Almost degenerate Rieman surfaces in the theory of anomalous waves in $1+1$ and $2+1$.

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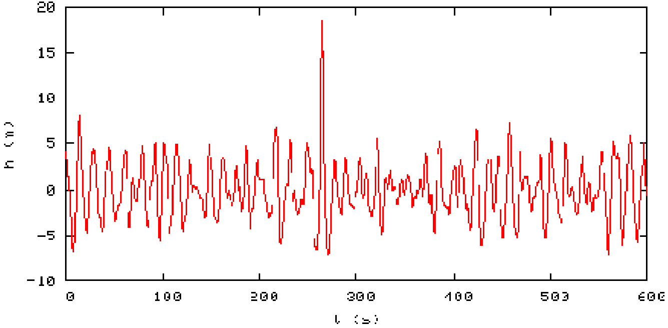
50 years of finite-gap integration, MIRAS, September 16, 2024

P. G. Grinevich, “Riemann surfaces close to degenerate ones in the theory of anomalous waves”, To appear in: Geometry, Topology, and Mathematical Physics, Collected papers. Dedicated to Academician Sergei Petrovich Novikov on the occasion of his 85th birthday, Trudy Mat. Inst. Steklova, 325, ed. V. M. Buchstaber, P. G. Grinevich, I. A. Dynnikov, O. K. Sheinman, Steklov Mathematical Institute of RAS, Moscow, 2024

The Draupner wave

Anomalous (rogue) waves were observed by sailors for long time, but these observations were treated as sailor's stories (like observations of mermaids).



The first confirmed observation of a rogue wave at the Draupner platform 160 kilometers southwest from the southern tip of Norway, January 1, 1995.

Rogue waves

Rogue waves were experimentally observed in

- Ocean;
- Water tanks;
- Photorefractive cristals;
- Optical fibers;
- Bose-Einstein condensates;
- Plasma physics;
- ...

Most popular point of view: generation of anomalous (rogue) waves is a non-linear phenomenon, involving modulation instability.
Alternative explanation of the Draupner wave origin:
F. Fedele, J. Brennan, S. Ponce de León, J. Dudley, F. Dias, "Real world ocean rogue waves explained without the modulational instability", *Scientific Reports*, **6** (2016), 27715, 11 pp.

Is it possible to use soliton integrable equations as rogue waves models?

3 classical paper about modulation instability in nonlinear medias

1) V. I. Bespalov and V. I. Talanov, “Filamentary structure of light beams in nonlinear liquids”, JETP Letters. **3** (12), 307, 1966.

2) T. B. Benjamin, J. E. Feir, “The disintegration of wave trains on deep water”. Part I. Theory, Journal of Fluid Mechanics 27 (1967) 417-430.

3) V. E. Zakharov, “Stability of period waves of finite amplitude on surface of a deep fluid”, Journal of Applied Mechanics and Technical Physics, 9(2) (1968) 190-194.

In papers 1) and 3) the Nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} \pm 2u^2\bar{u} = 0, \quad u = u(x,t) \in \mathbb{C}, \quad (x,t) \in \mathbb{R}^2, \tag{1}$$

was used as the basic mathematical model. If the sing “$+$”, we have self-focusing NLS, if the sign “$-$”, we have defocusing NLS. There is a big difference between this real forms from the analytic point of view.

Modulation instability is described by **self-focusing** NLS.

Zero-curvature representation

Integrability of self-focusing NLS equation (SfNLS)

$$iu_t + u_{xx} + 2u^2\bar{u} = 0, \quad u = u(x,t)$$

is based on the zero-curvature representation (Zakharov-Shabat):

$$\vec{\Psi}_x(\lambda,x,t) = U(\lambda,x,t)\vec{\Psi}(\lambda,x,t), \quad \vec{\Psi}_t(\lambda,x,t) = V(\lambda,x,t)\vec{\Psi}(\lambda,x,t),$$

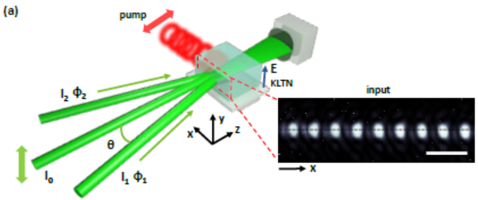
$$U = \begin{bmatrix} -i\lambda & iu(x,t) \\ i\overline{u(x,t)} & i\lambda \end{bmatrix},$$

$$V(\lambda,x,t) = \begin{bmatrix} -2i\lambda^2 + iu(x,t)\overline{u(x,t)} & 2i\lambda u(x,t) - u_x(x,t) \\ 2i\lambda\overline{u(x,t)} + \overline{u_x(x,t)} & 2i\lambda^2 - iu(x,t)\overline{u(x,t)} \end{bmatrix},$$

where

$$\vec{\Psi}(\lambda,x,t) = \begin{bmatrix} \Psi^1(\lambda,x,t) \\ \Psi^2(\lambda,x,t) \end{bmatrix}.$$

Pierangeli D., Flammini M., Zhang L., Marcucci G., Agranat A.J., Grinevich P.G., Santini P.M., Conti C., DelRe E. “Observation of Fermi-Pasta-Ulam-Tsingou recurrence and its exact dynamics”, *Physical Review X*, **8**:4 (2018), p. 041017 (9 pages);



The symmetric 3-wave interferometric scheme used to generate the background wave with a single-mode perturbation propagating in a pumped photorefractive KLTN (potassium-lithium-tantalate-niobate) crystal.
Since NLS is supposed to describe the above physics only at the leading order, one expects that the exact NLS RW recurrence be replaced by a “Fermi-Pasta-Ulam-Tsingou” - type recurrence, before thermalization destroys the pattern.

Periodic problem for the soliton equations. Finite-gap solutions.

The aim of this research was to develop some theory for the recurrence of anomalous waves and to compare it with the experiment.
Let us remark that in these experiments the anomalous waves are spatially-periodic.

The main method for constructing spatially-periodic solutions of the soliton equations is the finite-gap method.

S. P. Novikov, The periodic problem for the Korteweg-de Vries equation, Funct. Anal. Appl., **8**:3 (1974), 236–246.

B.A. Dubrovin, V.B. Matveev, A.R. Its, P. Lax, H. McKean, P. van Moerbeke, I.M. Krichever, ...
Korteweg-de Vries equation:

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x, \quad u = u(x,t), \quad u \in \mathbb{R}.$$

Auxiliary linear problem: the stationary 1-d Schrödinger operator

$$L\psi = E\psi, \quad L = -\partial_x^2 + u(x).$$

Periodic problem for the soliton equations. Finite-gap solutions.

Let $u(x)$ be spatially-periodic. We have quantum theory of an electron in 1-d crystals.

