule of M. Then the following statements hold:*

(*i*) *$N = eM$ for some idempotent $e \in End_\ell(M_R)$.*

(*ii*) *N is Baer semisimple.*

*Proof.* (*i*) By our hypothesis, $M$ is Baer semisimple and hence, $N$ is a direct summand of $M$. Now, by Lemma 4, $N = eM$, for some idempotent $e \in End_\ell(M_R)$.

(*ii*) It suffices to verify that any perpetual submodule of $N$ is still a perpetual submodule of $M$. In other words, we only need to prove that the smallest perpetual submodule $SM_M^*(x)$ of $M$ containing $x$ is the smallest perpetual submodule $SM_N^*(x)$ of $N$ containing $x$, for all $x \in N$. By Lemma 5, we have $N = eM$, for some idempotent $e \in End_\ell(M_R)$. Denote the left annihilator of $K$ related to the $R$-module $M$ and related to the $R$-module $N$ by $ann_\ell^M(K)$ and $ann_\ell^N(K)$, respectively. Now, by Lemma 3, $SM_M^*(x) \subseteq N$. Let $f$ be an idempotent endomorphism in $End_\ell(M_R)$ such that $SM_M^*(x) = fM$. Then, $fM \subseteq eM$. Thus, for any $x \in M$,we have

$$fx = ey = eey = efx \quad (y \in M),$$

and thereby, $f = ef$. Hence, $fe$ is an idempotent endomorphism in $End_\ell(M_R)$ and also

$$fM = ffM \subseteq fefM \subseteq feM \subseteq fM,$$

that is, $fM = feM$. On the other hand, since the restriction $fe|_{eM}$ of $fe(= efe)$ to $eM$ is an idempotent $R$-endomorphism which maps $eM$

into itself, we have $fM = feM = fe(eM)$ and hence, by Lemma 2, $fM$ is a perpetual submodule of $N$. Now, by the minimality of $SM_N^*(x)$, we have $SM_N^*(x) \subseteq SM_M^*(x)$.

Now let $\varphi \in ann_\ell^N(x)$. Then, it can be easily observed that $M = eM \oplus (1-e)M$. Hence, we can define a mapping

$$\overline{\varphi} : M \to M; \quad y \mapsto \varphi(ey),$$

which is a $R$-homomorphism of $M$ into itself with $\overline{\varphi}|_N = \varphi$. Clearly, $\overline{\varphi} \in ann_\ell^M(x)$. If $y \in SM_M^*(x)$, then by Lemma 3, $\psi(y) = 0$ for all $\psi \in ann_\ell^M(x)$, and furthermore, $\overline{\varphi}y = 0$, for all $\varphi \in ann_\ell^N(x)$. Note that $SM_M^*(x) \subseteq N$ and $\overline{\varphi}|_N = \varphi$. Thus $\overline{\varphi}y = 0$ implies that $\varphi y = 0$. This shows that $ann_\ell^N(x) \subseteq ann_\ell^N(y)$. Consequently, we can deduce $y \in SM_N^*(x)$, by Lemma 3. This leads to $SM_M^*(x) \subseteq SM_N^*(x)$. Thus, $SM_N^*(x) = SM_M^*(x)$, as required. □

The following is a characterization theorem for the Baer simple $R$-modules.

**Theorem 1.** *Let $M$ be a Baer semisimple $R$-module and $N$ a perpetual $R$-submodule of $M$. Then $N$ is Baer simple $R$-module if and only if $N = eM$, for some primitive idempotent $e \in End_\ell(M_R)$.*

*Proof.* Suppose that $N$ is a Baer simple $R$-submodule of $M$. Then, by Lemma 4, $N = eM$ for some idempotent $e \in End_\ell(M_R)$. Now let $f^2 = f \in End_\ell(M_R)$ such that $f \leq e$, i.e., $f = ef = fe$. Then $fM \subseteq eM$. Since $N$ is Baer simple, $fM = (0)$ or $fM = eM$.

- If $fM = (0)$, then $f = 0$.
- If $fM = eM$, then for all $x \in M$,

  $$e(x) = f(y) = ff(y) = fe(x) = f(x) \quad (y \in M),$$

  and whence $e = f$.

This shows that $e$ is a primitive idempotent of $End_\ell(M_R)$. Conversely, we assume that $N = eM$, where $e$ is a primitive idempotent of $End_\ell(M_R)$. Then $N$ is a perpetual submodule of $M$. Let $K$ be a perpetual submodule of $N$. Now, by using the proof of Proposition 1, we can show that $K$ is still a perpetual submodule of $M$, and by Lemma 4, $K = fM$ for some idempotent $f \in End_\ell(M_R)$. Now, $fM \subseteq eM$ implies that for all $x \in M$, we have

$$f(x) = e(y) = ee(y) = ef(x) \quad (y \in M),$$

and thereby, $f = ef$. By routine verification, $fe$ is an idempotent of $End_\ell(M_R)$, and $fe \leq e$. But since $e$ is primitive, $fe = e$ or $fe = 0$.

- If $fe = e$, then

$$fM \subseteq eM = feM \subseteq fM,$$

that is, $K = N$.

- If $fe = 0$, then

$$K = fM = f(fM) \subseteq f(eM) = (0).$$

This shows that the submodule $N$ is indeed Baer simple. □

We next establish a "Schur Lemma " for Baer simple modules.

