On Provability Logic of HA

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Propositional non-modal language

$$\mathcal{L}_0 : \quad \vee, \wedge, \to, \mathsf{par}, \mathsf{var}$$

- var and par are countably infinite sets of atomics and $\top, \bot \in \mathsf{par}$.
- $\neg A := A \to \bot$.
- $\mathsf{atom} := \mathsf{par} \cup \mathsf{var}$
- par stands for Σ_1-substitutions, var for arbitrary.
- For a propositional substitution θ, by default $\theta(p) := p$ for every $p \in \mathsf{par}$.
- $\mathcal{L}_0(X)$ indicates the set of all Boolean combinations of propositions in the set X.

Modal language

$$\mathcal{L}_\rhd := \mathcal{L}_0 + \rhd \quad \text{and} \quad \Box A := \top \rhd A \quad \text{and} \quad \mathcal{L}_\Box := \mathcal{L}_0 + \Box$$

- $\rhd$ is a binary modal operator.
- We usually consider $A \rhd B$ for preservativity.
- $\mathsf{B} := \{\Box A : A \in \mathcal{L}_\Box\}$.
- $\mathsf{parb} := \mathsf{par} \cup \mathsf{B}$.
- $\mathsf{atomb} := \mathsf{atom} \cup \mathsf{B}$.
- $\boxdot A := A \wedge \Box A$.

Logics

K: $\Box(A \to B) \to (\Box A \to \Box B)$.

4: $\Box A \to \Box\Box A$.

L: $\Box(\Box A \to A) \to \Box A$. (The Löb's axiom)

$\mathsf{C_p}$: $p \to \Box p$ for every $p \in \mathsf{par}$.

$\mathsf{C_a}$: $a \to \Box a$ for every $a \in \mathsf{atom}$.

Given a logic L and axiom-schemata $\mathsf{X_1}, \ldots, \mathsf{X_n}$, the logic $\mathsf{LX_1} \ldots \mathsf{X_n}$ is defined as L plus the axioms $\mathsf{X_1}, \ldots, \mathsf{X_n}$. Then we define following modal logics:

- i: IPC plus necessitation and $\mathsf{C_p}$.
- $\mathsf{iGL} := \mathsf{iK4L}$.

Propositional substitutions

- $\theta(x)$ is a proposition in the language $\mathcal{L}_\rhd$ for every $x \in \mathsf{var}$.
- $\theta(p) = p$ for every $p \in \mathsf{par}$.
- $\theta(B \circ C) = \theta(B) \circ \theta(C)$ for every $\circ \in \{\vee, \wedge, \rightarrow, \rhd\}$.

Given θ, define $\hat{\theta}$ same as θ except for boxed propositions for which $\hat{\theta}$ operates as identity:

$$\hat{\theta}(A \rhd B) := A \rhd B \text{ and hence } \hat{\theta}(\Box A) := \Box A.$$

Heyting Arithmetic

The Heyting arithmetic is defined as the intuitionistic fragment of first-oder Peano Arithmetic PA.

Arithmetical substitutions

A function α on atom such that $\alpha(a)$ is a first-order arithmetical sentence for every $a \in \mathsf{atom}$ and $\alpha(a) \in \Sigma_1$ for every $a \in \mathsf{par}$ and $\alpha(\bot) = \bot$ and $\alpha(\top) = \top$. Moreover α is called a Σ_1-substitution if $\alpha(a) \in \Sigma_1$ for every $a \in \mathsf{atom}$.

- $\alpha_{\mathsf{HA}}(a) := \alpha(a)$ for every $a \in \mathsf{atom}$, and $\alpha_{\mathsf{HA}}(\bot) = \bot$.
- α_{HA} commutes with boolean connectives: $\vee, \wedge$ and $\to$.
- $\alpha_{\mathsf{HA}}(A \rhd B)$ is defined as an arithmetization of Σ_1-preservativity: For every $E \in \Sigma_1$,

$$\text{if } \mathsf{HA} \vdash E \to \alpha_{\mathsf{HA}}(A) \text{ then } \mathsf{HA} \vdash E \to \alpha_{\mathsf{HA}}(B).$$

- $\alpha_{\mathsf{HA}}(\Box A)$ = an arithmetization of "A is provable in HA".