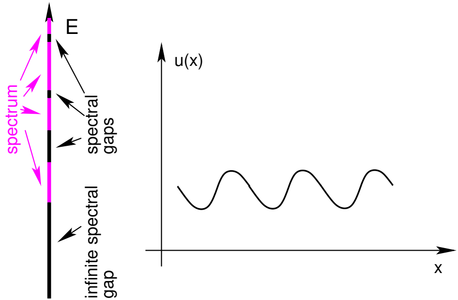


Figure: On the left: a periodic potential. On the right: the spectrum of the quantum particle in the periodic potential.

Periodic problem for the soliton equations. Finite-gap solutions.

The standard eigenfunctions basis in the solid state physics is formed by Bloch solutions

$$L\psi(x) = E\psi(x)$$
$$\psi(x+L) = \varkappa(E)\psi(x).$$

The energy E belongs to the spectrum iff $|\varkappa(E)| = 1$.

Novikov's starting point: the Bloch eigenfunctions are well-defined for complex energies, and they are meromorphic on Riemann surface Γ, which is a two-sheeted covering of the E-plane. The branch points are exactly the ends of the spectral gaps.

Periodic problem for the soliton equations. Finite-gap solutions.

For a generic smooth periodic potential $u(x)$, $u(x+L)=u(x)$

- The number of spectral gaps is infinite;
- The spectral gaps with large numbers are very small.

A potential $u(x)$ is called **finite-gap** if its spectrum has finite number of spectral gaps. Equivalently, the Bloch eigenfinction is meromorphic on a Riemann surface with finite number of branch points.

The spectral curve is invariant with respect to the Korteweg-de Vries dynamics. In particular, the branch points are conservation laws for KdV.

KdV solution corresponding to Riemann surface with finite number of branch points can be written explicitly in terms of the Riemann theta-functions associated to these surfaces. These solutions can be treated as nonlinear analogs of finite Fourier series.

The parameters in the theta-functional formulas are transcendental expressions in terms of the spectral data. Usually for practical applications these formulas require additional effectivization.

Anomalous (rogue) waves Cauchy problem for the Nonlinear Schrödinger equation

Fortunately, the generation of anomalous waves is described by special solutions, not by generic ones. For these solutions it is possible to derive good approximate formulas using the fact that the corresponding Riemann surfaces are close to the degenerate ones.

P.G. Grinevich, P.M. Santini "The finite gap method and the periodic NLS Cauchy problem of the anomalous waves, for a finite number of unstable modes", Russian Mathematical Surveys, 2019, v.74, No. 2.

Self-focusing NLS:

$$iu_t + u_{xx} + 2u^2\bar{u} = 0, \tag{2}$$

Anomalous waves Cauchy data

$$u(x,0) = a + \epsilon v(x), \quad v(x+L) \equiv v(x), \quad |\epsilon| \ll 1,$$

$$v(x) = \sum_{j\geq 1} \left(c_j e^{ik_j x} + c_{-j} e^{-ik_j x} \right), \quad k_j = \frac{2\pi}{L} j, \quad |c_j| = O(1),$$

Degeneration of Riemann surfaces

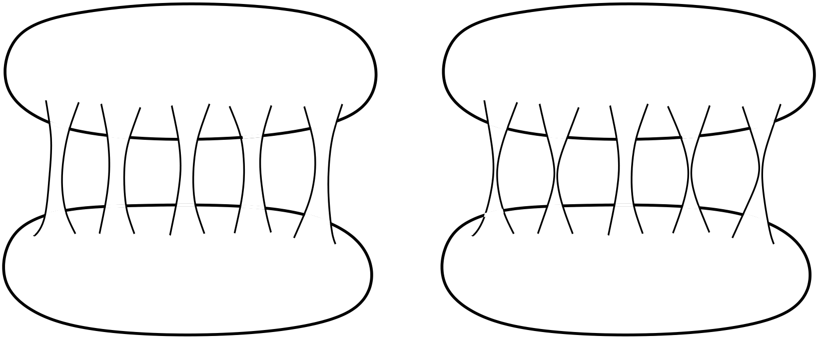


Figure: On the left: a regular Riemann surface. On the right: a degenerate Riemann surface.

Degenerate surfaces correspond to the N-breathers solutions.

We use Riemann surfaces, which are close to the degenerate ones, i.e. the handles are very thin.

Three steps

- We approximate generic periodic rogue wave type solution by a $2N$-gap one, where N the number of unstable modes.
- For any t we approximate the finite-gap solution up to $O(\epsilon)$ correction by N-soliton solutions, but this approximation depends on the time interval.
- We keep only “essential” solitons, i.e. we approximate our solution up to $O(\epsilon^p)$ error, $0 < p < 1$ by $\mathcal{N}(t)$ soliton solutions, $\mathcal{N}(t) \leq N$.

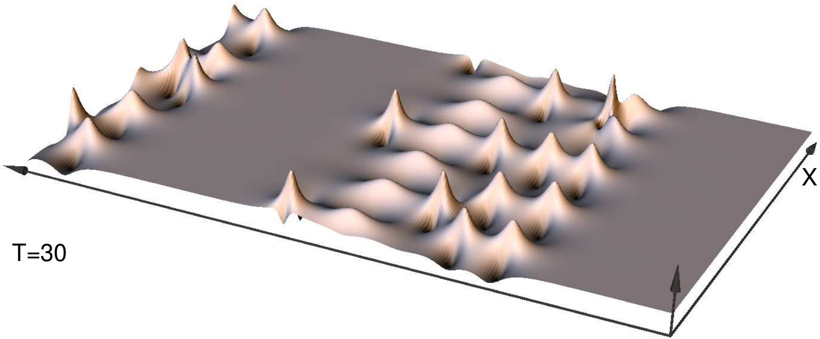


Figure: $L = 20$, $0 \leq t \leq 30$, 6 unstable modes, $\epsilon = 10^{-6}$, numeric simulation

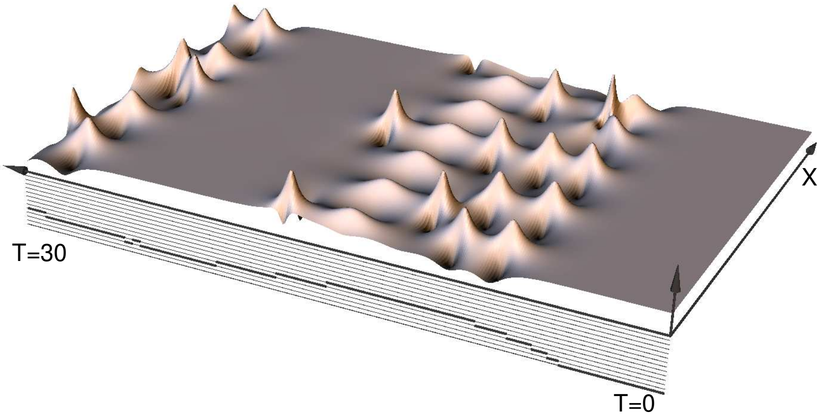


Figure: $L = 20$, $0 \leq t \leq 30$, 6 unstable modes, $\epsilon = 10^{-6}$, full hypercube finite-gap approximation

