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Contributions to the convolution and ΨDO's over ultradistribution spaces

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Convolution

Let $f, g \in \mathcal{D}'$.
V. S. Vladimirov: Let $\eta_k(x, y), k \in \mathbb{N}$, be a sequence in $\mathcal{D}(\mathbb{R}^{2d})$ so that η_k equals one on the ball centered at zero of radius k, for every $k \in \mathbb{N}$. Then the convolution is defined by

$$\langle f * g, \phi \rangle = \langle f(x)g(y), \eta_k(x, y)\phi(x + y) \rangle, \phi \in \mathcal{D}(\mathbb{R}^d)$$

if this limit exists. Modifications are given for $\mathcal{S}'$ and $\mathcal{D}_{L^p}$ spaces.

Convolutors-Multipliers

Recall that

$$M_\phi f(x) = \mathcal{F}^{-1}(\phi \mathcal{F}(f))(x) = \mathcal{F}^{-1}\phi * f$$

$$a(D)f(x) = M_a f(x) = \mathcal{F}^{-1}(a\mathcal{F}(f))(x) = \mathcal{F}^{-1}(a) * f(x)$$

If $f \in (S_\nu^\mu)'$ what we can say about the multiplier

$$M_f : (\mathcal{S}_\beta^\alpha)' \to (\mathcal{S}_q^p)'?$$

What we can say about convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(x-t)g(t)dt, x \in \mathbb{R}^n.$$

The product goes paralelly by the question concerning the convolution in the framework of $\mathcal{S}'(\mathbb{R}^d)$ via

$$\mathcal{F}(fg)(\xi) = (2\pi)^{d/2}\hat{f}(\xi) * \hat{g}(\xi)$$

ΨDO

Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$. Then Psudodifferential operator is defined by

$$Op_\tau(a)u(x) = \int e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi)u(y)d'yd'\xi, u \in \mathcal{S}(\mathbb{R}^d)$$

and a_τ is its τ symbol. It maps $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.
Left quantization is obtain for $\tau = 0$; then we put $a(x, D)$ (Hörmander's calculus). The short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to (the fixed window function) $\phi \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$(V_\phi f)(x, \xi) = \mathcal{F}(f \cdot \overline{\phi(\cdot - x)})(\xi) =$$

$$\int_{\mathbb{R}^d} e^{-it\xi} e^{-|t-x|^2} f(t)dt, \text{ where } \phi(t) = e^{-\frac{1}{2}|t|^2}.$$

$$(V_\gamma^* F(x, \xi))(t) = \int_{\mathbb{R}^{2d}} F(x, \xi)e^{2\pi i\xi \cdot t}\gamma(t - x)dxd\xi$$

Localization (Toeplitz) operators. Relation with the Weyl Quantization

We also have

$$A_a^{\phi_1,\phi_2} u = (2\pi)^{-d} V_{\phi_1}^*(a V_{\phi_2} u), u \in \mathcal{S}(\mathbb{R}^d).$$

It maps $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

$$\langle L_\sigma f(x), g \rangle = \langle \sigma, W(f,g) \rangle,$$

where W is the Wigner transformation

$$W(f,g)(x,\omega) = \int e^{-2\pi i \omega t} f(x - t/2) \overline{g(x + t/2)} dt.$$

Again we have, if $\sigma \in S'^{\mu}_{\mu}(\mathbf{R}^{2d})$ then

$$L_\sigma : S^{\mu}_{\mu}(\mathbf{R}^d) \to S'^{\mu}_{\mu}(\mathbf{R}^d)$$

continuously, $\mu \geq 1/2$.

Localization operators. Relation with the Weyl Quantization

Now we come to the identity

$$\sigma = a * W(\varphi_1, \varphi_2).$$

On the right hand side we have a convolutor $W(\varphi_1, \varphi_2)$ and we are aimed to determine what is the domain $D_{W(\varphi_1,\varphi_2)} \subset {S'}^{\mu}_{\mu}(\mathbf{R}^d)$ of

$$W(\varphi_1, \varphi_2) : a \to \sigma.$$

and what is the range in ${S'}^{\mu}_{\mu}(\mathbf{R}^d)$.

