On algebraic properties of the Krichever-Novikov equation

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On classification of integrable equations of KdV type

The classification of integrable equations of the form

$$u_t = u_{xxx} + f(u, u_x, u_{xx}), \tag{1}$$

was done in [1].

The complete (up to point and quasi-local transformations) list of equations (1) possessing an infinite series of conservation laws, can be written as:

$$u_t = u_{xxx} + u\,u_x, \tag{2}$$

$$u_t = u_{xxx} + u^2\,u_x,$$

$$u_t = u_{xxx} - \frac{1}{2}u_x^3 + (\alpha e^{2u} + \beta e^{-2u})u_x,$$

$$u_t = u_{xxx} - \frac{1}{2}Q''\,u_x + \frac{3}{8}\frac{(Q - u_x^2)_x^2}{u_x\,(Q - u_x^2)},$$

$$u_t = u_{xxx} - \frac{3}{2}\,\frac{u_{xx}^2 + Q(u)}{u_x}, \tag{3}$$

where $Q'''''(u) = 0$.

Equations

$$v_t = v_{xxx} + 3v\, v_x,$$

and the Schwarz–KdV equation

$$u_t = u_{xxx} - \frac{3}{2}\,\frac{u_{xx}^2}{u_x},$$

are related by the differential substitution

$$v = \frac{u_{xxx}}{u_x} - \frac{3}{2}\,\frac{u_{xx}^2}{u_x^2}.$$

The Schwarz–KdV equation admits a three-parameter symmetry group consisting of fractional-linear transformations of the form

$$\bar{u} \to \frac{\alpha\bar{u} + \beta}{\gamma\bar{u} + \delta}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \qquad \alpha\delta - \beta\gamma = 1.$$

The function v is the simplest invariants of the group.

Recursion operators.

Recursion and Hamiltonian operators relate symmetries and cosymmetries of evolution equations

$$u_t = F(u, u_1, u_2, \ldots, u_n), \qquad u_i = \frac{\partial^i u}{\partial x^i}. \tag{4}$$

The rigth hand side G of an infinitesimal symmetry satisfies the relation

$$D_t(G) - F_*(G) = 0, \qquad \text{where} \qquad F_* = \sum_{i=0}^{n} \frac{\partial F}{\partial u_i} D_x^i. \tag{5}$$

It is well known that $X = \dfrac{\delta \rho}{\delta u}$ satisfies the conjugate equation of (5):

$$D_t\,(X) + F_*^+\,(X) = 0\,. \tag{6}$$

Any solution $X \in \mathcal{F}$ of equation (6) is called *cosymmetry*.

Recall that the recursion operator is an operator $\mathcal{R}$ which satisfies relation

$$D_t(\mathcal{R}) = F_*\,\mathcal{R} - \mathcal{R}\,F_*. \tag{7}$$

Let us rewrite (7) as

$$[D_t - F_*,\, \mathcal{R}] = 0. \tag{8}$$

Then, for any symmetry of equation (4), the equation $u_\tau = \mathcal{R}(G)$ is a symmetry for (4). The usual way to obtain all symmetries of equation (4) is to apply a recursion operator to the simplest symmetry u_x.

For example, for the Korteweg-de Vries equation $u_t = u_{xxx} + 6\,u\,u_x$ the simplest recursion operator

$$\mathcal{R} = D_x^2 + 4u + 2u_x D_x^{-1} \tag{9}$$

is the ratio of two differential Hamiltonian operators

$$\mathcal{H}_1 = D_x, \qquad \mathcal{H}_2 = D_x^3 + 4uD_x + 2u_x.$$

The analogue of the operator identity (8) for Hamiltonian operators is the relation

$$(D_t - F_*)\,\mathcal{H} = \mathcal{H}(D_t + F_*^+), \tag{10}$$

which means that $\mathcal{H}$ maps cosymmetries (and, in particular, the variational derivatives of the conserved densities) to symmetries.

Most of known recursion operators have the following special form

$$\mathcal{R} = R + \sum_{i=1}^{k} G_i D_x^{-1} g_i, \tag{11}$$

where R is a differential operator, G_i and g_i are some fixed symmetries and cosymmetries common for all members of the hierarchy. We call recursion operators (11) *quasilocal.*

As the rule, the Hamiltonian operators are local (i.e. differential) or quasilocal operators. The latter means that

$$\mathcal{H} = H + \sum_{i=1}^{m} G_i D_x^{-1} \bar{G}_i, \tag{12}$$

where H is a differential operator and $G_i, \bar{G}_i$ are fixed symmetries.