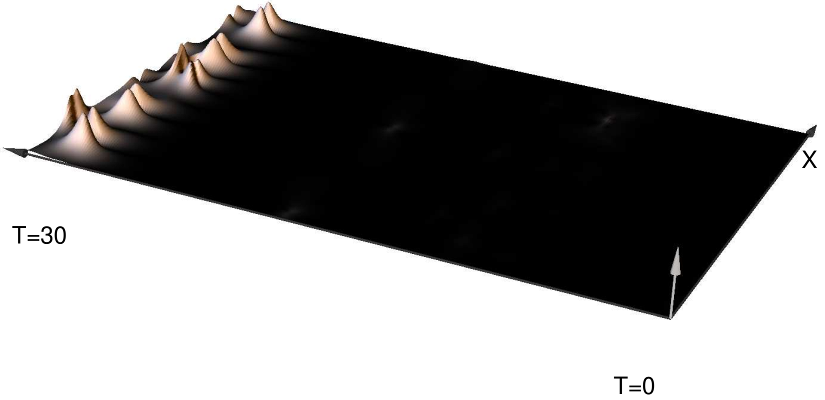


Figure: $L = 20$, $0 \le t \le 30$, 6 unstable modes, $\epsilon = 10^{-6}$, the difference between the numerics and full hypercube finite-gap approximation times 1000. The difference on the left of the picture is likely to be a numerical artifact

The auxiliary linear problem has the form:

$$\mathcal{L}\vec{\Psi}(\lambda, x, t) = \lambda\vec{\Psi}(\lambda, x, t),$$

$$\mathcal{L} = \left[\begin{array}{cc} i\partial_x & u(x,t) \\ \hline -u(x,t) & -i\partial_x \end{array} \right].$$

The operator $\mathcal{L}$ is not self-adjoint, and the spectrum of this problem typically contains complex points.

We consider the x-periodic problem for anomalous waves.:

$$u(x + L, t) = u(x, t).$$

$$u(x, 0) = a + \epsilon v(x), \quad v(x + L) \equiv v(x), \quad |\epsilon| \ll 1,$$

$$v(x) = \sum_{j\geq 1} \left(c_j e^{ik_j x} + c_{-j} e^{-ik_j x}\right), \quad k_j = \frac{2\pi}{L} j, \quad |c_j| = O(1),$$

The spectral data

In the periodic theory of the NLS equation the following two spectral problems are used to define the spectral data:

1. The spectral problem on the line, i.e. the spectral problem in $L^2(\mathbb{R})$. It is also called the **main spectrum**.
2. The spectral problem on the interval $[x_0, x_0 + L]$ with the following Dirichlet-type boundary conditions:

$$\Psi^1(\lambda, x_0, t) = \Psi^1(\lambda, x_0 + L, t) = 0.$$

   This spectrum is called the **auxiliary spectrum** or **divisor**.

Remark. Many authors use the following symmetric boundary condition:

$$\Psi^1(\lambda, x_0, t) + \Psi^2(\lambda, x_0, t) = \Psi^1(\lambda, x_0 + L, t) + \Psi^2(\lambda, x_0 + L, t) = 0.$$

This approach has the following advantage: all divisor points are located in a compact area of the spectral curve, but it requires one extra divisor point and increases the complexity of the formulas.

The spectral curve

To define the spectrum of the problem on the line, it is convenient to introduce the monodromy matrix. Consider the matrix equation

$$L\hat{\Psi}(\lambda, x, t_0) = \lambda\hat{\Psi}(\lambda, x, t_0),$$

here $\hat{\Psi}$ is a 2×2 matrix with the initial condition

$$\hat{\Psi}(\lambda, x_0, t_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

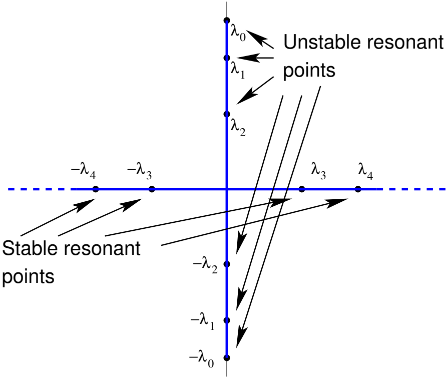
Then the monodromy matrix $\hat{T}(\lambda, x_0, t_0)$ is defined by:

$$\hat{T}(\lambda, x_0, t_0) = \hat{\Psi}(\lambda, x_0 + L, t_0).$$

The eigenvalues and eigenvectors of $T(\lambda, x_0, t_0)$ are defined on a two-sheeted covering of the λ-plane. This Riemann surface Γ is called the **spectral curve**.
The spectral curve Γ is well-defined and does not depend on time. The eigenvectors of $T(\lambda, x_0, t_0)$ are the Bloch eigenfunctions of L

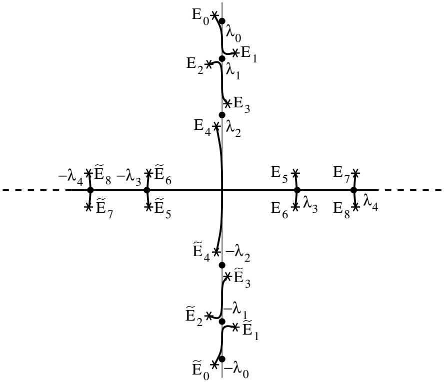
Unperturbed spectral curve

Let $\epsilon = 0$. Then the spectral curve has the following structure:



The number of resonant points on the imaginary axis, corresponding to the unstable modes of the linearised equation is **finite.** The number of resonant points on the real axis, corresponding to the unstable modes of the linearised equation is **infinite.**

Perturbed spectral curve

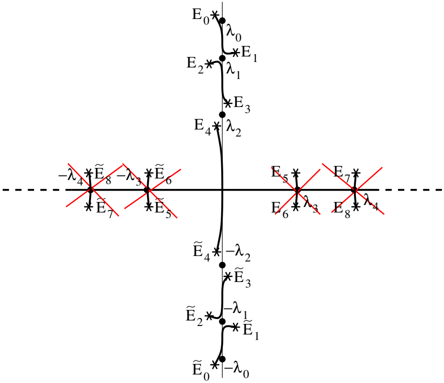


Γ is invariant with respect to the complex conjugation.

Perturbations of **real** resonant points generate **stable** perturbations of solutions. They can be neglected.

Perturbations of **imaginary** resonant points generate **exponentially growing** perturbations of solutions.