$\mathsf{PL}^{\Box}(\mathsf{HA})$, the provability logic of HA is defined as

$$\{A \in \mathcal{L}_\Box : \mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A) \text{ for every arithmetical substitution } \alpha\}$$

$\mathsf{PL}^{\rhd}(\mathsf{HA})$, the Preservativity logic of HA is defined

$$\{A \in \mathcal{L}_\rhd : \mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A) \text{ for every arithmetical substitution } \alpha\}$$

Similarly one may define $\mathsf{PL}^{\Box}_{\Sigma}(\mathsf{HA})$ and $\mathsf{PL}^{\rhd}_{\Sigma}(\mathsf{HA})$ as provability and preservativity logics for Σ_1-substitutions.

- $\mathsf{PL}^{\Box}(\mathsf{HA}) \nvdash \Box(A \vee B) \to (\Box A \vee \Box B)$, Myhill [1973]; Friedman [1975]
- $\mathsf{PL}^{\Box}(\mathsf{HA}) \vdash \Box(A \vee B) \to \Box(\boxdot A \vee \boxdot B)$, in which $\boxdot A$ is a shorthand for $A \wedge \Box A$, Leivant [1975]
- $\mathsf{PL}^{\Box}(\mathsf{HA}) \vdash \Box\neg\neg\Box A \to \Box\Box A$ and $\mathsf{PL}^{\Box}(\mathsf{HA}) \vdash \Box(\neg\neg\Box A \to \Box A) \to \Box(\Box A \vee \neg\Box A)$, Visser [1981, 1982]
- Decidability of letterless fragment of $\mathsf{PL}^{\Box}(\mathsf{HA})$. Visser [2002]
- Axiomatization and decidability of $\mathsf{PL}^{\Box}_{\Sigma}(\mathsf{HA})$. Ardeshir and Mojtahedi [2018]; Visser and Zoethout [2019]
- Axiomatization and decidability of $\mathsf{PL}^{\Box}_{\Sigma}(\mathsf{HA})$ relative in PA and $\mathbb{N}$. Mojtahedi [2021]

The following translation, is some variant of the Gödel's celebrated translation for the embedding of IPC in S4 [Gödel, 1933].

Definition

For every proposition $A \in \mathcal{L}_\Box$ define $A^\Box$ inductively as follows:

- $A^\Box := \boxdot A$, for $A \in \mathsf{var}$.
- $A^\Box := A$ for $A \in \mathsf{parb}$.
- $(B \circ C)^\Box := B^\Box \circ C^\Box$. for $\circ \in \{\vee, \wedge\}$.
- $(B \to C)^\Box := \boxdot (B^\Box \to C^\Box)$.

- $A \in \mathcal{L}_\Box$ is called self complete if there is some $B \in \mathcal{L}_\Box$ such that $A = B^\Box$:

$$\mathsf{S} := \{B^\Box : B \in \mathcal{L}_\Box\}.$$

- A is called T-complete if $\mathsf{T} \vdash A \to \Box A$:

$$\mathsf{C}^\mathsf{T} := \{A \in \mathcal{L}_\Box : \mathsf{T} \vdash A \to \Box A\}.$$

- If $\mathsf{T} \supseteq \mathsf{iK4}$ we have $\mathsf{S} \subseteq \mathsf{C}^\mathsf{T}$.
- We may omit the superscript T in the notation C^T and simply write C.

A Kripke model for the intuitionistic modal logic, is a combination of a Kripke model for intuitionistic logic and the classical modal logic. Let $\mathcal{K} = (W, \prec, \sqsubset, V)$:

- $W \neq \emptyset$.
- $(W, \prec)$ is a partial order (transitive and irreflexive). We write $\preccurlyeq$ for the reflexive closure of $\prec$.
- V is the valuation on atomics, i.e. $V \subseteq W \times \mathsf{atom}$.
- $w \preccurlyeq u$ and $w\ V\ a$ implies $u\ V\ a$ for every $w, u \in W$ and $a \in \mathsf{atom}$.
- $(\preccurlyeq; \sqsubset) \subseteq \sqsubset$, i.e. $w \preccurlyeq u \sqsubset v$ implies $w \sqsubset v$. This condition is assumed to ensure that the previous property holds for all modal propositions and not only for $a \in \mathsf{atom}$.

