On hyperbolic type equations with fifth-order generalized symmetries.

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Plan of the talk.

1. Tzitzeica equation. Some historical remarks.
2. Properties of equations with fifth-order generalized symmetries.
3. Integrability conditions.
4. Differential equations.
5. Differential-discrete equations.
6. Discrete equations.

Tzitzeica equation.

The celebrated Tzitzeica equation

$$u_{xy} = e^{u} + e^{-2u},$$

have interesting hystory. It was found in research as an integrable equation independently in two works: 1. [Dodd R. K. and Bullough R. K. 1977 *Proc. R. Soc. Lond. A* 352481-503]

Polynomial conserved densities for the sine–Gordon equations 489

Theorem 9. The equation $z_{,xt} = A\,e^{\alpha z} + B\,e^{\beta z}$ does not have any conserved densities for rank greater than 24 if $\alpha+\beta \neq 0$ or $\alpha \neq \beta$.

give a relation between α and β (namely that $\beta = -\frac{1}{2}\alpha$ or $\beta = -2\alpha$ for given A, B) and a condition on n for a density. This condition cannot be satisfied for any integral n.

2. [A. V. Zhiber, A. B. Shabat, *Dokl. Akad. Nauk SSSR*, 247:5 (1979), 1103-1107]

Т е о р е м а 1. *Уравнение (1) обладает нетривиальной группой тогда и только тогда, когда*

$$a(u) = c_1 e^{\lambda u} + c_2 e^{-\lambda u} \quad \text{или} \quad a(u) = c_1 e^{\lambda u} + c_2 e^{-2\lambda u}.$$

But this equation was known for a long time before from geometrical research:
[G. Tzitzeica, *Rendiconti del Circolo Matematico di Palermo* 25:1 (1908) 180-187.]

$$(13) \qquad \begin{cases} \dfrac{\partial^2 \omega}{\partial \alpha^2} = \dfrac{1}{h}\dfrac{\partial h}{\partial \alpha}\dfrac{\partial \omega}{\partial \alpha} + \dfrac{1}{h}\dfrac{\partial \omega}{\partial \beta}, \\ \dfrac{\partial^2 \omega}{\partial \beta^2} = \dfrac{1}{h}\dfrac{\partial \omega}{\partial \alpha} + \dfrac{1}{h}\dfrac{\partial h}{\partial \beta}\dfrac{\partial \omega}{\partial \beta}, \\ \dfrac{\partial^2 \omega}{\partial \alpha \partial \beta} = h\omega, \end{cases}$$

où h est une intégrale de l'équation

$$(14) \qquad \frac{\partial^2 \log h}{\partial \alpha \partial \beta} = h - \frac{1}{h^2}.$$

This explains its common name Tzitzeica equation.

Tzitzeica type equations has the following properties:

1. Infinitely many of nontrivial conservation laws of different orders
2. Lax pair in terms of 3×3 matrices
3. Hierarchies of generalized symmetries in both directions. The lowest ones are of the fifth order.

The second and third properties distinguish Tzitzeica type equations from other integrable sin-Gordon type equations.

