Constructive asymptotics for linear (pseudo)differential equations with localized right-hand sides.

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Formulation of the problem:

Let $\widehat{\mathcal{H}} = \mathcal{H}(x, \hat{p}, h), \qquad x = (x_1, \dots, x_n), \quad \hat{p} = (\hat{p}_1, \dots, \hat{p}_n), \quad \hat{p}_j = -ih\frac{\partial}{\partial x\, j}$

be a self adjoint operator with a real-valued symbol

$$\mathcal{H}(x, p, h) \in C^\infty([0,1]; S^\infty(\mathbb{R}^{2n}_{(x,p)})), \qquad \mathcal{H} = H(x, p) + hH_1(x, p) + \dots.$$

h be a small positive parameter.

The equation:

$$\widehat{\mathcal{H}}u = f \equiv f(\frac{x-\xi}{h}),$$

here

$$f(y, h) = \Big(\frac{i}{2\pi h}\Big)^{n/2} \int_{\mathbb{R}^n} e^{ip\cdot y} \tilde{f}(p) dp_1 \dots dp_1, \quad \arg i = \frac{\pi}{2}.$$

The function $A(p)$ is a smooth one.

If $\tilde{f} = 1$, than $f = \delta(x - \xi) \Longrightarrow \quad u$ is the Green function

Additional conditions may be those that arise when constructing the Green function: the principle of limiting absorption $\sim$ to Sommerfeld condition for the Helmholtz equation.

We do not use such conditions and we speak about asymptotic solutions only.

The main idea (closed to idea of V.P.Maslov and V.V.Kucherenko) is based on consideration of the non stationary Cauchy problem

$$-ihv_t = \hat{\mathcal{H}}v, \quad v|_{t=0} = F(\frac{x-\xi}{h}),$$

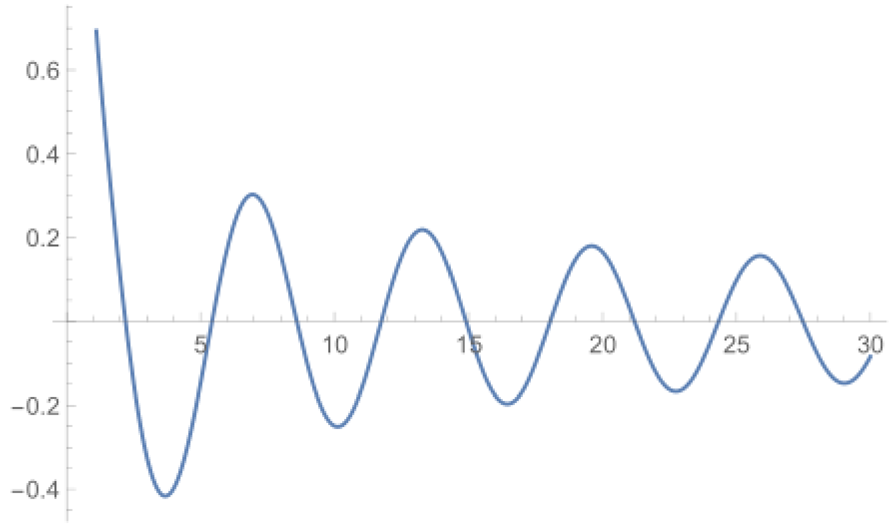
and representation of solutions in the form of Duhamel integral

$$u = \frac{i}{h}\int_0^\infty v(x,\tau)d\tau$$

The aim: to construct asymptotic solution as $h \to +0$ and our conditions are related to assumptions that ensure (in a sense), **convergence of this integral**.

The Helmholtz equation with constant n and the Dirac delta function in the right hand side:

u is the Green function = the Hankel function $G(x,k) = \frac{i}{4}H_0^{(1)}(\frac{|x|}{h})$. The function has singularity in the point $x = 0$ and has oscillations for large x.



The Helmholtz equation with variable coefficients:

J. B. Keller, 1962

V. M. Babich, 1964 (matching method , without focal points and caustics)

V. V. Kucherenko, 1968 (with focal points, the Maslov canonical operator)

Generalization:

L. Hörmander and J. J. Duistermaat, 1973, R. Melrose and G. A. Uhlmann, 1979 (pair of the manifolds, asymptotics with respect to smoothness)

B.Yu.Sternin, V.E.Shatalov, 1981 (some geometry)

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, M. Rouleux 2017, (Doklady Mathematics, Solutions of stationary equations with the localized right hand side)

S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. I. Shafarevich, 2016 (simplification of the canonical Maslov operator)

S. Yu. Dobrokhotov, D. S. Minenkov, M. Rouleux, 2018, (The Maupertuis-Jacobi correspondence in semiclassical stationary problems)

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, M. Rouleux, Lagrangian manifolds and the construction of asymptotics for (pseudo)differential equations with localized right-hand sides, Theoret. and Math. Phys., 214:1 (2023), 1-23

Examples of operators

(1) The Helmholtz equation: $H(p,x) = p^2 - n^2(x)$.

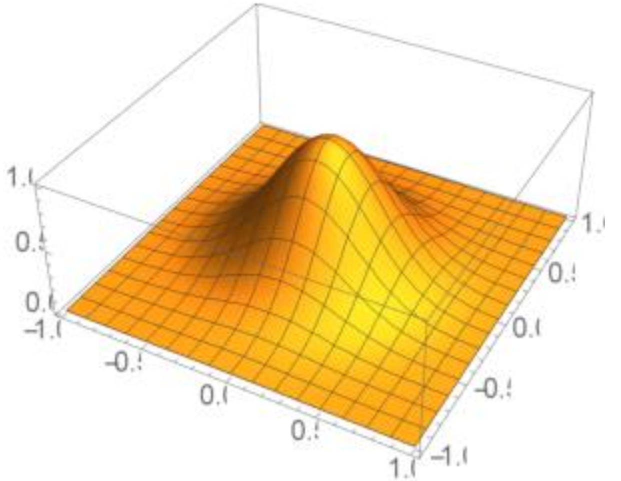
(2) The generalized Helmholtz equation: $H(p,x,h) = \langle p, A(x)p\rangle - n^2(x)$, $n^2(x)$ be smooth, $n^2 \to \text{const} > 0$, as $|x| \to \infty$, $g(x)$ be a positive smooth matrix, $g \to g_0 = \text{const} > 0$, as $|x| \to \infty$.

