Pricing FX Options in the Heston/CIR Jump-Diffusion Model International Conference Advanced Finance and Stochastics

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Abstract

- The goal is to derive a semi-analytical solution for the price of a Foreign exchange option under the Heston [21] stochastic volatility model for the exchange rate which includes jumps in both the spot exchange rate and the volatility dynamics, combined with the Cox et al. [14] dynamics for the domestic and foreign stochastic interest rates. The instantaneous volatility is correlated with the dynamics of the exchange rate return, whereas the domestic and foreign short-term rates are assumed to be independent of the dynamics of the exchange rate.
- The main result is an analytic formula for the price of a European call on the exchange rate.
- It is derived using martingale methods in arbitrage pricing of contingent claims and Fourier inversion. techniques.

The Model

The framework under which we will price FX options. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an underlying probability space. We postulate that the dynamics of the exchange rate $Q = (Q_t)_{t \in [0,T]}$,

$$\begin{cases} dQ_t = \left(r_t - \widehat{r}_t - \lambda \mu_Q\right) Q_t \, dt + Q_t \sqrt{v_t} \, dW_t^Q + Q_{t-} \, dZ_t^Q, \\ dv_t = \left(\theta - \kappa v_t\right) dt + \sigma_v \sqrt{v_t} \, dW_t^v + dZ_t^v, \\ dr_t = \left(a_d - b_d r_t\right) dt + \sigma_d \sqrt{r_t} \, dW_t^d, \\ d\widehat{r}_t = \left(a_f - b_f \widehat{r}_t\right) dt + \sigma_f \sqrt{\widehat{r}_t} \, dW_t^f, \end{cases} \tag{1}$$

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\end{cases}
\tag{1}
$$

- (A.1) $W^Q = (W_t^Q)_{t\in[0,T]}$, $W^v = (W_t^v)_{t\in[0,T]}$ are correlated Brownian motions with a constant correlation coefficient, so that the quadratic covariation between W^Q and W^v satisfies $d[W^Q, W^v]_t = \rho\, dt$ for some constant $\rho \in [-1, 1]$,
- (A.2) $W^d = (W_t^d)_{t\in[0,T]}$ and $W^f = (W_t^f)_{t\in[0,T]}$ are independent Brownian motions and they are also independent of the Brownian motions W^Q and W^v (hence the processes Q, r and $\widehat{r}$ are independent),
- (A.3) $Z_t^Q = \sum_{k=1}^{N_t^Q} J_k^Q$ is the compound Poisson process; specifically, the Poisson process N^Q has the intensity $\lambda_Q > 0$ and the random variables $\ln(1 + J_k^Q)$, $k = 1, 2, \dots$ have the probability distribution $N(\ln[1 + \mu_Q] - \frac{1}{2}\sigma_Q^2, \sigma_Q^2)$; hence the jump sizes J_k^Q, $k = 1, 2, \dots$ are lognormally distributed on $(-1, \infty)$ with mean $\mu_Q > -1$,

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- (A.3) $Z^Q_t = \sum_{k=1}^{N^Q_t} J^Q_k$ is the compound Poisson process; specifically, the Poisson process N^Q has the intensity $\lambda_Q > 0$ and the random variables $\ln(1 + J^Q_k)$, $k = 1, 2, \dots$ have the probability distribution $N(\ln[1 + \mu_Q] - \frac{1}{2}\sigma^2_Q, \sigma^2_Q)$; hence the jump sizes J^Q_k, $k = 1, 2, \dots$ are lognormally distributed on $(-1, \infty)$ with mean $\mu_Q > -1$,

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- (A.3) $Z_t^Q = \sum_{k=1}^{N_t^Q} J_k^Q$ is the compound Poisson process; specifically, the Poisson process N^Q has the intensity $\lambda_Q > 0$ and the random variables $\ln(1 + J_k^Q)$, $k = 1, 2, \dots$ have the probability distribution $N(\ln[1 + \mu_Q] - \frac{1}{2}\sigma_Q^2, \sigma_Q^2)$; hence the jump sizes J_k^Q, $k = 1, 2, \dots$ are lognormally distributed on $(-1, \infty)$ with mean $\mu_Q > -1$,

Assumptions

- (A.4) $Z_t^v = \sum_{k=1}^{N_t^v} J_k^v$ is the compound Poisson process; specifically, the Poisson process N^v has the intensity $\lambda_v > 0$ and the jump sizes J_k^v are exponentially distributed with mean μ_v,
- (A.5) the Poisson processes N^Q, N^v and sequences J_k^Q and J_k^v are mutually independent, as well as independent of the Brownian motions W^Q, W^v, W^d, W^f,
- (A.6)the model's parameters satisfy the stability conditions: $2\theta > \sigma_v^2 > 0$, $2a_d > \sigma_d^2 > 0$ and $2a_f > \sigma_f^2 > 0$.