Comparing of hyperbolic type equations

$u_{xy} = f(u, u_x, u_y)$ $D_y = \frac{d}{dy}$, $D_x = \frac{d}{dx}$	$u_{n+1,x} = f(u, u_{n,x}, u_{n+1})$ $D_x = \frac{d}{dx}$ $T_n g(n,m) = g(n+1,m)$,	$u_{n+1,m+1} = f(u_{n,m}, u_{n+1,m}, u_{n,m+1})$ $T_n g(n,m) = g(n+1,m)$, $T_m g(n,m) = g(n,m+1)$
$u_{xy} = 0$ $u = f(x) + g(y)$	$u_{n+1,x} - u_{n,x} = 0$ $u = f(x) + \alpha_n$	$u_{n+1,m+1} - u_{n+1,m} - u_{n,m+1} + u_{n,m} = 0$ $u = \alpha_n + \beta_m$
$u_{xy} = e^u$ $W_1 = u_{xx} - u_x^2/2$, $W_2 = u_{yy} - u_y^2/2$,	$u_{n+1,x} u_{n,x} = u_{n+1} + u_n$ $W_1 = (u_{n,xx} - 1)^2 / u_{n,x}^2$ $W_2 = \frac{(u_{n+3} - u_{n+1})(u_{n+2} - u_n)}{u_{n+2} + u_{n+1}}$	$u_{n+1,m+1} u_{n,m} - u_{n+1,m} u_{n,m+1} = 1$ $W_1 = \frac{u_{n+1,m} + u_{n-1,m}}{u_{n,m}}$, $W_2 = \frac{u_{n,m+1} + u_{n,m-1}}{u_{n,m}}$,
$u_{xy} = \sin u$ $\partial_t u = u_{xxx} + u_x^3/2$ $\partial_\tau u = u_{yyy} + u_y^3/2$	$u_{n+1,x} - u_{n,x} = u_{n+1}^2 - u_n^2$ $\partial_t u_n = u_{n,xxx} - 6 u_n u_{n,x}$ $\partial_\tau u_n = \frac{(u_{n+1} - u_n)(u_n - u_{n-1})}{u_{n+1} - u_{n-1}}$	$(u_{n+1,m+1} - u_{n,m})(u_{n+1,m} - u_{n,m+1}) = 1$ $\partial_t u_{n,m} = 1/(u_{n+1,m} - u_{n-1,m})$ $\partial_\tau u_{n,m} = 1/(u_{n,m+1} - u_{n,m-1})$
Tzitzeica equation $u_{xy} = e^u - e^{-2u}$ $\partial_t u = u_{xxxxx} + \dots$ $\partial_\tau u = u_{yyyyy} + \dots$	$u_{n+1,x} - u_{n,x} =$ $= e^{n+1} + e^{u_n} + e^{-u_{n+1} - u_n}$ $\partial_t u_n = u_{n,xxxxx} + \dots$ $\partial_\tau u_n = \Phi(u_{n-2}, \dots, u_{n+2})$	$(u_{n,m} + u_{n+1,m+1}) u_{n,m+1} u_{n+1,m} = 1$ Mikhailov&Xenitidis 13 $\partial_t u_{n,m} = \Psi(u_{n-2,m}, \dots, u_{n+2,m})$ $\partial_\tau u_{n,m} = \Phi(u_{n,m-2}, \dots, u_{n,m+2})$

Integrability conditions. The case of continious symmetry.

Let's consider nonlinear differential-difference equations of the form

$$\frac{du_{n+1}}{dx} = f\left(\frac{du_n}{dx}, u_{n+1}, u_n, x\right) \tag{1}$$

where the sought function $u_n = u_n(x)$ depends on integer n and real x variables and $u_{n,x}$ denotes the derivative of $u_n(x)$ with respect to x.
Let us given an evolutionary type PDE of the order N, which solution depends on n:

$$u_{n,t} = g\left(x, u_n, u_{n,x}, u_{n,xx}, \ldots, \frac{\partial^N u_n}{\partial x^N}\right). \tag{2}$$

Equation (2) is called a *symmetry* of the lattice (1) in the direction of x if the flows defined by the equations (1) and (2) commute identically. In other words the following relation holds

$$D_x D_t u_{n+1} = D_x T_n g = D_t f \qquad \mod (1), (2) \tag{3}$$

Here T_n is the shift operator of the discrete argument n and D_x, D_t – operators of the total derivative with respect to the variables x and t.

Compatibility condition (3) has a form of linearisation of eqiation (1) for right hand side of equation (2):

$$g_{n+1,x} = \frac{\partial f}{\partial u_{n,x}} g_{n,x} + \frac{\partial f}{\partial u_{n+1}} g_{n+1} + \frac{\partial f}{\partial u_n} g_n. \tag{4}$$

The problem to find both functions f and g from this equation is very difficult.
The first consequence from this equation reads

$$(T-1)\frac{\partial}{\partial u_{0,N}} g(x, u_0, u_{0,1}, u_{0,2}, \ldots, u_{0,N}) = 0 \tag{5}$$

and can be easily solved (if we suppose that (1) is not Darboux integrable)

$$\frac{\partial}{\partial u_{n,N}} g = C(x), \quad C(x) \neq 0. \tag{6}$$

We see that the right hand side of the symmetry on x direction should has the form:

$$u_{n,t} = \frac{\partial^N u_n}{\partial x^N} + g^{(1)}\left(x, u_n, u_{n,x}, u_{n,xx}, \ldots, \frac{\partial^{N-1} u_n}{\partial x^{N-1}}\right). \tag{7}$$

Here we put $C(x) = 1$ by rescalling of x. This type equation is called as equations with a constant separant.

