QUADRATIC EXPONENTIALS
and its applications: finite sums, integral transforms, series.

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Plan of the lecture

QUADRATIC EXPONENTIALS:

1. Finite sums
2. Integral transform
3. Series

$$L_1(x, y) = \exp(axy)$$

$$Q_2(x, y) = \exp(ax^2 + by^2 + cxy)$$

Gauss, Fresnel

QUADRATIC EXPONENTIALS: Finite sums

Carl Friedrich Gauss (1777–1855)

QUADRATIC EXPONENTIALS: Finite sums

Geometric Series: $L_n(x) = \sum_{k=0}^{n} \exp(ixk)$

Gauss Sum: $Q_n(x) = \sum_{k=0}^{n} \exp(ixk^2)$

Just find a sign...11 years. Law of quadratic reciprocity.
Connections: Jacobi theta functions, Legendre symbol.

Qubic Sum: $S_n(x) = \sum_{k=0}^{n} \exp(ixk^3)$

Berndt, B. C.; Evans, R. J.; Williams, K. S. (1998). Gauss and Jacobi Sums. Canadian Mathematical Society Series of Monographs and Advanced Texts. Wiley.

QUADRATIC EXPONENTIALS: Finite sums

Discrete Fourier Transform

$$f_{kj} = \frac{1}{\sqrt{n}} \exp(-i\frac{2\pi kj}{n}),\ 0 \le k \le n-1,\ 0 \le j \le n-1.$$

Trace=Gauss sum!!!

$n = 4$

$$1/2 \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

QUADRATIC EXPONENTIALS: Finite sums

n	***1***	***i***	***-1***	***-i***
2	1	0	1	0
3	1	1+	1	0
4	2	0	1	**1**+
5	2	**1**+	1	1
6	2	1	**2**+	1
7	2	1	2	**2**+
8	**3**+	1	2	2
9	3	**2**+	2	2
10	3	2	**3**+	2
11	3	2	3	**3**+
12	**4**+	2	3	3
13	4	**3**+	3	3
14	4	3	**4**+	3
15	4	3	4	**4**+
16	**5**+	3	4	4

QUADRATIC EXPONENTIALS: Finite sums

n	1	i	-1	- i
4N	***N+1 (+)***	N-1	N	N
4N+1	N+1	***N (+)***	N	N
4N+2	N+1	N	***N+1 (+)***	N
4N+3	N+1	N	N+1	***N+1 (+)***

Eigenvalues multiplicities

Symmetry breaking in finite case!

QUADRATIC EXPONENTIALS: Finite sums

Modified Discrete Fourier Transform:

$$F_r = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_n \\ r_1^2 & r_2^2 & \dots & r_n^2 \\ \dots & \dots & & \dots \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} \end{pmatrix}.$$

$r_1^{k-1}, r_2^{k-1}, \dots, r_n^{k-1}$ — roots of unity.
$n!$ variants.

QUADRATIC EXPONENTIALS: Finite sums

MDFT examples: 5) $r = (-1, i, 1, -i)$, 6) $r = (-1, -i, 1, i)$, 7) $r = (i, -1, -i, 1)$, 8)$r = (-i, -1, i, 1)$.
SIMPLE SPECTRUM!!!
$x^4 - i$.
Eigenvector:

$$\{s, (1-\sqrt{2}+\sqrt{4-2\sqrt{2}})i, \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i, 1\},$$

s root of 8th power equation

$$s^8 - 8s^7 + 32s^6 - 24s^5 + 2s^4 + 24s^3 + 32s^2 + 8s + 1 = 0.$$

QUADRATIC EXPONENTIALS: Finite sums

9) $r = \{1, -1, i, -i\}$, 10) $r = \{1, -1, -i, i\}$, 11) $r = \{1, i, -i, -1\}$,
12) $r = \{1, -i, i, -1\}$.
SIMPLE SPECTRUM!!!:

$$\{-\frac{\sqrt{7}+1}{4} - \frac{\sqrt{7}-1}{4}i, \frac{\sqrt{7}-1}{4} + \frac{\sqrt{7}+1}{4}i, -1, 1\}.$$

CP:

$$x^4 + (\frac{1}{2} - \frac{i}{2})x^3 - (1+i)x^2 - (\frac{1}{2} - \frac{i}{2})x + i.$$

EV: $\{0, \frac{1}{2} i \left((2+i) + \sqrt{7}\right), -\frac{1}{2} i \left((2-i) + \sqrt{7}\right), 1\}$,
$\{0, -\frac{1}{2} i \left((-2-i) + \sqrt{7}\right), \frac{1}{2} i \left((-2+i) + \sqrt{7}\right), 1\}$,
$\{-1, 1, 1, 1\}, \{3, 1, 1, 1\}$.

