Algebraic Groups with Good Reduction

(joint work with V. Chernousov and I. Rapinchuk)

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Let

- K be a field equipped with a set V of discrete valuations;
- G be an absolutely almost simple algebraic K-group (typically simply connected or adjoint)

We are interested in

K-forms of G that have good reduction at all $v \in V$.

- a K-group G' is a *K-form* (or $\overline{K}/K$-form) of G if

$$G' \otimes_K \overline{K} \simeq G \otimes_K \overline{K}.$$

Examples.

1. If A is a central simple algebra of degree n over K, **then** $G' = \mathrm{SL}_{1,A}$ is a K-form of $G = \mathrm{SL}_n$.

2. If q is a nondegenerate quadratic form in n variables over K (char $K \neq 2$) and

$$G = \mathrm{Spin}_n(q),$$

then for any other nondegenerate quadratic form q' in n variables,

$$G' = \mathrm{Spin}_n(q')$$

is a K-form of G.

If n is *odd* then these are **all** K-forms.

Otherwise, there may be K-forms coming from hermitian forms over noncommutative division algebras.

- G has good reduction at a discrete valuation v of K if there exists a reductive group scheme $\mathcal{G}$ over valuation ring $\mathcal{O}_v \subset K_v$ such that

1. *generic fiber* $\mathcal{G} \otimes_{\mathcal{O}_v} K_v$ is isomorphic to $G \otimes_K K_v$;
2. *special fiber* (reduction) $\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$ is *a connected simple group (of same type as G)* ($K^{(v)}$ residue field)

Remark. Similarly one defines good reduction for arbitrary reductive groups (in particular, for tori).

Examples.

0. If G is K-split then G has a good reduction at any v, with $\mathcal{G}$ provided by Chevalley construction.

1. $G = \mathrm{SL}_{1,A}$ has good reduction at v if there exists an Azumaya algebra $\mathcal{A}$ over $\mathcal{O}_v$ such that
$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$
(in other words, A is *unramified* at v).

2. $G = \mathrm{Spin}_n(q)$ has good reduction at v if
$$q \sim \lambda(a_1x_1^2 + \cdots + a_nx_n^2) \quad \text{with} \quad \lambda \in K_v^{\times}, \quad a_i \in \mathcal{O}_v^{\times}$$
(assuming that $\mathrm{char}\, K^{(v)} \neq 2$).

To make problem of characterizing forms with good reduction *meaningful* one needs to specify K, V and/or G.

Most popular case: K field of fractions of Dedekind ring R, and V consists of places associated with maximal ideals of R.

In this context problem was considered by Harder and Colliot-Thèléne.

Inventiones math. 4, 165–191 (1967)

Halbeinfache Gruppenschemata über Dedekindringen

GÜNTER HARDER (Heidelberg)

Einleitung

Sei A ein Dedekindring ([4], S. 22), sei $S=\mathrm{Spec}(A)$. Mit K wollen wir den Quotientenkörper von A bezeichnen, und wir setzen $s=\mathrm{Spec}(K)$. Sei ferner $i: s \hookrightarrow S$ die natürliche Inklusion. Sei G ein halbeinfaches Gruppenschema über S (s. [6], S. 382), wir wollen dafür G/S oder auch G/A schreiben. Mit G_s bezeichnen wir die „allgemeine Faser" von G, d.h.

$$G_s = G \underset{S}{\times} s.$$

Mit anderen Worten, G_s ist die Konstantenerweiterung von G/A mit K, wir wollen daher auch G_K für G_s schreiben. Wir nennen das halbeinfache Gruppenschema G/A *rational quasitrivial*, wenn G_K eine Boreluntergruppe über K besitzt (s. [6], S. 392), es heiße *rational trivial*, wenn G_K ein Chevalleyschema über K ist (s. [6], S. 408). Ein Ziel dieser Arbeit ist, Strukturaussagen über solche Gruppenschemata zu gewinnen und sie zu klassifizieren.

Später wollen wir dann die Voraussetzung machen, daß K ein algebraischer Zahlkörper ist und A der affine Ring einer offenen Teilmenge $U \subset \mathrm{Spec}(A_0)$, wobei A_0 der Ring der ganzen Zahlen von K ist.

In diesem Fall können wir dann die Voraussetzung, daß G/A rational quasitrivial ist, durch eine wesentlich schwächere Voraussetzung ersetzen. Da wir aber vom Hasseprinzip für $H^1(K, G)$ Gebrauch machen müssen, werden wir den Fall, daß G Faktoren vom Typ E_8 enthält, ausschließen müssen.

