Prior-to-default equivalent supermartingale measures

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Monday 24th of June, 2013

Outline

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Beginnings

Economic idea—qualitative version of the Fundamental Theorem of Asset Pricing (FTAP). In a market without frictions, there is equivalence between the following statements:

1. There are no possibilities for free lunch.
2. Agents' utility maximisation problems have solutions.
3. There exists a linear valuation rule which is *consistent*:
   - Liquid assets are valued correctly;
   - Non-zero claims have non-zero value.

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Questions: How are the previous made precise?

1. How should free lunch be defined? (Further, should we always forbid its existence, even if we cannot construct one?)
2. How should utility maximisation problems be formulated?
3. How do linear consistent valuation rules look like?

Classical theory — brief selected bibliography

Discrete-time models:

- Ross '73.
- Harrison and Kreps '79.
- Dalang, Morton and Willinger '90.
- Rogers '94.
- Jacod and Shiryaev '98.
- Kabanov and Stricker '01.

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Continuous-time models:

- Harrison and Pliska '81.
- Kreps '81.
- ...
- Delbaen and Schachermayer — loads of work from '91 to '98.

Further steps — in a nutshell

After the '98 D&S FTAP, viability in frictionless models seemed well-understood. However, in the 00's, interest in models that allowed for certain "free lunches" started to form:

- Stochastic Portfolio (Optimisation) Theory of R. Fernholz. Aim: *discover* profits rather than banning the model.
- Benchmark approach of E. Platen builds a valuation framework using a very special portfolio as numéraire.
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Further research on viability and valuation was carried out, utilising a *descriptive*, rather than *normative* approach.

- Deflators, not risk-neutral probabilities, became important.
- Exact connections to *numéraire portfolios* were established.
- Existence of arbitrage was characterised in terms of *predictable characteristics* of the model, making it easily verifiable.
- Valuation ideas were extended, covering "imperfect" markets.

Extending the modelling framework — natural questions

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3. Possibility of default for the whole economy?
4. Model uncertainty?
   - Slew of (potentially mutually singular) underlying probabilities.

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Stochastic framework

State space E (Polish), denoting possible states in economy.

- Append cemetery state $\triangle$ to E.
- For right-continuous $\omega : \mathbb{R}_+ \mapsto E \cup \{\triangle\}$, define

$$\zeta(\omega) := \inf \{t \in \mathbb{R}_+ \mid \omega_t = \triangle\} .$$

  to be the economy's *default* time (or *life*time).

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Underlying stochastic basis: $(\Omega, \mathbf{F})$, where

- Ω: set of all right-continuous $\omega : \mathbb{R}_+ \mapsto E \cup \{\triangle\}$ such that $\omega_0 \in E$ and $\omega_t = \triangle$ holds for all $t \in [\zeta(\omega), \infty)$.
- $\mathbf{F}$: right-continuous enlargement of natural filtration of ω.

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Notation: $\mathcal{O}$: optional processes, $\mathcal{T}$: stopping times.

Prior-to-ζ equivalent probability measures

Definition: Two probabilities $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are called *prior-to-ζ equivalent*, denoted by $\widetilde{\mathbb{P}} \overset{<\zeta}{\sim} \mathbb{P}$, if

$$\widetilde{\mathbb{P}}[A_T \cap \{T < \zeta\}] = 0 \quad \Longleftrightarrow \quad \mathbb{P}[A_T \cap \{T < \zeta\}] = 0$$

holds for all $T \in \mathcal{T}$ and $A_T \in \mathcal{F}_T$.

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Proposition. $\widetilde{\mathbb{P}} \overset{<\zeta}{\sim} \mathbb{P} \iff \exists\ \mathbb{P}$-supermartingale Y with $Y_0 = 1$, $[\![0, \zeta[\![\subseteq \{Y > 0\}$ (up to $\mathbb{P}$-evanescence), such that

$$\widetilde{\mathbb{P}}\,[A_T \cap \{T < \zeta\}] = \mathbb{E}_{\mathbb{P}}\,[Y_T; A_T \cap \{T < \zeta\}] \qquad \text{(DENS)}$$

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holds for all $T \in \mathcal{T}$ and $A_T \in \mathcal{F}_T$.