Consider the Krichever-Novikov equation [1]

$$u_{t_1} = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x} = G_1, \qquad P^{(V)} = 0. \tag{13}$$

The fifth order symmetry of (13) is given by

$$G_2 = u_5 - 5\frac{u_4 u_2}{u_1} - \frac{5}{2}\frac{u_3^2}{u_1} + \frac{25}{2}\frac{u_3 u_2^2}{u_1^2} - \frac{45}{8}\frac{u_2^4}{u_1^3} - \frac{5}{3}P\frac{u_3}{u_1^2} + \frac{25}{6}P\frac{u_2^2}{u_1^3} - \frac{5}{3}P'\frac{u_2}{u_1} - \frac{5}{18}\frac{P^2}{u_1^3} + \frac{5}{9}u_1 P''.$$

The simplest three conserved densities of (13) are

$$\rho_1 = -\frac{1}{2}\frac{u_2^2}{u_1^2} - \frac{1}{3}\frac{P}{u_1^2},$$

$$\rho_2 = \frac{1}{2}\frac{u_3^2}{u_1^2} - \frac{3}{8}\frac{u_2^4}{u_1^4} + \frac{5}{6}P\frac{u_2^2}{u_1^4} + \frac{1}{18}\frac{P^2}{u_1^4} - \frac{5}{9}P'',$$

$$\rho_3 = \frac{u_4^2}{u_1^2} + 3\frac{u_3^3}{u_1^3} - \frac{19}{2}\frac{u_3^2 u_2^2}{u_1^4} + \frac{7}{3}P\frac{u_3^2}{u_1^4} + \frac{35}{9}P'\frac{u_2^3}{u_1^4} + \frac{45}{8}\frac{u_2^6}{u_1^6} -$$
$$\frac{259}{36}\frac{u_2^4 P}{u_1^6} + \frac{35}{18}P^2\frac{u_2^2}{u_1^6} - \frac{14}{9}P''\frac{u_2^2}{u_1^2} + \frac{1}{27}\frac{P^3}{u_1^6} -$$
$$\frac{14}{27}\frac{P''P}{u_1^2} - \frac{7}{27}\frac{P'^2}{u_1^2} - \frac{14}{9}P^{(IV)}u_1^2.$$

In the paper [2] the forth order quasilocal recursion operator of the form

$$\mathcal{R}_1 = D_x^4 + a_1 D_x^3 + a_2 D_x^2 + a_3 D_x + a_4 + G_1 D_x^{-1} \frac{\delta \rho_1}{\delta u} + u_x D_x^{-1} \frac{\delta \rho_2}{\delta u},$$

was found. Here the coefficients a_i are given by

$$a_1 = -4\frac{u_2}{u_1}, \qquad a_2 = 6\frac{u_2^2}{u_1^2} - 2\frac{u_3}{u_1} - \frac{4}{3}\frac{P}{u_1^2},$$

$$a_3 = -2\frac{u_4}{u_1} + 8\frac{u_3 u_2}{u_1^2} - 6\frac{u_2^3}{u_1^3} + 4P\frac{u_2}{u_1^3} - \frac{2}{3}\frac{P'}{u_1},$$

$$a_4 = \frac{u_5}{u_1} - 2\frac{u_3^2}{u_1^2} + 8\frac{u_3 u_2^2}{u_1^3} - 4\frac{u_4 u_2}{u_1^2} - 3\frac{u_2^4}{u_1^4} + \frac{4}{9}\frac{P^2}{u_1^4} + \frac{4}{3}P\frac{u_2^2}{u_1^4} + \frac{10}{9}P'' - \frac{8}{3}P'\frac{u_2}{u_1^2}.$$

It turns our that there exists one more quasilocal recursion operator for (13) of the form

$$\mathcal{R}_2 = D_x^6 + b_1 D_x^5 + b_2 D_x^4 + b_3 D_x^3 + b_4 D_x^2 + b_5 D_x + b_6 - \frac{1}{2} u_x D_x^{-1} \frac{\delta \rho_3}{\delta u} + G_1 D_x^{-1} \frac{\delta \rho_2}{\delta u} + G_2 D_x^{-1} \frac{\delta \rho_1}{\delta u},$$

The operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are related by the following elliptic curve

$$\mathcal{R}_2^2 = \mathcal{R}_1^3 - \phi \mathcal{R}_1 - \theta, \qquad (14)$$

where

$$\phi = \frac{16}{27}\Big((P'')^2 - 2P'''P' + 2P^{(IV)}P\Big),$$

$$\theta = \frac{128}{243}\Big(- \frac{1}{3}(P'')^3 - \frac{3}{2}(P')^2 P^{(IV)} + P'P''P''' + 2P^{(IV)}P''P - (P''')^2\Big).$$

It is possible to prove that for the Korteweg-de Vries equation the associative algebra **A** of all quasilocal recursion operators is generated by one operator (9). In other words, this algebra is isomorphic to the algebra of all polynomials in one variable.

For the Krichever-Novikov equation the algebra **A** is isomorphic to the coordinate ring of the elliptic curve. The same is true for the Landau-Lifshitz equation, where the recursion operators have orders 2 and 3.

Hamiltonian operators.

The simplest quasilocal Hamiltonian operator of order -1

$$\mathcal{H}_0 = u_x D_x^{-1} u_x$$

for the Krichever-Novikov equation was found in [2].

The recursion operators presented above appear to be ratios

$$\mathcal{R}_1 = \mathcal{H}_1\mathcal{H}_0^{-1}, \qquad \mathcal{R}_2 = \mathcal{H}_2\mathcal{H}_0^{-1}$$

of the following quasilocal Hamiltonian operators

$$\begin{aligned}\mathcal{H}_1 = {} & \frac{1}{2}(u_x^2D_x^3 + D_x^3u_x^2) + \\ & (2u_{xxx}u_x - \frac{9}{2}u_{xx}^2 - \frac{2}{3}P)D_x + D_x(2u_{xxx}u_x - \frac{9}{2}u_{xx}^2 - \frac{2}{3}P) + \\ & G_1D_x^{-1}G_1 + u_xD_x^{-1}G_2 + G_2D_x^{-1}u_x,\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_2 = {} & \frac{1}{2}(u_x^2D_x^5 + D_x^5u_x^2) + \\ & (3u_{xxx}u_x - \frac{19}{2}u_{xx}^2 - P)D_x^3 + D_x^3(3u_{xxx}u_x - \frac{19}{2}u_{xx}^2 - P) + \\ & hD_x + D_xh + G_1D_x^{-1}G_2 + G_2D_x^{-1}G_1 + u_xD_x^{-1}G_3 + G_3D_x^{-1}u_x,\end{aligned}$$

where

$$h = u_{xxxxx}u_x - 9u_{xxxx}u_{xx} + \frac{19}{2}u_{xxx}^2 - \frac{2}{3}\frac{u_{xxx}}{u_x}(5P - 39u_{xx}^2) + \frac{u_{xx}^2}{u_x^2}(5P - 9u_{xx}^2) + \frac{2}{3}\frac{P^2}{u_x^2} + u_x^2 P'',$$

and $G_3 = \mathcal{R}_1(G_1) = \mathcal{R}_2(u_x)$ is the seventh order symmetry of (13)

Landau–Lifschitz equation. Lax pair

Let $\mathfrak{g}$ be a Lie algebra with a basis $\mathbf{e}_i$, $i = 1, \dots, n$. The Lie algebra $\mathfrak{g}((\lambda))$ of formal series of the form

$$\sum_{i=-n}^{\infty} g_i \lambda^i \quad | \quad g_i \in \mathfrak{g}, \quad n \in \mathbb{Z} \tag{15}$$

is called the (extended) *loop algebra* over $\mathfrak{g}$.

Consider decompositions

$$\mathfrak{g}((\lambda)) = \mathfrak{g}[[\lambda]] \oplus \mathcal{U} \tag{16}$$

of the loop algebra into a direct sum of vector subspaces, the first of which is the Lie subalgebra $\mathfrak{g}[[\lambda]]$ of all Taylor series, and the second one is a Lie subalgebra. The Lie algebra $\mathcal{U}$ is called *factoring*, or *complementary*.

