# An Equilibrium Model for Commodity Prices

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# Outline

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1. Motivation & Results
2. The Market and the Participants
3. Market Equilibrium
4. A Dynamic Version

# Motivation

## Facts

- Financialization of commodity markets is claimed to influence heavily the spot commodity prices.
- Producers use commodity financial derivatives to hedge the price uncertainty (it is estimated that more than 88% participated in these markets, see Acharya et al (2013)).
- Speculators satisfy the producers' hedging demand and hence they become part of the equilibrium spot/forward commodity prices (according to CFTC report (2008), institutional investors' positions in commodity futures have increased from $15 bil. in 2003 to $200 bil. in 2008).

## Questions/Tasks

- Under exogenously given commodity demand shocks, how does the interaction of producers and speculators result in equilibrium prices?
- How do the speculators/producers' characteristics affect the spot/forward equilibrium commodity price?
- How about the time evolution of the equilibrium prices?

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# Relative Literature/Contributions

## Literature

- Recent active literature on forward commodity pricing in finance (see among others, Irwin and Sanders (2011), Singleton (2012), Tang & Xiong (2012), and Fattouh et al (2013), Acharya et al (2013), Basak & Pavlova (2013), etc).
- A variety of equilibrium pricing models for financial markets in mathematical finance literature (see among others, Filipovic & Kupper (2008), Horst et al (2010), A. & Žitković (2010), Kramkov (2013) etc).

## Contributions

- Assuming that participants are utility maximizers, we propose a sport/forward equilibrium model of commodity markets.
- When consumers' demand and the exogenously priced financial market are driven by a Lévy process, we prove the existence and the uniqueness of the equilibrium prices.
- For specific market examples, we exploit the explicit equilibrium formulas to check the relation between the prices and the model parameters.
- A dynamic version of the equilibrium arguments can also be provided.

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# The Set Up

First, we consider two periods of commodity price settlement: today and some given terminal time horizon $T$. Here is the notation used:

- $\pi_0$ and $\pi_T$ are the producers' *deterministic* production outputs.
- $P_0$ and $P_T$ are the spot prices of the commodity.
  (prices that make consumers' demand equal to the producers' supply).
- $\alpha$ stands for the commodity units producers store during the period $[0, T)$.
- $c \in (0, 1)$ is the cost of storage (considered as fraction of the production).
- $F$ is the forward price of one commodity unit, with maturity at time $T$.
- $h^p$ and $h^s$ are the positions on forward contracts taken by producers and speculators, respectively.
- $(S_t)_{t\in[0,T]}$ is the stock price process (exogenously given).
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# The Market Dynamics

We assume that the stochastic factors that drive the market are modelled by a $d$-dimensional Lévy process $X = (X_t)_{t \in [0,T]}$.

- The exogenously given financial asset (stock) price process $S$ is modelled as

$$S_t = S_0 e^{X_t^1}, \quad 0 \le t \le T, S_0 \in \mathbb{R}_+,$$

  where $X^1 := \langle u_1, X \rangle$ and $u_1 \in \mathbb{R}^d$ is a given vector.
- The consumers' demand function is *linear* and at the initial time is given as

$$Q_0(P) := \mu - mP, \quad m \in \mathbb{R}_+.$$

- We assume stable elasticity but a **random shift of the demand** function

$$Q_T(P) := \mu - mP + X_T^2 = Q_0(P) + X_T^2,$$

where $X^2 := \langle u_2, X \rangle$ and $u_2 \in \mathbb{R}^d$. Hence, commodity spot prices are

$$\begin{aligned} P_0 &= \phi_0(\pi_0 - \alpha) = \phi_0(\pi_0) + \frac{\alpha}{m}, \\ P_T &= \phi_T(\pi_T + \alpha(1-c)) = \phi_0(\pi_T) - \frac{\alpha(1-c)}{m} + \frac{X_T^2}{m}, \end{aligned}$$

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# The Producers & Prices

## The representative producer's choices

1. How much production is going to supply in the market (choose units to store).
2. How much future price uncertainty is going to hedge (choose the position in the forward contract).