Theorem [Föllmer '72]. For a $\mathbb{P}$-supermartingale Y with $Y_0 = 1$, $[\![0, \zeta[\![\subseteq \{Y > 0\}$, there exists (!) $\widetilde{\mathbb{P}}$ such that (DENS) holds.

Foretellability of ζ

Definition: Say that ζ is foretellable under $\mathbb{P}$ if there exists a *nondecreasing* sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times such that:

- $\tau_n \leq \zeta$ and $\mathbb{P}[\tau_n < \zeta] = 1$ holds for all $n \in \mathbb{N}$, and
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- $\mathbb{P}$ is constructed in a (maximal) way so that the "density" process $(\mathrm{d}\overline{\mathbb{P}}/\mathrm{d}\mathbb{P})|_{\mathcal{F}_\cdot}$ is non-increasing.
- This result allows use of localising techniques later on.

Wealth processes: the guiding example

Liquid assets: $S \equiv (S^i)_{i \in I}$, where I is an *arbitrary* index set.

- S^i right-continuous and $S^i = S^i \mathbb{I}_{[\![0,\zeta[\![}$ holds for all $i \in I$.
- All assets and wealth denominated in units of $S^0 \equiv \mathbb{I}_{[\![0,\zeta[\![}$.

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Wealth process-set $\mathcal{X}$ with (normalised) unit initial capital : all *nonnegative* processes of the form

$$X = \left(1 + \int_0^{\cdot} H_t \mathrm{d}S_t\right) \mathbb{I}_{[\![0,\zeta[\![},$$

where H predictable, R^I-valued, whenever integral is well-defined:

- always for H being simple (buy-and-hold, finite holdings);
- if I is finite and S semimartingale, for all S-integrable H.

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Constraints: ask that $(H^i S^i_-)_{i \in I} \in X_- \mathcal{C}$, where $\mathcal{C}$ is a predictable process with values in convex subsets of R^I such that $0 \in \mathcal{C}$.

Wealth-process sets: abstract definition

Wealth-process set. The class $\mathcal{X}$ of wealth processes starting from normalized unit capital is such that:

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2. $\mathbb{I}_{[\![0,\zeta[\![} \in \mathcal{X}$.
3. $\mathcal{X}$ is *fork-convex*: $\forall \tau \in \mathbb{R}_+$, $X \in \mathcal{X}$, $\forall X' \in \mathcal{X}$, $X'' \in \mathcal{X}$ with $[\![0,\zeta[\![\subseteq \{X' > 0, \ X'' > 0\}$, and $\forall [0,1]$-valued $\mathcal{F}_\tau$-measurable α_τ, the process defined below is also an element of $\mathcal{X}$:

$$t \mapsto \begin{cases} X_t, & \text{if } 0 \leq t < \zeta \wedge \tau, \\ \alpha_\tau \left(X_\tau / X'_\tau\right) X'_t + (1 - \alpha_\tau) \left(X_\tau / X''_\tau\right) X''_t, & \text{if } \zeta \wedge \tau \leq t < \zeta, \\ 0, & \text{if } \zeta \leq t, \end{cases}$$

Super-hedging

Super-hedging: for $V \in \mathcal{O}_+$ and $T \in \mathcal{T}$, define

$$\overline{p}(V, T) := \inf \{x > 0 \mid \exists X^x \in x\mathcal{X} \text{ with } \mathbb{P}[X_T^x < V_T,\ T < \zeta] = 0\},$$

where $\overline{p}(V, T) = \infty$ when the last set is empty.

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Remarks:

- Both quantities are invariant under prior-to-ζ equivalent probability changes.
- Clearly, $p \leq \overline{p}$.
- For $\mathcal{X}$ without "closedness" properties, p is more appropriate (for example, it is continuous from below).