The simplest factoring subalgebra consists of polynomials in $\frac{1}{\lambda}$ with a zero free term:

$$\mathcal{U}^{st} = \Big\{ \sum_{i=1}^{n} g_i \lambda^{-i} \quad | \quad g_i \in \mathfrak{g}, \quad n \in \mathbb{N} \Big\}. \tag{17}$$

Example 1. Let $\mathfrak{g} = \mathfrak{so}_3$ with the basis

$$\mathbf{e_1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{e_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then the elements

$$\mathbf{E}_1 = \frac{\sqrt{1 - p\lambda^2}}{\lambda}\,\mathbf{e}_1, \qquad \mathbf{E}_2 = \frac{\sqrt{1 - q\lambda^2}}{\lambda}\,\mathbf{e}_2,$$

$$\mathbf{E}_3 = \frac{\sqrt{1 - r\lambda^2}}{\lambda}\,\mathbf{e}_3.$$

generate a factoring subalgebra for any parameters p, q, r.

Any factoring subalgebra $\mathcal{U}$ in $\mathfrak{so}_3$ generates a Lax pair of the form

$$L = \frac{d}{dx} + U, \qquad U = \sum_{i=1}^{3} s_i \, \mathbf{E}_i, \qquad s_1^2 + s_2^2 + s_3^2 = 1, \quad (18)$$

$$A = \sum_i s_i \, [\mathbf{E}_j, \, \mathbf{E}_k] + \sum_i t_i \, \mathbf{E}_i, \quad (19)$$

leading to a nonlinear integrable PDE of the Landau–Lifschitz type. For this special case the Lax equation can be written as

$$U_t - A_x + [U, \, A] = 0. \quad (20)$$

We can find the coefficients t_i and the corresponding nonlinear system of the form

$$\mathbf{s}_t = \vec{F}(\mathbf{s}, \, \mathbf{s}_x, \, \mathbf{s}_{xx}), \qquad \text{where} \quad \mathbf{s} = (s_1, s_2, s_3), \qquad \mathbf{s}^2 = 1,$$

using a direct calculation.

Namely, comparing the coefficients of λ^{-2} in the relation (20), we express t_i in terms of $\mathbf{s}, \mathbf{s}_x$. And then, equating the coefficients of λ^{-1}, we get a system of evolution equations for $\mathbf{s}$.

Example 1 (continuation). Equating to zero the coefficient of λ^{-2} in (20), we get $\mathbf{s}_x = \mathbf{s} \times \mathbf{t}$, where $\mathbf{t} = (t_1, t_2, t_3)$. Since $\mathbf{s}^2 = 1$, we find that $\mathbf{t} = \mathbf{s}_x \times \mathbf{s} + \mu\, \mathbf{s}$.

Comparing the coefficients of λ^{-1}, we arrive at the equation $\mathbf{s}_t = \mathbf{t}_x - \mathbf{s} \times \mathbf{V}\mathbf{s}$, where $\mathbf{V} = \operatorname{diag}(p, q, r)$. Substituting the expression for $\mathbf{t}$, we obtain

$$\mathbf{s}_t = \mathbf{s}_{xx} \times \mathbf{s} + \mu_x\, \mathbf{s} + \mu\, \mathbf{s}_x - \mathbf{s} \times \mathbf{V}\mathbf{s}.$$

Since the scalar product $(\mathbf{s},\, \mathbf{s}_t)$ has to be zero, we find that $\mu = \text{const}$. The resulting equation coincides (up to the involution $t \to -t$, a trivial additional term $\mu\, \mathbf{s}_x$ and a change of notation) with the Landau–Lifschitz equation

$$\mathbf{u}_t = \mathbf{u} \times \mathbf{u}_{xx} + \mathbf{R}(\mathbf{u}) \times \mathbf{u}, \qquad |\mathbf{u}| = 1.$$

Here $\times$ stands for the cross product.

A scalar Laurent series

$$\mathbf{m} = \sum_{i=-n}^{\infty} c_i \lambda^i, \qquad c_i \in \mathbb{C},$$

is called a *multiplicand* of a factoring subalgebra $\mathcal{U}$ if $\mathbf{m}\mathcal{U} \subset \mathcal{U}$. Obviously, the set of all multiplicands forms a (commutative) associative algebra with unity over $\mathbb{C}$.

In the case of Example 1 the functions

$$x = \frac{1}{\lambda^2}, \qquad y = \frac{\sqrt{(1-p\lambda^2)(1-q\lambda^2)(1-r\lambda^2)}}{\lambda^3}$$

are multiplicands of $\mathcal{U}$ of order 2 and 3, respectively. They are related by the elliptic algebraic curve:

$$y^2 = (x-p)(x-q)(x-r).$$