The same form (7) of x-symmetry is valid for continious hyperbolic equation of the form:

$$u_{xy} = f(u, u_x, u_y) \tag{8}$$

To find all continious or semidiscrete Tzitzeica type equation, we need to take one by one integrable evolutionary equations of fifth order and try to find rhs of equation. The list of such equation with constant separant is known [Drinveld, Sokolov, Svinolupov 1985, Meshvov Sokolov 2012] and contain 15 equations.
Analogous procedure for third order symmetry was done for continious case in [Meshkov Sokolov 2011] for semidiscrete case in [GRN 2023].

List of fifth-order symmetries. (Here $u_i = \frac{\partial^i}{\partial x^i} u$)

$$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1, \tag{9}$$

$$u_t = u_5 + 5uu_3 + \frac{25}{2}u_1u_2 + 5u^2u_1, \tag{10}$$

$$u_t = u_5 + 5u_1u_3 + \frac{5}{3}u_1^3, \tag{11}$$

$$u_t = u_5 + 5u_1u_3 + \frac{15}{4}u_2^2 + \frac{5}{3}u_1^3, \tag{12}$$

$$u_t = u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2 - 5u_1^3 + 5u^4u_1, \tag{13}$$

$$u_t = u_5 + 5(u_2 - u_1^2)u_3 - 5u_1u_2^2 + u_1^5, \tag{14}$$

$$\begin{aligned} u_t = u_5 &+ 5(u_2 - u_1^2 - \lambda_1^2 e^{2u} - \lambda_2^2 e^{-4u})\, u_3 - 5u_1u_2^2 \\ &+ 15(-\lambda_1^2 e^{2u} + 4\lambda_2^2 e^{-4u})\, u_1u_2 \\ &+ u_1^5 - 90\lambda_2^2 e^{-4u}\, u_1^3 - 5(\lambda_1^2 e^{2u} + \lambda_2^2 e^{-4u})^2\, u_1, \end{aligned} \tag{15}$$

