Semimartingale models with additional information and their applications in Mathematical Finance

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"Advanced Finance and Stochastics",June 24-28, Moscow, 2013

Modelling

- An investor carry out the trading of risky asset $S(\xi) = \mathcal{E}(X(\xi))$, depending on random parameter ξ,
- $X(\xi)$ is a semi-martingale in its natural filtration
- ξ is random factor which can be a random variable or random process.
- ξ can represent the additional economical information, for example the price of row materials, some political changes, exchange rates,price process of a correlated risky asset.

Let us give some examples

Example : Default models

In default models ξ is default time and semi-martingale $X(\xi)$ which is stochastic logarithm of the price process, has a structure:

$$X_t(\xi) = X_{t \wedge \xi}$$

- the factor ξ is supposed not to be a stopping time in natural filtration of the process X.
- Eliott, Jeanblanc, Yor (2001)
- El Karoui, Jeanblanc, Jiao (2009)

Example : Change-point model

In change-point model ξ is the change point for the characteristics of the process $X(\xi)$. More precisely stochastic logarithm of the price process is obtained by pasting together L and $\tilde{L}$ at ξ:

$$X_t(\xi) = L_t \mathbf{1}_{\{\xi > t\}} + (L_\xi + \tilde{L}_t - \tilde{L}_\xi)\mathbf{1}_{\{\xi \leq t\}}$$

- Cawston, Vostrikova (2012)The case when L, $\tilde{L}$ are independent Levy processes, which are independent of ξ.
- Gapeev, Jeanblanc (2010) Models with random dividends which is a particular case of such a model.
- Fontana, Grbac, Jeanblanc, Li (2013) No arbitrage and completeness for diffusion type change-point models.

Indifference pricing

- An investor carry out the trading of risky asset $S(\xi) = \mathcal{E}(X(\xi))$
- The same investor holds a European type option with pay-off function $G_T = g(\xi)$ which he can not trade because of lack of liquidity or legal restrictions.
- Let U be utility function satisfying usual properties: concave, strictly increasing, verifying Inada conditions.

What is indifference price for buyer and seller of the option, i.e. what is the amount of money which buyer would like to pay today (and seller would like to receive today) for the right to receive (to transmit) the option at time T?

Utility optimisation

Optimal expected utility with option:

$$V_T(x,g) = \sup_{\phi\in\Pi} E[U(x + \int_0^T \phi_s \, dS_s(\xi) + g(\xi))]$$

- x is initial capital
- π class of self-financing admissible strategies

Indifference price for buyer p_T^b is a solution of

$$V_T(x - p_T^b, g) = V_T(x, 0)$$

Indifference price for seller p_T^s is a solution of

$$V_T(x + p_T^s, -g) = V_T(x, 0)$$

Level of information

Level of information about ξ change the class of self-financing admissible strategies which we use for maximisation.

- For non-informed agents, the class self-financing admissible strategies Π related with natural filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ generated by risky asset $S(\xi)$.
- for partially informed agents the class of self-financing admissible strategies will be related with progressively enlarged filtration with the process corresponding to ξ.
- For perfectly informed agents the class of self-financing admissible strategies will be related with initially enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$

$$\mathcal{G}_t = \cap_{s>t}(\mathcal{F}_s \otimes \sigma(\xi))$$

Some remarks

- Often it is sufficient to consider the case of initial enlargement since for $t \in [0, T]$
$$\mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t \subseteq \mathcal{G}_t$$
and
$$\tilde{\mathcal{F}}_T = \mathcal{G}_T$$
- Enlargement of filtration increase the set of martingale measures to consider. How to choose the "best"?
- Manipulation of semi-martingales depending on a parameter give a number of measurability problems mentioned in Stricker, Yor (1978)
- For initial enlargement Jacod's lemma (1980) is useful.

Main Assumptions

We denote

- P is the law of $X(\xi)$
- P^u is the regular conditional law of $X(\xi)$ given $\xi = u$

ASSUMPTION 1 For $t \in]0, T]$,

$$\mathcal{L}(\xi|\mathcal{F}_t) \ll \mathcal{L}(\xi)$$

ASSUMPION 2 For all $u \in \Xi$

$$P^u \overset{loc}{\ll} P$$

Reduction to conditional utility maximisation problem

We define also conditional maximal utility

$$V^u(x,g) = \sup_{\varphi \in \Pi^u(\mathbf{F})} E_{P^u}\left[U\left(x + \int_0^T \varphi_s(u) dS_s(u) + g(u) \right) \right]$$

THEOREM 1 *Let us suppose that Assumptions 1 and 2 holds. Then we can reduce classical utility maximisation problem to the corresponding conditional utility maximisation problem in the sense that*

$$V(x,g) = \int_{\Xi} V^u(x,g) d\alpha(u)$$

where α is the law of ξ.