Especially, we will do this in the case $\varphi_1(t) = \varphi_2(t) = 2^{d/4} e^{-\pi |t|^2}$, when $W(\varphi_1, \varphi_2)(x, \omega) = 2^d e^{-2\pi(|x|^2+|\omega|^2)}$ and

$$\sigma = a * e^{-c\pi(|x|^2+|\omega|^2)} \Rightarrow \hat{\sigma} = c_1 \hat{a} e^{-c_2(|x|^2+|\omega|^2)}.$$

$$e^{c_2(|x|^2+|\omega|^2)}\hat{\sigma} = \hat{a} \Rightarrow \mathcal{F}^{-1}(e^{c_2(|x|^2+|\omega|^2)}) * \sigma = a?$$

Ultradistributions; Preliminaries

BEURLING ULTRADISTRIBUTION SPACES,
ROUMIEU ULTRADISTRIBUTION SPACE

$M_p = p!$ -QUASU-ANALYTIC CLASS;
$M_p = p!^s, s > 1$-NON-QUASY-ANALYTIC CLASS

$\mathfrak{R}$ denotes the set of sequences increasing to ∞.
The common notation for the symbols (M_p) and $\{M_p\}$ will be *.
For $h > 0$ we denote by $\mathcal{S}^{M_p,h}_{A_p,h}$ the (B)-space of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ for which the norm

$$\sigma_h(\varphi) = \sup_\alpha \frac{h^{|\alpha|} \left\| e^{A(h|\cdot|)} D^\alpha \varphi \right\|_{L^\infty(\mathbb{R}^d)}}{M_\alpha} < \infty.$$

Ultradistributions; Preliminaries

For $h_1 < h_2$ the canonical inclusion $\mathcal{S}^{M_p,h_2}_{A_p,h_2} \to \mathcal{S}^{M_p,h_1}_{A_p,h_1}$ is compact.
$\mathcal{S}^{(M_p)}_{(A_p)} = \varprojlim_{h\to\infty} \mathcal{S}^{M_p,h}_{A_p,h}$, resp. $\mathcal{S}^{\{M_p\}}_{\{A_p\}} = \varinjlim_{h\to 0} \mathcal{S}^{M_p,h}_{A_p,h}$.
$\mathcal{S}^{(M_p)}_{(A_p)}$ is an (FS)-space and $\mathcal{S}^{\{M_p\}}_{\{A_p\}}$ is a (DFS)-space; both Montel spaces.
Moreover, $\mathcal{S}^{\{M_p\}}_{\{A_p\}} = \varprojlim_{r_p\in\mathfrak{R}} \mathcal{S}^{M_p,r_p}_{A_p,r_p}$.
We use $\mathcal{S}^*_\dagger$ as a common notation for these spaces.
For each $h > 0$, resp. $(r_p) \in \mathfrak{R}$, $\mathcal{S}^{M_p,h}_{A_p,h}$, resp. $\mathcal{S}^{M_p,(r_p)}_{A_p,(r_p)}$, is continuously injected into $\mathcal{S}(\mathbb{R}^d)$.

$$\mathcal{E}^{(M_p)}(U) = \varprojlim_{K\subset\subset U} \varprojlim_{h\to 0} \mathcal{E}^{\{M_p\},h}(K), \mathcal{E}^{\{M_p\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}(K),$$

$$\mathcal{D}^{(M_p)}(U) = \varinjlim_{K\subset\subset U} \varprojlim_{h\to 0} \mathcal{D}^{\{M_p\},h}_K, \quad \mathcal{D}^{\{M_p\}}(U) = \varinjlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{D}^{\{M_p\},h}_K.$$

Assumptions and the Idea

In the first part we assume

$$(M.1)\ \ M_p^2 \leq M_{p-1}M_{p+1},$$

$$(M.2)\ \ M_p \leq Ch^p \min_{r+l=p} M_r M_l$$

$$(M.3)\ \ \sum_{p>k+1} M_{p-1}/M_p \leq CH^k M_{k+1}/M_k;$$

Recall the non quasi-analicity condition is

$$(M.3)' : \ \ \sum_p M_p/M_{p+1} < \infty$$

Topological properties to be used: Weak approximation property and the ε tensor product.

