Real normalized differentials

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Publications

I. M. Krichever, *Spectral theory of finite-zone nonstationary Schrödinger operators. A nonstationary Peierls model*, Functional Analysis and Its Applications, 1986, Volume 20, Issue 3, Pages 203–214

S. Grushevsky, I. Krichever, *The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces*, in: Surveys in Differential Geometry Vol. 14: Geometry of Riemann surfaces and their moduli spaces, International Press of Boston Inc, 2010, pp. 111–130

I. M. Krichever, *Real normalized differentials and Arbarello's conjecture*, Functional Analysis and Its Applications, 2012, Volume 46, Issue 2, Pages 110–120

Publications

I. Krichever, S. Lando, A. Skripchenko, *Real-normalized differentials with a single order 2 pole*, Letters in Mathematical Physics. 2021. Vol. 111. No. 2. Article 36

M. Nenasheva *About the isoperiodic foliation on the stratum of codimension 1 in the space of real-normalized differentials*, 2024, St.-Petersburg Mathematical Journal, 36:2, pp. 93–107

M. Nenasheva, *Principal stratum in the moduli space of real-normalized differentials with a single pole*, to appear in Functional Analysis and Its Applications

Definition

A *period* of a meromorphic 1-form on a Riemann surface is its integral over a closed path not passing through the poles.
All the periods of a meromorphic 1-form ω constitute an additive subgroup in $\mathbb{C}$, the *group of periods* P_ω.

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A meromorphic 1-form on a Riemann surface X is called a *real normalized differential* if all its periods are real. The group of periods of a real normalized differential is an additive subgroup in $\mathbb{R}$.

The only holomorphic real normalized differential on a compact Riemann surface is 0, and all the residues of a meromorphic real normalized differential are purely imaginary.

Uniqueness of real normalized differentials

Theorem

Let X be a compact Riemann surface, $x_1 \in X$. Then there is a unique, up to multiplication by a real constant, real normalized differential ω on X having a pole of order 2 at x_1 and no other poles.

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This theorem is a special case of

Theorem (Krichever, 86)

Let X be a compact Riemann surface, $x_1, \dots, x_n \in X$, $p_1, \dots, p_n$ — principal parts of 1-forms at $x_1, \dots, x_n$, respectively, such that all the residues are purely imaginary and their sum is zero. Then there is a unique real normalized differential ω on X having poles only at $x_1, \dots, x_n$ and principal parts $p_1, \dots, p_n$ at these points.

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These theorems relate spaces of meromorphic differentials directly to moduli spaces of complex curves.

Real-valued solutions to Schrödinger equations

I. Krichever used real normalized differentials to construct real-valued solutions to Schrödinger equations: for a given pair of real-normalized differentials dp, dE, both having no poles other than at a given point $P \in X$ and such that the order of the pole of dp is 2 and that of dE is 3, the Baker–Akhiezer function

$$\psi(t_0, t_1, Q) = \exp\left(i\left(t_0\int^Q dp + t_1\int^Q dE\right)\right)\cdot(1+\xi_1(t_0,t_1)z+\dots)$$

$(z = z(Q))$ is a real valued solution to the nonlinear Schrödinger equation

$$\left(i\frac{\partial}{\partial t_1} - \frac{\partial^2}{\partial \bar{t}_0^2} + 2i\frac{\partial \xi_1}{\partial t_0}\right)\psi = 0$$

in the variables t_0, t_1.

If the group of periods $L_\omega \subset \mathbb{R}$ $(\omega = dp, dE)$ has rank 1, then the potential is periodic in the specified variable (rather than just quasiperiodic).

Diaz's theorem

Theorem (Diaz, 1986)

There is no complete complex subvariety of dimension greater than $g-2$ *in the moduli space* $\mathcal{M}_g$ *of genus* g *curves.*

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I. Krichever and S. Grushevsky's **proof** of Diaz's theorem (2010) is based on the study of isoperiodic foliation on the moduli spaces of real normalized differentials having two simple poles. The moduli space of such differentials having residues $\pm i$ is identified with $\mathcal{M}_{g;2}$. They argue by contradiction: assuming there is a complete subvariety of dimension $\geq g-1$ in $\mathcal{M}_g$, it then can be pulled back to a complete subvariety of dimension $g+1$ in $\mathcal{M}_{g;2}$. Then, a contradiction is obtained by studying the restrictions of the imaginary parts of the integrals of the differentials to the pullback.

Arbarello's conjecture

Theorem (Krichever, 2012)

Any compact complex cycle in $\mathcal{M}_g$ of dimension at least $g-m$ intersects the irreducible subvariety $W_m \subset \mathcal{M}_g$.

Here the subvarieties $W_2, W_3, \dots, W_g$ form a flag $W_2 \subset W_3 \subset \cdots \subset W_g = \mathcal{M}_g$, $\dim\ W_m = 2g+m-3$ (the *Weierstrass flag*); W_m consists of curves carrying a meromorphic function with a single pole of order m.