Finite-gap approximation



We keep all branch points and all divisor points located near the **imaginary** axis, and we neglect all branch points and all divisor points located near the **real** axis.

Finite-gap approximation

Using the scaling $x \to \alpha^2 x$, $t \to \alpha t$ the generic case can be reduced to the case $u_0(x,t) = e^{2it}$.

The number N of unstable modes is given by:

$$N = \left[\frac{L}{\pi}\right]$$

Let us denote

$$\phi_j = \arccos\left(\frac{\pi}{L} j\right) = \arccos\left(\frac{k_j}{2}\right), \quad j = 1, \ldots, N.$$

Then

$$\lambda_j = i \sin(\phi_j), \quad \mu_j = \cos(\phi_j), \quad \sigma_j = 2\sin(2\phi_j),$$

$$\alpha_j = \epsilon\left(\overline{c_j} - e^{2i\phi_j} c_{-j}\right), \quad \beta_j = \epsilon\left(\overline{c_{-j}} - e^{-2i\phi_j} c_j\right),$$

where σ_j is the linear increment of the unstable mode:

$$\sigma_j = k_j \sqrt{4 - k_j^2}, \quad 1 \le j \le N,$$

Spectral data for a small perturbation of the constant solution

Let us define:

$$\alpha_j = \left(\overline{c_j} - (\mu_j + \lambda_j)^2 c_{-j}\right), \quad \beta_j = \left(\overline{c_{-j}} - (\mu_j - \lambda_j)^2 c_j\right),$$

$$\mu_j = \frac{\pi j}{L}, \quad \lambda_j = \pm\sqrt{\mu_j^2 - 1}, \quad \mathrm{Re}\,\lambda_j + \mathrm{Im}\,\lambda_j > 0, \quad j = 1, 2, \ldots \infty.$$

Theorem (Direct spectral transform):

$$E_0 = i + O(\epsilon^2), \quad E_l = \lambda_j \mp \frac{\epsilon}{2\lambda_j}\sqrt{\alpha_j \beta_j} + O(\epsilon^2), \quad l = 2j-1, 2j,$$

$$\lambda(\gamma_n) = \lambda_n + \frac{\epsilon}{4\lambda_n}\left[\alpha_n + \beta_n\right] + O(\epsilon^2), \quad p(\gamma_n) = \frac{\epsilon}{4\mu_n}\left[\alpha_n - \beta_n\right] + O(\epsilon^2)$$

Here $\exp(\pm i L p(\gamma_n))$ are the Bloch multiplies for the Dirichlet spectrum.
Homoclinic orbit in this approximation: $\alpha_j \neq 0$, $\beta_j = 0$.

The theta-functional solutions

A. R. Its., V. P. Kotljarov "Explicit formulas for solutions of the Nonlinear Schrödinger equation" — Dokl. Ukrain. SSR, Ser. A, no. 11, (1976), 965–968.

$$u(x,t) = \frac{\theta(\vec{A}(\infty_-) - \vec{U}_1 x - \vec{U}_2 t - \vec{A}(D) - \vec{K})}{\theta(\vec{A}(\infty_+) - \vec{U}_1 x - \vec{U}_2 t - \vec{A}(D) - \vec{K})} \times \qquad (3)$$

$$\times \frac{\theta(\vec{A}(\infty_+) - \vec{A}(D) - \vec{K})}{\theta(\vec{A}(\infty_-) - \vec{A}(D) - \vec{K})} \cdot u(0,0) \cdot \exp(2it)(1 + O(\epsilon^2)),$$

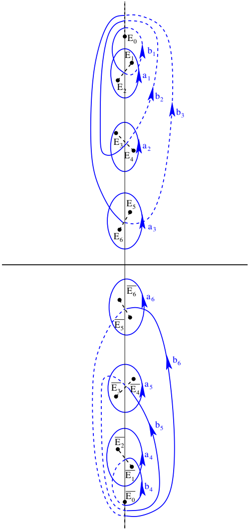
$$\theta(z|B) = \sum_{n_j} \exp\left[2\pi i \sum_j n_j z_j + \pi i \sum_{j,k} b_{jk} n_j n_k\right], \quad j,k = 1,\ldots,g.$$

Here $\vec{A}(\gamma)$ denotes the Abel transform, $\vec{K}$ is the vector of Riemann constants, B denotes the matrix of periods, $\vec{U}_1$, $\vec{U}_2$ are some periods of meromorphic differentials.

We need some explicit approximate formulas.

Approximation of spectral curve

We use the following basis of cycles:



The Riemann period matrix

For the matrix of period we obtain (here $g = 2N$):

$$b_{jj} = \frac{1}{\pi i}\log\left[\frac{\epsilon\sqrt{\alpha_j\beta_j}}{4\sin(2\phi_j)\cos(\phi_j)}\right] + O(\epsilon),\quad b_{j+N,j+N} = -\overline{b_{j,j}},\quad 1\le j\le N,$$

$$b_{kj} = \frac{1}{\pi i}\log\left|\frac{\sin\left(\frac{\hat\phi_l-\hat\phi_s}{2}\right)}{\cos\left(\frac{\hat\phi_l+\hat\phi_s}{2}\right)}\right| + O(\epsilon^2)\ \text{ for all }\ j\neq k.$$

$$\exp\left(\pi i b_{jj}\right) = \frac{\epsilon\sqrt{\alpha_j\beta_j}}{4\sin(2\phi_j)\cos(\phi_j)} + O(\epsilon^2),\quad 1\le j\le N,$$

Final finite-gap formulas

One can choose integration path so that

$$u(x,t) = \exp(2it)\frac{\theta(\vec{z}_+(x,t)|B)}{\theta(\vec{z}_-(x,t)|B)} \times (1 + O(\epsilon)) \qquad (4)$$

$$\vec{z}_\pm(x,t) = \mp\vec{A}(\infty_-) - \vec{U}_1 x - \vec{U}_2 t - \vec{A}(D)$$

$$\vec{A}(\infty_-) = \begin{bmatrix} -\frac{1}{4} - \frac{\phi_1}{2\pi} \\ \vdots \\ -\frac{1}{4} - \frac{\phi_N}{2\pi} \\ \hline -\frac{1}{4} + \frac{\phi_1}{2\pi} \\ \vdots \\ -\frac{1}{4} + \frac{\phi_N}{2\pi} \end{bmatrix} + O(\epsilon^2), \quad \vec{A}(D) = \begin{bmatrix} \frac{1}{2\pi i}\log\left[\frac{\alpha_1}{\sqrt{\alpha_1\beta_1}}\right] \\ \vdots \\ \frac{1}{2\pi i}\log\left[\frac{\alpha_N}{\sqrt{\alpha_N\beta_N}}\right] \\ \hline \frac{1}{2\pi i}\log\left[\frac{e^{-2i\phi_1}\overline{\beta_1}}{\sqrt{\overline{\alpha_1\beta_1}}}\right] \\ \vdots \\ \frac{1}{2\pi i}\log\left[\frac{e^{-2i\phi_N}\overline{\beta_N}}{\sqrt{\overline{\alpha_N\beta_N}}}\right] \end{bmatrix} + \ldots,$$