Assumptions and the Idea

The main idea is to show that

$$\langle T * S(x), \phi(x) \rangle =$$

$$_{\tilde{\tilde{\mathcal{D}}}'^{\{M_p\}}_{L^1(\mathbb{R}^d)}} \langle T(x)S(y)\phi(x+y), 1_{x,y} \rangle_{\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty(\mathbb{R}^d)}}$$

with suitable assumptions on ultradistributions S, T and test function ϕ

ε-tensor product

$E\varepsilon F$, is the space of all bilinear functionals on $E'_c \times F'_c$ which are hypocontinuous with respect to the equicontinuous subsets of E' and F'. It is equipped with the topology of uniform convergence on products of equicontinuous subsets of E' and F'.

$$E\varepsilon F \cong \mathcal{L}_\epsilon\left(E'_c, F\right) \cong \mathcal{L}_\epsilon\left(F'_c, E\right),$$

If both E and F are complete then $E\varepsilon F$ is complete. The tensor product $E \otimes F$ is injected in $E\varepsilon F$ under $(e \otimes f)(e', f') = \langle e, e'\rangle\langle f, f'\rangle$. The induced topology on $E \otimes F$ is the ϵ topology and we have the topological imbedding $E \otimes_\epsilon F \hookrightarrow E\varepsilon F$.

ε-tensor product

The l.c.s. E is said to have the sequential approximation property (resp. the weak sequential approximation property) if the identity mapping $\mathrm{Id} : E \longrightarrow E$ is in the sequential limit set (resp. the sequential closure) of $E' \otimes E$ in $\mathcal{L}_c(E, E)$.

Theorem

If E and F are complete l.c.s. and if either E or F has the weak approximation property then $E\varepsilon F$ is isomorphic to $E\hat{\otimes}_\epsilon F$.

Lebessgue's type spaces of ultradistributions

Let $p \in [1, \infty]$.

$$\mathcal{D}_{L^p}^{(M_p)}(\mathbb{R}^d) = \varprojlim_{h \to \infty} \mathcal{D}_{L^p}^{M_p,h}(\mathbb{R}^d)$$

resp.

$$\mathcal{D}_{L^p}^{\{M_p\}}(\mathbb{R}^d) = \varinjlim_{h \to 0} \mathcal{D}^{M_p,h}(\mathbb{R}^d)$$

where $\mathcal{D}_{L^p}^{M_p,h}(\mathbb{R}^d)$ is the space of smooth functions ϕ so that

$$\sup_{p \in \mathbb{N}_0^d, x \in \mathbb{R}^d} \frac{|\phi^{(p)}(x)|}{h^p M_p} < \infty.$$

Convolution. I case

We know that $\mathcal{D}_{L^\infty}^{\{M_p\}}$ is the inductive limit of spaces $\mathcal{D}_{L^\infty,h}^{\{M_p\}}$. Recall $\widetilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$ is the projective limit of
$\widetilde{\mathcal{D}}_{L^\infty,r_p}^{\{M_p\}} = \{\phi : ||\frac{D^\alpha \phi}{M_p \prod_{i=1}^{p} r_i}||_{L^\infty} < \infty\}$.
For $g \in C_0\left(\mathbb{R}^d\right)$ (the space of all continuous functions that vanish at infinity) and $(t_j) \in \mathfrak{R}$, consider the seminorms

$$p_{g,(t_j)}(\varphi) = \sup_{\alpha\in\mathbb{N}^d} \sup_{x} \frac{|g(x) D^\alpha \varphi(x)|}{T_\alpha M_\alpha}, \ \varphi \in \tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}.$$

They generate Hausdorff locally convex topology on $\tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$ and this space with this topology will be denoted by $\tilde{\tilde{\mathcal{D}}}_{L^\infty}^{\{M_p\}}$. Note that the inclusions $\tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}} \longrightarrow \tilde{\tilde{\mathcal{D}}}_{L^\infty}^{\{M_p\}}$ and $\mathcal{D}^{\{M_p\}} \longrightarrow \tilde{\tilde{\mathcal{D}}}_{L^\infty}^{\{M_p\}} \longrightarrow \mathcal{E}^{\{M_p\}}$ are continuous.

Convolution. I case

$\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right)$ has the weak approximation property.