$$
\begin{aligned}
u_t &= u_5 + 5\left(u_2 - u_1^2 - \lambda_1^2 e^{2u} + \lambda_2 e^{-u}\right) u_3 - 5 u_1 u_2^2 \\
&\quad - 15\lambda_1^2 e^{2u}\, u_1 u_2 + u_1^5 + 5(\lambda_1^2 e^{2u} - \lambda_2 e^{-u})^2\, u_1, \quad \lambda_2 \neq 0,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
u_t &= u_5 - 5\frac{u_2 u_4}{u_1} + 5\frac{u_2^2 u_3}{u_1^2} + 5\left(\frac{\mu_1}{u_1} + \mu_2 u_1^2\right) u_3 \\
&\quad - 5\left(\frac{\mu_1}{u_1^2} + \mu_2 u_1\right) u_2^2 - 5\frac{\mu_1^2}{u_1} + 5\mu_1\mu_2 u_1^2 + \mu_2^2 u_1^5,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
u_t &= u_5 - 5\frac{u_2 u_4}{u_1} - \frac{15}{4}\,\frac{u_3^2}{u_1} + \frac{65}{4}\,\frac{u_2^2 u_3}{u_1^2} + 5\left(\frac{\mu_1}{u_1} + \mu_2 u_1^2\right)\, u_3 \\
&\quad - \frac{135}{16}\frac{u_2^4}{u_1^3} - 5\left(\frac{7\mu_1}{4u_1^2} - \frac{\mu_2 u_1}{2}\right) u_2^2 - 5\frac{\mu_1^2}{u_1} + 5\mu_1\mu_2 u_1^2 + \mu_2^2 u_1^5,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
u_t = u_5 &- \frac{5}{2}\,\frac{u_2 u_4}{u_1} - \frac{5}{4}\,\frac{u_3^2}{u_1} + 5\,\frac{u_2^2 u_3}{u_1^2} + \frac{5\,u_2 u_3}{2\sqrt{u_1}} - 5(u_1 - 2\mu u_1^{1/2} + \mu^2)\,u_3 - \frac{35}{16}\,\frac{u_2^4}{u_1^3} \\
&- \frac{5}{3}\,\frac{u_2^3}{u_1^{3/2}} + 5\Big(\frac{3\mu^2}{4u_1} - \frac{\mu}{\sqrt{u_1}} + \frac{1}{4}\Big)\,u_2^2 + \frac{5}{3}\,u_1^3 - 8\mu u_1^{5/2} + 15\mu^2 u_1^2 - \frac{40}{3}\,\mu^3 u_1^{3/2},
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
u_t = u_5 &+ \frac{5}{2}\,\frac{f - u_1}{f^2}\,u_2 u_4 + \frac{5}{4}\,\frac{2\,f - u_1}{f^2}\,u_3^2 + 5\,\mu\,(u_1 + f)^2 u_3 + \frac{5}{4}\,\frac{4\,{u_1}^2 - 8\,u_1 f + f^2}{f^4}\,u_2^2 u_3 \\
&+ \frac{5}{16}\,\frac{2 - 9\,u_1^3 + 18\,u_1^2 f}{f^6}\,u_2^4 + \frac{5\mu}{4}\,\frac{(4\,f - 3\,u_1)(u_1 + f)^2}{f^2}\,u_2^2 + \mu^2 (u_1 + f)^2 \big(2\,f (u_1 + f)^2 - 1\big),
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
u_t = u_5 &+ \frac{5}{2}\frac{f - u_1}{f^2} u_2 u_4 + \frac{5}{4}\frac{2f - u_1}{f^2} u_3^2 - 5\omega(f^2 + u_1^2) u_3 + \frac{5}{4}\frac{4u_1^2 - 8u_1 f + f^2}{f^4} u_2^2 u_3 \\
&+ \frac{5}{16}\,\frac{2 - 9\,u_1^3 + 18\,u_1^2 f}{f^6}\,u_2^4 + \frac{5}{4}\,\omega\,\frac{5\,u_1^3 - 2\,u_1^2 f - 11\,u_1 f^2 - 2}{f^2}\,u_2^2 \\
&- \frac{5}{2}\,\omega'\,(u_1^2 - 2\,u_1 f + 5\,f^2) u_1 u_2 + 5\,\omega^2 u_1 f^2 (3\,u_1 + f)(f - u_1),
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
u_t = u_5 &+ \frac{5}{2}\,\frac{f-u_1}{f^2}\,u_2u_4 + \frac{5}{4}\,\frac{2\,f-u_1}{f^2}\,u_3^2 + \frac{5}{4}\,\frac{4\,u_1^2-8\,u_1f+f^2}{f^4}\,u_2^2u_3 \\
&+ \frac{5}{16}\,\frac{2-9\,{u_1}^3+18\,{u_1}^2f}{f^6}\,u_2^4 + 5\,\omega\,\frac{2\,{u_1}^3+{u_1}^2f-2\,u_1f^2+1}{f^2}\,u_2^2 \\
&- 10\,\omega\,u_3(3\,u_1f+2\,u_1^2+2\,f^2) - 10\,\omega'(2\,f^2+u_1f+{u_1}^2)\,u_1u_2 \\
&+ 20\,\omega^2u_1({u_1}^3-1)(u_1+2\,f),
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
u_t = u_5 &+ \frac{5}{2}\,\frac{f-u_1}{f^2}\,u_2u_4 + \frac{5}{4}\,\frac{2\,f-u_1}{f^2}\,u_3^2 - 5\,c\,\frac{f^2+u_1^2}{\omega^2}\,u_3 + \frac{5}{4}\,\frac{4\,u_1^2-8\,u_1f+f^2}{f^4}\,u_2^2u_3 \\
&+ \frac{5}{16}\,\frac{2-9\,{u_1}^3+18\,u_1^2f}{f^6}\,u_2^4 - 10\,\omega\,(3\,u_1f+2\,u_1^2+2\,f^2)\,u_3 \\
&- \frac{5}{4}\,c\,\frac{11\,u_1f^2+2\,u_1^2f+2-5\,u_1^3}{\omega^2f^2}\,u_2^2 + 5\,\omega\,\frac{2\,u_1^3+u_1^2f-2\,u_1f^2+1}{f^2}\,u_2^2 \\
&+ 5\,c\,\omega'\,\frac{u_1^2+5\,f^2-2\,u_1f}{\omega^3}\,u_1u_2 - 10\,\omega'\,(2\,f^2+u_1f+{u_1}^2)\,u_1u_2 \\
&+ 20\,\omega^2u_1(u_1^3-1)(u_1+2\,f) + 40\,\frac{c\,u_1f^3(2\,u_1+f)}{\omega} + 5\,\frac{c^2u_1f^2(3\,u_1+f)(f-u_1)}{\omega^4}, \quad c\neq 0.
\end{aligned}
$$