In beiden Fällen wird der starke Approximationssatz (KNESER [15, 16]) eine entscheidende Rolle spielen. Einige unserer Hauptresultate erweisen sich als im wesentlichen äquivalent mit der Berechnung von Klassenzahlen, die KNESER in [15] auf Grund des starken Approximationssatzes andeutungsweise durchführt. Die Verbindung zwischen unserem Problem und dem der Bestimmung der Klassenzahl wird durch Satz 2.3.1 gegeben.

Wir werden uns in hohem Maße auf das Séminaire géometrie algébrique 1963/64 von GROTHENDIECK und DEMAZURE stützen (zitiert SGAD) und auch die dort entwickelte Terminologie ausgiebig verwenden. In der Thèse von DEMAZURE (zitiert TD) sind die für uns wesentlichen Begriffe

Lemma 4.1.3. *Sei G/A flach von endlichem Typ, sei G_K/K eine reduktive Gruppe, dann ist*

$$H^1_A(K,G)=\{\xi \mid \xi_{\mathfrak p} \in \mathrm{Im}\, H^1(\hat A_{\mathfrak p}, G) \to H^1(\hat K_{\mathfrak p}, G) \text{ für alle } \mathfrak p \in \mathrm{Spec}(A)\}.$$

Beweis. Es ist klar, daß die linke Seite in der rechten Seite enthalten ist. Sei $\xi \in H^1(K,G)$, und für alle $\mathfrak p$ sei

$$\xi_{\mathfrak p} \in \mathrm{Im}\big(H^1(\hat A_{\mathfrak p}, G) \to H^1(\hat K_{\mathfrak p}, G)\big).$$

Es gibt eine endliche Erweiterung K'/K, so daß $\xi \in H^1(K'/K, G)$. Wir können wegen der Sätze in § 2 die Erweiterung K' so groß wählen, daß für

$$G \underset{A}{\times} A'$$

der schwache Approximationssatz gilt. (Man wähle K' so groß, daß das Radikal R von G über K' auflösbar wird und G/R zerfällt.) Wir betrachten das Diagramm

$$1 \longrightarrow G(K) \xrightarrow{q_0} G(K') \underset{q_2}{\overset{q_1}{\rightrightarrows}} G\big(K' \underset{K}{\otimes} K'\big)$$

und repräsentieren ξ durch einen Kozyklus

$$a \in G\big(K' \underset{K}{\otimes} K'\big).$$

Ist $U \subset \mathrm{Spec}(A)$ offen, so sei $A'(U)$ der Abschluß von $A(U)$ in K'. Es gibt jetzt sicher eine offene, nicht leere Menge $U_1 \subset \mathrm{Spec}(A)$, so daß a im Bild der Abbildung

$$j_1:\ G\big(A'(U_1) \underset{A(U_1)}{\otimes} A'(U_1)\big) \to G\big(K' \underset{K}{\otimes} K'\big)$$

liegt, also ist $a=j_1(a_1)$, und weil G/A flach ist, ergibt sich, daß a_1 ein Kozyklus ist. Sei S die Menge der abgeschlossenen Primideale von A, die nicht in U_1 liegen, für jedes $\mathfrak p \in S$ gibt es nach Voraussetzung über ξ ein Element $b_{\mathfrak p} \in G'(\hat K_{\mathfrak p})$, so daß

$$a_{\mathfrak p} = p_1(b_{\mathfrak p}) \cdot a \cdot p_2(b_{\mathfrak p})^{-1} \in \mathrm{Im}\big(G(\hat A'_{\mathfrak p} \underset{\hat A_{\mathfrak p}}{\otimes} \hat A'_{\mathfrak p}) \to G(\hat K'_{\mathfrak p} \underset{\hat K_{\mathfrak p}}{\otimes} \hat K'_{\mathfrak p})\big).$$

Dafür muß man eventuell K' ein wenig größer machen, und es ist

$$\hat A'_{\mathfrak p} = \hat A_{\mathfrak p} \underset{A_{\mathfrak p}}{\otimes} A',$$

entsprechend ist $\hat K'_{\mathfrak p}$ definiert.

Nun gibt es wegen der Wahl von K' ein Element $b \in G(K')$, das an den endlich vielen Stellen $\mathfrak p \in S$ sehr dicht bei $b_{\mathfrak p}$ liegt, wir setzen

$$a' = p_1(b) \cdot a \cdot p_2(b)^{-1}.$$

Basic case $R = \mathbb{Z}$:

B.H. Gross (Invent. math. **124**(1996), 263-279) and B. Conrad.