Comments

Note that we postulate here that the:

- Volatility process v, the domestic short-term interest rate, denoted as r, and its foreign counterpart, denoted as $\widehat{r}$, are independent stochastic processes.
- This assumption is indeed crucial(for details see [4]).

Bond Prices Under CIR Dynamics

Lemma

The discount bonds prices maturing at time $T > t$ are given by:

$$B_d(t,T) = \exp\left(m_d(t,T) - n_d(t,T)r_t\right),$$
$$B_f(t,T) = \exp\left(m_f(t,T) - n_f(t,T)\widehat{r}_t\right),$$

where, for $i \in \{d, f\}$,

$$m_i(t,T) = \frac{2a_i}{\sigma_i^2}\log\left[\frac{\gamma_i\, e^{\frac{1}{2}b_i(T-t)}}{\gamma_i \cosh(\gamma_i(T-t)) + \frac{1}{2}\, b_i \sinh(\gamma_i(T-t))}\right],$$
$$n_i(t,T) = \frac{\sinh(\gamma_i(T-t))}{\gamma_i \cosh(\gamma_i(T-t)) + \frac{1}{2}\, b_i \sinh(\gamma_i(T-t))},$$

and

$$\gamma_i = \frac{1}{2}\sqrt{b_i^2 + 2\sigma_i^2}.$$

Foreign Exchange Call Option

The arbitrage price $C_t(T,K)$ of the FX call option for every time $t \in [0, T]$, is given by the conditional expectation with respect to the σ-field $\mathcal{F}_t$ of the option's payoff at expiration discounted by the domestic money market account,

$$\begin{aligned} C_t(T,K) &= \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)C_T(T,K)\right\} \\ &= \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)(Q_T-K)^+\right\} \end{aligned}$$

or, equivalently,

$$\begin{aligned} C_t(T,K) &= \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)Q_T\mathbb{1}_{\{Q_T>K\}}\right\} \\ &\quad -K\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)\mathbb{1}_{\{Q_T>K\}}\right\}. \end{aligned}$$

Option Price In Terms Forward Exchange Rate

Lemma

The forward exchange rate $F(t,T)$ at time t for settlement date T equals

$$F(t,T) = \frac{B_f(t,T)}{B_d(t,T)}\, Q_t. \tag{2}$$

$Q_T = F(T,T)$, the option's payoff at expiry

$$C_T(T,K) = F(T,T)\mathbb{1}_{\{F(T,T)>K\}} - K\mathbb{1}_{\{F(T,T)>K\}}.$$

Hence, the option's value at time $t \in [0,T]$

$$\begin{aligned} C_t(T,K) \;&=\; \mathbb{E}_t^{\mathbb{P}}\left\{ \exp\left(-\int_t^T r_u\,du \right) F(T,T)\mathbb{1}_{\{F(T,T)>K\}} \right\} \\ &\quad - K\mathbb{E}_t^{\mathbb{P}}\left\{ \exp\left(-\int_t^T r_u\,du \right) \mathbb{1}_{\{F(T,T)>K\}} \right\}. \end{aligned}$$

Notation $x_t = \ln F(t,T)$ for every $t \in [0,T]$.

Main Result:Pricing Formula for the FX Call Option

Theorem

Assume that the foreign exchange model is given by SDEs (1) under Assumptions (A.1)–(A.6). Then the price of a European currency option is given by, for every $t \in [0, T]$,

$$\begin{aligned} C_t(T,K) &= Q_t B_f(t,T) P_1\big(t, Q_t, v_t, r_t, \widehat{r}_t, K\big) \\ &\quad - K B_d(t,T) P_2\big(t, Q_t, v_t, r_t, \widehat{r}_t, K\big). \end{aligned}$$