Final finite-gap formulas

$$\vec{U}_1 = \begin{bmatrix} -\frac{\cos(\phi_1)}{\pi} \\ -\frac{\cos(\phi_2)}{\pi} \\ \vdots \\ -\frac{\cos(\phi_N)}{\pi} \\ \hline -\frac{\cos(\phi_1)}{\pi} \\ -\frac{\cos(\phi_2)}{\pi} \\ \vdots \\ -\frac{\cos(\phi_N)}{\pi} \end{bmatrix} + O(\epsilon^2), \quad \vec{U}_2 = \begin{bmatrix} -\frac{\sin(2\phi_1)}{\pi} \\ -\frac{\sin(2\phi_2)}{\pi} \\ \vdots \\ -\frac{\sin(2\phi_N)}{\pi} \\ \hline \frac{\sin(2\phi_1)}{\pi} \\ \frac{\sin(2\phi_2)}{\pi} \\ \vdots \\ \frac{\sin(2\phi_N)}{\pi} \end{bmatrix} + O(\epsilon^2),$$

$$\theta(z|B) = \sum_{n_j} \exp\left[2\pi i \sum_j n_j z_j + \pi i \sum_{j,k} b_{jk} n_j n_k\right], \quad j,k = 1,\ldots,2N.$$

Final finite-gap formulas

Here:
N is the number of unstable modes:

$$N = \left[\frac{L}{\pi}\right],$$

$$\phi_j = \arccos\left(\frac{\pi}{L} j\right) = \arccos\left(\frac{k_j}{2}\right), \quad j = 1, \ldots, N,$$

$$\lambda_j = i\sin(\phi_j), \quad \mu_j = \cos(\phi_j), \quad \sigma_j = 2\sin(2\phi_j),$$

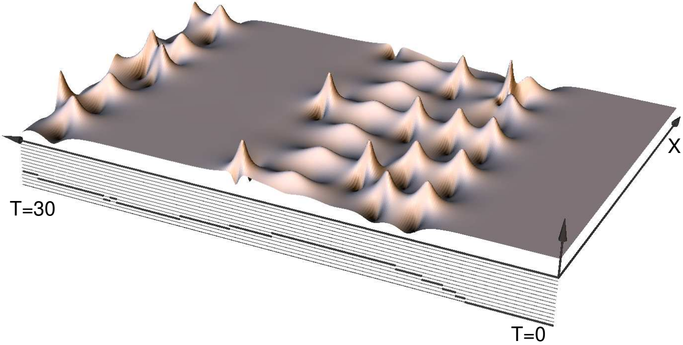
$$\alpha_j = \epsilon\left(\overline{c_j} - e^{2i\phi_j} c_{-j}\right), \quad \beta_j = \epsilon\left(\overline{c_{-j}} - e^{-2i\phi_j} c_j\right),$$

σ_j is the linear increment of the unstable mode:

$$\sigma_j = k_j \sqrt{4 - k_j^2}, \quad 1 \leq j \leq N,$$

Final finite-gap formulas

For each t only one hypercube in the sum for the theta-function is essential, and this hypercube can be calculated explicitly. Therefore the time line is covered by intervals, and at each interval the solution can be approximate by the N-breather one.



Remark. The appearance of “whiskered” tori in this problem was discussed in the literature, see, for example:

McLaughlin, D., Overman, E. A. “Whiskered tori for integrable PDEs: Chaotic behavior in near integrable PDEs”, In Surveys in applied mathematics, Vol. 1, pp. 83-203 (1994). Plenum Press.

A. Calini, N. M. Ercolani, D. W. McLaughlin, C. M. Schober, “Mel’nikov analysis of numerically induced chaos in the nonlinear Schrödinger equation”, *Phys. D,* **89**:3-4 (1996), 227–260.

In particular, it was shown that the lengths of tori increases as $\log(|\epsilon|)$, but constants in the formulas were not calculated.

One unstable mode

We derived approximate formulas for finite number N of unstable mode. In particular, we proved that solutions at each time are well-approximated by n-breather solutions ($n \leq N$), but different approximations at different time intervals shall be used.

Let us discuss the first non-trivial case $N = 1$.

We assume: $\pi/|a| < L < 2\pi/|a|$ i.e. we have exactly one unstable mode. It corresponds to the following perturbation of the background:

$$u(x,0) = a\left(1 + \epsilon\left(c_1 e^{k_1 x} + c_{-1} e^{-ik_1 x}\right)\right), \quad k_1 = \frac{2\pi}{L}, \quad \epsilon \ll 1,$$

where c_1 and c_{-1} are arbitrary $O(1)$ complex parameters.

Problem: Calculate the time of the first rogue wave appearance and its position. Calculate the periodicity of appearances in terms of the Cauchy data.

The unstable mode is described by Riemann theta functions of 2 variables.

But for this special Cauchy data it admits a good approximation as a sequence of Akhmediev breathers (Grinevich–Santini).

Akhmediev breathers:

N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, “Exact first order solutions of the Nonlinear Schödinger equation”, *Theor. Math. Phys,* **72**, 809 (1987).

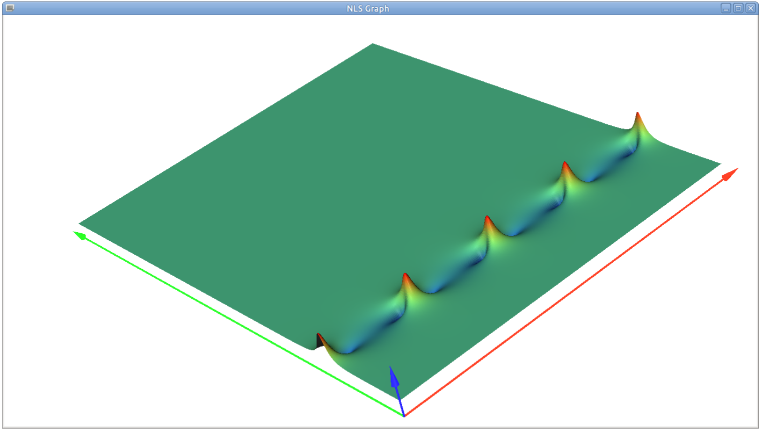
$$\mathcal{A}(x,t;\theta,X,T)=$$

$$= a\, e^{2i|a|^2 t}\cdot\frac{\cosh[\sigma(\theta)(t-T)+2i\theta]+\sin\theta\cos[k(\theta)(x-X)]}{\cosh[\sigma(\theta)(t-T)]-\sin\theta\cos[k(\theta)(x-X)]},$$