(3) 3-D stationary Schrodinger equation with a Coulomb potential $H(p,x) = p^2 + \gamma/|x| - E$, E is the energy.

(4) The water wave equation: $H(p,x,h) = |p| \tanh(|p|D(x)| - \omega^2$, $D(x)$ is a smooth depth, ω is a frequency.

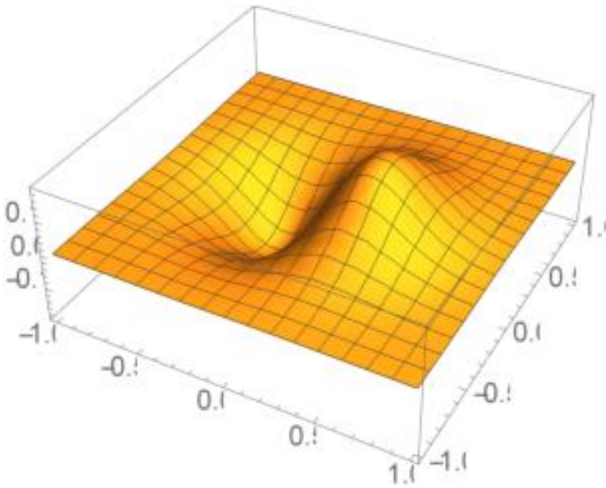
Examples of right hand side.

Directional radiation: let $n = 2$, $\rho = |p|$ and ψ be the polar angle of the momentum p,



$$(1) \qquad V(y) = \exp\left[-\frac{1}{2}\left(\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2}\right)\right] \qquad \text{then}$$

$$A(p) = ab\exp\left[-\frac{a^2p_1^2 + b^2p_2^2}{2}\right],$$



$$(2) \qquad (a_1y_1 + b_1y_2)\exp\left[-\frac{1}{2}\left(\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2}\right)\right] \qquad \text{then}$$

$$A(p) = iab(a^2a_1p_1 + b^2b_1p_2)\exp\left[-\frac{a^2p_1^2 + b^2p_2^2}{2}\right].$$

(3) antenna type $\qquad A(p) = ab\exp\left[-\frac{a^2p_1^2 + b^2p_2^2}{2}\right]\mathbf{e}(\psi), \quad \operatorname{supp}\mathbf{e}(\psi) \in [\psi^1, \psi^2].$

$$\mathcal{H}(x,p,h), \quad H(x,p)=\mathcal{H}(x,p,0), \quad H_{\text{sub}}(x,p)=\frac{\partial \mathcal{H}}{\partial h}(x,p,0)+\frac{i}{2}\sum_{i=1}^{n}\frac{\partial^2 H}{\partial x_j \partial p_j}(x,p)$$

symbol principle symbol subprinciple symbol

Assertions

I. $|H(\xi,p)| \geqslant C|p|^{-N}$, $C,N>0$, $|p|$ is large

II. Solutions $(x,p)=(X(q,\tau),P(q,\tau))$ of the Cauchy problem for the Hamilton system

$$\dot{x}=H_p(x,p), \quad \dot{p}=-H_x(x,p), \quad x|_{\tau=0}=\xi, \quad p|_{\tau=0}=q\in\mathbb{R}^n,$$

defined for all $\tau\in[0,\infty)$ with $q\in L_0$, where $L_0=\{q\in\mathbb{R}^n | H(\xi,q)=0\}$.

III. For any $R>0$, there exists such a t_R that $|X(q,\tau)|>R$ for $\tau>t_R$ and $q\in L_0$.

Lagrangian manifolds Λ_0 and Λ

$$\Lambda_0 = \{x = \xi, p = q \in \mathbb{R}^n\} \in \mathbb{R}^{2n}_{x,p},$$

$$\Lambda = \{(x = X(q,\tau), t = \tau, p = P(q,\tau), E(q) - H(\xi,q))\} \in \mathbb{R}^{2n+2}_{x,t,p,E},$$

Expression of the right-hand side via the canonical operator on Λ_0

$$f\left(\frac{x-\xi}{h}\right) = \frac{e^{i\pi n/4}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle p, x-\xi\rangle}\widetilde{f}(p)\,dp = h^{n/2}[\mathcal{K}_0\widetilde{f}](x,h),$$

where $\mathcal{K}_0$ is the Maslov canonical operator on the Lagrangian manifold Λ_0 with the measure $d\mu_0 = dq_1 \wedge \cdots \wedge dq_n$ and the Fourier transform

$$\widetilde{f}(p) = \frac{e^{i\pi n/4}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\langle p,y\rangle} f(y)\,dy$$

Construction of an asymptotic solution

A partition of unity on Λ_0 permits one to write the solution as the sum of solutions of two such equations in one of which $\operatorname{supp} \widetilde{f} \cap L_0 = \varnothing$, while in the other the function $\widetilde{f}$ is compactly supported.

Theorem 1. *(**"Elliptical part" of the "near field"**). If* $\operatorname{supp} \widetilde{f} \cap L_0 = \varnothing$*, then there exists an asymptotic solution*

$$\psi(x,h) = h^{n/2}[\mathcal{K}_0 B](x,h)$$

where the amplitude B is an element of the space $\mathcal{S}(\Lambda_0; h)$ of smooth functions on the parameter $h \in [0,1]$ ranging in the space $\mathcal{S}(\Lambda_0)$, and its leading term has the form

$$B_0(q) \equiv B(q,0) = \frac{\widetilde{f}(q)}{H(\xi,q)}.$$

Lemma 1. *Let $\widetilde{f}(q)$ be a compactly supported function on Λ_0. Then on the manifold Λ there exists a smooth cutoff function χ such that*

(i) $\chi = 1$ *in some neighborhood of the set* $M = \big(L_0 \times [0,\infty)\big) \cup \big(\operatorname{supp} \widetilde{f} \times \{0\}\big)$.