Hence, his position is:

$$w(\alpha, h^p) = P_0(\pi_0 - \alpha)(1+R) + P_T(\pi_T + \alpha(1-c)) + h^p(F - P_T)$$

and his optimization problem:

$$\sup_{\alpha\in[0,\pi_0],\, h^p\in\mathbb{R}} \mathbb{U}_p(w(\alpha, h^p)) = \sup_{\alpha\in[0,\pi_0],\, h^p\in\mathbb{R}} \left(-\frac{1}{\gamma_p}\log \mathbb{E}[e^{-\gamma_p w(\alpha,h^p)}]\right).$$

Given $\pi_0$, $\pi_T$ and the producers' choice $\hat{\alpha}$, the prices are

$$\hat{P}_0 = \phi_0(\pi_0) + \frac{\hat{\alpha}}{m} \quad \text{and} \quad \hat{P}_T = \phi_0(\pi_T) - \frac{\hat{\alpha}(1-c)}{m} + \frac{X_T^2}{m}.$$

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1. Optimal position in the stock market (optimal dynamic portfolio).
2. How many commodity contracts are needed for good diversification (choose the position in the forward contract)

Hence, his optimization problem:

$$\sup_{\theta\in\Theta, h^s\in\mathbb{R}} \mathbb{U}_s\left(h^s(P_T - F) + \int_0^T \theta_u dS_u\right) = \sup_{\theta\in\Theta, h^s\in\mathbb{R}}\left(-\frac{1}{\gamma_s}\log\mathbb{E}[e^{-\gamma_s\left(h^s(P_T-F)+\int_0^T\theta_u dS_u\right)}]\right),$$

where $\Theta$ is the set of uniformly bounded from below investment strategies.

## Equilibrium prices

A price $\hat{F}\in\mathbb{R}_+$ is an *equilibrium commodity forward price* if it satisfies the clearing condition

$$\hat{h}^p(\hat{F}) = \hat{h}^s(\hat{F}).$$

The induced price $\hat{P}_0 := \hat{P}_0(\hat{F})$ is the corresponding *equilibrium commodity spot price*.

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# Standing Assumptions

Let $(b, c, \nu)$ be the characteristic triplet of the $d$-dimensional Lévy process that drives the market. We define the sets:

$$\mathcal{U} := \left\{ u \in \mathbb{R}^d : \int_{|x|>1} e^{\langle u,x \rangle} \nu(dx) < \infty \right\} \text{ and } \mathcal{U}_2 := \left\{ z \in \mathbb{R} : \int_{|x|>1} e^{z\langle u_2,x \rangle} \nu(dx) < \infty \right\}$$

Throughout this work we impose the following:

## Assumption ($\mathbb{EM}$)

$$0 \in \mathcal{U}^o.$$

## Assumption ($\mathbb{CO}$)

In the case where $\partial\mathcal{U}_2 = \pm\infty$, the following limit holds

$$\kappa_2(z) \xrightarrow[|z|\to\infty]{} +\infty.$$

where $\kappa_2(z) := z\langle b, u_2 \rangle + z^2 \frac{\langle u_2, cu_2 \rangle}{2} + \int_{\mathbb{R}^d} (e^{z\langle u_2, x \rangle} - 1 - z\langle u_2, x \rangle)\nu(dx).$

# The Optimization Problems

## Proposition (Producers)

- The producers' optimization problem can be written as

$$\sup_{\alpha\in[0,\pi_0]} \sup_{h^p\in\mathbb{R}} \{h^p F - g_\alpha(h^p)\} = \sup_{\alpha\in[0,\pi_0]} g^*_\alpha(F)$$

  where $g_\alpha(h^p)$ is convex and lower semi-continuous w.r.t. $h^p$ and $\alpha$.
- Under $(\mathbb{EM})$ and $(\mathbb{CO})$, for every $F \in \mathbb{R}$, there exist unique maximizers $(\hat{\alpha}, \hat{h}^p) \in [0, \pi_0] \times \mathbb{R}$.