Here $\lambda_1, \lambda_2, \mu, \mu_1, \mu_2$ *and* c *— are parameters, function* $f(u_1)$ *is solution of cubic algebraic equation*

$$(f + u_1)^2(2f - u_1) + 1 = 0, \tag{24}$$

and function $\omega(u)$ *— a nonconstant solution of differential eqiation*

$$\omega'^2 = 4\,\omega^3 + c. \tag{25}$$

Results for continious case.

Theorem

If hyperbolic type equation (8) *has a fifth order generalized symmetry* (9)-(23) *and not Darboux integrable then up to some point transformations it has the form* (26)-(31)*.*

★ For equation (13) we have the answer:

$$u_{xy} = 2f(u_y)u \qquad (26)$$

this equation can be found in [Sokolov Svinolupov 1995, Zhiber Sokolov 2001] For this equation (26) we can show a transformation:

$$f(u_y) + u_y = e^v, \quad u = 2v_x, \quad v_{xy} = e^v/3 + e^{-2v}/6$$

★ For equation(14) we have Tzitzeica equation:

$$u_{xy} = ae^u + be^{-2u}. \qquad (27)$$

★ For equation (19) with $\mu = 0$:

$$u_{xy} = 2f(u_y)\sqrt{u_x}. \tag{28}$$

In this case the transformations read

$$u_y = (2e^v + e^{-2v})/3, \quad u_x = v_x^2, \quad v_{xy} = (e^v - e^{-2v})/3;$$
$$u_x = w^2, \quad f(u_y) = w_y, \quad w_{xy} = 2f(w_y)w.$$

★ For equation (20) we have two answers:

$$u_{xy} = 2f(u_x)u, \tag{29}$$

$$u_{xy} = 2f(u_x)\sqrt{u_y}. \tag{30}$$

★ For equation (23):

$$u_{xy} = -2f(u_x)f(u_y)\frac{\omega'(u)}{\omega(u)}, \tag{31}$$

this equation appear in [Zhiber Sokolov 2001, Borisov Zykov Pavlov 2002]. Here the transformation to Tzitzeica equation is:

$$e^v = \frac{c - \sqrt{c}\omega'(u)}{4c(f(u_x) + u_x)(f(u_y) + u_y)\omega(u)}$$

Results for semidiscrete case.