Market viability via condition NA_1

Arbitrage of the first kind: $V \in \mathcal{O}_+$ and $T \in \mathbb{R}_+$ with

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If *no* opportunities for such arbitrage exist, say that NA_1 holds.

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- In classical theory, it is weaker than the NFLVR condition of Delbaen and Schachermayer. It is equivalent to Kabanov's BK condition and Kar(atz+dar)as' NUPBR condition.
- NA_1 is a numéraire-invariant notion.

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Fundamental Theorem of Asset Pricing

Prior-to-ζ equivalent supermartingale measure: A probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ with the following properties:

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Remark: No hope for local martingales in this framework. However, if $\mathcal{X}$ is generated by finite-dimensional S without constraints, then $\mathsf{NA}_1 \iff \exists$ prior-to-ζ ELMM.

NA_1 and (log-)utility maximisation

Back to economic formulation of FTAP: Validity of NA_1 is connected to the ability to solve the log-utility maximisation problem, through the numéraire portfolio. More precisely, and in order to show the difficult implication $\Longrightarrow$:

- Assume w.l.o.g. that $(\tau_n)_{n \in \mathbb{N}}$ foretells ζ under $\mathbb{P}$.

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- For each $n \in \mathbb{N}$, there exists $\chi_n \in \overline{\{X_{\tau_n} \mid X \in \mathcal{X}\}}^{\mathbb{L}^0}$ such that $\mathbb{E}_{\mathbb{P}}\left[X_{\tau_n}/\chi_n\right] = \mathbb{E}_{\mathbb{P}}\left[y_n X_{\tau_n}\right] \leq 1$ (with $y_n := 1/\chi_n$) for all $n \in \mathbb{N}$. In effect, χ_n is the expected-log-optimal element.

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- By consistency (myopic property of log-optimality) and regularisation, one constructs a process $\widehat{Y}$ with $\widehat{Y}_0 = 1$, $\{\widehat{Y} > 0\} = [\![0, \zeta[\![$, with the property that $\widehat{Y} X$ is a supermartingale under $\mathbb{P}$ for all $X \in \mathcal{X}$.

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- Using $\widehat{Y}$ as density, define $\mathbb{Q} \in \mathcal{Q}$ via Föllmer's construction.

Semimartingales and the numéraire portfolio

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Prior-to-ζ semimartingales: In particular, the previous argument shows that $X \in \mathcal{S}_{[\![0,\zeta[\![}(\mathbb{P})$ whenever ζ is foretellable under $\mathbb{P}$.

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Enlargement of the wealth process set: define $\overline{\mathcal{X}}$ as the closure in the (local) semimartingale topology on $\mathcal{S}_{[\![0,\zeta[\![}(\mathbb{P})$.

- topology invariant under prior-to-ζ equivalent probability changes, when ζ is foretellable under the probabilities involved.
- $\overline{\mathcal{X}}$ is also a wealth-process set according to present definition. Furthermore, condition NA_1 is still valid for $\overline{\mathcal{X}}$.

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The numéraire portfolio: $\widehat{X} := (1/\widehat{Y})\mathbb{I}_{[\![0,\zeta[\![} \in \overline{\mathcal{X}}$ has the property that $(X/\widehat{X})\mathbb{I}_{[\![0,\zeta[\![}$ is a $\mathbb{P}$-supermartingale for all $X \in \overline{\mathcal{X}}$.