QUADRATIC EXPONENTIALS: Finite sums

Conclusions:

Let $n = 4k$

1. From eigenvalue symmetry point of view classical DFT are the worst.
2. MDFT have better spectral properties.
3. Hypothesis:
4. Million $ problem: find modified Gauss sums.
5. Applications — cryptography algorithms. $n!$ – variants.

QUADRATIC EXPONENTIALS: Integral transforms

$$L_1(x, y) = \exp(axy)$$

$$Q_2(x, y) = \exp(ax^2 + by^2 + cxy)$$

Augustin–Jean Fresnel (1788 — 1827)

QUADRATIC EXPONENTIALS: Integral transforms

Fractional/Quadratic/Fresnel/Bohr/Wiener Fourier transform.

$$(F^{\alpha} f)(y) = \frac{1}{\sqrt{\pi(1-e^{2i\alpha})}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}i(x^2+y^2)\operatorname{ctg}\alpha} e^{ixy\operatorname{cosec}\alpha} f(x)\,dx; \tag{1}$$

$$(H_{\nu}^{\alpha} f)(y) = \frac{2\left(-e^{i\alpha}\right)^{-\frac{\nu}{2}}}{1-e^{i\alpha}} \int_{0}^{\infty} e^{-\frac{1}{2}i\operatorname{ctg}\frac{\alpha}{2}(x^2+y^2)} (xy)^{\frac{1}{2}} J_{\nu}\left(\frac{2xy\sqrt{-e^{i\alpha}}}{1-e^{i\alpha}}\right) f(x)\,dx \tag{2}$$

QUADRATIC EXPONENTIALS: Integral transforms

Fractional/Quadratic/Fresnel/Bohr/Wiener Fourier transform.
Definition by series – Hermite functions.
Ahiezer's Problem.

$$A_\nu^- = x^{\nu+\frac{1}{2}} e^{-\frac{x^2}{2}} \frac{d}{dx} x^{-\nu-\frac{1}{2}} e^{\frac{x^2}{2}} = -\frac{\nu+\frac{1}{2}}{x} + x + \frac{d}{dx},$$

$$A_\nu^+ = x^{-\nu-\frac{1}{2}} e^{\frac{x^2}{2}} \frac{d}{dx} x^{\nu+\frac{1}{2}} e^{-\frac{x^2}{2}} = \frac{\nu+\frac{1}{2}}{x} - x + \frac{d}{dx},$$

$$N_\nu = x^{\nu+\frac{1}{2}} \frac{d}{dx} x^{-\nu-\frac{1}{2}} = -\frac{\nu+\frac{1}{2}}{x} + \frac{d}{dx},$$

$$M_\nu = x^{-\nu-\frac{1}{2}} \frac{d}{dx} x^{\nu+\frac{1}{2}} = \frac{\nu+\frac{1}{2}}{x} + \frac{d}{dx}.$$

QUADRATIC EXPONENTIALS: Integral transforms

Fractional/Quadratic/Fresnel/Bohr/Wiener Fourier transform.

$$L_\nu = -\frac{1}{4}D^2 - \frac{\nu^2 - 1/4}{x^2} + \frac{1}{4}x^2 - \frac{\nu+1}{2} = -\frac{1}{4}A_\nu^+ A_\nu^-,$$
$$A_\nu^- = N_\nu + x, \qquad A_\nu^+ = M_\nu - x, \tag{3}$$

$$L_\nu = M_\nu N_\nu, \qquad x\frac{d}{dx} + \frac{d}{dx}x = N_\nu x + x M_\nu = M_\nu x + x N_\nu, \tag{4}$$

Fractional/Quadratic/Fresnel/Bohr/Wiener Fourier transform.

References:

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2. Alexander D. Poularikas. Transforms and Applications Handbook, Third Edition. 2010 by CRC Press, 911 p.
3. Sergei M. Sitnik. Transmutations and Applications: a survey. 2012, arXiv:1012.3741, 141 p.

QUADRATIC EXPONENTIALS: Series

$$g(x) = \sum_{k=-\infty}^{k=\infty} f_k \exp\left(-\frac{(x-k)^2}{2s^2}\right),$$

$$g(m) = f(m), m \in \mathbb{Z}.$$

Why use Gaussians?

1. Fresnel waves.
2. Gabor frames. Nobel prize for holography.
3. GAUSSIAN package. Nobel prize for computer package!!!

(in fact for applications in biology, chemistry, quantum phasics...)

QUADRATIC EXPONENTIALS: Series

Three main approaches:

1. Special functions — theta functions.
2. Use of DFT and Jacobi–Poisson summation.
3. Linear systems.

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THANK YOU FOR ATTENTION!