(ii) The set $\pi_\Lambda^{-1}(K) \cap \operatorname{supp} \chi$ *is compact for any compact set* $K \subset \mathbb{R}^{n+1}_{x,t}$.

(iii) The set $\pi_\Lambda(\operatorname{supp} \chi)$ *has the following property: if* $(x,t) \in \pi_\Lambda(\operatorname{supp} \chi)$ *and* $|x| \le R$, *then* $-1 \le t \le \tau_{R+1} + 1$.

$$\Lambda = \{(x = X(q,\tau), t = \tau, p = P(q,\tau), E(q) - H(\xi, q))\} \in \mathbb{R}^{2n+2}_{x,t,p,E},$$

Let $\mathcal{K}\colon C^\infty_{pr}(\Lambda; h) \to \mathcal{F}^h(\mathbb{R}^{n+1}_{(x,t)})$ be the canonical operator on the Lagrangian manifold Λ with the measure $d\mu = d\mu_0(q) \wedge d\tau$.

Theorem 2. *(**The first main Theorem**) If $\widetilde{f}$ is compactly supported, then the initial equation has an asymptotic solution of the form*

$$\psi(x, h) = i h^{\frac{n}{2}-1} \int_0^\infty [\mathcal{K}(\chi A)](x, t, h)\, dt,$$

where χ is the function described in Lemma 1 and the leading term of the amplitude $A \in C^\infty(\Lambda \times [0, 1])$ has the form

$$A_0(q, \tau) \equiv A(q, \tau, 0) = e^{-i \int_0^\tau H_{\mathrm{sub}}(X(q,\tau'), P(q,\tau'))\, d\tau'} \widetilde{f}(q).$$

Theorem 3. *If the function $\widetilde{f}$ is compactly supported and $\operatorname{supp} \widetilde{f} \cap L_0 = \varnothing$, then the asymptotic solutions constructed in Theorems 1 and 2 coincide modulo $O(h^\infty)$.*

Principal type condition, the "main" Lagrangian manifold Λ_+, and the form of the asymptotic solution outside of the vicinity of the point $x = \xi$

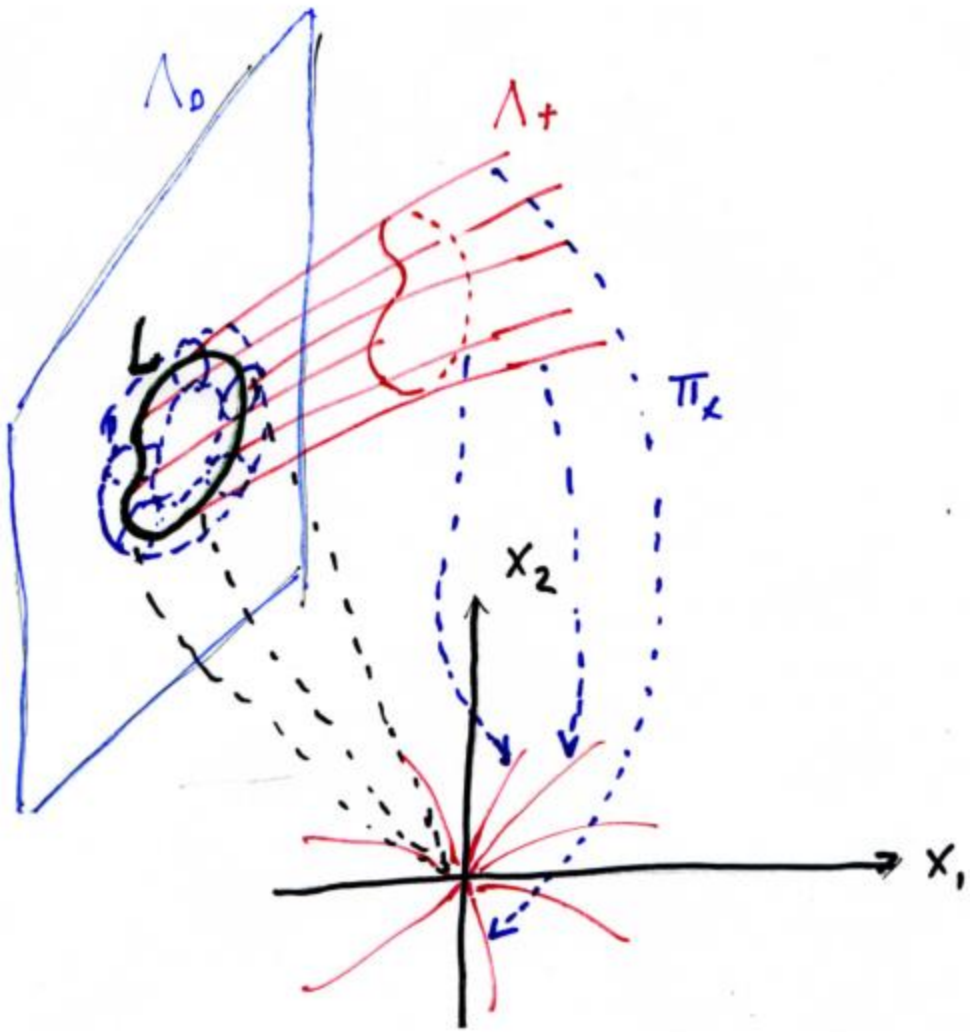
Assume that the following condition is satisfied:

IV. $H_p(\xi, p) \neq 0$ for $p \in L_0$.

Then L_0 is an $(n-1)$-dimensional submanifold in Λ_0. The points of L_0 will be denoted by α, and the corresponding values of $q \in \Lambda_0$, by $q(\alpha)$. We construct $n - D$ Lagrangian manifold

$$\Lambda_+ = \{x = X(q(\alpha), \tau), p = P(q(\alpha), \tau)\} \in \mathbb{R}^{2n}_{x,p}.$$

We equip Λ_+ with the measure $d\mu_+ = d\sigma_0(\alpha) \wedge d\tau$, where $d\sigma_0$ is the volume form on L_0 uniquely determined by the equality $d\mu_0 = d\sigma_0 \wedge dH$ on the tangent spaces to Λ_0 at the points of L_0.



