Combinatorial statistics of Dyson and Andrews–Garvan modulo 11

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Rank and crank

A partition of a positive integer n is a weakly-decreasing sequence of positive integers whose sum is n. We denote the number of partitions of n by $p(n)$. Among the most famous results in the theory of partitions are Ramanujan's congruences:

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$


Dyson's rank

In 1944, Dyson conjectured combinatorial interpretations of the first two congruences. He defined the rank of a partition as the largest part minus the number of parts and conjectured that the rank modulo 5 divided the partitions of $5n+4$ into 5 equal classes and that the rank modulo 7 divided the partitions of $7n+5$ into 7 equal classes. For example the partitions of the number 4 are $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$ and their ranks are $3, 1, 0, -1, -3$ respectively, giving an equinumerous distribution of the partitions of 4 into the five residue classes modulo 5. Let $N(a,r,n)$ denote the number of partitions of n with rank $\equiv a \pmod{r}$. In 1954 Dyson's modulo 5 and modulo 7 rank conjectures were proved by Atkin and Swinnerton-Dyer, they showed

$$N(a,5,5n+4) = \frac{p(5n+4)}{5}, \quad \text{for } 0 \le a \le 4,$$
$$N(a,7,7n+5) = \frac{p(7n+5)}{7}, \quad \text{for } 0 \le a \le 6.$$

Andrews–Garvan's crank

Although the rank does not explain Ramanujan's third congruence, Dyson conjectured another function, which he called the crank, that would divide the partitions of $11n+6$ into 11 equal classes. Andrews and Garvan later discovered the crank. For a partition π, let $\lambda(\pi)$ denote the largest part, $\vartheta(\pi)$ the number of ones, and $\mu(\pi)$ the number of parts larger than $\vartheta(\pi)$. The crank of π, denoted $c(\pi)$, is defined as follow

$$c(\pi) := \begin{cases} \lambda(\pi), & \text{when } \vartheta(\pi) = 0, \\ \mu(\pi) - \vartheta(\pi), & \text{otherwise}. \end{cases}$$

The cranks of the five partitions of 4 are $4, 0, 2, -2, -4$ respectively, giving an equinumerous distribution of the partitions of 4 into the five residue classes modulo 5. Let $M(a,r,n)$ denote the number of partitions of n with crank $\equiv a \pmod{r}$. In 1988 Andrews and Garvan showed

$$M(a,5,5n+4) = \frac{p(5n+4)}{5}, \quad \text{for } 0 \le a \le 4,$$
$$M(a,7,7n+5) = \frac{p(7n+5)}{7}, \quad \text{for } 0 \le a \le 6,$$
$$M(a,11,11n+6) = \frac{p(11n+6)}{11}, \quad \text{for } 0 \le a \le 10.$$

How does the rank fail to explain Ramanujan's congruence modulo 11?

Let $q := e^{2\pi i z}$ be a nonzero complex number with $\mathrm{Im}(z) > 0$. Recall the q-Pochhammer notation, defined by

$$(x;q)_\infty := \prod_{i=0}^{\infty} (1 - xq^i).$$

and the definition of theta function

$$j(x;q) := (x;q)_\infty (q/x;q)_\infty (q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^k,$$

where the equivalence of product and sum follows from Jacobi's triple product identity. For $1 \le i \le 5$ we introduce

$$J_i := (q^i;q^i)_\infty \quad \text{and} \quad P_i := j(q^i;q^{11}).$$

Deviations of the rank and crank modulo 11

For $0 \le a \le 10$ and $0 \le m \le 10$ define $Q_{a,m}(q)$ to be the elements of the 11-dissection of the deviation of the rank from the expected value:

$$Q_{a,m}(q) := \sum_{n=0}^{\infty} \left(N(a,11,11n+m) - \frac{p(11n+m)}{11} \right) q^n.$$

Using recent results of Frank Garvan and Rishabh Sarma [3] we calculated all the $Q_{a,m}(q)$ in terms of theta functions and so-called mock theta functions [2], originally considered by Ramanujan, which are close to ordinary theta functions, but they are not theta functions [5]. For example, we deduced

$$\sum_{n=0}^{\infty} \left(N(0,11,11n+6) - \frac{p(11n+6)}{11} \right) q^n = \frac{J_{11}^6}{J_1^2} \left(\frac{2q}{P_1P_4^2} + \frac{2q}{P_2^2P_5} - \frac{2q}{P_2P_3^2} \right),$$

which shows how rank fails to explain Ramanujan's congruence $p(11n+6) \pmod{11}$. Also we calculated 11-dissections of the deviations of the crank from the expected value in terms of theta functions [2].

"The garden of Ramanujan"

Due to his early death Ramanujan was unable to complete his investigations of ranks and cranks of partitions [5] and Ramanujan's work [4] has been continued by many researchers. As Dyson astutely observed, "That was the wonderful thing about Ramanujan. He discovered so much, and yet he left so much more in his garden for other people to discover". In recent decades, there have been many papers published on rank and crank inequalities, congruences, asymptotics, combinatorics, generalizations, etc.

Inequalities

Using 11-dissections of the deviations of crank we re-derived crank-crank inequalities [2], which were first proved by Ekin, Berkovich and Garvan, such as

$$M(1,11,11n+1) \ge \frac{p(11n+1)}{11} \ge M(2,11,11n+1) \ge M(0,11,11n+1).$$

By developing and exploiting positivity conditions for quotients of theta functions, we derived new rank and rank-crank inequalities [2], such as

$$M(1,11,11n) \ge N(4,11,11n),$$
$$N(2,11,11n) + N(3,11,11n) \ge N(4,11,11n) + M(1,11,11n),$$
$$N(2,11,11n) + 2N(3,11,11n) \ge N(5,11,11n) + 2M(1,11,11n).$$

It is still an open question how to prove these inequalities combinatorially. Also we stated conjectural rank-crank inequalities [2], which were recently fully solved by Bringmann and Pandey using the Circle method [1].

Congruences

Using 11-dissections of the deviations of the rank we re-derived congruences for the partition function [2], which were established by Atkin and Swinnerton-Dyer, such as

$$\sum_{n \ge 0} p(11n) q^n \equiv \frac{J_{11}^2}{P_1} \pmod{11}.$$

Let $N(m,n)$ denote the number of partitions of n with rank m. Using 11-dissections of the deviations of the rank we derived new congruences for so-called rank moments [2], such as

$$\sum_{n=0}^{\infty} \left(\sum_{m=-\infty}^{\infty} m^2 N(m,11n+6) \right) q^n \equiv$$
$$\equiv \frac{J_{11}^6}{J_1^2} \left(-\frac{4q^2}{P_4P_5^2} + \frac{3}{P_1^2P_3} + \frac{q}{P_1P_4^2} + \frac{5q}{P_2^2P_5} - \frac{2q}{P_2P_3^2} \right) \pmod{11}.$$

References

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