Theorem (Gross)

Let G be an absolutely almost simple simply connected algebraic group over $\mathbb{Q}$. **Then** *G has good reduction at all primes p if and only if G is split over all $\mathbb{Q}_p$.*

Nonsplit groups with good reduction can be constructed explicitly and in some cases even classified.

Proposition

Let G be an absolutely almost simple simply connected algebraic group over a number field K, and assume that V contains almost all places of K. **Then** *the number of K-forms of G that have good reduction at all $v \in V$ is finite.*

Case $R = k[x]$, $K = k(x)$, and

$$V = \{ v_{p(x)} \mid p(x) \in k[x] \text{ irreducible } \}.$$

Theorem (Raghunathan–Ramanathan, 1984)

Let k be a field of characteristic zero, and let G_0 be a connected reductive group over k. **If** *G' is a K-form of $G_0 \otimes_k K$ that has good reduction at all $v \in V$* **then**

$$G' = G'_0 \otimes_k K$$

for some k-form G'_0 of G_0.

Case $R = k[x, x^{-1}]$, $K = k(x)$, and

$$V = \{ v_{p(x)} \mid p(x) \in k[x] \text{ irreducible, } \neq x \}.$$

Theorem (Chernousov–Gille–Pianzola, 2012)

Let k be a field of characteristic zero, and let G_0 be a connected reductive group over k. **Then** *K-forms of $G_0 \otimes_k K$ that have good reduction at all $v \in V$ are in bijection with $H^1(k((x)), G_0)$.*

This was used to prove conjugacy of Cartan subalgebras in some infinite-dimensional Lie algebras.

We initiated analysis of higher-dimensional situation.

- Let K be a *finitely generated* field,
- Pick a *model* $X = \operatorname{Spec} A$ for K, where A is a finitely generated integrally closed $\mathbb{Z}$-algebra with fraction field K,
- Let V be set of places associated with prime divisors on X (divisorial set).

Finiteness conjecture

Let G be an absolutely almost simple algebraic group over a finitely generated field K, and let V be a divisorial set of places of K.

Set of isomorphism classes of K-forms of G that have good reduction at all $v \in V$ is finite (at least when char k is "good").

This conjecture has implications for

- analysis of algebraic K-groups with same isomorphism classes of maximal K-tori (*genus problem*)
- analysis of weakly commensurable Zariski-dense subgroups

  (these techniques were used in work with Prasad (Publ. math. IHES **109**(2009), 113-184) to prove commensurability of some isospectral locally symmetric spaces)
- properness of global-to-local map

$$H^1(K, \overline{G}) \longrightarrow \prod_{v \in V} H^1(K_v, \overline{G})$$

  for *adjoint* groups

Finiteness conjecture is also related to finiteness of *unramified cohomology.*

Consider the following question:

($*$) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related* **if** *they have same maximal subfields?*

- D_1 and D_2 have same maximal subfields **if**
  - $\deg D_1 = \deg D_2 =: n$;
  - for P/K of degree n, $P \hookrightarrow D_1 \Leftrightarrow P \hookrightarrow D_2$.

Geometry

Prasad-A.R.: *In many (although not all) situations, two arithmetically defined locally symmetric spaces having same lengths of closed geodesics are commensurable.*

Arithmetic Riemann surfaces were considered by A. Reid.

Underlying algebraic fact:

Let D_1 *and* D_2 *be two quaternion division algebras over a number field* K. **If** D_1 *and* D_2 *have same maximal subfields* **then** $D_1 \simeq D_2$.

However, most Riemann surfaces are not arithmetic $\Rightarrow$ One needs to understand to what degree this fact extends to more general fields

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a *discrete torsion free subgroup.*

- Some properties of M can be understood in terms of *associated quaternion algebra.*

Let

- $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$;
- $\tilde{\Gamma} = \pi^{-1}(\Gamma) \subset \mathrm{M}_2(\mathbb{R})$.

Set $A_\Gamma = \mathbb{Q}[\tilde{\Gamma}^{(2)}]$, $\tilde{\Gamma}^{(2)} \subset \tilde{\Gamma}$ generated by squares.

One shows: A_Γ is a *quaternion algebra* with center

$$K_\Gamma = \mathbb{Q}(\mathrm{tr}\,\gamma \mid \gamma \in \Gamma^{(2)})$$

(trace field).