Dual approach for conditional maximisation problem

Let us denote by f the convex conjugate of U obtained by Frenchel-Legendre transform of U:

$$f(y) = \sup_{x>0} \left(U(x) - yx \right).$$

We say that $Q^{u,*}$ is f-divergence minimal equivalent martingale measure for conditional problem if under $Q^{u,*}$ the process $S(\xi)$ given $\xi = u$ is a martingale and

$$E_{P^u}\left[f\left(\frac{dQ^{u,*}_T}{dP^u_T}\right)\right] = \inf_{Q^u} E_{P^u}\left[f\left(\frac{dQ^u_T}{dP^u_T}\right)\right]$$

Dual approach for conditional maximisation problem

THEOREM 2 *Suppose that there exists an equivalent f-divergence minimal martingale measure $Q^{u,*}$ for conditional problem and $x > \underline{x}$ and $g > 0$, then*

$$V^u(x,g) = E_{P^u}\left[U\left(-f'\left(\lambda_g(u)\frac{dQ_T^{u,*}}{dP_T^u}\right)\right)\right]$$

and $\lambda_g(u)$ is a unique solution of the equation

$$E_{Q^{u,*}}\left[-f'\left(\lambda_g(u)\frac{dQ_T^{u,*}}{dP_T^u}\right)\right] = x + g(u)$$

We introduce three important quantities related with P^u_T and $Q^{u,*}_T$ namely the entropy of P^u with respect to $Q^{u,*}_T$,

$$\mathbf{I}(P^u_T|Q^{u,*}_T) = -E_{P^u}\left[\ln(\frac{dQ^{u,*}_T}{dP^u_T})\right],$$

the entropy of $Q^{u,*}_T$ with respect to P^u_T,

$$\mathbf{I}(Q^{u,*}_T|P^u_T) = E_{P^u}\left[\frac{dQ^{u,*}_T}{dP^u_T}\ln(\frac{dQ^{u,*}_T}{dP^u_T})\right],$$

and Hellinger type integrals

$$\mathbf{H}^{(q),*}_T(u) = E_{P^u}\left[(\frac{dQ^{u,*}_T}{dP^u_T})^q\right],$$

where $q = \frac{p}{p-1}$ and $p < 1$.

Final result for maximisation for HARA utilities

THEOREM 3 *Under the Assumptions 1 and 2 we have the following expressions for* $V_T(x,g)$:

- *If* $U(x) = \ln x$ *then*

$$V_T(x,g) = \int_{\Xi} [\ln(x+g(u)) + \mathbf{I}(P^u_T | Q^{u,*}_T)] d\alpha(u)$$

- *If* $U(x) = \frac{x^p}{p}$ *with* $p < 1, p \neq 0$ *then*

$$V_T(x,g) = \frac{1}{p} \int_{\Xi} (x+g(u))^p \left(\mathbf{H}^{(q),*}_T(u) \right)^{1-p} d\alpha(u)$$

- *If* $U(x) = 1 - e^{-\gamma x}$ *with* $\gamma > 0$ *then*

$$V_T(x,g) = 1 - \int_{\Xi} \exp\{-[\gamma(x+g(u)) + \mathbf{I}(Q^{u,*}_T | P^u_T)]\} d\alpha(u)$$

Information process for logarithmic utility

For logarithmic utility, we introduce the corresponding Information process:

$$\mathcal{I}_t^*(u) = \frac{1}{2}\int_0^t (\beta_s^{u,*})^2 dC_s - \int_0^t \int_{\mathbb{R}} (\ln(Y_s^{u,*}(x)) - Y_s^{u,*}(x) + 1)\, \nu^u(ds, dx).$$

where $\beta^{u,*}$ and $Y^{u,*}(x)$ are Girsanov parameters for the changing of measure from P^u into $Q^{u,*}$.