$\dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d, E\right) = \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \varepsilon E$

$\mathcal{D}^{\{M_p\}}$ is dense in $\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}$. $(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty})' = \widetilde{D}'^{\{M_p\}}_{L^1}$

Convolution-Definition

Let $S, T \in \mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^d\right)$ are such that for all $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $(S \otimes T)\varphi^{\Delta} \in \tilde{\mathcal{D}}\left(\mathbb{R}^{2d}\right)$.
Define the convolution of S and T, $S * T \in \mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^d\right)$, by

$$\langle T * S(x), \phi(x)\rangle =$$

$$_{\tilde{\tilde{\mathcal{D}}}'^{\{M_p\}}_{L^1(\mathbb{R}^d)}}\langle T(x)S(y)\phi(x+y), 1_{x,y}\rangle_{\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty(\mathbb{R}^d)}}$$

Convolution. Non-quasi-analytic case

Theorem

Let $S, T \in \mathcal{D}'^{\{M_p\}}\left(\mathbb{R}^d\right)$. The following are equivalent:

i) *the convolution of S and T exists;*

ii) *....;*

iii) *....;*

iv) *for all $\varphi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{T}\right) S \in \left(\tilde{\tilde{\mathcal{D}}}^{\{M_p\}}_{L^\infty}\right)'\left(\mathbb{R}^d\right)$ and for every compact subset K of $\mathbb{R}^d$, the bilinear mapping $(\varphi, \chi) \mapsto \left\langle \left(\varphi * \check{T}\right) S, \chi \right\rangle$, $\mathcal{D}^{\{M_p\}}_K \times \dot{\tilde{\mathcal{B}}}^{\{M_p\}}\left(\mathbb{R}^d\right) \longrightarrow \mathbb{C}$, is continuous;*

v) *for all $\varphi, \psi \in \mathcal{D}^{\{M_p\}}\left(\mathbb{R}^d\right)$, $\left(\varphi * \check{S}\right)\left(\psi * T\right) \in L^1\left(\mathbb{R}^d\right)$.*

Convolution with the gaussian kernel

$$B^* = \{S \in \mathcal{D}^* : \cosh(k|x|)S \in \mathcal{S}^*, k > 0\}, \quad B_s^* = e^{-s|x|^2} B^*$$

$$A^* = \{f \in \mathcal{O} : \forall K \exists h, C(\text{ resp. })\forall h \exists C, |f(\xi + i\eta)| \leq Ce^{M(h|\eta|)}\}$$

$$A_s^* = e^{s|x|^2} A_{real}^*.$$

Theorem

Let $s \in \mathbb{R}$, $s \neq 0$. Then

a) *The convolution of $S \in \mathcal{D}'^*$ and $e^{s|x|^2}$ exists if and only if $S \in B_s^*$.*

c) *The mapping $B_s^* \longrightarrow A_s^*$, $S \mapsto S * e^{s|x|^2}$ is bijective and for $S \in B_s^*$, $\left(S * e^{s|\cdot|^2}\right)(x) = e^{s|x|^2} \mathcal{L}\left(e^{s|\cdot|^2} S\right)(2sx)$.*

A new class of Anti-Wick operators

This Theorem allows us to define Anti-Wick operators $A_a : \mathcal{D}^* \left(\mathbb{R}^d\right) \longrightarrow \mathcal{D}'^* \left(\mathbb{R}^d\right)$, when a is not necessary in $\mathcal{S}'^* \left(\mathbb{R}^{2d}\right)$. Let $a \in B^*_{-1}$ and $b(x,\xi) = \pi^{-d} \left(a(\cdot,\cdot) * e^{-|\cdot|^2-|\cdot|^2}\right)(x,\xi)$ be such that for every $\varphi \in \mathcal{D}^* \left(\mathbb{R}^{2d}\right)$ the integral

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} b\left(\frac{x+y}{2}, \xi\right) \varphi(x,y) dx dy d\xi \qquad (1)$$

is well defined as oscillatory integral defining an ultradistribution. Then the operator associated to the corresponding kernel is the Anti-Wick operator with symbol a. This is appropriate generalization of Anti-Wick operators.

