Definable Combinatorial Principles in Fragments of Arithmetic

Wang Wei

Sun Yat-sen University

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The Language

L_A, the language of arithmetic, is a first order language with two constants $(0, 1)$, two binary operations $(+, \times)$ and a binary relation $(<)$.

The Axioms

PA^- is a finite L_A-theory, saying that a desired model is the non-negative part of a discrete ordered ring.

For each L_A-formula $\varphi(x, \vec{y})$, $I\varphi$ is the following formula

$$\forall \vec{y}\big(\varphi(0, \vec{y}) \land \forall x(\varphi(x, \vec{y}) \to \varphi(x+1, \vec{y})) \to \forall x\, \varphi(x, \vec{y})\big).$$

$I\Sigma_n$ is the following L_A-theory

$$\mathrm{PA}^- + \{I\varphi : \varphi \in \Sigma_n\},$$

while

$$\mathrm{PA} = \bigcup_n I\Sigma_n.$$

For each L_A-formula $\varphi(x, y, \vec{z})$, $B\varphi$ is the following formula

$$\forall u, \vec{z}\big(\forall x < u \exists y \varphi \to \exists v \forall x < u \exists y < u \varphi\big).$$

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Relations between the Fragments

Theorem (Parsons; Paris and Kirby; Ryll-Nardzewski)

The following implications are all provable over $I\Sigma_0$, but none of them is reversible

$$I\Sigma_{n+1} \to B\Sigma_{n+1} \to I\Sigma_n.$$

Thus PA is not finitely axiomatizible.

Pigeonhole Principle

Pigeonhole Principle: There is no injection from any $b+1=\{0,1,\dots,b\}$ to $b=\{0,1,\dots,b-1\}$.

Since PA is some sort of finite set theory, Pigeonhole Principle can be formalized in PA.

$\mathrm{PHP}(\Sigma_n)$: There is no Σ_n-injection from any $b+1$ to b.

Theorem (Dimitracopoulos and Paris, 1986)

Over $\mathrm{PA}^- + I\Sigma_0$, for $n>0$, $B\Sigma_n$ and $\mathrm{PHP}(\Sigma_n)$ are equivalent.

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Some Variants of Pigeonhole Principle

For a cardinal κ, a model M (of PA^-) is κ-like iff $|M| = \kappa > |a| = |[0, a-1]^M|$ for every $a \in M$.

Theorem (Richard Kaye, 1995)

Let $M \models \mathrm{PA}$.

(1) *M is κ like, iff $(M, \mathcal{P}(M))$ satisfies CARD_2, which states that there is no injection $M \to a \in M$.*

(2) *M is κ like for some limit cardinal κ, iff $(M, \mathcal{P}(M))$ satisfies GPHP_2, which states that for any $a \in M$ there is $b \in M$ s.t. there is no injection $b \to a$.*

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Fragments of PA and Variants of PHP

$\exp$: The function $x \mapsto 2^x$ is total. $\exp$ is provable in $I\Sigma_1$ but not in $I\Sigma_0$.

Let GPHP be the first-order part of GPHP_2 which consists of infinitely many axioms.

Theorem (Kaye, 1996)

Let κ be a singular cardinal and $n > 0$. Every model of $B\Sigma_n + \exp + \neg I\Sigma_n$ is elementarily equivalent to a κ-like model.

Thus, $B\Sigma_n + \exp + \neg I\Sigma_n \vdash \mathrm{GPHP}$.

This may seem very interesting, because 'moderately' large infinity is involved in solving some 'elementary' problem of arithmetic.

Fragments of PHP

CARD is the first order part of CARD_2, so CARD says that there is no first order injection sending the whole model into a proper initial segment.

GPHP is the first order part of GPHP_2 just mentioned.

$\mathrm{CARD}(\Gamma)$ ($\mathrm{GPHP}(\Gamma)$) is CARD (resp. GPHP) restricted to Γ-definable maps.

It is easy to see that

$$B\Sigma_{n+1} \vdash \mathrm{GPHP}(\Sigma_{n+1}), \quad \mathrm{PA}^- + \mathrm{GPHP}(\Sigma_n) \vdash \mathrm{CARD}(\Sigma_n).$$

Quetion (Kaye, 1995)

Does CARD *imply* GPHP *over* PA^- *(*$+I\Sigma_1$*)?*

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Fragments of PHP and Reverse Math

$\mathrm{CARD}(\Sigma^0_2)$ has proved useful in reverse mathematics. E.g.,

Theorem (Groszek and Slaman; Kaye)

$I\Sigma_n \nvdash \mathrm{CARD}(\Sigma_{n+1})$.

A coloring $C : [\mathbb{N}]^2 \to \mathbb{N}$ is 2-bounded iff $|C^{-1}(k)| \leq 2$ for all k; a set R is a rainbow for C iff C is injective on $[R]^2$.