## Proposition (Speculators)

- The speculators' optimization problem can be written as

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- Under $(\mathbb{EM})$ and $(\mathbb{CO})$, for every $\alpha \in [0, \pi_0]$ and $F \in (\inf_{\mathbb{Q}\in\mathcal{M}_a} \mathbb{E}_{\mathbb{Q}}[P_T], \sup_{\mathbb{Q}\in\mathcal{M}_a} \mathbb{E}_{\mathbb{Q}}[P_T])$, there exists a unique maximizer $\hat{h}^s \in \mathbb{R}$.

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# Existence and Uniqueness of the Equilibrium Prices

## Theorem

Under $(\mathbb{EM})$ and $(\mathbb{CO})$, for every $\alpha \in [0, \pi_0]$ there exists a unique forward price $\hat{F}(\alpha)$ such that

$$\hat{F}(\alpha) \in \partial(g_\alpha^* \circledast f_\alpha^*)(0) \cap \left( \inf_{\mathbb{Q} \in \mathcal{M}_a} \mathbb{E}_{\mathbb{Q}}[P_T], \sup_{\mathbb{Q} \in \mathcal{M}_a} \mathbb{E}_{\mathbb{Q}}[P_T] \right),$$

and $\hat{F}(\alpha)$ is a continuous function of $\alpha$.
Then, the **unique** equilibrium commodity forward price is given by

$$\hat{F} := \hat{F}(\hat{\alpha}) := \max_{\alpha \in [0, \pi_0]} g_\alpha^*(\hat{F}(\alpha)).$$

The equilibrium spot price is given by

$$\hat{P}_0 = \phi_0(\pi_0 - \hat{\alpha}),$$

i.e., the inverse consumers' demand function with commodity supply equal to $\pi_0 - \hat{\alpha}$.

# An Example

$$b = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} \sigma^2 & \rho\sigma y \\ \rho\sigma y & y^2 \end{pmatrix} \quad \text{and} \quad \nu = 0$$

and

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

That is, $X_t^1 = b_1 t + \sigma W_t^1$ and $X_T^2 = yW_T^2$ where $W^1$ and $W^2$ are correlated Brownian motions, with correlation equal to $\rho \in (-1, 1)$.
The speculators' position is given by

$$\hat{h}^s(F) = \frac{\mathbb{E}[P_T] - F}{\mathbb{V}\mathrm{ar}[P_T]\tilde{\gamma}_s} + \frac{\lambda\rho m}{\tilde{\gamma}_s},$$

where $\tilde{\gamma}_s = \gamma_s(\rho^2 y^2 + 1)$, $\mathbb{E}[P_T] = \phi_0(\pi_T) - \frac{\alpha(1-c)}{m}$ and $\lambda$ is the stock market price of risk.
For the producers' problem the solution is given by

$$\hat{\alpha} = (\alpha^* \vee 0) \wedge \pi_0 \quad \text{and} \quad \hat{h}^p = \frac{c_1 + c_2\hat{\alpha}}{c_3},$$

for some explicit constants $\alpha^*, c_1, c_2$ and $c_3$.

# Some Illustration

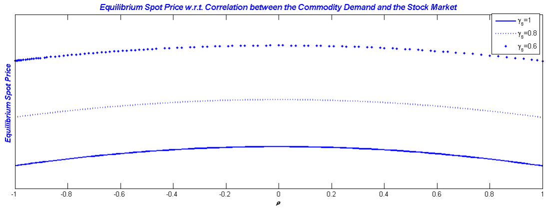


Figure: Spot equilibrium price is an increasing function of $\gamma_s$

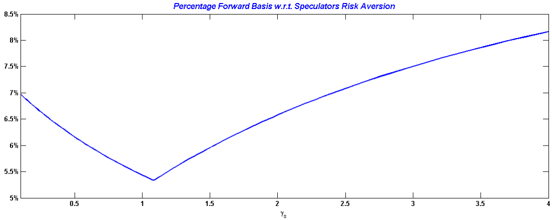


Figure: Percentage Forward Basis $= \frac{\hat{P}_0 - \hat{F}}{\hat{P}_0}$

# Outline

# Producers' Case

Consider the set up of the previous example and assume for simplicity that there is no storage option.