Λ_0
Λ_+
L
π_x
x_2
x_1

Let

$$\mathcal{K}_+ \colon C^\infty_{pr}(\Lambda_+, \mathbb{R}^n_x \setminus \{\xi\}; h) \to \mathcal{F}^h(\mathbb{R}^n_x \setminus \{\xi\})$$

be the Maslov canonical operator on the manifold Λ_+ with the measure $d\mu_+$.

Theorem 4. *(**The main second theorem**). Assume that the function $\widetilde{f}$ is compactly supported and condition* (IV) *is satisfied. Then the asymptotic solution can be represented in the domain $\mathbb{R}^n_x \setminus \{\xi\}$ as*

$$\psi(x,h) = (2\pi)^{\frac{1}{2}} e^{\frac{\pi i}{4}} h^{\frac{n-1}{2}} [\mathcal{K}_+ Q](x,h) + O(h^\infty),$$

where the leading term of the amplitude $Q \in C^\infty_{pr}(\Lambda_+, \mathbb{R}^n_x \setminus \{\xi\}; h)$ has the form

$$Q_0(\alpha,\tau) \equiv Q(\alpha,\tau,0) = A_0(q(\alpha),\tau).$$

+ Simplifications, computational tricks and representations in the form of special functions of a complex argument

Wave part outside the focal points

$$u(x,h)=\frac{\sqrt{2\pi}A_0(q(\alpha),\tau)}{\sqrt{h|J(\alpha,\tau)|}}e^{\frac{i}{h}S(\alpha,\tau)-\frac{i\pi}{2}(m-\frac{1}{2})}\Big|_{(\alpha,\tau)=(\alpha(x),\tau(x))},$$

$$S(\alpha,\tau)=\int_0^\tau p\cdot H_p\Big|_{x=X(q(\alpha),\eta),p=P(q(\alpha),\eta)}d\eta,$$

m is the corresponding Maslov index,

$J=\det(\frac{\partial X}{\partial(\alpha,\tau)}$,

$(\alpha(x),\tau(x))$ is the solution to equations $X(q(\alpha),\eta)=x$

Example: the inhomogeneous Schrödinger equation with the Coulomb repulsive potential and localised right hand side (the Green function type problem)

S. Yu. Dobrokhotov, A. A. Tolchennikov, Keplerian Trajectories and an Asymptotic Solution of the Schroedinger Equation with Repulsive Coulomb Potential and a Localized Right-Hand Side, Russ.J.Math. Phys., Vol. 29, No. 4, 2022

$$\left(-h^2\Delta + \frac{\gamma}{|x|} - E\right)\psi(x) = F\left(\frac{x - x^0}{h}\right), \quad x \in \mathbb{R}^3,$$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, γ, E, h are positive constants, $h \ll 1$,

$F(y)$ is a smooth fast-decreasing function $\quad x_0 = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}, b > 0$

$+$ absorption conditions at infinity

The algorithm:

1) Construction the invariant Lagrangian manifold

$$\Lambda^3_+ = g^t_H\{p^2 + \frac{\gamma}{|x_0|} = E, \quad t \geq 0\}, \qquad H = p^2 + \frac{\gamma}{|x|}$$

(via Keplerian trajectories)

2) Construction the phase, the invariant measure, the Maslov indices, the Maslov canonical operator $\psi = K^h_{\Lambda^3} \cdot 1$

3) Simplification and global representation in the form of an Airy function of a complex argument

The Lagrangian manifold via Keplerian trajectories

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{\gamma^2}{4bE^2} \begin{pmatrix} (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)(\operatorname{ch}\xi + \operatorname{ch}\beta) + \operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}^2\beta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\beta(\operatorname{ch}\xi + \operatorname{ch}\beta) - \operatorname{sh}\xi \operatorname{sh}\beta(\operatorname{ch}\xi_0 + \operatorname{ch}\beta)]\cos\theta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\beta(\operatorname{ch}\xi + \operatorname{ch}\beta) - \operatorname{sh}\xi \operatorname{sh}\beta(\operatorname{ch}\xi_0 + \operatorname{ch}\beta)]\sin\theta \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{\gamma}{2b\sqrt{E}} \frac{1}{\operatorname{ch}\beta \operatorname{ch}\xi + 1} \begin{pmatrix} (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{sh}\xi + \operatorname{sh}\xi_0 \operatorname{ch}\xi \operatorname{sh}^2\beta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}\beta - (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{ch}\xi \operatorname{sh}\beta]\cos\theta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}\beta - (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{ch}\xi \operatorname{sh}\beta]\sin\theta \end{pmatrix}$$

$$t = q(\operatorname{ch}\beta \operatorname{sh}\xi + \xi - t_0) \qquad q = \tfrac{a^{\frac{3}{2}}}{\sqrt{2\gamma}},\ t_0 = \operatorname{ch}\beta \operatorname{sh}\xi_0 + \xi_0.$$

$$\operatorname{sh}\beta = -A\sin\psi,\ \operatorname{ch}\beta = \sqrt{1 + A^2\sin^2\psi}$$

$$A = \tfrac{2Fb\sqrt{E}}{\gamma}$$

$$\operatorname{sh}\xi_0 = \frac{A\cos\psi}{\sqrt{1 + A^2\sin^2\psi}},\ \operatorname{ch}\xi_0 = \frac{\sqrt{1 + A^2}}{\operatorname{ch}\beta}$$