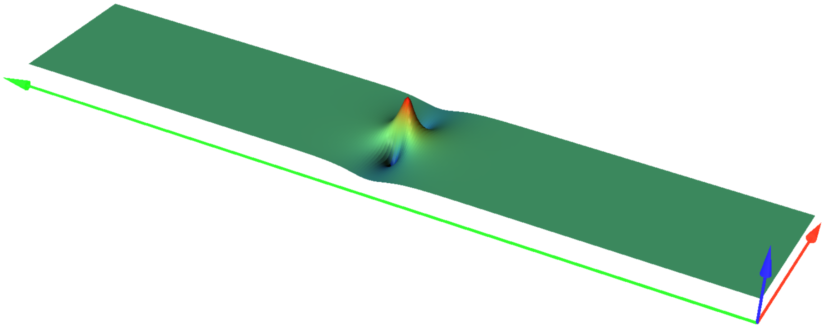
$$k_1=k(\theta)=2|a|\cos\theta,\ \sigma(\theta)=k(\theta)\sqrt{4|a|^2-k^2(\theta)}=2|a|^2\sin(2\theta),$$

Akhmediev breathers

They are spatially periodic and localized in time:



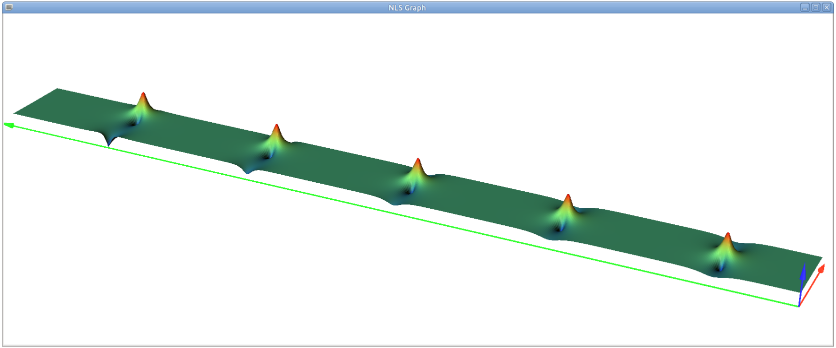
The x coordinate axis marked red, the t coordinate axis marked green. In the future we draw only one period of solution with respect to x.



Single Akhmediev (Akhmediev-Eleonskii-Kulagin) breather ($L = 6$).

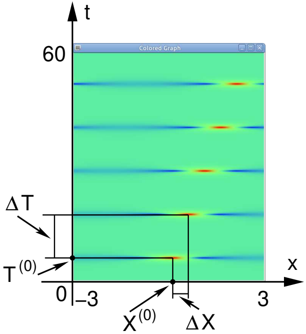
One unstable mode

Generic solution for one unstable mode is well-approximated by a sequence of Akhmediev breathers:



Recurrence of Akhmediev breathers for one unstable mode ($L = 6$). Here we draw exactly one period in the x-variable.

One unstable mode



Recurrence of Akhmediev breathers for one unstable mode ($L = 6$).

Essential parameters:

1. First appearance time $T^{(0)}$;
2. position of maximum at first appearance $X^{(0)}$;
3. interval between subsequent appearances ΔT;
4. phase shift between subsequent appearances ΔX.

Approximation of the genus 2 solution:

$$u(x,t) = \sum_{m=0}^{n} \mathcal{A}\left(x,t;\phi_1,x^{(m)},t^{(m)}\right)e^{i\rho^{(m)}} - \frac{1-e^{4in\phi_1}}{1-e^{4i\phi_1}}ae^{2i|a|^2 t}, \quad x\in[0,L],$$

where:

$$x^{(m)} = X^{(1)} + (m-1)\Delta X, \quad t^{(m)} = T^{(1)} + (m-1)\Delta T,$$

$$X^{(1)} = \frac{\arg\alpha}{k_1} + \frac{L}{4}, \quad \Delta X = \frac{\arg(\alpha\beta)}{k_1}, \quad (\bmod\ L),$$

$$T^{(1)} = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2|a|^4\epsilon|\alpha|}\right), \quad \Delta T = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4|a|^8\epsilon^2|\alpha\beta|}\right),$$

$$\rho^{(m)} = 2\phi_1 + (m-1)4\phi_1, \quad n = \left\lfloor \frac{T-T^{(1)}}{\Delta T} + \frac{1}{2}\right\rfloor,$$

$$\cos\phi_1 = \frac{\pi}{L|a|}, \quad k_1 = \frac{2\pi}{L} = 2|a|\cos(\phi_1), \quad \sigma_1 = k_1\sqrt{4|a|^2 - k_1^2} = 2|a|^2\sin(2\phi_1),$$

$$\alpha = e^{-i\phi_1}\overline{c_1} - e^{i\phi_1}c_{-1}, \quad \beta = e^{i\phi_1}\overline{c_{-1}} - e^{-i\phi_1}c_1.$$

The spectra curve has genus $g = 2$ and 6 branch points: E_0, E_1, E_2, $\bar{E}_0$, $\bar{E}_1$, $\bar{E}_2$. The pair E_1, E_2 is obtained as a results of splitting the resonant point $\lambda_1 = i|a|\sin\phi_1$:

$$E_l = \lambda_1 + (-1)^l \frac{\epsilon|a|^2}{2\lambda_1}\sqrt{\alpha\beta} + O(\epsilon^2), \quad l = 1, 2,$$

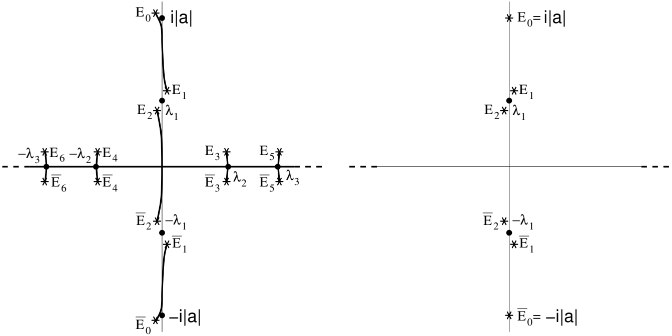


Figure: Right: the exact spectrum; Left: the approximating curve.

Extension to two spatial dimensions

In some situations, the one-dimensional approximation may be not relevant from the physical point of view. For example, ocean waves are essentially two-dimensional.

Problems:

- The list of integrable equations in 2+1 is much shorter;
- The multidimensional soliton systems are usually non-local;
- In the finite-gap approach one has to work with generic Riemann surfaces instead of ramified coverings (Krichever). It results in additional technical difficulties.

Our aim: to extend the aforementioned results to 2 spatial dimensions.

Why Davey-Stewardson equation?