RRT^2_2 (RRT for Rainbow Ramsey Theorem): every 2-bounded coloring on $[\mathbb{N}]^2$ admits an infinite rainbow.

Theorem (Conidis and Slaman, 2013)

$\mathsf{RCA}_0 + \mathsf{RRT}^2_2 \vdash \mathrm{CARD}(\Sigma^0_2)$.

So, RRT^2_2 *is not arithmetically conservative over* RCA_0.

Another Variant of PHP

$\mathrm{WPHP}(\Sigma_n)$: There is no Σ_n-injection from any $2x$ to $x > 0$.

Clearly,

$$B\Sigma_{n+1} \vdash \mathrm{WPHP}(\Sigma_{n+1}), \quad \mathrm{PA}^- + \mathrm{WPHP}(\Sigma_n) \vdash \mathrm{GPHP}(\Sigma_n).$$

$\mathrm{WPHP}(\Sigma_n)$ arises from the reverse mathematics of some analysis propositions related to Dominated Convergence Theorem.

Dominated Convergence Theorem

DCT': Suppose that

- $f, (f_n)_n$ and g are measurable functions defined on $[0, 1]$;
- f_n's are dominated by g and converge almost everywhere to f.

Then $\lim_n \int f_n = \int f$.

$2-\mathrm{WWKL}$: Every Π^0_2 set in Cantor space with positive measure is non-empty.

$2-\mathrm{RAN}$: There is a random real relative to the halting problem.

Theorem (Avigad, Dean and Rute)

Over RCA_0, the followings are equivalent

$$\mathrm{DCT}', \quad 2-\mathrm{WWKL}, \quad B\Sigma^0_2 + 2-\mathrm{RAN}\,.$$

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$2 - \mathrm{WWKL}$ and $2 - \mathrm{RAN}$ over $\mathbb{N}$

Theorem (Kučera)

If X is Martin-Löf random relative to the halting problem then every Π^0_2 subset of 2^ω with positive measure contains an element like

$$\langle b_0 b_1 \ldots b_k X(0) X(1) \ldots X(n) \ldots \rangle.$$

As a consequence,

$$(\mathbb{N}, \mathcal{S}) \models 2 - \mathrm{WWKL} \;\Leftrightarrow\; (\mathbb{N}, \mathcal{S}) \models 2 - \mathrm{RAN}\,.$$

Weak PHP and Reverse Mathematics

$2-\mathrm{WWKL}(1/2)$: Every Π_2 set in Cantor space with measure greater than $1/2$ is non-empty.

Theorem (Belanger, Chong, W., Wong and Yang)

(1) $\mathsf{RCA}_0 + 2-\mathrm{WWKL}(1/2) \vdash 2-\mathrm{RAN}$;

(2) *The first order theory of* $2-\mathrm{WWKL}(1/2)$ *is* $I\Sigma^0_1 + \mathrm{WPHP}(\Sigma^0_2)$;

(3) *Over* $I\Sigma^0_1$, *the following implications are strict*

$$B\Sigma^0_{n+1} \to \mathrm{WPHP}(\Sigma^0_{n+1}) \to \mathrm{CARD}(\Sigma^0_{n+1}).$$

Corollary (Slaman; BCWWY)

$\mathsf{RCA}_0 + 2-\mathrm{RAN}$ *is strictly weaker than* $2-\mathrm{WWKL}$ *or* DCT'.

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New Variants of PHP

$\mathrm{WPHP}(\Sigma_n)$: There is no Σ_n-injection from any $2x$ to $x > 0$.

$\mathrm{GPHP}(\Sigma_n)$: For each a, there is b s.t. there is no Σ_n-injection from any b to a.

$\mathrm{FRT}^e_k(\Sigma_n)$: For each a, there is b s.t. every Σ_n coloring $C : [b]^e \to k$ admits a homogeneous set of size a.

$\mathrm{FRT}^e(\Sigma_n)$ is $\forall k\ \mathrm{FRT}^e_k(\Sigma_n)$, and $\mathrm{FRT}(\Sigma_n)$ is $\forall e\ \mathrm{FRT}^e(\Sigma_n)$.

$\mathrm{fairFRT}^e_k(\Sigma_n)$: For each a, there is b s.t. every Σ_n coloring $C : [b]^e \to k$ corresponds to some $i < k$ s.t. $F^{-1}(i)$ has Σ_n-cardinality $\geq a$ (denoted by $|F^{-1}(i)|_{\Sigma_n} \geq a$), i.e., there is a Σ_n-injection $F : a \to b$ s.t. if $X \in [a]^e$ then

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Some Simple Implications

Over PA^-,

(1) $I\Sigma_1 \vdash \mathrm{FRT}(\Sigma_1)$.