Theorem

If semidiscrete hyperbolic type equation (1) *has a fifth order generalized symmetry* (9)-(23) *and not Darboux integrable then it has the form (S1)-(S4).*

★ For equation (15) semidiscrete hyperbolic type equation reads:

$$u_{n+1,x} - u_{n,x} = \lambda_1(e^{-2u_n} + e^{-2u_{n+1}}) + \lambda_2\sqrt{e^{2u_n} + e^{2u_{n+1}}} \tag{S1}$$

It was found in [G, Habibullin 2021]. Its disrete symmetry reads:

$$\begin{aligned} \partial_\tau u_n &= \left((v_n^2 - 1)^2 - 4v_{n-1}^2 T^{-1}\right)\frac{(v_{n+1}^2 + 1)(v_{n-1}^2 + 1)}{q_{n+1}q_n}, \\ q_n &= v_n^2(v_{n-1} + 1)^2 + (v_{n-1} - 1)^2, \\ v_n &= \sqrt{1 + e^{2(u_n - u_{n+1})}} + e^{u_n - u_{n+1}}. \end{aligned} \tag{32}$$

★ For equation (16) semidiscrete hyperbolic type equation is found:

$$u_{n+1,x} - u_{n,x} = \lambda_1(e^{u_n} + e^{u_{n+1}}) + \frac{\lambda_2}{\lambda_1}e^{-u_n - u_{n+1}}. \tag{S2}$$

Its discrete symmetry reads:

$$\frac{du_n}{d\tau} = (v_n - T^{-1})\frac{1}{v_n v_{n-1} + v_{n-1} + 1}\left(\frac{1}{v_{n+1}v_n + v_n + 1} + v_{n-1}\right), \qquad v_n = e^{u_{n+1} - u_n}. \tag{33}$$

★ For equation (21) semidiscrete hyperbolic type equation is found:

$$(f(u_{n+1,x}) + u_{n+1,x})\alpha^{1/2}(u_{n+1,0}) = (f(u_{n,x}) + u_{n,x})\beta^{1/2}(u_n). \tag{S3}$$

Its discrete symmetry has a rational form in variable z_n related to u_n by point transformation

$$\frac{(z_n^2 - 1)^2}{4z_n^2} = \frac{\alpha(u_n)}{\beta(u_n)} \tag{34}$$

Disrete equation has the form

$$\frac{dz_n}{d\tau} = \frac{z_n(z_n^2 - 1)}{z_n^2 + 1}(T - 1)((z_n^2 - 1)^2 - 4z_{n-1}^2 T^{-1})\frac{(z_{n+1}^2 + 1)(z_{n-1}^2 + 1)}{q_{n+1}q_n}, \qquad q_n = z_n^2(z_{n-1} + 1)^2 + (z_{n-1} - 1)^2. \tag{35}$$

★ For equation (22) semidiscrete hyperbolic type equation is found:

$$\frac{f(u_{n+1,x}) + u_{n+1,x}}{\alpha(u_{n+1,0})} = \frac{f(u_{n,x}) + u_{n,x}}{\beta(u_n)}. \tag{S4}$$

In this case discrete symmetry is rational in variable y_n:

$$y_n = \frac{\alpha(u_n)}{\beta(u_n)} \tag{36}$$

and reads:

$$\frac{dy_n}{d\tau} = y_n(T-1)(y_n - T^{-1})\frac{1}{q_n}\left(\frac{y_n}{q_{n+1}} + y_{n-1}\right), \qquad q_n = y_n y_{n-1} + y_{n-1} + 1. \tag{37}$$

In equations (S3),(S4) function $f(x)$ is solution of cubic equation (24), functions $\alpha(x), \beta(x)$ are diferent solutions of cubic equation:

$$\alpha(x) + \alpha^{-2}(x) = -3\omega(x)c^{-1/3}. \tag{38}$$

We can write function $\beta(x)$ in terms of $\alpha(x)$

$$\beta(x) = \frac{1 \pm (-c/27)^{-1/6}\alpha'(x)}{2\alpha^2(x)}.$$

Using a cubic equation for function f allows us to express $u_{n+1,x}$ or $u_{n,x}$ via other functions. For example from (S4) one has:

$$3u_{n+1,x} = 2(u_{n,x} + f(u_{n,x}))\frac{\alpha(u_{n+1})}{\beta(u_n)} + (u_{n,x} - 2f(u_{n,x}))\frac{\beta^2(u_n)}{\alpha^2(u_{n+1})},$$

$$3u_{n,x} = 2(u_{n+1,x} + f(u_{n+1,x}))\frac{\beta(u_n)}{\alpha(u_{n+1})} + (u_{n,x} - 2f(u_{n,x}))\frac{\alpha^2(u_{n+1})}{\beta^2(u_n)}.$$

On the relationship between the equations (S1)-(S4).