V is extended to all modal propositions:

- $\mathcal{K}, w \Vdash a$ iff $w \; V \; a$, for $a \in \mathsf{atom}$.
- $\mathcal{K}, w \Vdash A \wedge B$ iff $\mathcal{K}, w \Vdash A$ and $\mathcal{K}, w \Vdash B$.
- $\mathcal{K}, w \Vdash A \vee B$ iff $\mathcal{K}, w \Vdash A$ or $\mathcal{K}, w \Vdash B$.
- $\mathcal{K}, w \Vdash A \to B$ iff for every $u \succcurlyeq w$ if we have $\mathcal{K}, w \Vdash A$ then $\mathcal{K}, w \Vdash B$.
- $\mathcal{K}, w \Vdash A \rhd B$ iff for every $u \sqsupset w$ with $\mathcal{K}, u \Vdash A$ we have $\mathcal{K}, w \Vdash B$.
- $\mathcal{K}, w \Vdash \Box A$ iff for every $u \sqsupset w$ we have $\mathcal{K}, w \Vdash A$.

- We say that u is a successor of w if $w \sqsubset u$.
- We say that u is a predecessor of w if $u \sqsubset w$.
- We say that u is above w if $w \preccurlyeq u$.
- We say that u is beneath w if $u \preccurlyeq w$.
- We say that u is generated by w if $w \ (\overline{\sqsubset \cup \preccurlyeq})\ u$.
- $\overline{S}$ indicates the reflexive transitive closure of S.

◂ iGL/iGL$\mathsf{C_a}$

Theorem

iGL *is sound and complete for good Kripke models. Also* iGL$\mathsf{C_a}$ *is sound and complete for good* $\mathsf{C_a}$ *Kripke models.*

- The class of *No Nested Implications to the Left*, NNIL formulae, for the nonmodal language $\mathcal{L}_0$, was introduced in [Visser et al., 1995], and more explored in [Visser, 2002].
- Visser et al. [1995] chracterize the NNIL via Kripke semantics.
- $A \in \mathsf{NNIL}$ and $A \in \mathsf{NI}$ for every $A \in \mathsf{atomb}$.
- $B \circ C \in \mathsf{NNIL}$ if $B, C \in \mathsf{NNIL}$. Also $B \circ C \in \mathsf{NI}$ if $B, C \in \mathsf{NI}$. $(\circ \in \{\vee, \wedge\})$
- $B \to C \in \mathsf{NNIL}$ if $B \in \mathsf{NI}$ and $C \in \mathsf{NNIL}$.

- One of the most interesting features of the Gödel-Löb axiom, is the fixed-point theorem.
- It is the propositional remainder of the Gödels diagnalization lemma.
- It says that if x only appears in the scope of $\Box$ in A, then there is some D such that $\mathsf{GL} \vdash D \leftrightarrow A[x : D]$. [Smoryński, 1985]
- One may generalize the same fixed-point theorem to iGL. [Iemhoff et al., 2005]
- It is well-known that one may generalize this fixed-point theorem to a simultaneous version.

Theorem

Let $\vec{E} := \{E_1, \ldots, E_m\}$ and $\vec{a} = \{a_1, \ldots, a_m\}$ such that every occurrences of a_i in E_j is in the scope of some $\Box$. Then there is a substitution τ which is the simultaneous fixed point of $\vec{a}$ with respect to $\vec{E}$ in iGL, *i.e.*

- iGL $\vdash \tau(E_i) \leftrightarrow \tau(a_i)$ *for every* $1 \leq i \leq m$.

Unification

- Unification problem (in propositional Logic L) asks for substitutions θ which unify A, i.e. $\mathsf{L} \vdash \theta(A)$.
- More ambitiously: describe the set of all unifiers for A.
- $\theta \leq \gamma$ iff there is some λ s.t. $\mathsf{L} \vdash \theta(x) \leftrightarrow \lambda\gamma(x)$.
- Classical logic: every unifiable proposition has a most general unifier.

If θ is a unifier of A then χ_θ is a most general one:

$$\chi_\theta(x) := (A \wedge x) \vee (\neg A \wedge \theta(x))$$

χ_θ is a unifier indeed:

$$A \vdash \chi_\theta(A) \leftrightarrow \top \quad \text{and} \quad \neg A \vdash \chi_\theta(A) \leftrightarrow \top.$$

χ_θ is more general than every other unifier γ:

$$A \vdash \chi_\theta(x) \leftrightarrow x \Longrightarrow \gamma(A) \vdash \gamma\chi_\theta(x) \leftrightarrow \gamma(x)$$

Definition

A is called projective (in L) if there is some unifier θ for A s.t. for every $x \in \mathsf{var}$:

$$A \vdash_{\mathsf{L}} \theta(x) \leftrightarrow x$$

- $x \vee \neg x$ does not have a most general unifier in IPC.
- Ghilardi [1999] answered to the unification problem for L = IPC and par = $\emptyset$ (Elementary unification or E-unification).
- Ghilardi [1999] first characterized projectives via Kripke semantics.
- Then with the aid of projective approximations he proved that IPC is finitary, i.e. every unifiable A has a finite set of unifiers which are more general than every unifier of A.