(**Note** that for general Fuchsian groups, K_Γ is not necessarily a number field.)

- **If** Γ is *arithmetic*, **then** A_{Γ} is the quaternion algebra involved in its description;

- In general, A_{Γ} **does not** determine Γ, **but** is an invariant of the commensurability class of Γ.

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds:

- *geometrically*: a closed geodesic $c_{\gamma} \subset M$,
  if $\gamma \sim \pm \begin{pmatrix} t_{\gamma} & 0 \\ 0 & t_{\gamma}^{-1} \end{pmatrix}$ $(t_{\gamma} > 1)$ then *length* $\ell(c_{\gamma}) = 2 \log t_{\gamma}$;

- *algebraically*: a maximal etale subalgebra $K_{\Gamma}[\gamma] \subset A_{\Gamma}$.

For a Riemannian manifold M:

$L(M)$ = set of lengths of closed geodesics in M
((weak) length spectrum of M)

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$

   elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

   $K[\gamma_1] \subset A_{\Gamma_1}$ and $K[\gamma_2] \subset A_{\Gamma_2}$ are isomorphic.

So, A_{Γ_1} and A_{Γ_2} share "lots" of maximal etale subalgebras.

(Not all – but we will ignore it for now ...)

- For M_1 and M_2 to be commensurable, A_{Γ_1} and A_{Γ_2} must be isomorphic.

So, proving that length-commensurable M_1 and M_2 are commensurable implicitly involves answering a version of $(*)$.

Theorem

Let $M_i = \mathbb{H}/\Gamma_i$ $(i \in I)$ *be a family of* length-commensurable *Riemann surfaces where* $\Gamma_i \subset \mathrm{PSL}_2(\mathbb{R})$ *is finitely generated and Zariski-dense.* **Then** *quaternion algebras* A_{Γ_i} $(i \in I)$ *split into finitely many isomorphism classes (over common center).*

Algebra

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K. **If** D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties), which are infinite extensions of K.

Can one prove Amitsur's Theorem using only splitting fields of finite degree, or just maximal subfields?

- Amitsur's Theorem is no longer true in this setting.
  (Counterexamples can be found using cubic algebras over number fields.)

This leads to question $(*)$ and its variations.

Question (Prasad-A.R.)

Are quaternion algebras over $K = \mathbb{Q}(x)$ determined by their maximal subfields?

- Yes – D. Saltman

- Same over $K = k(x)$, k a number field

  (S. Garibaldi - D. Saltman)

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

- genus of every quaternion algebra reduces to one element;
- genus of every division algebra is finite.

(Follows from Albert-Hasse-Brauer-Noether Theorem.)

Theorem 1 (Stability Theorem)

Let $\mathrm{char}\, k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for every quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2.*
- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.
- $\mathbf{gen}(D)$ can be infinite.

Construction yields examples over fields that are infinitely generated

(in fact, HUGE)

Theorem 2.

Let K be a finitely generated *field. Then for any central division K-algebra D the genus* $\mathbf{gen}(D)$ *is finite.*

- Proofs of both theorems use *analysis of ramification* and info about *unramified Brauer group.*

BASIC FACT: *Let v be a discrete valuation of K, and n be prime to characteristic of residue field $K^{(v)}$. If D_1 and D_2 are central division K-algebras of degree n having same maximal subfields, then either both algebras are* ramified *at v or both are* unramified.

(When n is divisible by $\mathrm{char}\, K^{(v)}$, we need some additional assumptions)

- Recall that a c. s. a. A over K (or its class $[A] \in \mathrm{Br}(K)$) is *unramified* at v if there exists Azumaya algebra $\mathcal{A}/\mathcal{O}_v$ such that

$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v.$$

If $(n\,, \mathrm{char}\, K^{(v)}) = 1$ or $K^{(v)}$ is perfect, there is a *residue map*

$$r_v \colon {}_n\mathrm{Br}(K) \longrightarrow H^1(\mathcal{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}),$$

where $\mathcal{G}^{(v)}$ is absolute Galois group of $K^{(v)}$.

- **Then** $x \in {}_n\mathrm{Br}(K)$ is unramified at $v \Leftrightarrow r_v(x) = 0$.