Pairs of equations (S1),(S3) and (S2),(S4) are connected by transformations invertible on solutions of discrete equtions, see [Yamilov 90, Startsev 2010]. To construct these transformations we use a form of equations (S1)-(S4). In first two equations lhs is full derivative with respect to x, while rhs doesn't depend on derivative; in the last two equations lhs contains only shifted variables, while rhs only variables without shifts. Using this properties we can construct a transformation.

Let us denote both sides of equation (4) by new shifted variable, in our case exact formula is:

$$\widetilde{u}_{n+1} = \log \frac{f(u_{n,x}) + u_{n,x}}{\beta(u_n)} - \log\gamma, \quad \gamma^3 = \frac{\lambda_1^2}{2\lambda_2}, \tag{39}$$

then $\widetilde{u}_n$ without shift is defined by:

$$\widetilde{u}_n = \log \frac{f(u_{n,x}) + u_{n,x}}{\alpha(u_n)} - \log\gamma. \tag{40}$$

Equations (39),(40) can be solve to $u_n, u_{n,x}$:

$$\alpha^3(u_n) = \frac{e^{\widetilde{u}_n - \widetilde{u}_{n+1}} + 1}{e^{2\widetilde{u}_n - 2\widetilde{u}_{n+1}}}, \quad u_{n,x} = \frac{2}{3}\gamma\alpha(u_n)e^{\widetilde{u}_n} + \frac{1}{3}\left(\gamma\alpha(u_n)e^{\widetilde{u}_n}\right)^{-2}. \tag{41}$$

Equating expressions for $u_{n,x}$ we get semidiscrete equation for $\widetilde{u}_n$. In case

$$c = -\frac{27\lambda_1^2\lambda_2^2}{16},$$

we obtain for $\widetilde{u}_n$ equation (S2).
Equations (S1),(S3) are connected by transformation:

$$\begin{aligned} \widetilde{u}_n = &\quad \log((f(u_{n,x}) + u_{n,x})\alpha^{1/2}(u_n)) - \log\gamma, \\ \widetilde{u}_{n+1} = &\quad \log((f(u_{n,x}) + u_{n,x})\beta^{1/2}(u_n)) - \log\gamma, \quad \gamma^3 = \frac{2\lambda_1}{\lambda_2}. \end{aligned} \tag{42}$$

Here u_n is unknown of equation (S3), and $\widetilde{u}_n$ is unknown of equation (S1). Inverse transform read:

$$\alpha^3(u_n) = \frac{e^{2\widetilde{u}_{n+1}-2\widetilde{u}_n} + 1}{e^{4\widetilde{u}_{n+1}-4\widetilde{u}_n}}, \quad u_{n,x} = \frac{2}{3}\gamma e^{\widetilde{u}_n}\alpha^{-1/2}(u_n) + \frac{1}{3}\left(\gamma e^{\widetilde{u}_n}\alpha^{-1/2}(u_n)\right)^{-2}. \tag{43}$$

Constants of equations must satisfy to relation:

$$c = -108\lambda_1^2\lambda_2^4.$$

Lax representation for (16), (S2).

We construct Lax representation for (16), (S2):

Lemma

Equations (16), (S2) have Lax representations

$$L_t = [L, A], \quad (T_n L)M = ML \tag{44}$$

with operators

$$L = \partial^3 - 3u_{n,x}\partial^2 - (u_{n,xx} - 2u_{n,x}^2 + \lambda_1^2 e^{2u_n} - \lambda_2 e^{-u_n})\partial. \tag{45}$$

$$A = 9(L^{5/3})_+ = 9\partial^5 - 45u_{n,x}\partial^4 - 15(4u_{n,xx} - 5u_{n,x}^2 + \lambda_1^2 e^{2u_n} - \lambda_2 e^{-u_n})\partial^3 + \dots, \tag{46}$$

$$\begin{gathered} M = e^{u_{n+1}-u_n}\left(\partial - 2u_{n,x} + \frac{\lambda_2}{\lambda_1} e^{-u_{n+1}-u_n} + \lambda_1 e^{u_{n+1}}\right) \\ *(\partial^2 - (u_{n,x} + \lambda_1 e^{u_{n+1}})\partial - \lambda_1^2(e^{2u_n} + e^{u_{n+1}+u_n})) \end{gathered} \tag{47}$$

Discrete Tzitzeica type equation.