Action function, measure and Jacobian on Λ_+

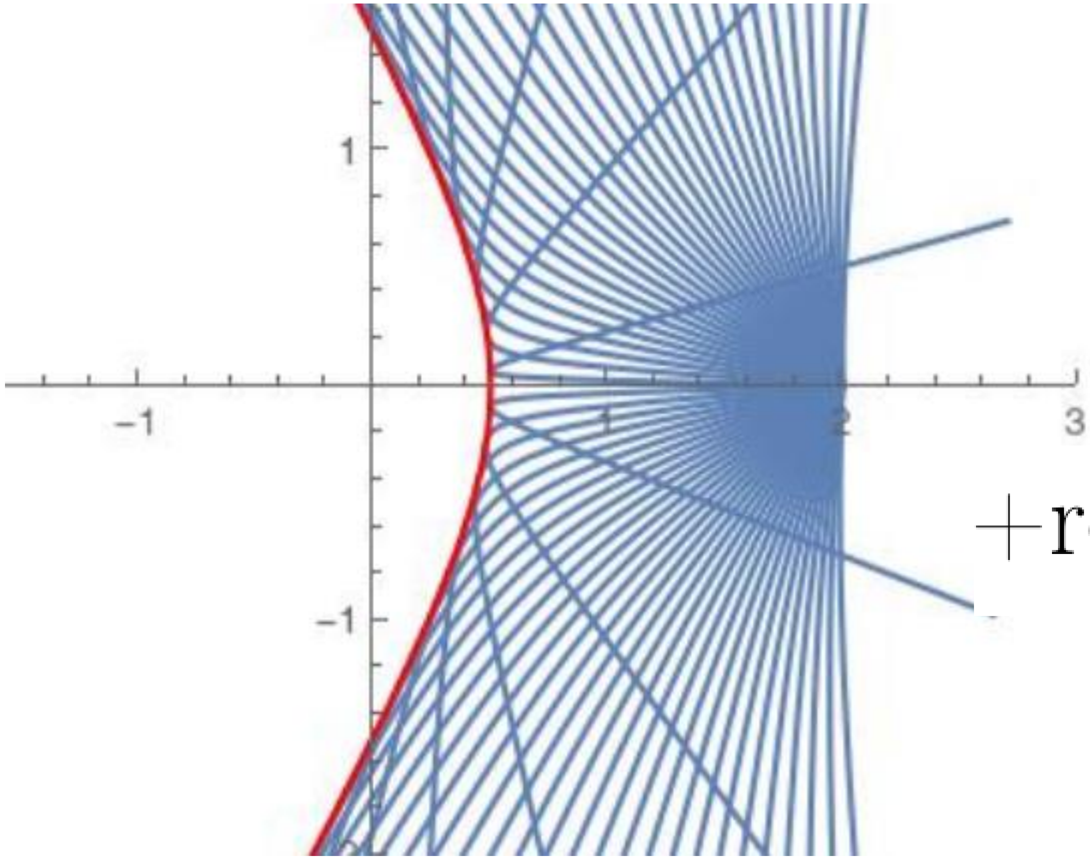
$$S(\xi,\psi)=\int_{\xi_0}^{\xi} p\,dx=\int_{\xi_0}^{\xi}\frac{1}{2t'_\xi}|y'_\xi|^2 d\xi=\frac{\gamma}{2\sqrt{E}}\left(\operatorname{ch}\beta(\operatorname{sh}\xi-\operatorname{sh}\xi_0)-(\xi-\xi_0)\right)$$

$$d\mu_+=\frac{1}{2}F\sin\psi d\psi\wedge d\theta\wedge dt=\frac{qF}{2}\sin\psi\left(\operatorname{ch}\beta\operatorname{ch}\xi+1\right)d\xi\wedge d\psi\wedge d\theta$$

$$J=\frac{dx_1\wedge dx_2\wedge dx_3}{d\mu_+}=\frac{F\gamma^2}{E^2}\left(\operatorname{ch}^2\beta(\operatorname{sh}\xi-\operatorname{sh}\xi_0)^2-\operatorname{sh}^2(\xi-\xi_0)\right)$$

The caustic = the half of a bicuspid hyperboloid

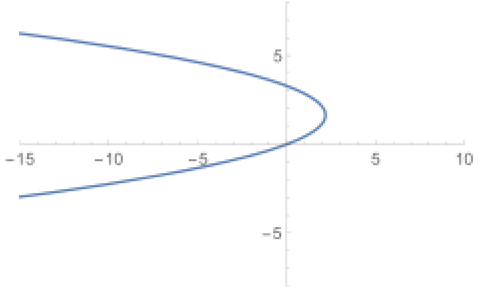
$$\frac{\left(x_1 - \frac{b}{2}\right)^2}{\left(\frac{b}{2} - \frac{\gamma}{E}\right)^2} - \frac{x_2^2}{\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2} - \frac{\gamma}{E}\right)^2} = 1$$



$+$rotation around the axis x_1

The focal points, "simple caustics" and uniform asymptotics in the form of Airy function: *"naive" constructive approach*

Let $x = X(\alpha^*)$ be nondegenerated focal point ($1D$-case) :



$$\frac{\partial X}{\partial \alpha}(\alpha^*) = 0, \quad \frac{\partial^2 X}{\partial \alpha^2}(\alpha^*) \neq 0.$$

$$\Lambda = \{p = P^* + P'^*(\alpha - \alpha^*) + O((\alpha - \alpha^*)^2), x = X^* + \frac{1}{2}X''^*(\alpha - \alpha^*)^2 + O((\alpha - \alpha^*)^3)$$

looks like a "horizontal" parabola.

