Faddeev's quantum dilogarithm and 3-manifold invariants

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Faddeev's quantum dilogarithm

For $\hbar \in \mathbb{R}_{>0}$, Faddeev's quantum dilogarithm function is defined by

$$\Phi_\hbar(z) = (\bar{\Phi}_\hbar(z))^{-1} = \exp\left(\int_{\mathbb{R}+i\epsilon} \frac{e^{-i2xz}}{4\sinh(xb)\sinh(xb^{-1})x}dx\right)$$

in the strip $|\Im z| < \frac{1}{2\sqrt{\hbar}}$, where $\hbar = (b + b^{-1})^{-2}$, and extended to the whole complex plane through the functional equations

$$\Phi_\hbar(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\Phi_\hbar(z + ib^{\pm 1}/2)$$

One can choose $\Re b > 0$ and $\Im b \geq 0$. If $\Im b > 0$ (i.e. $\hbar > 1/4$), then one can show that

$$\Phi_\hbar(z) = \frac{(-qe^{2\pi bz}; q^2)_\infty}{(-\bar{q}e^{2\pi b^{-1}z}; \bar{q}^2)_\infty}$$

where $q := e^{i\pi b^2}$, $\bar{q} := e^{-i\pi b^{-2}}$, and

$$(x; y)_\infty := (1 - x)(1 - xy)(1 - xy^2)\dots$$

Analytical properties of Faddeev's quantum dilogarithm

Zeros and poles:

$$(\Phi_\hbar(z))^{\pm 1} = 0 \;\Leftrightarrow\; z = \mp\left(\frac{i}{2\sqrt{\hbar}} + mib + nib^{-1}\right), \;\; m, n \in \mathbb{Z}_{\geq 0}$$

Behavior at infinity:

$$\Phi_\hbar(z)\Big|_{|z|\to\infty} \approx \begin{cases} 1 & |\arg z| > \frac{\pi}{2} + \arg b \\ \zeta_{inv}^{-1} e^{i\pi z^2} & |\arg z| < \frac{\pi}{2} - \arg b \\ \frac{(\bar{q}^2;\bar{q}^2)_\infty}{\Theta(ib^{-1}z;-b^{-2})} & |\arg z - \frac{\pi}{2}| < \arg b \\ \frac{\Theta(ibz;b^2)}{(q^2;q^2)_\infty} & |\arg z + \frac{\pi}{2}| < \arg b \end{cases}$$

where $\zeta_{inv} := e^{\pi i(2+\hbar^{-1})/12}$, $\Theta(z;\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i z n}$, $\Im\tau > 0$.
Inversion relation:

$$\Phi_\hbar(z)\Phi_\hbar(-z) = \zeta_{inv}^{-1} e^{i\pi z^2}.$$

Complex conjugation:

$$\overline{\Phi_\hbar(z)}\Phi_\hbar(\bar{z}) = 1.$$

Quantum five term identity

Heisenberg's (normalized) selfadjoint operators in $L^2(\mathbb{R})$

$$\mathbf{p}f(x) := \frac{1}{2\pi i} f'(x), \quad \mathbf{q}f(x) := xf(x)$$

Quantum five term identity for unitary operators

$$\Phi_\hbar(\mathbf{p})\Phi_\hbar(\mathbf{q}) = \Phi_\hbar(\mathbf{q})\Phi_\hbar(\mathbf{p}+\mathbf{q})\Phi_\hbar(\mathbf{p})$$

The ultralocal form

$$\mathbf{T}_{12}\mathbf{T}_{13}\mathbf{T}_{23} = \mathbf{T}_{23}\mathbf{T}_{12}$$

where

$$\mathbf{T}_{kl} = e^{2\pi i \mathbf{p}_k \mathbf{q}_l} \bar{\Phi}_\hbar(\mathbf{q}_k + \mathbf{p}_l - \mathbf{q}_l)$$

$$\mathbf{p}_k\mathbf{q}_l - \mathbf{q}_l\mathbf{p}_k = \frac{\delta_{kl}}{2\pi i}, \quad \mathbf{p}_k\mathbf{p}_l - \mathbf{p}_l\mathbf{p}_k = \mathbf{q}_k\mathbf{q}_l - \mathbf{q}_l\mathbf{q}_k = 0$$

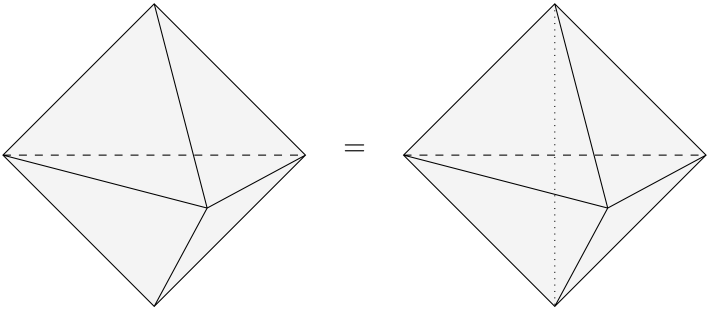
Fourier transformations

$$\int_{\mathbb{R}} \frac{\Phi_\hbar(x+u)}{\Phi_\hbar\left(x - \frac{i}{2\sqrt{\hbar}} + i0\right)} e^{-2\pi i w x}\, dx = \zeta_o \frac{\Phi_\hbar(u)\,\Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right)}{\Phi_\hbar(u-w)}$$