Projectivity: relativised

- Instead of unification ($\top$-fication) we consider Γ-fication for $\Gamma \subseteq \mathcal{L}_0(\mathsf{par})$.
- This means that we ask for all θ's such that $\mathsf{L} \vdash \theta(A) \in \Gamma$, i.e. $\mathsf{L} \vdash \theta(A) \leftrightarrow E$ for some $E \in \Gamma$.
- In this setting, we say that A is Γ-projective iff there is a Γ-fier θ for A which is projective:

$$A \vdash_{\mathsf{L}} \theta(x) \leftrightarrow x$$

- $\downarrow\Gamma :=$ the set of all Γ-projective propositions.

In the first of two consecutive manuscripts on provability logic of HA we considered the case $\Gamma = \mathsf{NNIL}(\mathsf{par})$ and $\mathsf{L} = \mathsf{IPC}$. We followed Ghilardi [1999] to

- characterize NNIL(par)-projectivity via Kripke semantics,
- and then for a given A, compute a finite NNIL(par)-projective approximation.

Admissible rules

- The problem of admissibility (Friedman 1975) asks for the characterization and decidability of all inference rules A/B which are admissible to the logic L, i.e. for every substitution θ if we have $\mathsf{L} \vdash \theta(A)$ then $\mathsf{L} \vdash \theta(B)$.
- The classical case is trivial: A/B is admissible iff $A \to B$ is derivable.
- $\neg x \to (y \vee z)/(\neg x \to y) \vee (\neg x \to z)$ is admissible to IPC. [Harrop, 1960]
- Rybakov [1987] showed that admissibility for IPC is decidable.

Admissible rules of IPC

- de Jongh and Visser provided a base for all known admissible rules of IPC and conjectured it to be complete.
- Iemhoff [2001b] with the aid of [Ghilardi, 1999] proved the completeness of the base.

Relative admissibility

- In the first manuscript, we considered a relative version of admissibility.
- We say that A/B is admissible relative in Γ if

$$\forall E \in \Gamma \, \forall \theta \; (\vdash \theta(E \to A) \Longrightarrow \vdash \theta(E \to B)).$$

- Following the tools and methods in [Iemhoff, 2001b] we found a base for the admissibility relative in NNIL(par).

NNIL(par)-projective approximation

($\downarrow$N($\Box$)$^\vee$, iGL) rsdc

Theorem

Given $A \in \mathcal{L}_0$, *we may effectively compute a finite set* $\Pi \subseteq \downarrow\mathsf{N}(\mathsf{par})$ *such that*

1. $\mathsf{IPC} \vdash \bigvee \Pi \to A$.
2. $[\![\mathsf{IPC}, \mathsf{par}]\!] \vdash A \rhd \bigvee \Pi$.
3. Π *is computable as a function of* A.

$[\![\mathsf{T},\Delta]\!]$ has following axioms and rules:

Ax : $A \rhd B$, for every $\mathsf{T} \vdash A \to B$.

V(Δ) : $B \to C \rhd \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$, and

$$\frac{A \rhd B \qquad A \rhd C}{A \rhd B \wedge C}\ \text{Conj} \qquad \frac{A \rhd B \qquad B \rhd C}{A \rhd C}\ \text{Cut}$$

$$\frac{B \rhd A \qquad C \rhd A}{B \vee C \rhd A}\ \text{Disj} \qquad \frac{A \rhd B \qquad (C \in \Delta)}{C \to A \rhd C \to B}\ \text{Mont}(\Delta)$$

$$\{A\}_\Delta(B) := \begin{cases} B & : B \in \Delta \\ A \to B & : \text{otherwise} \end{cases}$$

Elevating projectivity to the modal language I

Let $A \in \mathcal{L}_\Box$ and $\Gamma \subseteq \mathcal{L}_0(\mathsf{parb})$. A substitution θ is called *A-projective* (in T) if

$$\text{For all atomic } a \text{ we have} \quad \mathsf{T} \vdash A \to (a \leftrightarrow \theta(a)). \tag{3.1}$$

A substitution θ, is a Γ-*fier* for $A \in \mathcal{L}_\Box$ (notation $A \overset{\theta}{\underset{\mathsf{T}}{\twoheadrightarrow}} \Gamma$), if

$$\mathsf{T} \vdash \hat{\theta}(A) \in \Gamma \quad \text{i.e. } \hat{\theta}(A) \text{ is } \mathsf{T}\text{-equivalent to some } A' \in \Gamma.$$

θ is a unifier for A if it is $\{\top\}$-fier for A.