The bond prices $B_d(t,T)$ *and* $B_f(t,T)$ *are given in lemma 1.*

Functions P_1, P_2

Functions P_1 and P_2 are given by, for $j = 1, 2$,

$$P_j(t, Q_t, v_t, r_t, \widehat{r}_t, K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left(f_j(\phi) \frac{\exp(-i\phi \ln K)}{i\phi} \right) d\phi, \tag{3}$$

where the $\mathcal{F}_t$-conditional characteristic function $f_j(\phi) = f_j(\phi, t, Q_t, v_t, r_t, \widehat{r}_t)$, $j = 1, 2$ of the random variable $x_T = \ln(Q_T)$ under the probability measure $\widehat{\mathbb{P}}_T$ (cf. Definition 10) and $\mathbb{P}_T$ (cf. Definition 7), respectively, are given by

Characteristic Function- f_1

$$
\begin{aligned}
f_1(\phi) \;=\;& c_t \exp\left[\lambda_Q \tau (1+\mu_Q)\left((1+\mu_Q)^{i\phi} e^{-\frac{1}{2}(1+i\phi)\sigma_Q^2} - 1\right)\right] \\
& \times \exp\left[-\left(i\phi\lambda_Q\mu_Q\tau - \lambda_v\tau\left(\frac{\sigma_v - \rho(1+i\phi)\mu_v}{\sigma_v + \rho(1+i\phi)\mu_v}\right) + \frac{(1+i\phi)}{\sigma_v} \right.\right. \\
& \times \exp\left[-i\phi\left(n_d(t,T)r_t + \int_t^T a_d n_d(u,T)\,du\right)\right] \\
& \times \exp\left[(1+i\phi)\left(n_f(t,T)\widehat{r}_t + \int_t^T a_f n_f(u,T)\,du\right)\right] \\
& \times \exp\left[-G_1(\tau,s_1,s_2)v_t - G_2(\tau,s_3,s_4)r_t - G_3(\tau,s_5,s_6)\widehat{r}_t\right] \\
& \times \exp\left[-\theta H_1(\tau,s_1,s_2) - a_d H_2(\tau,s_3,s_4) - a_f H_3(\tau,s_5,s_6)\right]
\end{aligned}
$$

$$
\begin{aligned}
f_2(\phi) \;=\;& c_t \exp\left[\lambda_Q \tau (1+\mu_Q)\left((1+\mu_Q)^{i\phi} e^{-\frac{1}{2}(1+i\phi)\sigma_Q^2} - 1\right)\right] \\
&\times \exp\left[-\left(i\phi\lambda_Q\mu_Q\tau - \lambda_v\tau\left(\frac{\sigma_v - i\phi\rho\mu_v}{\sigma_v + i\phi\rho\mu_v}\right) + \frac{i\phi\rho}{\sigma_v}(v_t + \theta\tau)\right.\right. \\
&\times \exp\left[(1-i\phi)\left(n_d(t,T)r_t + \int_t^T a_d n_d(u,T)\,du\right)\right] \\
&\times \exp\left[i\phi\left(n_f(t,T)\widehat{r}_t + \int_t^T a_f n_f(u,T)\,du\right)\right] \qquad (\\
&\times \exp\left[-G_1(\tau,q_1,q_2)v_t - G_2(\tau,q_3,q_4)r_t - G_3(\tau,q_5,q_6)\right] \\
&\times \exp\left[-\theta H_1(\tau,q_1,q_2) - a_d H_2(\tau,q_3,q_4) - a_f H_3(\tau,q_5,q_6)\right]
\end{aligned}
$$

where the functions $G_1, G_2, G_3, H_1, H_2, H_3$ are given in Lemma 5 and c_t equals

The Constants: C_t, s_1, s_2, s_3, s_4, s_5, s_6

$$c_t = \exp\left(i\phi x_t\right) = \exp\left(i\phi \ln F(t,T)\right).$$

$$\begin{aligned}
s_1 &= -\frac{(1+i\phi)\rho}{\sigma_v}, \\
s_2 &= -\frac{(1+i\phi)^2(1-\rho^2)}{2} - \frac{(1+i\phi)\rho\kappa}{\sigma_v} + \frac{1+i\phi}{2}, \\
s_3 &= 0, \\
s_4 &= -i\phi, \\
s_5 &= 0, \\
s_6 &= 1+i\phi,
\end{aligned} \tag{6}$$

Constants: $q_1, q_2, q_3, q_4, q_5, q_6$

$$
\begin{aligned}
q_1 &= -\frac{i\phi\rho}{\sigma_v}, \\
q_2 &= -\frac{i\phi\rho\kappa}{\sigma_v} - \frac{(i\phi)^2(1-\rho^2)}{2} + \frac{i\phi}{2}, \\
q_3 &= 0, \\
q_4 &= 1 - i\phi, \\
q_5 &= 0, \\
q_6 &= i\phi.
\end{aligned}
\tag{7}
$$