- The spot and forward commodity transactions occur at times $t_k = kh$, $k = 0, \dots, N$ where $h = T/N$ for some finite time horizon $T > 0$.
- $P_k, F_k$ are the commodity spot, forward prices at time $t_k$, with $F_N = P_N$.
- $\pi_k$ is the output production at time $t_k$.

Producers' position:

$$w(h^p) = \sum_{n=1}^{N} (\pi_n P_n + h_n^p (F_n - F_{n-1})) = \beta_0 + \sum_{n=1}^{N} \left( \beta_n (W_{t_n}^2 - W_{t_{n-1}}^2) + h_n^p (F_n - F_{n-1}) \right),$$

for some constants $\beta_n$.

- We then define for each $n$:

$$\mathbb{U}_n^p = -\frac{1}{\gamma_p} \log \mathbb{E}_n \left[ \exp \left( -\gamma_p \left( \sum_{k=n+1}^{N} \beta_k \left( W_{t_k}^2 - W_{t_k-1}^2 \right) + h_k^p (F_k - F_{k-1}) \right) \right) \right]$$

and make the Ansatz

$$F_n - F_{n-1} = \frac{y}{m} \left( W_{t_n}^2 - W_{t_{n-1}}^2 \right) + \Delta\varphi_n,$$

where $\Delta\varphi_n$ is are real valued constants.

# Producers' Case, *cont'd*

- We get

$$\mathbb{U}_n^p = -\frac{T\gamma_p}{2N}\left(\beta_{n+1} + h_{n+1}^p \frac{y}{m}\right)^2 + h_{n+1}^p \Delta\varphi_{n+1} + \mathrm{const}$$

- Hence, we can get the optimal hedging position:

$$\hat{h}_{n+1}^p = \frac{Nm^2\Delta\varphi_{n+1}}{T\gamma_p y^2} - \frac{m\beta_{n+1}}{y}.$$

# Speculators' Case

- We have to maximize

$$\mathbb{U}^s\left(w(\theta,h^s)\right) = -\frac{1}{\gamma_s}\log\mathbb{E}\left[\exp\left\{-\gamma_s\left(\int_0^T \theta_u dS_u + \sum_{n=1}^N h_n^s(F_n - F_{n-1})\right)\right\}\right].$$

- We work backwards:

$$\begin{aligned}
\mathbb{U}_n^s &= \mathbb{U}_n^s\left(\int_{t_n}^{t_{n+1}} \theta_u dS_u + h_{n+1}^s \frac{y}{m}\left(W_{t_{n+1}}^2 - W_{t_n}^2\right) + h_{n+1}^s \Delta\varphi_{n+1} + U_{n+1}^s\right) \\
&= \mathbb{U}_n^s\left(\int_{t_n}^{t_{n+1}} \theta_u dS_u + h_{n+1}^s \frac{y}{m}\left(W_{t_{n+1}}^2 - W_{t_n}^2\right)\right) + h_{n+1}^s \Delta\varphi_{n+1} + \text{const.} \\
&= -\frac{Ty^2\gamma_s}{2Nm^2}(h_{n+1}^s)^2 - \frac{T}{2N\gamma_s}\left(\lambda - \frac{\rho\gamma_s h_{n+1}^s y^2}{m}\right)^2 + h_{n+1}^s\Delta\varphi_{n+1} + \text{const.}
\end{aligned}$$

- Hence, we can get the optimal forward position:

$$\hat{h}_{n+1}^s = \frac{\lambda m\rho}{\tilde{\gamma}_s} + \frac{Nm^2\Delta\varphi_{n+1}}{T\tilde{\gamma}_s}.$$

Matching $h_n^s = h_n^p$ gives the forward commodity price at $\hat{F}_n$.

# The End

THANK YOU FOR YOUR ATTENTION