Elevating projectivity to the modal language II

- We say that a substitution θ projects A to Γ in T (notation: $A \underset{\mathsf{T}}{\overset{\theta}{\rightarrowtail\!\!\!\twoheadrightarrow}} \Gamma$) if θ is A-projective in T and $A \underset{\mathsf{T}}{\overset{\theta}{\twoheadrightarrow}} \Gamma$.
- We say that A is Γ-projective in T if there is some θ such that $A \underset{\mathsf{T}}{\overset{\theta}{\rightarrowtail\!\!\!\twoheadrightarrow}} \Gamma$.
- $\overset{\mathsf{T}}{\downarrow}\Gamma$ indicates the set of all propositions which are Γ-projective in T.
- A is projective, if it is $\{\top\}$-projective.

Some notations on sets of propositions

- We write $\mathsf{X_1 \ldots X_n}$ for $\mathsf{X_1} \cap \ldots \cap \mathsf{X_n}$.
- $\Gamma^\vee := \{\bigvee \Delta : \Delta \subseteq_{\text{fin}} \Gamma \text{ and } \Delta \neq \emptyset\}$.
- $\Gamma(X) := \Gamma \cap \mathcal{L}_0(X)$ and $\Gamma(\Box) := \Gamma(\mathsf{parb})$.
- $\overset{\mathsf{T}}{\downarrow}\Gamma :=$ the set of all Γ-projective propositions in the logic T.
- $(.)^\vee$ has the lowest precedence and $\downarrow(.)$ has the second lowest precedence. This means that

$$\downarrow\mathsf{SN}(\Box)^\vee := (\downarrow(\mathsf{SN}(\Box)))^\vee \quad \text{and} \quad \mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee := (\mathsf{C}(\downarrow(\mathsf{SN}(\Box))))^\vee.$$

Definitions of admissibility and preservativity

$$A \mathrel{|\!\!\sim^{\mathsf{T}}_{\Gamma}} B \text{ iff for every substitution } \theta \text{ and } C \in \Gamma\text{:}$$
$$\mathsf{T} \vdash \hat{\theta}(C \to A) \Rightarrow \mathsf{T} \vdash \hat{\theta}(C \to B).$$

$$A \mathrel{|\!\!\approx^{\mathsf{T}}_{\Gamma}} B \quad \text{iff} \quad \forall\, E \in \Gamma(\mathsf{T} \vdash E \to A \Rightarrow \mathsf{T} \vdash E \to B).$$

$\hat{\theta}$ is same as θ on the non-modal language and $\hat{\theta}(\Box B) := \Box B$.

$\mathrel{|\!\sim}^{\mathsf{T}}_{\Gamma} \subseteq \mathrel{|\!\approx}^{\mathsf{T}}_{\downarrow\Gamma}$

By definition it can be inferred that $A \mathrel{|\!\sim}^{\mathsf{T}}_{\Gamma} B$ implies $A \mathrel{|\!\approx}^{\mathsf{T}}_{\Gamma} B$, however the converse may not hold. As a counterexample let A and B two different variables and $\Gamma := \{\top\}$ and $\mathsf{T} = \mathsf{IPC}$. Then we have $A \mathrel{|\!\approx}^{\mathsf{T}}_{\Gamma} B$ and not $A \mathrel{|\!\sim}^{\mathsf{T}}_{\Gamma} B$.

Theorem

Let T be a logic which is closed under outer substitutions. Then $A \mathrel{|\!\sim}^{\mathsf{T}}_{\Gamma} B$ implies $A \mathrel{|\!\approx}^{\mathsf{T}}_{\downarrow\Gamma} B$.

Some more notations

- $\mathsf{P}^{\mathsf{T}} := \{A : \mathsf{T} \vdash A \to B \vee C \Rightarrow \mathsf{T} \vdash A \to B \text{ or } \mathsf{T} \vdash A \to C\}$.
- $\Gamma^{\vee} := \{\bigvee \Delta : \emptyset \neq \Delta \subseteq_{\text{fin}} \Gamma\}$.

We may omit T from notations P^{T} and C^{T}.