PROPOSITION 1 *We suppose that* $E_{P^u}|\ln \frac{dQ_T^{u,*}}{dP_T^u}| < \infty$ *and that* $X(u)$ *has no predictable jumps. Then*

$$\mathbf{I}(P_T^u \,|\, Q_T^{u,*}) = E_{P^u}\mathcal{I}_T^*(u).$$

Information process for exponential utility

In the case of exponential utility, we introduce the corresponding Kullback-Leiber process with

$$I_t^*(u) = \frac{1}{2}\int_0^t (\beta_s^{u,*})^2 dC_s$$

$$+ \int_0^t \int_{\mathbb{R}} [Y_s^{u,*}(x) \ln(Y_s^{u,*}(x)) - Y_s^{u,*}(x) + 1]\, \nu^u(ds, dx).$$

PROPOSITION 2 *We suppose that* $E_{P^u}|\frac{dQ_T^{u,*}}{dP_T^u} \ln \frac{dQ_T^{u,*}}{dP_T^u}| < \infty$ *and that* $X(u)$ *has no predictable jumps. Then,*

$$\mathbf{I}(\,Q_T^{u,*} \mid P_T^u\,) = E_{Q^{u,*}}\, I_T^*(u)$$

For the case of power utility we consider the corresponding Hellinger type process:

$$h_t^{(q),*}(u) = \frac{1}{2}q(q-1)\int_0^t (\beta_s^{u,*})^2 dC_s + \int_0^t \int_{\mathbb{R}} \left[(Y_s^{u,*}(x))^q - q(Y_s^{u,*}(x) - 1) - 1\right] \nu^u(ds, dx),$$

PROPOSITION 3 *Suppose that* $\mathbf{H}_T^{(q),*}(u) < \infty$ *and that* $X(u)$ *has no predictable jumps. Then,*

$$\mathbf{H}_T^{(q),*}(u) = E_{P^u}\left[\mathcal{E}\left(h^{(q),*}\right)_T\right].$$

Indifference price for logarithmic utility

PROPOSITION 4 *In the case of logarithmic utility the buyer's and seller's indifference price satisfy:*

$$\int_{\Xi} \ln\left[1 - \frac{p_T^b}{x} + \frac{g(u)}{x}\right] d\alpha(u) = 0$$

and

$$\int_{\Xi} \ln\left[1 + \frac{p_T^s}{x} - \frac{g(u)}{x}\right] d\alpha(u) = 0.$$

If $g(\xi) \in]0, x[$ *(*α*-a.s.) and* $\ln(g(\xi))$, $\ln(x - g(\xi))$ *are integrable functions then the solutions of indifference price equations exist, they are unique and* $p_T^b, p_T^s \in [0, x]$.

PROPOSITION 5 *In the case of the power utility, the buyer's and seller's indifference prices are defined respectively from the equations:*

$$\int_{\Xi} [(1 - \frac{p_T^b}{x} + \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u)\right)^{1-p} d\alpha(u) = 0 \qquad (1)$$

and

$$\int_{\Xi} [(1 + \frac{p_T^s}{x} - \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u)\right)^{1-p} d\alpha(u) = 0 \qquad (2)$$

Moreover, under $g(\xi) \in]0, x[$ (α-a.s.) and some integrability conditions, the above equations have unique solutions.

Indifference price for exponential utility

PROPOSITION 6 *In the case of the exponential utility the buyer's and seller's indifference prices verify:*

$$p_T^b = \frac{1}{\gamma} \ln \left[\frac{\int_\Xi \exp\left\{ - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)}{\int_\Xi \exp\left\{ -\gamma g(u) - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)} \right] \tag{3}$$

and

$$p_T^s = -\frac{1}{\gamma} \ln \left[\frac{\int_\Xi \exp\left\{ - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)}{\int_\Xi \exp\left\{ \gamma g(u) - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)} \right] \tag{4}$$

Risk measures

The application $\rho : \mathcal{F}_T \to \mathbb{R}^+$ is convex risk measure if for all contingent claims $C_T^{(1)}, C_T^{(2)} \in \mathcal{F}_T$ and all $0 < \gamma < 1$ we have:

1. convexity of ρ with respect to the claims:

$$\rho(\gamma\, C_T^{(1)} + (1-\gamma)\, C_T^{(2)}) \leq \gamma\rho(C_T^{(1)}) + (1-\gamma)\rho(C_T^{(2)})$$

2. it is increasing function with respect to the claim:

$$\text{for } C_T^{(1)} \leq C_T^{(2)}, \text{ we have } \rho(C_T^{(1)}) \leq \rho(C_T^{(2)})$$