Given a set V of discrete valuations of K, one defines corresponding *unramified Brauer group:*

$$\mathrm{Br}(K)_V = \{\, x \in \mathrm{Br}(K) \mid x \text{ unramified at all } v \in V \,\}.$$

- To prove Theorem 1 (Stability Theorem) we use: *if* $K = k(x)$ *and* V = *set of geometric places, then*
$$ {}_n\mathrm{Br}(K)_V = {}_n\mathrm{Br}(k) $$
when $(n\,, \mathrm{char}\, k) = 1$ (Faddeev)

- There are **two** proofs of Theorem 2. **Both** show that for a divisorial set of places of a finitely generated field K one can make some finiteness statements about unramified Brauer group.

  - More recent argument works in all characteristics, **but** gives no estimate of size of $\mathbf{gen}(D)$.
  - Earlier argument works when $(n\,, \mathrm{char}\, K) = 1$, gives finiteness of ${}_n\mathrm{Br}(K)_V$ and estimate
  $$ |\,\mathbf{gen}(D)\,| \leqslant |\,{}_n\mathrm{Br}(K)_V\,| \cdot \varphi(n)^r $$
  where r is number of $v \in V$ that ramify in D.

Question. *Does there exist a quaternion division algebra D over $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, such that*

$$|\mathbf{gen}(D)| > 1?$$

- The answer is **not** known for *any* finitely generated K.
- One can construct examples where ${}_2\mathrm{Br}(K)_V$ is “large.”

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

Let G_1 and G_2 be semi-simple groups over a field K. G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

Let G be an absolutely almost simple K-group.

$$\mathbf{gen}_K(G) = \text{set of isomorphism classes of } K\text{-forms } G' \text{ of } G \text{ having same } K\text{-isomorphism classes of maximal } K\text{-tori as } G.$$

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 3 (Prasad-A.R.)

Let G *be an absolutely almost simple simply connected algebraic group over a number field* K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If* G *is not of type* A_n, D_{2n+1} $(n > 1)$ *or* E_6, *then* $|\mathbf{gen}_K(G)| = 1$.

Conjecture. (1) *For* $K = k(x)$, k *a number field, and* G *an absolutely almost simple simply connected* K*-group with* $|Z(G)| \leqslant 2$, *we have* $|\mathbf{gen}_K(G)| = 1$;
(2) *If* G *is an absolutely almost simple group over a finitely generated field* K *of "good" characteristic then* $\mathbf{gen}_K(G)$ *is finite.*

"Unramified division algebras" $\rightsquigarrow$ "groups with good reduction"

Theorem 3.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K. Assume that $K^{(v)}$ is finitely generated, and $\operatorname{char} K^{(v)} \neq 2$ if G is of type B_ℓ.

If *G has good reduction at v* **then** *every $G' \in \mathbf{gen}_K(G)$ has good reduction at v.*

V.I. Chernousov, A.S. Rapinchuk, I.A. Rapinchuk, *Simple algebraic groups with the same tori, weakly commensurable Zariski-dense subgroups, and good reduction*, arXiv:2112.04315

Let K be a finitely generated field, and V be a divisorial set of places of K.

Corollary.

Let G be an absolutely almost simple simply connected K-group. There exists a finite subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.

So, truth of Finiteness Conjecture for a given G and *any* divisorial V implies finiteness of $\mathbf{gen}_K(G)$.

A theorem of Raghunathan-Ramanathan extends Faddeev's result to reductive algebraic groups.

Applying it in conjunction with Theorem 3, we obtain

Theorem 4.

Let G be an absolutely almost simple algebraic group over a finitely generated field k of $\mathrm{char} \neq 2$, *and let* $K = k(x)$. *Then any* $H \in \mathbf{gen}_K(G \times_k K)$ *is of the form* $H = H_0 \times_k K$ *for some* $H_0 \in \mathbf{gen}_k(G)$.

In particular, if k is a number field then for

$$L = k(x_1, \ldots, x_r),$$

genus $\mathbf{gen}_L(G \times_k L)$ is *finite*, and in fact is *trivial* if G is of type different from A_n, D_{2n+1} $(n > 1)$ or E_6.

• Finiteness Conjecture is true for inner forms of type A_n over any finitely generated field provided that

$$(n+1\,,\,\mathrm{char}\,k) \;=\; 1.$$

For other types there are *additional challenges*:

– it is not known how to classify all forms in terms of cohomological invariants

– cohomological approach depends on finiteness of unramified cohomology, which is not known in general case in dimension > 2.

Following Kato, by *2-dimensional global field* we mean function field of:

- smooth curve defined over a number field, or
- smooth surface defined over a finite field.