**Theorem 2.** *(Schur Lemma) If $M$ is a Baer simple $R$-module, then $End_\ell(M_R)$ is a domain (such a ring satisfies the cancellative law).*

*Proof.* It suffices to show that any $\varphi \in End_\ell(M_R)\backslash\{0\}$ is injective. For this purpose, we only need to prove that $ann_r(\varphi) = (0)$. By Lemma 2, $ann_r(\varphi)$ is a perpetual submodule of $M$ and, since $M$ is Baer simple, $ann_r(\varphi) = M$ or $ann_r(\varphi) = (0)$. But since $\varphi \neq 0$, it is clear that $ann_r(\varphi) \neq M$. Thus $ann_r(\varphi) = (0)$ and hence $\varphi$ is injective. □

**Lemma 5.** *Any nonzero Baer semisimple $R$-module $M$ contains a Baer simple $R$-module.*

*Proof.* Without loss of generality, we may assume that $M$ is not a Baer simple $R$-module. Then we can pick a nonzero element $x$ of $M$ such that $SM^*(x) \subset M$. By Lemma 4, $SM^*(x) = eM$ for some idempotent endomorphism $e \in End_\ell(M_R)$. By Lemma 1, $K = (1-e)M$ is a perpetual submodule of $M$ not containing $x$. Now, by Zorn's lemma, there exists a perpetual submodule $N$ of $M$ which is maximal with respect to the property that $x \notin N$. Choose a perpetual submodule $N'$ of $M$ such that $M = N \oplus N'$ (by Lemma 4). Then, we can finish our proof by showing that $N'$ is Baer simple. Indeed, if $N''$ is a nonzero perpetual submodule of $N'$, then by Proposition 1, $N'$ is Baer semisimple and $N' = N'' \oplus N'''$, where $N'''$ is a submodule of $N'$. Thus $N \oplus N''$ is a direct summand of $M$. Again by Lemma 4, $N \oplus N'' = fM$ for some idempotent $f \in End_\ell(M_R)$ and by Lemma 2, $N \oplus N''$ is a perpetual submodule of $M$ containing $x$ (by the maximality of $N$) and $N \oplus N'' = M$, which implies that $N'' = N'$, as desired. □

**Proposition 2.** *A Baer semisimple module is the direct sum of a family of Baer simple submodules.*

*Proof.* Assume that $M$ is a Baer semisimple module. Denote by $A$ the set of Baer simple submodules of $M$. Then, we consider the subset $B \subset A$ with the following conditions:

- $\sum_{J \in B} J$ is a direct sum.
- $\sum_{J \in B} J$ is a perpetual submodule of $M$.

By Lemma 5, $A \neq \emptyset$. Now, by Zorn's lemma, we can consider the family of all the above $B$'s with respect to the set inclusion. Thus we can pick a $B$ to be the maximal element. For such a $B$, we can construct a perpetual submodule $M_1 := \oplus_{J \in B} J$. Now, by our hypothesis, $M = M_1 \oplus M_2$, where $M_2$ is a submodule of $M$. By Lemma 4, $M_2$ is a perpetual submodule of $M$ and by Proposition 1, $M_2$ is a Baer semisimple, and hence by Lemma 5 again, $M_2 = K \oplus Q$, where $K$ is a Baer simple submodule of $M_2$ and $Q$ a submodule of $M_2$. Thus $M_1 \oplus K$ is a direct summand of $M$ and of course, $M_1 \oplus K$ is a perpetual submodule of $M$, by Lemma 4. On the other hand, by using the proof of Proposition 1(ii), we can show that $K$ is Baer simple in $M$. This contradicts the maximality of $B$. Therefore $M = M_1 = \oplus_{J \in B} J$. □

**Theorem 3.** *Let $M$ be a $R$-module and $P$ the set of submodules of the form $eM$, with $e \in E(End_\ell(M))$. Order the set $P$ by set inclusion. Then the following statements are equivalent:*
*$(i)$ $M$ is Baer semisimple.*
*$(ii)$ The following two conditions hold:*
*$(a)$ For any $x \in M$, $SM^*(x)$ is a direct summand of $M$.*
*$(b)$ $P$ forms a complete lattice.*

*Proof.* $(i) \Rightarrow (ii)$ Since condition $(a)$ holds trivially, we need only to show that condition $(b)$ holds. Let $T \subseteq P$. Since every element of $P$ is a perpetual submodule of $M$, $\bigcap_{J \in T} J$ is a perpetual submodule of $M$. By our hypothesis, $\bigcap_{J \in T} J$ is a direct summand of $M$. By Lemma 4, we have $\bigcap_{J \in T} J \in P$. Consider the smallest perpetual submodule $K$ of $M$ containing $J$ with $J \in T$. It is clear that $K$ is a direct summand of $M$, and whence $K \in P$. Thus $K$ can be viewed as $sup_{J \in T}(J)$. Thus, $P$ indeed forms a complete lattice.

$(ii) \Rightarrow (i)$ Assume that $(ii)$ holds. Let $I$ be a perpetual submodule of $M$. Consider $I = \bigcup_{x \in I} SM^*(x)$. Then by condition $(a)$, $I = \bigcup_{x \in I} e_x M$, where $e_x$ is the idempotent of $End_\ell(M)$ such that $SM^*(x) = e_x M$, for any $x \in I$. By condition $(b)$, $I = eM$ for some $e \in E(End_\ell(M))$, that is, $I$ is a direct summand of $M$. Consequently, $M$ is a Baer semisimple module. □

Recall that [14, Lemma 2.3] a ring $R$ is Baer if and only if $R$ is lpp and under set inclusion, the set of all idempotent-generated principal right ideals forms a complete lattice. By using Lemma 1 and Theorem 3, we deduce the following characterization theorem of Baer rings.

**Theorem 4.** *A ring $R$ is a Baer ring if and only if $R$ itself, regarded as a regular $R$-module, is a Baer semisimple module.*

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