3. it is invariant with respect to the translation: for $m > 0$

$$\rho(C_T^{(1)} + m) = \rho(C_T^{(1)}) + m$$

PROPOSITION 7 *For HARA utilities the indifference prices for sellers* $p_T^s(g)$ *and* $(-p_T^b)$ *for buyers are risk measures.*

How it works: BS models

- two risky assets

$$S_t^{(1)} = \exp\{(\mu_1 - \frac{\sigma_1^2}{2})t + \sigma_1 W_t^{(1)}\}$$

$$S_t^{(2)} = \exp\{(\mu_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_t^{(2)}\}$$

with $(W^{(1)}, W^{(2)})$ bi-dimensional standard Brownian motions with correlation ρ, $|\rho| < 1$ on $[0, T]$.

- What is ξ?

$$\xi = W_{T'}^{(2)}$$

- What is $X(\xi)$?

$$X_t(\xi) = \mu_1 t + \sigma_1 W_t^{(1)}$$

Conditional law of X : Assumption 2

- The conditional law of X given $\xi = u$ coincide with the law of

$$X_t(u) = \mu_1 t + \sigma_1 \rho V_t(u) + \sigma_1 \sqrt{1-\rho^2}\gamma_t$$

where $V(u)$ is a Brownian bridge starting from 0 at $t=0$ and ending in u at $t = T'$ which is independent from Brownian motion γ.

- As known,

$$V_t(u) = \int_0^T \frac{u - V_s(u)}{T' - s} ds + \eta_t$$

where η is standard Brownian motion independent from γ.

- Since $\hat{\gamma} = \rho\eta + \sqrt{1-\rho^2}\gamma$ is again standard Brownian motion, we get:

$$X_t(u) = \mu_1 t + \sigma_1 \rho \int_0^t \frac{u - V_s(u)}{T' - s} ds + \sigma_1 \hat{\gamma}_t$$

- Hence, $P_t^u \ll P_t$ for all $u \in \mathbb{R}$ and $t \in [0, T]$.

Conditional law of ξ : Assumption 1

- We recall that $\xi = W^{(2)}_{T'}$ and $\mathcal{F}_t = \sigma(W^{(1)}_s, s \leq \ t)$.
- By Markov property we get: for $A \in \mathcal{B}(\mathbb{R})$

$$P(\xi \in A \,|\, \mathcal{F}_t) = P(W^{(2)}_{T'} \in A \,|\, \mathcal{F}_t) = P(W^{(2)}_{T'} \in A \,|\, W^{(1)}_t)$$

$$= P(W^{(2)}_{T'} - W^{(2)}_t + W^{(2)}_t \in A \,|\, W^{(1)}_t)$$

- Finally,

$$P(\xi \,|\, \mathcal{F}_t) = \mathcal{N}(\rho\, x,\, T' - \rho^2 t)$$

and since $T' - \rho^2 t \neq 0$ for $t \in [0, T]$, it is equivalent to the law of $W^{(2)}_{T'}$ being $\mathcal{N}(0, T')$.

PROPOSITION 8 *For mentioned three information quantities we have the following result:*

$$\mathbf{I}(P^u \mid Q^{*,u}) = \frac{\sigma_1^2}{2}\left[\left(\mu_1 - \frac{\sigma_1 \rho u}{T'}\right)^2 T + \frac{\sigma_1^2 \rho^2}{T'}\left(T' \ln\left(\frac{T'}{T'-T}\right) - T\right)\right],$$

$$\mathbf{I}(Q^{*,u} \mid P^u) = \frac{\sigma_1^2}{2}\left\{\mu_1^2\, T + 2\sigma_1\, \mu_1\, \rho\, u\, \ln\left(\frac{T'}{T'-T}\right) + \sigma_1^2 \rho^2\, u^2 \frac{T}{T'(T'-T)} + \sigma_1^2 \rho^2 \left[\frac{T}{T'-T} - \ln\left(\frac{T'}{T'-T}\right)\right]\right\},$$

$$\mathbf{H}_T^{(q)}(u) = \left(\frac{T'}{T'-T+qT}\right)^{1/2} \exp\left\{-\frac{(1-q)}{2}\left[\frac{u^2}{T'} - \frac{(u+cT)^2}{T'-T+qT}\right]\right\}$$

with $q > -(\frac{T'}{T} - 1)$ *and* $c = \frac{\mu_1}{\sigma_1 \sqrt{1-\rho^2}}$

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