Super-hedging duality

Super-hedging duality. Assume that NA_1 holds; equivalently, that $\mathcal{Q} \neq \emptyset$. Then, for all $V \in \mathcal{O}_+$ and $T \in \mathcal{T}$:

$$p(V, T) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[V_T; T < \zeta].$$

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Case of not "nice" probabilistic structure? Above theory still works, with $\mathcal{Q}$ replaced by the class *prior-to-*ζ *strictly positive supermartingale deflators* $\mathcal{Y}$; to wit,

$$\mathrm{NA}_1 \quad \Longleftrightarrow \quad \mathcal{Y} \neq \emptyset.$$

Furthermore, under condition NA_1,

$$p(V, T) = \sup_{Y \in \mathcal{Y}} \mathbb{E}_{\mathbb{P}}[Y_T V_T; T < \zeta].$$

Utility maximisation

The problem. Assume NA_1, fix $T \in \mathcal{T}$ with $\mathbb{P}[T < \zeta] = 1$. Let:

$$u_T(x) = \sup_{X \in x\mathcal{X}} \mathbb{E}_{\mathbb{P}}[U(X_T)], \quad \overline{u}_T(x) = \sup_{X \in x\overline{\mathcal{X}}} \mathbb{E}_{\mathbb{P}}[U(X_T)].$$

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Duality. With $V(y) = \sup_{x>0}(U(x) - xy)$ for $y \in \mathbb{R}_+$, define

$$v(y) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{P}}\Big[V\big(y(\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})|_{\mathcal{F}_T}\big)\Big], \quad \forall y \in (0, \infty).$$

Then, in fact, $v(y) = \sup_{x>0}(u(x) - xy)$, for all $y > 0$.

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Then, in fact, $v(y) = \sup_{x>0}(u(x) - xy)$, for all $y > 0$.

Theorem [Kramkov & Schachermayer, '99, '01]. Assume that $v(y) < \infty$ for all $y > 0$. Then:

1. $u = \overline{u}$.
2. For all $x > 0$, $\exists \overline{X} \equiv \overline{X}(x) \in x\overline{\mathcal{X}}$ with $\mathbb{E}_{\mathbb{P}}[U(\overline{X}_T)] = u(x)$.
3. For all $x > 0$, there exists an $x\mathcal{X}$-valued sequence $(X^n)_{n \in \mathbb{N}}$ such that $\mathcal{S}\text{-}\lim_{n \to \infty} X^n = \overline{X}$ and $\lim_{n \to \infty} \mathbb{E}_{\mathbb{P}}[U(X^n_T)] = u(x)$.

Outline

NA_1 for models based on Itô-processes

Claim: In classical theory, existence of the numéraire portfolio (therefore, NA_1) can be directly validated from the model.

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Theorem. The numéraire portfolio exists if and only if:

1. $a = c\rho$ for some d-dimensional predictable ρ. ($\rho = c^\dagger a$.)
2. $\int_0^T \left(\rho_t^\top c_t \rho_t \right) \mathrm{d}t = \int_0^T \left(a_t^\top c_t^\dagger a_t \right) \mathrm{d}t < \infty$, $\mathbb{P}$-a.s.

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Indeed, the following hold:

1. If 1 fails, there are opportunities for *completely riskless profit*.
2. If 1 holds but 2 fails, following ρ closely enough one can create arbitrage of the first kind.

If 1 and 2 hold, ρ is the building block for the numéraire portfolio.

Non-constructibility of FLVR

Elusiveness of free lunch with vanishing risk. On the same probability space that affords two independent Brownian motions B and W, consider two single-asset models:

$$\begin{aligned}
\textbf{Model } \widehat{\ } : \quad \mathrm{d}\widehat{S}_t/\widehat{S}_t &= (1/\widehat{S}_t)^2 \mathrm{d}t + (1/\widehat{S}_t)\mathrm{d}B_t, \\
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Local drift and volatility are *exactly* the same in both models.

- Both models satisfy NA_1. However...
- An ELMM exists for Model $\widetilde{\ }$ (that is, NFLVR holds), but *not* for Model $\widehat{\ }$ (that is, NFLVR fails).

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NA$_1$: $\overline{p}(V, T) = 0 \implies \mathbb{P}\left[V_T > 0, T < \zeta\right] = 0,\ \forall \mathbb{P} \in \mathcal{P}$.

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Super-hedging duality: ...

The End

Thank You!