- It is a completely integrable 2+1 sysytem;
- At least some of real forms of this equation arise in physics;
- One of the real forms admits rogue waves type solutions. We hope this real form is also physically relevant.

Davey-Stewardson equation

Davey-Stewardson equation - an integrable 2+1 system:

$$
\begin{aligned}
& iu_t + u_{xx} - \nu^2 u_{yy} + 2\eta q u = 0, \quad \eta = \pm 1, \quad \nu^2 = \pm 1, \\
& q_{xx} + \nu^2 q_{yy} = \left(|u|^2\right)_{xx} - \nu^2 \left(|u|^2\right)_{yy}, \\
& u = u(x,y,t) \in \mathbb{C}, \quad q = q(x,y,t) \in \mathbb{R},
\end{aligned}
$$

Zero-curvature representation:

$$
\begin{aligned}
\nu \vec{\psi}_y &= i\sigma_3 \vec{\psi}_x + U\vec{\psi}, \\
\vec{\psi}_t &= 2i\sigma_3 \vec{\psi}_{xx} + 2U\vec{\psi}_x + V\vec{\psi},
\end{aligned}
$$

where

$$
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & u \\ -\eta\bar{u} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -\eta(w - iq) & u_x - i\nu u_y \\ -\eta(\bar{u}_x + i\nu\bar{u}_y) & -\eta(w + iq) \end{pmatrix},
$$

$$
\nu w_y = (q - |u|^2)_x, \quad w_x = -\nu(q + |u|^2)_y.
$$

NLS:

$$iu_t + u_{xx} \pm 2u^2\bar{u} = 0, \quad u = u(x,t),$$

DS:

$$iu_t + u_{xx} - \nu^2 u_{yy} + 2\eta q u = 0, \quad \eta = \pm 1, \quad \nu^2 = \pm 1,$$
$$q_{xx} + \nu^2 q_{yy} = \left(|u|^2\right)_{xx} - \nu^2 \left(|u|^2\right)_{yy}, \quad u = u(x,y,t).$$

If $u(x,y,t)$ does not depend on y, DS coincide with NLS.

- DS1: $\nu = i$, the spectral problem is hyperbolic;
- Defocusing DS2: $\nu = 1$, $\eta = -1$ the spectral problem is elliptic;
- Focusing DS2: $\nu = 1$, $\eta = 1$ the spectral problem is elliptic;

For the focusing DS-2 with the doubly-periodic boundary conditions our program can be fulfilled (Grinevich-Santini).

Effect of non-locality

DS2 equation is non-local, and the non-local term is defined by:

$$q_{xx} + q_{yy} = \left(|u|^2\right)_{xx} - \left(|u|^2\right)_{yy},$$

therefore the function q is defined up to an arbitrary integration constant

$$q(x, y, t) \to q(x, y, t) + f(t).$$

This change of integration constant corresponds to the standard gauge transformation:

$$u(x, y, t) \to u(x, y, t) \exp\left(-i\frac{\eta}{2}\int^{t} f(\tau)d\tau\right).$$

Without loss of generality we may assume

$$\iint_{T^2} q(x, y, t)dxdy = 0, \text{ for all } t.$$

Periodic problem for 2-D Dirac operator and DS2 solutions were studied in a series of works by Taimanov (partially in joint works with S.P. Tzarev and R.M. Matuev).

- Application to the geometry of tori in $\mathbb{R}^3$ and $\mathbb{R}^4$;
- Proof of the existence of spectral curve using Keldysh theorem;
- Generation of singular solutions as a result of conformal transformations of $\mathbb{R}^4$;
- Construction of DS2 solutions using Moutard transformations;
- The role of curves with double points in the theory of regular doubly-periodic potential. We show that these points correspond to unstable modes/

Direct spectral transform for doubly-periodic 2-D Schrödinger operator was developed by Krichever. We essentially follow his approach.

We assume that the Cauchy data is a small perturbation of a constant solution

$$u(x,y,0) = a + \varepsilon v_0(x,y), \quad \varepsilon \in \mathbb{R}, \quad \varepsilon \ll 1,$$
$$v_0(x+L_x,y) = v_0(x,y+L_y) = v_0(x,y).$$

Decompose $v_0(x,y)$ into Fourier series:

$$v_0(x,y) = \sum_{n_x,n_y \neq (0,0)} c_{n_x,n_y} e^{i(k_x x + k_y y)},$$

where

$$k_x = n_x \frac{2\pi}{L_x}, \quad k_y = n_x \frac{2\pi}{L_y}, \quad n_x, n_x \in \mathbb{Z}.$$

The spectral curve cam be obtained by attaching thin handles to resonant pairs:

1. Resonant pairs corresponding to the unstable modes $k_x^2 + k_y^2 < 4$:

$$\tau_1 = \frac{k_x + ik_y}{2}\left[-1 \pm i\sqrt{\frac{4 - k_x^2 - k_y^2}{k_x^2 + k_y^2}}\right], \tag{5}$$

$$\tau_2 = \frac{k_x + ik_y}{2}\left[1 \pm i\sqrt{\frac{4 - k_x^2 - k_y^2}{k_x^2 + k_y^2}}\right], \quad |\tau_1| = |\tau_2| = 1$$

2. Resonant pairs corresponding to the stable modes $k_x^2 + k_y^2 > 4$;

$$\tau_1 = \frac{k_x + ik_y}{2}\left[-1 + \sqrt{\frac{k_x^2 + k_y^2 - 4}{k_x^2 + k_y^2}}\right], \quad \tau_2 = -\frac{1}{\bar{\tau}_1}. \tag{6}$$

In our approximation we keep only the unstable modes.

Example

Example: Let $L_x = 2\pi/1.2$, $L_y = 2\pi/1.4$. Then $k_x = 1.2n_x$, $k_y = 1.4n_y$:

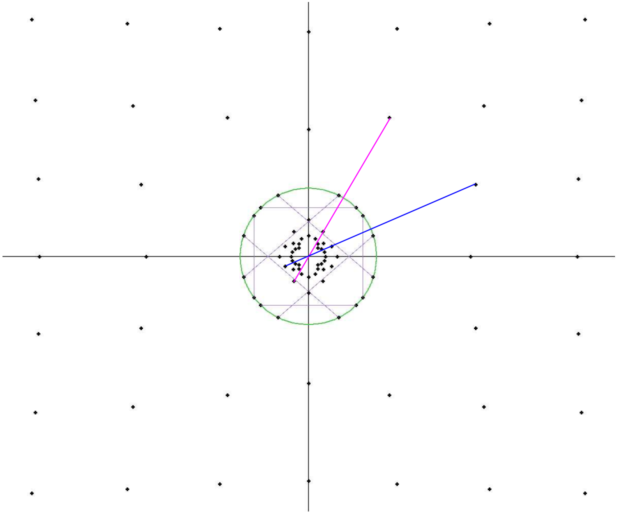


Figure: Resonant pairs for $-3 \le n_x \le 3$, $-3 \le n_y \le 3$.