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The **proof** uses a new concept of cycles dual to critical points of real normalized differentials on a Riemann surface. It is shown that the homology classes of these dual cycles generate the first homology group $H_1(X, \mathbb{Z})$ of the given Riemann surface X.

A combinatorial model

Using real normalized differentials allows one to construct a cell decomposition of the moduli space of such differentials, whence a cell decomposition of the moduli space of curves.

To each meromorphic differential ω on X, one associates an oriented line field L_ω on X. At a point $q \in X$ the line $L_\omega(q)$ of the line field L_ω is the tangent line to the curve given by the equation $\mathrm{Im} \int_q^x \omega = 0$ and oriented in the direction of increasing of the real part of the integral.

The line field L_ω has singularities at the zeroes and poles of ω. Integral curves of L_ω connecting poles to zeroes are *separatrices*.

A combinatorial model

A general real normalized differential ω on a genus g Riemann surface X having a single pole of order 2 has $2g$ simple zeroes, which are connected to the pole by $4g$ separatrices entering the zeroes. For a chosen point $A \in X$ that does not belong to the separatrices, the mapping $\Psi_A : x \mapsto \int_A^x \omega$ identifies the complement to the separatrices in X with the complex line cut along $4g$ vertical half lines. These $4g$ half lines split into $2g$ pairs, the two lines in each pair starting at the same height (*cut diagram*).
Conversely, each cut diagram determines a Riemann surface: the result of gluing the opposite sides in the half lines in each pair. The differential dz then defines a real normalized meromorphic differential on X.
Associate to a cut diagram a combinatorial object, an arc diagram,
Hence, an open dense subset of the moduli space $\mathcal{M}_{g;1}$ obtains a cell decomposition where cells are enumerated by arc diargams.

A combinatorial model

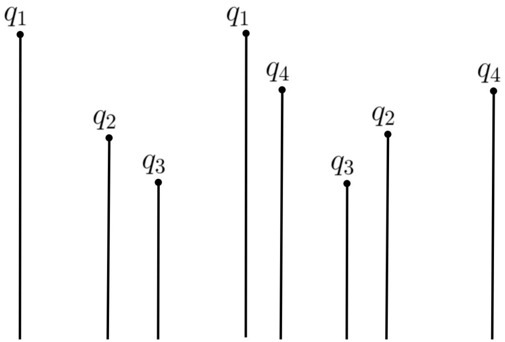

Рис.: A cut diagram

A combinatorial model

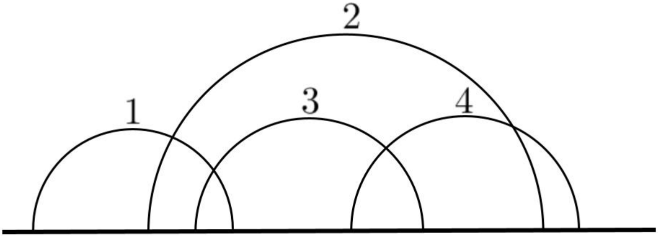

Рис.: The corresponding arc diagram

Absolute period foliation on the moduli spaces

Real normilized differentials whose group of periods coincides with the group of periods P_ω of a given 1-form ω form a leaf of the absolute period foliation.

Theorem (I. Krichever, S. Lando, A. Skripchenko, 2021)

If the group of periods P_ω is dense in $\mathbb{R}$, then the leaf of the absolute period foliation is dense in the moduli space of real normalized differential.

Absolute period foliation on the moduli spaces

Theorem (I. Krichever, S. Lando, A. Skripchenko, 2021)

Each generic leaf of the absolute period foliation contains at least $\mathrm{SL}(2g,\mathbb{Z})/\mathrm{Sp}(2g,\mathbb{Z})$ *connected components.*

Here 'generic' means that the group of periods P_ω is a subgroup of rank $2g$ in $\mathbb{R}$.

The proof is achieved by showing that each generic leaf contains a differential with a standard arc diagram, a g-caravan. Then one proves that any two such real normalized differentials with coinciding groups of periods can be connected by an arc inside the leaf.

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Later this theorem has been made more precise by M. Nenasheva:

Theorem (M. Nenasheva, 2024)

The connected components of a generic leaf are in one-to-one correspondence with $\mathrm{SL}(2g, \mathbb{Z})/\mathrm{Sp}(2g, \mathbb{Z})$.

Stratification of the moduli spaces

Real normalized differentials may be degenerate: separatrices may connect different zeroes and/or zeroes can have higher orders. Such degenerate differentials also determine cut diagrams. The corresponding combinatorial encoding proceeds in terms of permutations (M. Nenasheva, 2024): to each stratum a permutation of the set of cuts is associated.

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Simultaneously, permutations appeared (unexpected and naturally) in the theory of finite type knot invariants. They also arise as generalizations of arc diagrams (which are involutions without fixed points). They proved to be an effective tool in computing weight systems associated to classical Lie algebras (M. Kazarian, Z. Yang).

Thank you
for your attention