Then for small $x - X^*$ one can show (at least on the physical level of rigor) that the asymptotic of the integral is expressed in terms of the **Airy function and its derivative**. This implies the following anzats:

$$\psi \approx e^{i\frac{Q(x)}{h}} \big(A_1(x,h) \mathrm{Ai}(\Phi(x,h)) + A_2(x,h) \mathrm{Ai}'(\Phi(x,h)) \big),$$

here phases $Q(x)$, $\Phi(x,h)$ and amplitudes $A_j(x,h)$ are unknown functions,

not the matching method but a simplification of a solution

Airy Airy'

Why Ai$'$? $\alpha^* = 0$ $A(\alpha) = \frac{A(\alpha)+A(-\alpha)}{2} + \alpha \frac{A(\alpha)-A(-\alpha)}{2\alpha} = g_1(X) + \alpha g_2(X)$

This gives

$$Q = \frac{1}{2}(S_+ + S_-), \quad \Phi = -\frac{3(S_+ - S_-)^{2/3}}{2h^{2/3}},$$

$$A_1 = -\frac{e^{\frac{-i\pi}{4}}}{\sqrt[6]{h}\sqrt{\pi}}(a_+ - a_-)\sqrt[3]{S_+ - S_-}, \quad A_2 = -\frac{\sqrt[6]{h}e^{\frac{i\pi}{4}}}{\sqrt{\pi}}\frac{(a_+ + a_-)}{\sqrt[3]{S_+ - S_-}}$$

$$\psi \approx e^{i\frac{Q(x)}{h}}\Big(A_1(x,h)\mathrm{Ai}(\Phi(x,h)) + A_2(x,h)\mathrm{Ai}'(\Phi(x,h))\Big)$$

(Anikin, Dobrokhotov, Nazaikinskii, Tsvetkova, 2019)

The global uniform asymptotics via Airy function outside of some neighborhood of the point x_0 (“Far field ”)

$$\psi = K^h_{\Lambda_+}\widetilde{V} = \sqrt{\pi}e^{-\frac{i\pi}{4}}e^{\frac{i\Theta}{h}}\left[h^{-\frac{1}{6}}Ai\left(-\frac{\Phi}{h^{\frac{2}{3}}}\right)B_+ + i\,h^{\frac{1}{6}}Ai'\left(-\frac{\Phi}{h^{\frac{2}{3}}}\right)B_-\right],$$

Parabolic coordinates

$$\sigma = |x|-x_1,\ \eta = |x|+x_1,\ v = (|x|(1-c^2)+(1+c)^2x_1-2bc,\ c = \frac{\gamma}{E}\frac{1}{b-\frac{\gamma}{E}}$$

$\Theta, \Psi, B_{\pm}$ are defined to the right of caustics ($v > 0$)

$$\Theta = \frac{1}{2}\left(S_{+} + S_{-}\right), \; \Phi = \left(\frac{3\Psi}{2}\right)^{\frac{2}{3}}, \; \Psi = \frac{1}{2}\left(S_{+} - S_{-}\right)$$

$$B_{+} = \left(\frac{3\Psi}{2}\right)^{\frac{1}{6}}\left(\frac{\widetilde{V}_{+}}{\sqrt{|J_{+}|}} + \frac{\widetilde{V}_{-}}{\sqrt{|J_{-}|}}\right), \; B_{-} = \left(\frac{3\Psi}{2}\right)^{\frac{1}{6}}\left(\frac{\widetilde{V}_{+}}{\sqrt{|J_{+}|}} - \frac{\widetilde{V}_{-}}{\sqrt{|J_{-}|}}\right)$$

$$J_{\pm} = \mp 4F\frac{\sqrt{\eta\, v}}{c^2(1+c)}\left(\left((1+c)\sqrt{\eta} \pm \sqrt{v}\right)^2 - 2bc^2\right).$$

$$S_{\pm} = \frac{\gamma}{2\sqrt{E}}\left[\frac{1}{bc^2}\left(\sqrt{\eta}(1+c) \mp \sqrt{v}\right)\sqrt{-2bc^2 + \left((1+c)\sqrt{\eta} \pm \sqrt{v}\right)^2}\right.$$
$$\left. - \operatorname{arcsh}\left(\frac{1}{bc^2}\left(\sqrt{\eta}(1+c) \pm \sqrt{v}\right)\sqrt{-2bc^2 + \left((1+c)\sqrt{\eta} \pm \sqrt{v}\right)^2}\right)\right]$$

To the left of caustics ($v < 0$)

$$\Psi = \frac{2\gamma}{3\sqrt{E}} \left(-2bc^2 + (1+c)^2\eta\right)^{-\frac{3}{2}} v^{\frac{3}{2}} + O(v^{\frac{5}{2}})$$

$$\Phi = \frac{\gamma^{\frac{2}{3}}}{E^{\frac{1}{3}}} \left(-2bc^2 + (1+c)^2\eta\right)^{-1} v + O(v^2)$$

$$B_+ \to \frac{\gamma^{\frac{7}{6}} E^{\frac{5}{12}} b^{\frac{3}{4}}}{(Eb-\gamma)^{\frac{7}{4}}} \left(-2bc^2 + (1+c)^2\eta\right)^{-\frac{3}{4}} \eta^{-\frac{1}{4}} \widetilde{V}_0$$

$$B_- \to -\frac{E^{\frac{19}{12}} b^{\frac{7}{4}} \gamma^{\frac{5}{6}}}{(Eb-\gamma)^{\frac{11}{4}}} (-2bc^2 + (1+c)^2\eta)^{-\frac{5}{4}} \eta^{\frac{1}{4}} \widetilde{V}_0,$$