The basic cycles on the unperturbed curve:

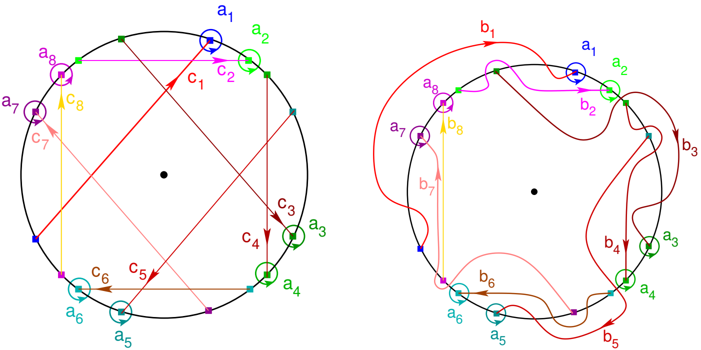


Figure: *On the right: the system of a and c-cycles for this example. On the left: the corresponding system of a and b cycles. Here:*

$$b_j = c_j - \sum_{k>j} (c_j \circ c_k) c_k,$$

Direct spectral transform

The perturbed curve: we cut the τ-plane along the intervals (E_{2j-1}, E_{2j}). For each resonant pair (τ_{2j-1}, τ_{2j}) we glue the borders of the cuts (E_{4j-3}, E_{4j-2}) and (E_{4j-2}, E_{4j}). The point E_{4j-3} is glued to E_{4j-1}, and the point E_{4j-2} is glued to E_{4j}. The cycle a_j is the oval surrounding the cut (E_{4j-2}, E_{4j}) and oriented counterclockwise, the cycle c_j is the union of oriented intervals $[E_{4j-3}, 0]$ and $[0, E_{4j-1}]$, the cycles b_j are as above.

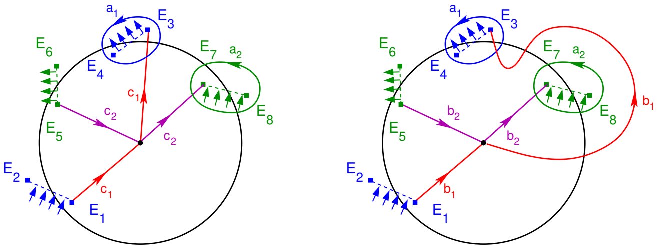


Figure: *The perturbed curve. The borders of the cuts (E_1, E_2) and (E_5, E_6) are glued to the borders of the cuts (E_3, E_4) and (E_7, E_8) respectively.*

Denote:

$$-\alpha_j = \frac{\bar{c}_j + \bar{\tau}_{2j-1}\tau_{2j}c_{-j}}{2q_{2j-1}}, \quad \beta_j = \frac{\bar{c}_{-j} + \bar{\tau}_{2j}\tau_{2j-1}c_j}{2q_{2j}}. \qquad (7)$$

Theorem:

$$E_{4j-4+k} = \tau_{2j-1} + (-1)^{k-1}\frac{2\tau_{2j-1}q_{2j}}{i\,\mathrm{Im}(\tau_{2j}\bar{\tau}_{2j-1})}\varepsilon\sqrt{\alpha_j\beta_j}, \qquad (8)$$

$$E_{4j-2+k} = \tau_{2j-1} + (-1)^{k-1}\frac{2\tau_{2j}q_{2j-1}}{i\,\mathrm{Im}(\tau_{2j}\bar{\tau}_{2j-1})}\varepsilon\sqrt{\alpha_j\beta_j},,$$

Theorem: In the leading order:

$$b_{jj} = \log\left[\varepsilon^2\frac{\tau_{2j-1}\tau_{2j}q_{2j-1}q_{2j}}{\mathrm{Im}^2(\tau_{2j}\tau_{2j-1}^{-1})(\tau_{2j-1}-\tau_{2j})^2}\alpha_j\beta_j\right],$$

$$b_{jk} = \log\left[\frac{(\tau_{2j}-\tau_{2k})(\tau_{2j-1}-\tau_{2k-1})}{(\tau_{2j}-\tau_{2k-1})(\tau_{2j-1}-\tau_{2k})}\right], \quad k\neq j.$$

Leading order solution

$$u(z,t)=\frac{\theta(\vec{A}(\infty_2)+\vec{W}_z z+\vec{W}_{\bar z}\bar z+\vec{W}_t t-\vec{A}(\mathcal{D})-\vec{K})}{\theta(\vec{A}(\infty_1)+\vec{W}_z z+\vec{W}_{\bar z}\bar z+\vec{W}_t t-\vec{A}(\mathcal{D})-\vec{K})},$$

where

$$\left[\vec{A}_{E_{4j-3}}(\gamma_j)\right]_k=\begin{cases}0 & k\neq j\\ \log\left[\frac{\alpha_j}{\sqrt{\alpha_j\beta_j}}\right] & k=j.\end{cases}$$

$$(W_z)_j=\frac{i}{2}\left[\bar\tau_{2j}-\bar\tau_{2j-1}\right],\ \ (W_{\bar z})_j=\frac{i}{2}\left[\tau_{2j}-\tau_{2j-1}\right], \tag{9}$$

$$(W_t)_j=\mathrm{Im}(\tau_{2j-1}^2-\tau_{2j}^2), \tag{10}$$

$$A_j(\infty_2)-A_j(\infty_1)=\log\left[\frac{\tau_{2j-1}}{\tau_{2j}}\right]=\log\left[\tau_{2j-1}\bar\tau_{2j}\right]. \tag{11}$$

Vector of Riemann constants

$$K_j = \frac{b_{jj}}{2} - \pi i + A_j(E_{4j-3}) + O(\epsilon).$$

$$A_j(E_{4j-3}) = -\log\left[\varepsilon \frac{\tau_{2j} q_{2j}}{i\,\mathrm{Im}(\tau_{2j}\tau_{2j-1}^{-1})(\tau_{2j-1} - \tau_{2j})} \sqrt{\alpha_j \beta_j}\right],$$

$$b_{jj} = \log\left[\varepsilon^2 \frac{\tau_{2j-1}\tau_{2j} q_{2j-1} q_{2j}}{\mathrm{Im}^2(\tau_{2j}\tau_{2j-1}^{-1})(\tau_{2j-1} - \tau_{2j})^2} \alpha_j \beta_j\right].$$

Remark. We used the non-symmetric normalization of the wave function $\Psi^1(\gamma, 0, 0) \equiv 1$. The potential is defined by the spectral data up to an arbitrary phase factor $e^{i\phi(t)}$. It is in good agreement with the DS-2 gauge freedom.