(2) $I\Sigma_1$ proves the following implications

$$\mathrm{WPHP}(\Gamma) \to \mathrm{GPHP}(\Gamma) \to \mathrm{CARD}(\Gamma).$$

(3) $I\Sigma_1$ proves that if $0 < e \leq e' \leq e''$ and $0 < k \leq k' \leq k''$ then

$$\mathrm{FRT}^{e''}_{k''}(\Gamma) \to \mathrm{FRT}^{e'}_{k'}(\Gamma) \to \mathrm{fairFRT}^{e'}_{k'}(\Gamma) \to \mathrm{fairFRT}^{e}_{k}(\Gamma).$$

(4) $I\Sigma_1 \vdash \mathrm{fairFRT}^1(\Gamma) \to \mathrm{GPHP}(\Gamma)$.

(5) $B\Sigma_{n+1} \vdash \mathrm{FRT}(\Sigma_{n+1})$.

Some Easy Equivalence

Theorem

Over $I\Sigma_n$ *where* $n > 0$*,*

$$\mathrm{GPHP}(\Sigma_{n+1}) \;\leftrightarrow\; \mathrm{FRT}^1(\Sigma_{n+1}) \;\leftrightarrow\; \mathrm{fairFRT}^1(\Sigma_{n+1}) \;\leftrightarrow\; \mathrm{FRT}^1_2(\Sigma_{n+1}).$$

An Interesting Case

Theorem

For $n > 0$, $I\Sigma_n \vdash \mathrm{fairFRT}^1_2(\Sigma_{n+1})$.

(1) We already have $I\Sigma_n + \mathrm{GPHP}(\Sigma_{n+1}) \vdash \mathrm{fairFRT}^1(\Sigma_{n+1})$;

(2) So we work in a model $M \models I\Sigma_n + \neg\,\mathrm{GPHP}(\Sigma_{n+1})$;

(3) The following is a proper additive cut,

$$I = \{a : \text{ for some } b, \text{ there is no injection } F \in \Sigma_{n+1}, F : b \to a\}.$$

If $a \in I < b$ then $|a|_{\Sigma_{n+1}} < b$.

(4) Fix a and pick $b > I$, also fix a Σ_{n+1}-coloring $F : b \to 2$;

(5) By $I\Sigma_n$, for each $i < 2$ there exists a Σ_{n+1}-bijection (called collapse) between $F^{-1}(i)$ and a proper cut J_i. Either

- One of J_i contains some $c > I$, so $|F^{-1}(i)|_{\Sigma_{n+1}} \geq a$;
- Both $J_i = I$. The collapses together produce a Σ_{n+1}-bijection from b to I; thus $|F^{-1}(i)|_{\Sigma_{n+1}} \geq a$ for both $i < 2$.

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If $a \in I < b$ then $|a|_{\Sigma_{n+1}} < b$.

(4) Fix a and pick $b > I$, also fix a Σ_{n+1}-coloring $F : b \to 2$;

(5) By $I\Sigma_n$, for each $i < 2$ there exists a Σ_{n+1}-bijection (called collapse) between $F^{-1}(i)$ and a proper cut J_i. Either

- One of J_i contains some $c > I$, so $|F^{-1}(i)|_{\Sigma_{n+1}} \geq a$;
- Both $J_i = I$. The collapses together produce a Σ_{n+1}-bijection from b to I; thus $|F^{-1}(i)|_{\Sigma_{n+1}} \geq a$ for both $i < 2$.

Separating CARD and GPHP

Theorem (W)

$\mathrm{PA}^- + I\Sigma_n + \mathrm{CARD}(\Sigma_{n+1}) \nvdash \mathrm{GPHP}(\Sigma_{n+1})$.

Lemma (Lifting)

Let M be a model of $\mathrm{PA}^- + I\Sigma_n$ for some $n > 0$. Suppose that

- *There exists a Σ_{n+1}^M-injection from M into some $a \in M$;*
- *For some $c < a$ in a monster model containing M,*

$$M(c) = \{f(c) : f \text{ is an } M\text{-finite function with domain } = a\}.$$

Then there also exists a $\Sigma_{n+1}^{M(c)}$-injection from $M(c)$ into a.

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Comparing Fragments of PHP: Sketch

Let $M \models \mathrm{PA}^- + I\Sigma_n + \neg\Sigma_{n+1} - \mathrm{CARD}$ be countable. Fix $a \in M$ and a Σ_{n+1}^M-definable 1-1 map $f\colon M \to [0,a]$. Also fix $(b_k : k \in \mathbb{N})$ cofinal in M.

We build a sequence $(M_k : k \in \mathbb{N})$ s.t.