TWO LEMMAS

Lemma

(i) Under Assumptions (A.3) and (A.5), the following equalities are valid

$$\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(i\phi J^Q(t,T)\right)\right\} = \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(i\phi \sum_{k=N_t^Q+1}^{N_T^Q} \ln(1+J_k^Q)\right)\right\}$$

$$= \exp\left[\lambda_Q \tau \int_{-\infty}^{+\infty} (e^{i\phi z} - 1)\,\nu_1(dz)\right]$$

$$= \exp\left[\lambda_Q \tau (1+\mu_Q)\left(\left((1+\mu_Q)^{i\phi} e^{-\frac{1}{2}\sigma_Q^2(1+i\phi)} - \right.\right.\right.$$

(ii) Under Assumptions (A.4) and (A.5), the following equalities are valid for $c = a + bi$ with $a \le 0$

$$\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(c\,(Z_T^v - Z_t^v)\right)\right\} = \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(c \sum^{N_T^v} J_k^v\right)\right\}$$

Lemma

The next result is an extension of Lemma 6.1 in Ahlip and Rutkowski [2] (see also Duffie et al. [15]) in which the model without the jump component in the dynamics of v was examined.

Lemma

Let the dynamics of processes v, r and $\widehat{r}$ be given by SDEs (1) with independent Brownian motions W^v, W^d and W^f. For any complex numbers $\mu, \lambda, \widetilde{\mu}, \widetilde{\lambda}, \widehat{\mu}, \widehat{\lambda}$, we set

$$F(\tau, v_t, r_t, \widehat{r}_t) = \mathbb{E}_t^{\mathbb{P}}\Big\{ \exp\Big(-\lambda v_T - \mu \int_t^T v_u\,du - \widetilde{\lambda} r_T - \widetilde{\mu} \int_t^T r_u\,du - \widehat{\lambda}$$

Then

$$\begin{aligned} F(\tau, v_t, r_t, \widehat{r}_t) &= \exp\big[-G_1(\tau,\lambda,\mu)v_t - G_2(\tau,\widetilde{\lambda},\widetilde{\mu})r_t \\ &\quad - G_3(\tau,\widehat{\lambda},\widehat{\mu})\widehat{r}_t - \theta H_1(\tau,\lambda,\mu) - a_d H_2(\tau,\widetilde{\lambda},\widetilde{\mu}) \end{aligned}$$

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$$= \exp\left[\lambda_Q \tau \int_{-\infty}^{+\infty} (e^{i\phi z} - 1)\,\nu_1(dz)\right]$$

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$$\mathbb{E}_t^{\mathbb{P}}\Big\{ \exp\big(c\,(Z_T^v - Z_t^v)\big)\Big\} = \mathbb{E}_t^{\mathbb{P}}\Bigg\{ \exp\Bigg(c \sum^{N_T^v} J_k^v \Bigg)\Bigg\}$$

Functions-G_1, G_2, G_3

$$G_1(\tau,\lambda,\mu) = \frac{\lambda[(\gamma+\kappa)+e^{\gamma\tau}(\gamma-\kappa)] + 2\mu(1-e^{\gamma t})}{\sigma_v^2\,\lambda\,(e^{\gamma\tau}-1)+\gamma-\kappa+e^{\gamma\tau}\,(\gamma+\kappa)},$$

$$G_2(\tau,\widetilde{\lambda},\widetilde{\mu}) = \frac{\widetilde{\lambda}[(\widetilde{\gamma}+b_d)+e^{\widetilde{\gamma}\tau}(\widetilde{\gamma}-b_d)] + 2\widetilde{\mu}(1-e^{\widetilde{\gamma}t})}{\sigma_d^2\,\widetilde{\lambda}\,(e^{\widetilde{\gamma}\tau}-1)+\widetilde{\gamma}-b_d+e^{\widetilde{\gamma}\tau}\,(\widetilde{\gamma}+b_d)},$$

$$G_3(\tau,\widehat{\lambda},\widehat{\mu}) = \frac{\widehat{\lambda}[(\widehat{\gamma}+b_f)+e^{\widehat{\gamma}\tau}(\widehat{\gamma}-b_f)] + 2\widehat{\mu}(1-e^{\widehat{\gamma}t})}{\sigma_f^2\,\widehat{\lambda}\,(e^{\widehat{\gamma}\tau}-1)+\widehat{\gamma}-b_f+e^{\widehat{\gamma}\tau}\,(\widehat{\gamma}+b_f)},$$