• Finiteness Conjecture is true for spinor groups $\mathrm{Spin}_n(q)$ of quadratic forms over 2-dimensional global fields of characteristic $\neq 2$.

PROOF consists of two parts:

- Using Milnor's conjecture proved by Voevodsky, one reduces problem to proving finiteness of unramified cohomology groups $H^i(K, \mu_2)_V$;
- Proof of finiteness of $H^i(K, \mu_2)_V$ for all $i \geqslant 1$ for 2-dimensional global field K and divisorial set of places V.

• Finiteness conjecture is also true for simple groups of types A_n, C_n, D_n, F_4 and G_2 over 2-dimensional global fields that split over a quadratic extension of base field.

• There are also finiteness results over purely transcendental extensions of global fields and function fields of Severi-Brauer varieties.

We will now discuss some applications of these results to finiteness of genus and properness of global-to-local map in Galois cohomology.

Theorem 5

Let $G = \mathrm{Spin}_n(q)$ $(n \geqslant 5)$ *where* q *is a nondegenerate quadratic form over a 2-dimensional global field* K *with* $\mathrm{char}\, K \neq 2$. **Then** $\mathbf{gen}_K(G)$ *is finite.*

Case of n odd follows directly from Theorem 4.

Case of n even was considered by I. Rapinchuk.

Theorem 6

Let G be a simple algebraic K-group of type G_2.

(1) *If $K = k(x)$ where k is a number field then $\mathbf{gen}_K(G)$ is trivial.*

(2) *If $K = k(x_1, \ldots, x_r)$ where k is a number field then $\mathbf{gen}_K(G)$ is finite.*

Similar results are available for other types.

E.g., for K-forms of type F_4, genus $\mathbf{gen}_K(G)$ is *trivial* if $K = k(x)$ where k is a number field, and is *finite* if K is any 2-dimensional global field of characteristic $\neq 2, 3$.

Let G be a linear algebraic group defined over a field K, and let V be a set of valuations of K.

One considers global-to-local map in Galois cohomology:

$$\theta_{G,V} \colon H^1(K,G) \longrightarrow \prod_{v \in V} H^1(K_v, G)$$

We say that Hasse principle holds if $\theta_{G,V}$ is *injective.*

- HP is known to hold when K is a number field and G is either *simply connected* or *adjoint.*

- HP may fail for arbitrary semi-simple groups over number fields, **but** here $\theta_{G,V}$ is always proper, i.e. has *finite* fibers.

In recent years a lot of attention has been given to HP over fields other than global. Colliot-Thèléne, Parimala ... analyzed HP over function fields of p-adic curves.

Conjecture

Let G be a reductive algebraic group over a finitely generated field K with a divisorial set of places V. **Then** *$\theta_{G,V}$ is proper.*

Finiteness Conjecture for forms with good reduction for a given *adjoint* $\overline{G}$ and any divisorial V $\Rightarrow$ properness of $\theta_{\overline{G},V}$ for any divisorial V.

It follows that $\theta_{G,V}$ is proper when:

1. $G = \mathrm{PSL}_{1,A}$ where A is a c.s.a. of degree n over a f.g. field K with $\mathrm{char}\, K$ prime to n
2. $G = \mathrm{SO}_n(q)$ where q is a nondegenerate quadratic form over a 2-dimensional global field K with $\mathrm{char}\, K \neq 2$
3. G is of type G_2 over a 2-dimensional global field K with $\mathrm{char}\, K \neq 2$

(where V is any divisorial set)

It turns out that θ is always proper for algebraic tori.

Theorem 7

Suppose K is a finitely generated field with a divisorial set of places V. **Then** *for any algebraic K-group D whose connected component is a torus, $\theta_{D,V}$ is proper. In particular, for any K-torus T, Tate-Shafarevich group*

$$Ш(T,V) := \mathrm{Ker}(H^1(K,T) \longrightarrow \prod_{v \in V} H^1(K_v,T))$$

is finite.

Corollary

Let G be a connected reductive algebraic group over a f.g. field K, and V be a divisorial set of places. Fix a maximal K-torus T of G and let $\mathcal{C}(T)$ denote set of all maximal K-tori T' of G such that T and T' are $G(K_v)$-conjugate for all $v \in V$. **Then** *consists of finitely many $G(K)$-conjugacy classes.*

A.R., I.A. Rapinchuk, *Linear algebraic groups with good reduction,* Res. Math. Sci. **7**(2020), article 28.