(1) $M_0 = M$;

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(4) For each k, there is a Σ_{n+1}-definable 1-1 map $f_k : M_k \to [0,a]$;

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Let $N = \bigcup_{k\in\mathbb{N}} M_k$.

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Separating FRT and WPHP

Theorem

For every countable $M \models I\Sigma_n$ *with* $n > 0$ *and any* $a \in M$*, there exists* N *s.t.* $M \preceq_{cf,\Sigma_{n+1}} N$*,* $[0,a]^M = [0,a]^N$ *and* $N \models I\Sigma_n + \mathrm{FRT}(\Sigma_{n+1})$*.*

Thus, $I\Sigma_n + \mathrm{FRT}(\Sigma_{n+1}) \nvdash \mathrm{WPHP}(\Sigma_{n+1})$*.*

Key Lemma

Lemma

Suppose that

(a) M *is a countable model of* $I\Sigma_n$ *with* $n > 0$,

(b) a, c, e, x *are elements of* M,

(c) φ *is a* $\Sigma_n(M)$*-formula defining* $C : [y]^e \times M \to c$ *with* y *sufficiently large, s.t.,* $\bar{C}(\vec{d}) = \lim_s C(\vec{d}, s)$ *exists for all* $\vec{d} \in [y]^e$.

Then either one of the followings holds,

(1) *There exist* $s \in M$, $H \in [y]^x \cap M$ *and* $i < c$, *s.t.* $\bar{C}(\vec{d}) = C(\vec{d}, t) = i$ *for all* $\vec{d} \in [H]^e$ *and* $t > s$.

(2) *There exists* N, *s.t.* $M \preceq_{cf,\Sigma_{n+1}} N$, $[0, a]^N = [0, a]^M$, $\lim_s C^N(\cdot, s)$ *is undefined on some* $\vec{d} \in [y]^e \cap N$ *where* C^N *is the map defined by* φ *in* N.

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By finite combinatorics, there exists a primitive recursive function r s.t.

$$r(x, e, k) = \min\{y : y \to (x)^e_k\}.$$

In $I\Sigma_1$, define

$$r^{(0)}(x, e, k) = x, \quad r^{(m+1)}(x, e, k) = r(r^{(m)}(x, e, k), e, k).$$

In the above lemma, for fixed a, c, e, x, we pick $b > \mathbb{N}$, $k = \max\{a, c\}$ and y s.t.

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FRT, WPHP and Collections

Combining the above proof and the proof that $I\Sigma_n + \mathrm{WPHP}(\Sigma_{n+1}) \nvdash B\Sigma_{n+1}$, we can get

Theorem

$I\Sigma_n + \mathrm{WPHP}(\Sigma_{n+1}) + \mathrm{FRT}(\Sigma_{n+1}) \nvdash B\Sigma_{n+1}$.

PHP and Fragments of PA

PHP and Reverse Mathematics

New Variants of PHP

Fragments of CARD and GPHP

Fragments of FRT and WPHP

Quetions

Definable Variants of Other Combinatorial Principles

We may formulate definable variants of other finite combinatorial principles, like Hales-Jewett theorem, Turán's Theorem.

Similar results can be proved. How about the relations between such definable finite combinatorial principles?

GPHP and $2-\mathrm{RAN}$

We mentioned that

- The first order theory of $\mathsf{RCA}_0 + 2 - \mathrm{WWKL}(1/2)$ is axiomatized by $I\Sigma_1 + \mathrm{WPHP}(\Sigma_2)$.
- $\mathsf{RCA}_0 + 2 - \mathrm{WWKL}(1/2) \vdash 2 - \mathrm{RAN}$.

Can these definable finite combinatorial principles help in understanding the first order theories of $2-\mathrm{RAN}$? E.g.,

$$\mathsf{RCA}_0 + 2 - \mathrm{RAN} \vdash \mathrm{GPHP}(\Sigma^0_2)?$$

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Variants of FRT and RT^2_2

- (Hirst) $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash B\Sigma^0_2$.
- (Chong, Slaman and Yang) $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash I\Sigma^0_2$.
- (Patey and Yokoyama) $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^0_3-conservative over $B\Sigma^0_2$.
- (Houéron, Patey and Yokoyama) $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $B\Sigma^0_2 + \mathsf{WF}(\epsilon_0)$ and over $B\Sigma^0_2 + \bigcup_n \mathsf{WF}(\omega^\omega_n)$.
- (Houéron, Patey and Yokoyama) $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^0_4-conservative over $B\Sigma^0_2$.

Quetion

- *Does* $B\Sigma_n$ *(*$n > 1$*) imply* $\mathrm{fairFRT}^1_2(\Sigma_{n+1})$*?*
- *Does* $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *imply* $\mathrm{fairFRT}^1_2(\Sigma_3)$*?*

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