Functions-H_1, H_2, H_3

$$H_1(\tau,\lambda,\mu) = \int_0^{\tau} \left(G_1(t,\lambda,\mu) + \frac{\lambda_v}{\theta\mu_v} \left(\frac{1-\mu_v G_1(t,\lambda,\mu)}{1+\mu_v G_1(t,\lambda,\mu)} \right) \right) dt,$$

$$H_2(\tau,\widetilde{\lambda},\widetilde{\mu}) = -\frac{2}{\sigma_d^2} \ln \left(\frac{2\widetilde{\gamma} e^{\frac{(\widetilde{\gamma}+b_d)\tau}{2}}}{\sigma_d^2 \widetilde{\lambda} \left(e^{\widetilde{\gamma}\tau} - 1\right) + \widetilde{\gamma} - b_d + e^{\widetilde{\gamma}\tau} \left(\widetilde{\gamma} + b_d\right)} \right),$$

$$H_3(\tau,\widehat{\lambda},\widehat{\mu}) = -\frac{2}{\sigma_f^2} \ln \left(\frac{2\widehat{\gamma} e^{\frac{(\widehat{\gamma}+b_f)\tau}{2}}}{\sigma_f^2 \widehat{\lambda} \left(e^{\widehat{\gamma}\tau} - 1\right) + \widehat{\gamma} - b_f + e^{\widehat{\gamma}\tau} \left(\widehat{\gamma} + b_f\right)} \right),$$

Change of measure from $\mathbb{P}$ to $\mathbb{P}_T$

Definition

The probability measure $\mathbb{P}_T$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, is defined by the Radon-Nikodým derivative process $\eta = (\eta_t)_{t\in[0,T]}$, where

$$\eta_t = \frac{d\mathbb{P}_T}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(- \int_0^t \sigma_d n_d(u, T)\sqrt{r_u}\, dW_u^d - \frac{1}{2}\int_0^t \sigma_d^2 n_d^2(u, T) r_u\, du\right). \tag{8}$$

$$W_t^T = W_t^d + \int_0^t \sigma_d n_d(u, T)\sqrt{r_u}\, du. \tag{9}$$

From Girsanov theorem $W^T = (W_t^T)_{t\in[0,T]}$ is the standard Brownian motion under the domestic forward martingale measure $\mathbb{P}_T$.

FX Call and Dynamics of $F(t,T)$ under $\mathbb{P}_T$

$$\begin{aligned} C_t(T,K) &= B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\left(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\right) \\ &\quad -KB_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\left(\mathbb{1}_{\{F(T,T)>K\}}\right). \end{aligned} \tag{10}$$

Lemma

$$\begin{aligned} dF(t,T) &= F(t,T)\,\Big(\sqrt{v_t}\,dW_t^Q + \sigma_d\, n_d(t,T)\sqrt{r_t}\,dW_t^T \\ &\quad -\sigma_f n_f(t,T)\sqrt{\widehat{r}_t}\,dW_t^f\Big) \end{aligned} \tag{11}$$

Solution: Of SDE and Remarks

Lemma

$$F(T,T) = F(t,T)\exp\left(\int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2\,du\right), \quad (12)$$

$$\widetilde{\sigma}_F(t,T) = \left[\sqrt{v_t},\ \sigma_d n_d(t,T)\sqrt{r_t},\ -\sigma_f n_f(t,T)\sqrt{\widehat{r}_t}\right]$$

$$\widetilde{W}_t^T = \left[W_t^Q,\ W_t^T,\ W_t^f\right]^*$$

The process $\widetilde{W}^T$ is the three-dimensional standard Brownian motion under $\mathbb{P}_T$.

In view of Lemma 8

$$
\begin{aligned}
&B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\big(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\big)\\
&\quad = Q_t B_f(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\bigg\{\exp\bigg(\int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T\\
&\qquad\qquad -\frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2\,du\bigg)\mathbb{1}_{\{F(T,T)>K\}}\bigg\}.
\end{aligned}
$$

To deal with the first term in the right-hand side in formula (10), we introduce another auxiliary probability measure, denoted by $\widehat{\mathbb{P}}_T$.