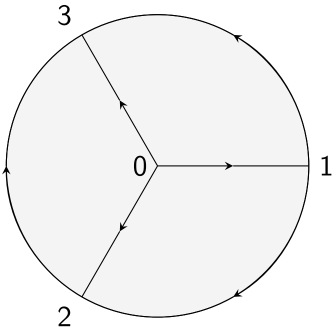
where $\zeta_o := \exp\left(\frac{\pi i}{12}\left(1+\frac{1}{\hbar}\right)\right)$, and $0 < \Im w < \Im u < \frac{1}{2\sqrt{\hbar}}$.
In particular,

$$\int_{\mathbb{R}} \Phi_\hbar(x+i0) e^{-2\pi i w x}\, dx = \zeta_o e^{-\pi i w^2} \Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right)$$

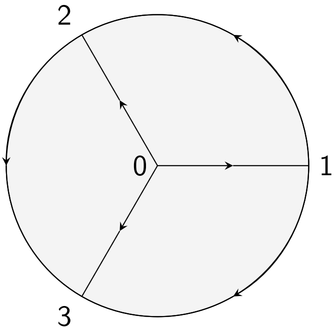
The Pentagon identity and 2 − 3 Pachner move



Oriented tetrahedra



positive tetrahedron



negative tetrahedron

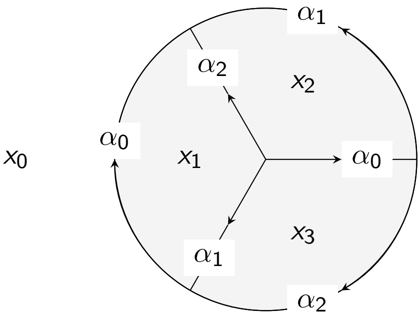
Notation:

- $\partial_i T$ = the face of T opposite to vertex i
- $\Delta_i(X)$ = the set of i-dimensional simplices of X
- $\Delta_i^j(X) = \{(a, b) |\ a \in \Delta_i(X),\ b \in \Delta_j(b)\}$

Labelled oriented tetrahedra

Two types of labellings:

- Real numbers on faces $x\colon \Delta_2(T) \to \mathbb{R}$
- Dihedral angles of ideal hyperbolic tetrahedra on edges $\alpha\colon \Delta_1(T) \to \mathbb{R}_{>0}$, $\sum_i \alpha(\partial_i\partial_j T) = \pi$



$$x_i := x(\partial_i T)$$
$$\alpha_i := \frac{1}{2\pi}\alpha(\partial_i\partial_0 T)$$
$$\alpha_0 + \alpha_1 + \alpha_2 = \frac{1}{2}$$

Neumann–Zagier symplectic structure: $\omega_{NZ} = d\alpha_0 \wedge d\alpha_2$

Weight functions of labelled oriented tetrahedra

Define $W_\hbar(s,t,x,y,u,v) := \delta(x+u-y)\phi_{s,t}(v-u)e^{i2\pi x(v-u)}$ where $\phi_{s,t}(z) := \bar{\Phi}_\hbar\left(z + \frac{1-2s}{2i\sqrt{\hbar}}\right) e^{2\pi tz/\sqrt{\hbar}}$.
To an oriented tetrahedron T with labellings x and α, associate the weight function $Z_\hbar(T,x,\alpha) = W_\hbar(\alpha_0,\alpha_2,x_0,x_1,x_2,x_3)$ if T is positive and complex conjugate otherwise.
Properties:

- shaped Pentagon identity
- full tetrahedral symmetry

Remark

In the case of a positive flat tetrahedron with dihedral angles $\alpha_0 = \alpha_2 = 0$, $\alpha_1 = 1/2$, the weight function is given by the matrix elements of the operator $\mathbf{T}$ in coordinate representation: $Z_\hbar(T,x,\alpha) = \langle x_0, x_2|\mathbf{T}|x_1,x_3\rangle$.

Shape structures

Let X be a closed ($\partial X = \emptyset$) oriented triangulated pseudo 3-manifold where all tetrahedra are oriented, and all gluings respect the orientations. Fix a shape structure $\alpha\colon \Delta_3^1(X) \to]0,\pi[$ i.e. each tetrahedron is provided with dihedral angles.

Gauge group action in the space of shape structures is generated by total dihedral angles around edges acting through the Neumann–Zagier Poisson bracket.