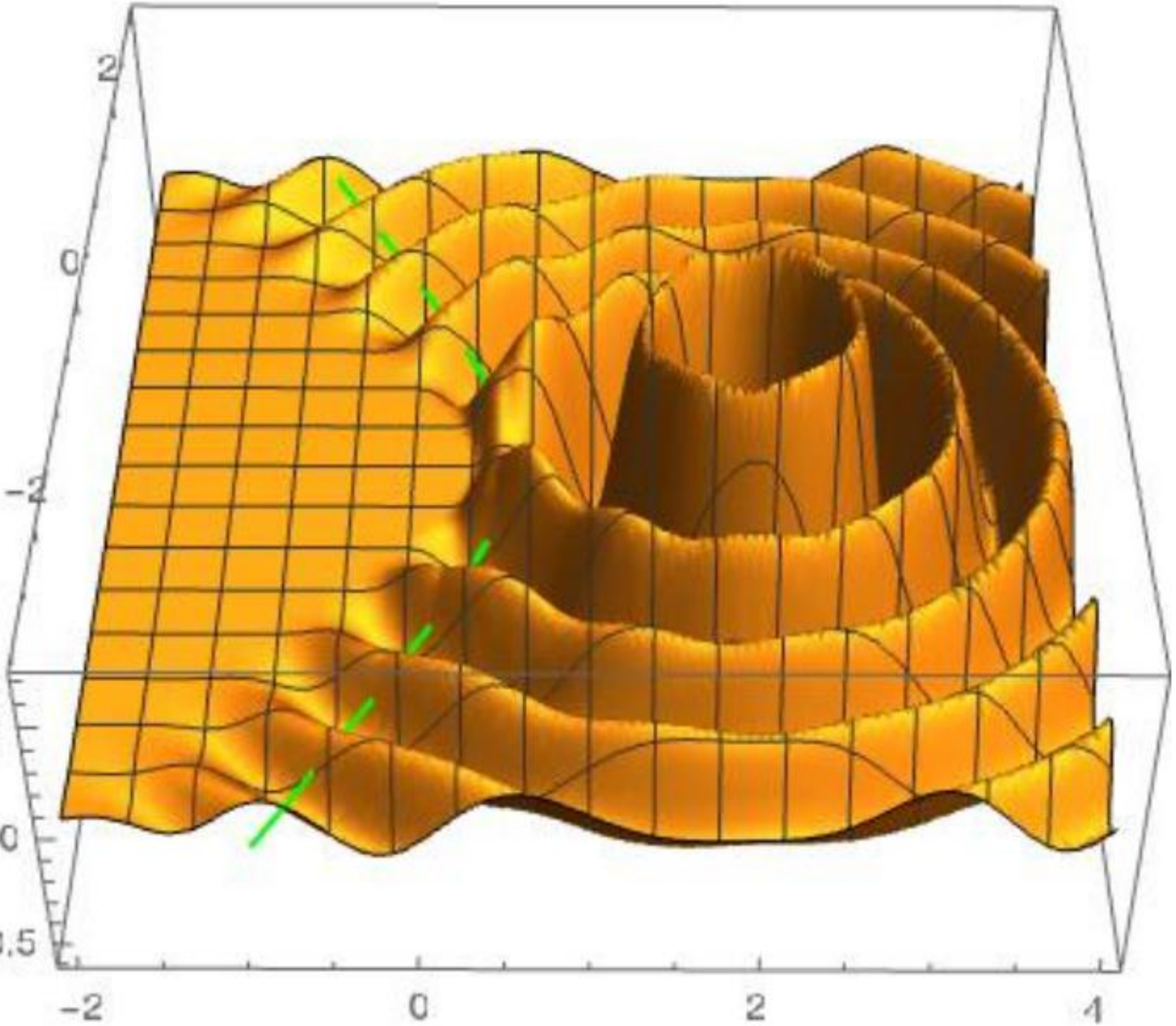
$$\widetilde{V}_0 = \widetilde{V}\left(\sqrt{E - \tfrac{\gamma}{b}}\cos\psi, \sqrt{E - \tfrac{\gamma}{b}}\sin\psi\cos\theta, \sqrt{E - \tfrac{\gamma}{b}}\sin\psi\sin\theta\right)|_{v=0}$$

$$\Phi = \begin{cases} \left(\frac{3}{4}\left(S_+ - S_-\right)\right)^{\frac{2}{3}} \text{ for } v > 0, \\ \gamma^{\frac{2}{3}} E^{-\frac{1}{3}} \left(-2bc^2 + (1+c)^2\eta\right)^{-1} v \text{ for } -\varepsilon < v \leq 0 \end{cases}$$

$$B_+ = \begin{cases} \left(\frac{3\Psi}{2}\right)^{\frac{1}{6}} \left(\frac{\widetilde{V}_+}{\sqrt{|J_+|}} + \frac{\widetilde{V}_-}{\sqrt{|J_-|}} \right) \text{ for } \; v > 0, \\ \frac{\gamma^{\frac{7}{6}} E^{\frac{5}{12}} b^{\frac{3}{4}}}{(Eb-\gamma)^{\frac{7}{4}}} \left(-2bc^2 + (1+c)^2\eta\right)^{-\frac{3}{4}} \eta^{-\frac{1}{4}} \widetilde{V}_0 \text{ for } -\varepsilon < v \leq 0 \end{cases}$$

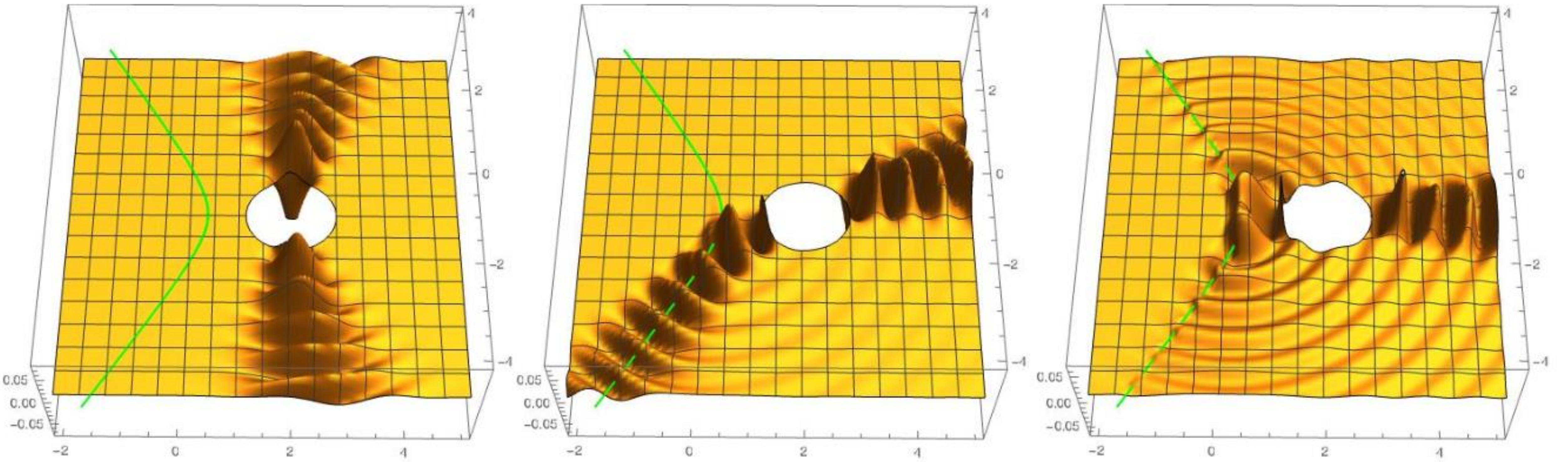
$$B_- = \begin{cases} \left(\frac{3\Psi}{2}\right)^{\frac{1}{6}} \left(\frac{\widetilde{V}_+}{\sqrt{|J_+|}} - \frac{\widetilde{V}_-}{\sqrt{|J_-|}} \right) \text{ for } v > 0, \\ -\frac{E^{\frac{19}{12}} b^{\frac{7}{4}} \gamma^{\frac{5}{6}}}{(Eb-\gamma)^{\frac{11}{4}}} (-2bc^2 + (1+c)^2\eta)^{-\frac{5}{4}} \eta^{\frac{1}{4}} \widetilde{V}_0 \text{for } \; -\varepsilon < v \leq 0. \end{cases}$$