Definition

The probability measure $\widehat{\mathbb{P}}_T$, equivalent to $\mathbb{P}_T$ on $(\Omega, \mathcal{F}_T)$, is defined by the Radon-Nikodým derivative process $\widehat{\eta} = (\widehat{\eta}_t)_{t \in [0,T]}$, where

$$\widehat{\eta}_t = \frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}_T}\Big|_{\mathcal{F}_t} = \exp\bigg(\int_0^t \widetilde{\sigma}_F(u,T) \cdot d\widetilde{W}_u^T - \frac{1}{2}\int_0^t \|\widetilde{\sigma}_F(u,T)\|^2 \, du \bigg).$$

RHS: Under $\widehat{P}_T$

Using equation (12) and the Bayes formula, we obtain

$$
\begin{aligned}
& B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\big(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\big) \\
&= Q_t B_f(t,T)\,\frac{\mathbb{E}_t^{\mathbb{P}_T}\big(\mathbb{1}_{\{F(T,T)>K\}}\widehat{\eta}_T\big)}{\mathbb{E}_t^{\mathbb{P}_T}(\widehat{\eta}_T)} \\
&= Q_t B_f(t,T)\,\mathbb{E}_t^{\widehat{\mathbb{P}}_T}\big(\mathbb{1}_{\{F(T,T)>K\}}\big).
\end{aligned}
$$

Pricing formula under $\widehat{\mathbb{P}}_T$ and $\mathbb{P}_T$

The price of the FX call option can be represented as follows

$$\begin{aligned} C_t(T,K) &= Q_t B_f(t,T)\,\widehat{\mathbb{P}}_T\big(Q_T > K \,|\, \mathcal{F}_t\big) \\ &\quad - K B_d(t,T)\,\mathbb{P}_T\big(Q_T > K \,|\, \mathcal{F}_t\big) \end{aligned}$$

or, equivalently,

$$\begin{aligned} C_t(T,K) &= Q_t B_f(t,T)\,\widehat{\mathbb{P}}_T\big(x_T > \ln K \,|\, \mathcal{F}_t\big) \\ &\quad - K B_d(t,T)\,\mathbb{P}_T\big(x_T > \ln K \,|\, \mathcal{F}_t\big). \end{aligned} \tag{13}$$

To complete the proof Theorem 3, it remains to evaluate the conditional probabilities in formula (13).

Markov Property

- The process $(Q, v, r, \widehat{r})$ has the Markov property both under the probability measures $\mathbb{P}_T$ and $\widehat{\mathbb{P}}_T$.
- In view of Lemma 1 and Lemma 2, the random variable x_T is a function of Q_T, r_T and $\widehat{r}_T$.
- $$\begin{aligned} C_t(T,K) &= Q_t B_f(t,T)\, P_1(t, Q_t, v_t, r_t, \widehat{r}_t, K) \\ &\quad - K B_d(t,T) P_2(t, Q_t, v_t, r_t, \widehat{r}_t, K), \quad (14) \end{aligned}$$
- where we denote

$$P_1(t, Q_t, v_t, r_t, \widehat{r}_t, K) = \widehat{\mathbb{P}}_T(x_T > \ln K \,|\, Q_t, v_t, r_t, \widehat{r}_t),$$
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-

$$
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$$
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P_1(t, Q_t, v_t, r_t, \widehat{r}_t, K) &= \widehat{\mathbb{P}}_T(x_T > \ln K \mid Q_t, v_t, r_t, \widehat{r}_t), \\
P_2(t, Q_t, v_t, r_t, \widehat{r}_t, K) &= \mathbb{P}_T(x_T > \ln K \mid Q_t, v_t, r_t, \widehat{r}_t).
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Current Research: Pricing with Stochastic Volatility and Jumps

Model dynamics are given the following SDEs:

$$
\begin{cases}
\frac{d\,Q(t)}{Q(t^-)} = \left(r_t - \widehat{r}_t - \lambda j_Q\right) dt + \sigma_Q \, dW_1^Q(t) + \sqrt{v_t}\, dZ_Q(t), \\
dv_t = (\theta - \kappa v_t)\, dt + \sigma_v \sqrt{v_t}\, dW_2^v(t) + dZ_v(t), \\
dr_t = \left(a_d - b_d r_t\right) dt + \sigma_d \sqrt{r_t}\, dW_t^d, \\
d\widehat{r}_t = \left(a_f - b_f \widehat{r}_t\right) dt + \sigma_f \sqrt{\widehat{r}_t}\, dW_t^f,
\end{cases}
\tag{15}
$$

Where, Z_Q, Z_v are independent compound Poisson processes with same intensity λ.

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