Definitions

Assumptions for $M_p = *$: $(M.1), (M.2)$ and $(M.5)$: M_p^r satisfies $(M.3)$
Assumptions for $A_p = \dagger$: $M.1, (M.2)$ and $(M.6)$: $p! \subset A_p$
$\mathcal{S}^{\{M_p\}}_{\{A_p\}} = \varprojlim_{(r_i),(s_j)\in\mathfrak{R}} \mathcal{S}^{M_{p,r_p}}_{A_{p,s_p}}$, where
$\mathcal{S}^{M_{p,s_p}}_{A_{p,s_p}} = \left\{\varphi \in \mathcal{C}^\infty\left(\mathbb{R}^d\right) | \|\varphi\|_{(r_p),(s_q)} < \infty\right\}$ and
$\|\varphi\|_{(r_p),(s_q)} = \sup_{\alpha\in\mathbb{N}^d} \frac{\left\|D^\alpha\varphi(x)e^{N_{s_p}(|x|)}\right\|_{L^\infty}}{M_\alpha \prod_{p=1}^{|\alpha|} r_p}$. Also, the Fourier transform is a topological automorphism of $\mathcal{S}^*$ and of $\mathcal{S}'^*$.

Definitions

Recall, $M(\rho) = \sup_p \ln_+ \frac{\rho^p}{M_p}, \rho > 0$ (for M_p), and for
$N_p = A_p \prod_{i=1}^p s_i$, $N_{s_p}(\rho) = \sup_p \ln_+ \frac{\rho^p}{N_p}$
It is said that $P(\xi) = \sum_{l \in \mathbb{N}_0^d} a_l \xi^l$, $\xi \in \mathbb{R}^d$, is an *ultrapolynomial of Beurling class* (*of Roumieu class*), if the coefficients a_l satisfy:

$(\exists a > 0,\ \exists C_a > 0)$ (resp. $\forall a > 0,\ \exists C_a > 0)$ $(\forall l \in \mathbb{N}_0^d)$ $|a_l| \leq C_a a^{|l|}/M_l$.

The corresponding operator $P(D) = \sum_{l \in \mathbb{N}_0^n} a_l D^l$ is an *ultradifferential operator* of Beurling class (resp. Roumieu class).

New class of sequences -test spaces

Let η be a weight $(\eta(x+h) \leq Ce^{a(\tau|h|)}\ \exists C, h;\ \forall\tau\exists C\ldots)$
$L^1_\eta = L^1/\eta$, $L^\infty_\eta = L^\infty\eta$.
We know that $\mathcal{S}^*_\dagger(\mathbb{R}^d)$ is not dense in $\mathcal{D}^*_{L^\infty_\eta}$ nor in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$.
$\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ is regular and complete.
$\mathcal{D}^{\{M_p\}}_{L^\infty_\eta}$ is injected continuously into $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$ and this inclusion is in fact surjective. As usual, we denote by $\mathcal{B}^*_\eta$ the space $\mathcal{D}^*_{L^\infty_\eta}$ and by $\dot{\mathcal{B}}^*_\eta$ the closure of $\mathcal{S}^*_\dagger(\mathbb{R}^d)$ in $\mathcal{B}^*_\eta$. We denote by $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}_\eta$ the closure of $\mathcal{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d)$ in $\tilde{\mathcal{D}}^{\{M_p\}}_{L^\infty_\eta}$.
$\mathcal{D}^*_E$ $(E = L^p_\eta)$ possesses the weak approximation property except $p = \infty$

Parametrix-Inluding quasy-analytic case

Lemma

Let $r' \geq 1$ and $k > 0$, resp. $(k_p) \in \mathfrak{R}$. There exists an ultrapolynomial $P(z)$ of class (M_p), resp. of class $\{M_p\}$, such that the function $x \mapsto 1/P(x)$ is in $\mathcal{C}^\infty(\mathbb{R}^d)$ and it satisfies the following estimate:
there exists $C > 0$ such that for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$

$$\left| D^\alpha \left(\frac{1}{P(x)} \right) \right| \leq C \frac{\alpha!}{r'^{|\alpha|}} e^{-M(k|x|)}, \text{ resp.} \tag{2}$$

$$\left| D^\alpha \left(\frac{1}{P(x)} \right) \right| \leq C \frac{\alpha!}{r'^{|\alpha|}} e^{-N_{k_p}(|x|)}. \tag{3}$$

Lemma

For every $t > 0$, resp. $(t_p) \in \mathfrak{R}$ there exists $G \in \mathcal{S}^{M_p,t}_{A_p,t}$, resp. $G \in \mathcal{S}^{M_p,(t_p)}_{A_p,(t_p)}$ and an ultradifferential operator $P(D)$ of class (M_p), resp. $\{M_p\}$, such that $P(D)G = \delta$.