Example $V = e^{-|x|^2},\ b = 2,\ E = 2,\ \gamma = 1, x_3 = 0.$

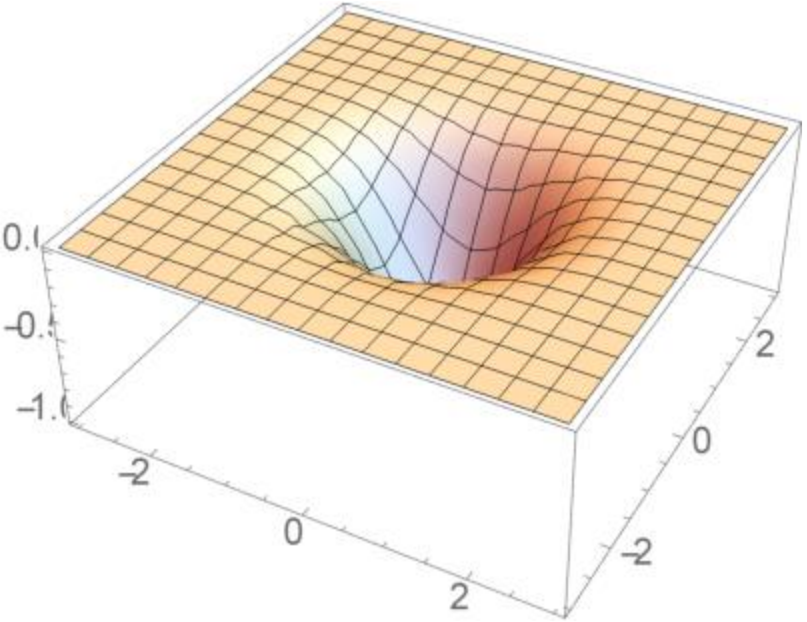


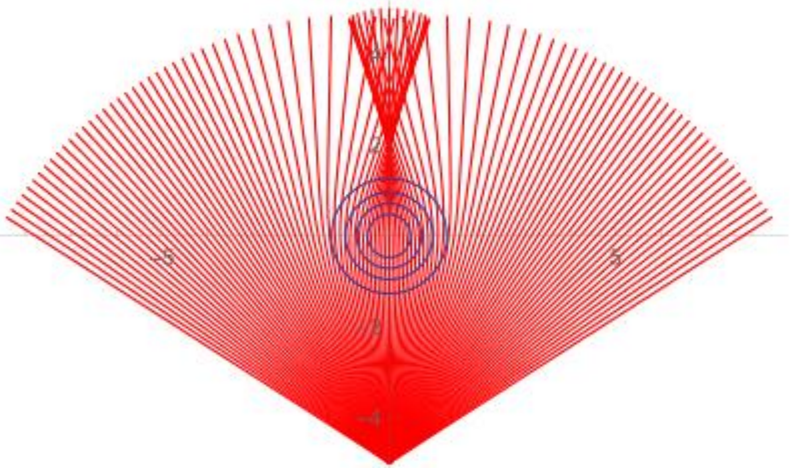
Example

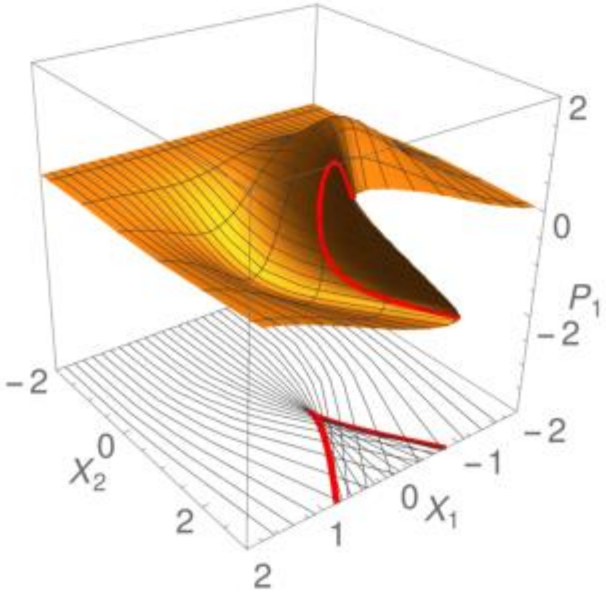
$$V(x) = e^{-\left(\frac{x_1 \cos\alpha + x_2 \sin\alpha}{a_1}\right)^2 - \left(\frac{-x_1 \sin\alpha + x_2 \cos\alpha}{a_2}\right)^2 - x_3^2}, \; a_1/a_2 = 1/3, \; \alpha \in [0, 2\pi]$$



More complicated example: cusp (in process)







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THANK YOU FOR YOUR ATTENTION!

Спасибо за внимание!