Stochastic Equilibrium in Congested Transportation Networks

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Outline

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$$p^{(i)} = \frac{e^{-c^{(i)}/\mu}}{\sum\limits_{j=1}^{k} e^{-c^{(j)}/\mu}}, \quad i = 1, \ldots, k.$$

(*Logit model.*)

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2. $\lim_{\mu\to 0} (-\mu\psi(c/\mu)) = \min_{1\le i\le k} c^{(i)}$.

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Since $\psi(t)$ is convex, we can go in both directions:

$$t \Rightarrow f(t), \quad f \Rightarrow t(f).$$

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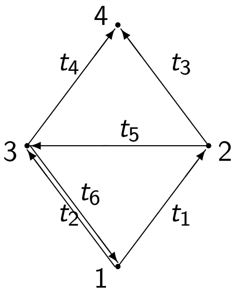

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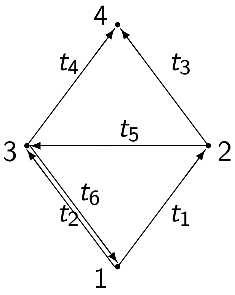

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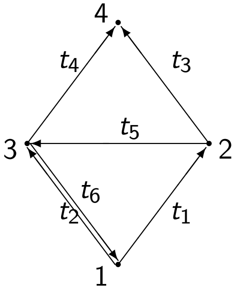

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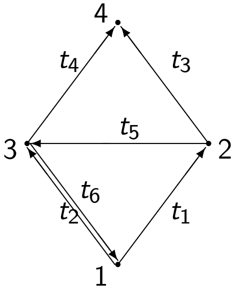

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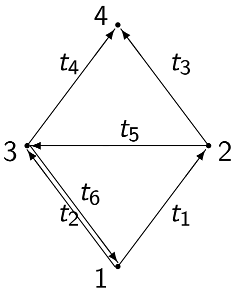

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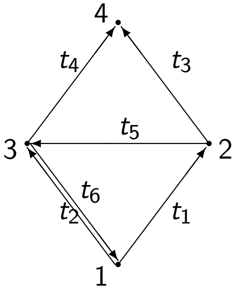

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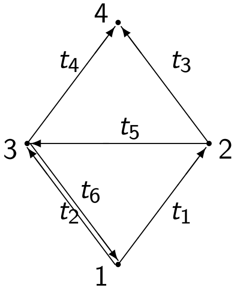

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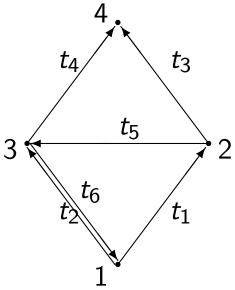

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is the *expected flow* on the arcs (with respect to given time t).

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$$\psi_{\mathcal{R}}(t) = \ln \left(e^{\psi_{\mathcal{R}_1}(t)} + e^{\psi_{\mathcal{R}_2}(t)} \right).$$

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Can we compute characteristic functions of some reasonably big sets of routes?

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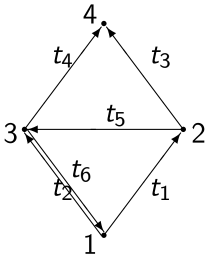

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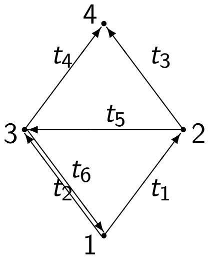

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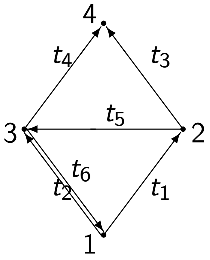

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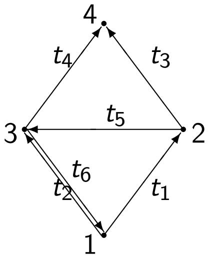

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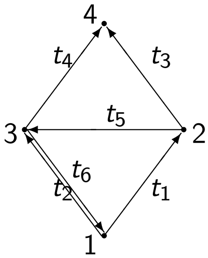

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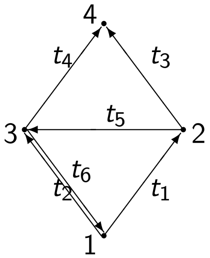

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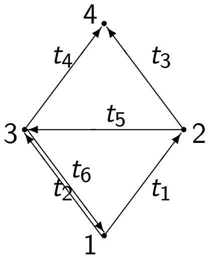

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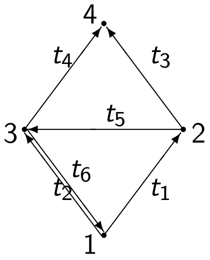
$E^2(t)$

Example



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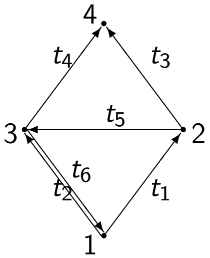

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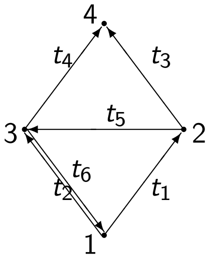

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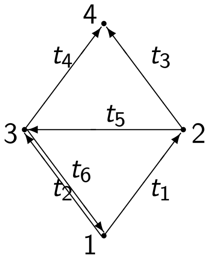

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Theorem. Each element of matrix function $E^p(t)$, $p \geq 1$, is the characteristic function for corresponding set of routs $\mathcal{R}^p_{i,j}$.

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This is the expected flow $j \to i$ passing through α.

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For the source j and length of the route p define the functions:

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Complexity: $O(m)$ operations per iteration,

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Interesting sets; $\widehat{\mathcal{R}}^L_{p,k}$, $\widetilde{\mathcal{R}}_{p,k}$.

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Expected OD-flows: can be computed by the gradients.

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- Network design (improve the structure, developments, long-run, etc.).

Thank you for your attention!