In the case of discrete hyperbolic type equations

$$u_{n+1,m+1} = f(u_{n+1,m}, u_{n,m+1}, u_{n,m}) \tag{48}$$

both generalized symmetries are discrete equation of the form:

$$u_t = G(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}). \tag{49}$$

From compatibility conditions we cann't specify its form like in previous cases. Moreover, there are no complete results we don't have complete results on classification equations of the form (49). This problem was intensively investigated by V.Adler [2016, 2018]. The result to some subcases can be found in [Adler 2016] for Mobious invariant equations, in [G Yamilov Levi 2017, 2018] for linear with respect to $u_{\pm 2}$ equations. It should be noted that classification problem for third order equation $u_t = G(u_{n+1}, u_n, u_{n-1})$ was solved in [Yamilov 1983, 2006].

Classification problem for quad equation (48) doesn't solve even for equations with third order generalized symmetries. The partly results with wide lists of equations can be found in [Adler Bobenko Suris 2003, Viallet 2009, G Yamilov 2012]. For equations with fifth order generalized symmetries there are only particular examples.

★ The first such type example can be found in [Adler 2011]:

$$\begin{aligned} u_{n,m}u_{n+1,m+1}\big(u_{n+1,m}u_{n,m+1}c^{-1} - u_{n+1,m} - u_{n,m+1}\big) \\ +u_{n+1,m+1} + u_{n,m} - c = 0,\ \ c \neq 0, c \neq \pm 1. \end{aligned} \tag{50}$$

Its symmetry in n direction reads:

$$u_t = \frac{u(c-u)}{u_1 u u_{-1} - c}\left(\frac{u(c-u_1)(c-u_{-1})(u_2 u_1 - u_{-1}u_{-2})}{(u_2 u_1 u - c)(u u_{-1} u_{-2} - c)} - u_1 + u_{-1}\right). \tag{51}$$

★ Next example can be find in [Mikhailov Xenitidis 2013]:

$$(u_{n+1,m+1} + u_{n,m})u_{n+1,m}u_{n,m+1} + 1 = 0 \tag{52}$$

with symmetry:

$$u_t = u(T_n - 1)\frac{1}{(u_1 u u_{-1} - 1)(u u_{-1} u_{-2} - 1)} \tag{53}$$

★ Two more examples are in [Scimiterna Hay Levi 2014]:

$$u_{n,m}u_{n+1,m} + u_{n+1,m}u_{n,m+1} + u_{n,m+1}u_{n+1,m+1} - 1 = 0 \tag{54}$$

$$u_t = u(u_1 u - 1)(uu_{-1} - 1)(u_2 u_1 - u_{-1}u_{-2}) \tag{55}$$

$$u_\tau = \frac{u(u + u_{0,-1})(u_{0,1} + u)(u_{0,2} + u_{0,1} - u_{0,-1} - u_{0,-2})}{(u_{0,2} + u_{0,1} + u)(u_{0,1} + u + u_{0,-1})(u + u_{0,-1} + u_{0,-2})} \tag{56}$$

★

$$u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1}(1 + u_{n,m} + u_{n+1,m+1}) + \chi = 0 \tag{57}$$

with n-symmetry [Xenitidis 2018]

$$u_t = \frac{u(u+1)(u_1 u + \chi)(uu_{-1} + \chi)(u_2 u_1 - u_{-1}u_{-2})}{(u_2 u_1 u - \chi)(u_1 uu_{-1} - \chi)(uu_{-1}u_{-2} - \chi)} \tag{58}$$

Thank you for attention