Convolution

Theorem

The spaces $\mathcal{D}^*_{L^\infty_\eta,c}$ and $\tilde{\tilde{\mathcal{D}}}^*_{L^\infty_\eta}$ are isomorphic as l.c.s.

Theorem

The spaces $\left(\mathcal{D}^*_{L^\infty_\eta,c}\right)'_b$ and $\mathcal{D}'^*_{L^1_\eta}$ are isomorphic as l.c.s.

Definition

If $f_1, f_2 \in \mathcal{S}'^*_\dagger(\mathbb{R}^d)$. We say that the convolution of f_1 and f_2 exists if for each $\varphi \in \mathcal{S}^*_\dagger(\mathbb{R}^d)$, $(f_1 \otimes f_2)\varphi^\Delta \in \mathcal{D}'^*_{L^1}(\mathbb{R}^{2d})$ and we define their convolution by

$$\langle f_1 * f_2, \varphi\rangle = {}_{\mathcal{D}'^*_{L^1}(\mathbb{R}^{2d})}\langle (f_1 \otimes f_2)\varphi^\Delta, 1_{x,y}\rangle_{\mathcal{D}^*_{L^\infty,c}(\mathbb{R}^{2d})}, \ \forall \varphi \in \mathcal{S}^*_\dagger(\mathbb{R}^d),$$

where $1_{x,y}$ is the functions that is identically equal to 1.

Convolution

Theorem

Let $f_1, f_2 \in \mathcal{S}'^{*}_{\dagger}(\mathbb{R}^d)$. *The following statements are equivalent*

i) *the convolution of* f_1 *and* f_2 *exists;*

ii) *for all* $\varphi \in \mathcal{S}^{*}_{\dagger}(\mathbb{R}^d)$, $(\varphi * \check{f}_1)f_2 \in \mathcal{D}'^{*}_{L^1}$;

iii) *for all* $\varphi \in \mathcal{S}^{*}_{\dagger}(\mathbb{R}^d)$, $(\varphi * \check{f}_2)f_1 \in \mathcal{D}'^{*}_{L^1}$;

iv) *for all* $\varphi, \psi \in \mathcal{S}^{*}_{\dagger}(\mathbb{R}^d)$, $(\varphi * \check{f}_1)(\psi * f_2) \in L^1(\mathbb{R}^d)$.

Sufficient conditions for the convolution with $e^{s\langle\cdot\rangle^q}$, $q \geq 1$, $s > 0$

Assume $(M.1), (M.2)$ and $(M.5)$ there exists $q > 0$ such that M_p^q is strongly non-quasianalytic , i.e., there exists $c_0 \geq 1$ such that $\sum_{j=p+1}^{\infty} M_{j-1}^q / M_j^q \leq c_0 p M_p^q / M_{p+1}^q$, $\forall p \in \mathbb{Z}_+$;
$(M.6)$ $p! \subset M_p$, i.e. there exist $C, L > 0$ such that $p! \leq CL^p M_p$, $\forall p \in \mathbb{N}$

Theorem

Let $q_1 > q \geq 1$, $s \in \mathbb{R} \backslash \{0\}$ and $S \in \mathcal{S}'^{\{M_p\}}_{\{p!^{1/q_1}\}}(\mathbb{R}^d)$. If

$$e^{s\langle\cdot\rangle^q} e^{k\langle\cdot\rangle^{(q-1)q_1/(q_1-1)}} S \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^d), \quad \text{for all } k \geq 0, \tag{4}$$

then the $\mathcal{S}'^{\{M_p\}}_{\{p!^{1/q_1}\}}$-convolution of S and $e^{s\langle\cdot\rangle^q}$ exists.

Necessary conditions for the convolution with $e^{s\langle\cdot\rangle^q}$, $q \geq 1$, $s > 0$

Let $q_1 > q \geq 1,\ s \in \mathbb{R}\backslash\{0\}$.