Gauge reduced shape structure $=$ the Hamiltonian reduction of a shape structure over fixed values of the total dihedral angles.

An edge is balanced if the total dihedral angle around it is 2π. A shape structure with all edges balanced is known as an angle structure (Casson, Lackenby, Rivin).

The gauge reduced angle structure is independent of triangulation.

Invariants of a restricted class of 3-manifolds

For a closed oriented triangulated pseudo 3-manifold X with shape structure α, associate the partition function $Z_\hbar(X,\alpha) := \int_{x\in\mathbb{R}^{\Delta_2(X)}} \prod_{T\in\Delta_3(X)} Z_\hbar(T,x,\alpha)dx.$

Theorem

If $H_2(X \setminus \Delta_0(X),\mathbb{Z}) = 0$, then the quantity $|Z_\hbar(X,\alpha)|$ is well defined (the integral is absolutely convergent), and it

- *depends on only the gauge reduced class of α;*
- *is invariant under shaped $3-2$ Pachner moves if α is an angle structure.*

Remark

This construction can be easily extended to manifolds with boundary eventually giving rize to a TQFT.

One vertex H-triangulations of knots in 3-manifolds

Let $K \subset M$ be a knot in an oriented closed compact 3-manifold. Let X be a one vertex H-triangulation of the pair (M, K), i.e. a one vertex triangulation of M where K is represented by an edge e_0 of X.
Fix another edge e_1, and for any small $\epsilon > 0$, consider a shape structure α_ϵ such that the total dihedral angle is ϵ around e_0, $2\pi - \epsilon$ around e_1, and 2π around any other edge.

Theorem

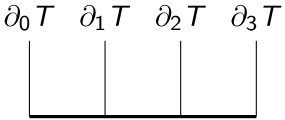
The limit

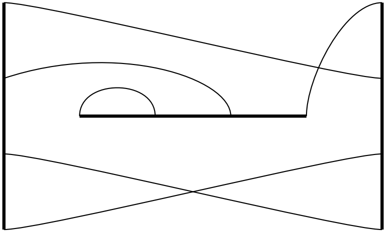
$$\tilde{Z}_\hbar(X) := \left| \lim_{\epsilon \to 0} Z_\hbar(X, \alpha_\epsilon) \Phi_\hbar\left(\frac{\pi - \epsilon}{2\pi i \sqrt{\hbar}}\right) \right|$$

is finite and is invariant under shaped $3 - 2$ *Pachner moves of triangulated pairs* (M, K).

An H-triangulation of the pair $(S^3, 4_1)$ (figure-eight knot)

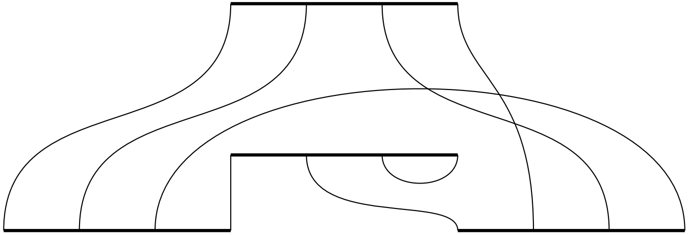
Graphical notation: T =





$$\tilde{Z}_\hbar(S^3, 4_1) = \int_{\mathbb{R}-i\epsilon} \frac{e^{i\pi z^2}}{\Phi_\hbar(z)^2} dz$$

An H-triangulation of the pair $(S^3, 5_2)$



$$\tilde{Z}_\hbar(S^3, 5_2) = \left| \int_{\mathbb{R}-i\epsilon} \frac{e^{i\pi z^2}}{\Phi_\hbar(z)^3} dz \right|$$

A version of the volume conjecture

Conjecture (Volume Conjecture for $\tilde{Z}_\hbar$)

For any hyperbolic knot $K \subset M$, one has

$$\lim_{\hbar \to 0} 2\pi\hbar \log |\tilde{Z}_\hbar(M, K)| = -\mathrm{vol}(M \setminus K)$$

Remark

Unlike the volume conjecture for the colored Jones polynomials, the invariant $\tilde{Z}_\hbar$ exponentially decays rather than grows.

Theorem

The volume conjecture for the invariant $\tilde{Z}_\hbar$ holds true in the case of the knots 4_1 and 5_2 in S^3.

Summary

- Faddeev's quantum dilogarithm satisfies the five term operator identity which can be interpreted as an anylitic realization of $3-2$ Pachner moves in triangulated 3-manifolds.
- The shape structures given by dihedral angles of ideal hyperbolic tetrahedra ensure the topological invariance and convergence.
- In the case of the knots 4_1 and 5_2 in S^3, the knot invariant $\tilde{Z}_\hbar$ exponentially decays for $\hbar \to 0$, the decay rate being given by the hyperbolic volume of the knot complement. This is consistent with the expected quasiclassical behavior of the partition function of $SL(2,\mathbb{C})$ Chern–Simons theory.