Theorem

(i) Assume $p!^{2-1/q} \subset M_p$, $S \in \mathcal{S}'^{\{M_p\}}_{\{p!^{1/q_1}\}}(\mathbb{R}^d$ *and that the convolution*

$$S * e^{s\langle\cdot\rangle^q} \quad \text{exists.}$$

Then

$$e^{s'\langle\cdot\rangle^q} S \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^d), \quad \text{for all } s' < s.$$

(ii) With the same assumptions

$$e^{s\langle\cdot\rangle^q} e^{k\langle\cdot\rangle^{(q-1)q_1/(q_1-1)}} S \in \mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^d), \quad \text{for all } k \geq 0.$$

Necessary conditions for the convolution with $e^{s\langle\cdot\rangle^q}$

Theorem

Let $q>1$, $s>0$ and $S\in\mathcal{S}'^{\{M_p\}}_{\{p!^{1/q}\}}(\mathbb{R}^d)$. The $\mathcal{S}'^{\{M_p\}}_{\{p!^{1/q}\}}$-convolution of S and $e^{s\langle\cdot\rangle}$ exists if and only if $e^{s\langle\cdot\rangle}S\in\mathcal{D}'^{\{M_p\}}_{L^1}(\mathbb{R}^d)$.

Theorem

Let $q_1>q\geq 1$, $s>0$, $p!^{2-1/q}\subset M_p$ and $S\in\mathcal{S}'^{\{M_p\}}_{\{p!^{1/q_1}\}}(\mathbb{R}^d)$. If the $\mathcal{S}'^{\{M_p\}}_{\{p!^{1/q_1}\}}$-convolution of S and $e^{s\langle\cdot\rangle^q}$ exists then the $\mathcal{S}'^{\{M_p\}}_{\{p!^{1/q_1}\}}$-convolution of S and $e^{s'\langle\cdot\rangle^q}$ also exists for all $s'<s$, $s'\neq 0$.

Sufficient condition for the extension

Suppose that for every $K \subset\subset \mathbb{R}^d$ there exist $\{k_p\}$, $h > 0$and $C_h > 0$ such that

$$\frac{h^\alpha}{\alpha!}|\partial_x^\alpha b(x,\xi)| \leq C_h e^{N_{k_p}(\langle\xi\rangle)}, x \in K, \xi \in \mathbb{R}^d. \quad (5)$$

Then we will be able to generalize the notion of localization operator. Let $b = \pi^{-d} a * W(e^{-r\langle x\rangle^q}, e^{-r\langle x\rangle^q})$ satisfy (8). Then the integral

$$(2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi} b(\frac{x+y}{2},\xi)\chi(x,y)dxdyd\xi = \langle K(x,y), \chi(x,y)$$

is a well defined oscillatory integral It defines the localization operator with symbol a and window φ.

So, if one has a symbol a belonging to a certain space of ultradistribution so that (5) holds, then by this integral is defined Weyl Hormander operator over the space of the corresponding test functions .

Extension

We define the Anti-Wick operators $A_a : \mathcal{D}^{\{M_p\}}(\mathbb{R}^d) \to \mathcal{D}^{\{M_p\}}(\mathbb{R}^d)$ when

$$a \in B_{-r}^{\{M_p\}} = \{S = e^{r|x|^2} \cosh(-k|x|)G(\xi); G(\xi) \in \mathcal{S}'^{\{M_p\}}_{\{M_p\}}(\mathbb{R}^d)\}(X = (x, \xi$$

$$a(x,\xi) = P_{r_p}(D_x)P_{l_p}(D_\xi)(e^{l|x|^q} P_{s_p}(\xi) f(x,\xi)) \qquad (6)$$

where $P_{r_p}(D_x), P_{l_p}(D_\xi)$ are pseudodifferential operators, $P_{s_p}(\xi)$ is ultrapolynomial, all of them corresponding to M_p, and $f(x,\xi)$ is an L^∞- function over $(\mathbb{R}^{2d})$. Clearly,

$$B_{-l,q}^{\{M_p\}}(\mathbb{R}^{2d}) \subset \mathcal{D}'^{\{M_p\}}(\mathbb{R}^{2d}) \setminus \mathcal{S}_q'^{\{M_p\}}(\mathbb{R}^{2d})$$

Theorem

Let $0 < l < r/2^q$ and $a(x,\xi) \in B_{-l,q}^{\{M_p\}}(\mathbb{R}^{2d})$, $a(x,\xi)$ be of the form (6). Let

$$b = a(x,\xi) * W(e^{-r\langle x\rangle^q}, e^{-r\langle x\rangle^q})(x,\xi).$$

Then b defines a symbol of a ΨDO which can be extended over elements of the given form in $\mathcal{D}'^{\{M_p\}}$

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