[T]

SN($\Box$)-rsdc
Projec-Res

Given a logic T, the logic $[\mathsf{T}]$ proves statements $A \rhd B$ for A and B in the language of T and has the following axioms and rules:

Aximos

Ax : $A \rhd B$, for every $\mathsf{T} \vdash A \to B$.

Rules

$$\frac{A \rhd B \qquad A \rhd C}{A \rhd B \wedge C}\ \text{Conj} \qquad\qquad \frac{A \rhd B \qquad B \rhd C}{A \rhd C}\ \text{Cut}$$

These axioms and rule are not interesting, because $[\mathsf{T}] \vdash A \rhd B$ iff $\mathsf{T} \vdash A \to B$.

Extra axioms for preservativity and admissibility

Le: $A \rhd \Box A$ for every $A \in \mathcal{L}_\Box$.

Le$^-$: $A \rhd \Box A$ for every $A \in \mathcal{L}_0(\mathsf{parb})$.

A: $A \rhd \hat{\theta}(A)$, for every substitution θ.

V(Δ) : $B \to C \rhd \bigvee_{i=1}^{n+m}\{B\}_\Delta(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$, and

$$\{A\}_\Delta(B) := \begin{cases} B & : B \in \Delta \\ A \to B & : \text{otherwise} \end{cases}$$

$$\frac{B \rhd A \qquad C \rhd A}{B \vee C \rhd A}\,\text{Disj} \qquad\qquad \frac{A \rhd B \qquad (C \in \Delta)}{C \to A \rhd C \to B}\,\text{Mont}(\Delta)$$

$$[\![\mathsf{T}, \Delta]\!] := [\mathsf{T}] + \text{Disj} + \text{Mont}(\Delta) + \mathsf{V}(\Delta),$$

$$[\![\mathsf{T}, \Delta]\!]\mathsf{Le} := [\![\mathsf{T}, \Delta]\!] + \mathsf{Le} \quad \text{and} \quad [\![\mathsf{T}, \Delta]\!]\mathsf{Le}^{-} := [\![\mathsf{T}, \Delta]\!] + \mathsf{Le}^{-}.$$

Lemma

$\mathsf{T} \subseteq \mathsf{T}'$ *and* $\Delta \subseteq \Delta'$ *implies* $[\![\mathsf{T}, \Delta]\!] \subseteq [\![\mathsf{T}', \Delta']\!]$.

Lemma

$\mathrel{|\!\!\approx}^{\mathsf{T}}_{\Gamma} = \mathrel{|\!\!\approx}^{\mathsf{T}}_{\Gamma^{\vee}}$ *and* $\mathrel{|\!\!\sim}^{\mathsf{T}}_{\Gamma} = \mathrel{|\!\!\sim}^{\mathsf{T}}_{\Gamma^{\vee}}$.

Intuitionistic submodel property

Given two Kripke models $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V)$ and $\mathcal{K}' = (W', \preccurlyeq', \sqsubset', V')$, we say that $\mathcal{K}'$ is an intuitionistic submodel of $\mathcal{K}$ (notation $\mathcal{K}' \leq \mathcal{K}$) iff $W = W'$, $\sqsubset = \sqsubset'$, $V = V'$ and $\preccurlyeq' \subseteq \preccurlyeq$. A class $\mathscr{K}$ of Kripke models has intuitionistic submodel property, if $\mathcal{K}' \leq \mathcal{K} \in \mathscr{K}$ implies $\mathcal{K}' \in \mathscr{K}$. A modal logic T is said to have intuitionistic submodel property iff it is sound and complete for some class $\mathscr{K}$ of good Kripke models with intuitionistic submodel property.

General soundness: preservativity

$\mathrel{|\!\!\approx}^{\mathsf{iGLC_a}}_{\mathsf{SN}}$ $\mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C\downarrow SN}(\Box)}$ $\mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$

Theorem (Soundness)

$[\mathsf{T}]$ is sound for preservativity interpretations, i.e. $[\mathsf{T}] \vdash A \rhd B$ implies $A \mathrel{|\!\!\approx}^{\mathsf{T}}_{\Gamma} B$ for every set Γ of propositions and every logic T. Moreover

- *if Γ is T-complete, then Le is sound,*
- *if Γ is T-prime, then Disj is also sound,*
- *if Γ is closed under Δ-conjunctions, then $\mathrm{Mont}(\Delta)$ is sound.*
- *if T has intuitionistic submodel property and $\Gamma \subseteq \mathsf{NNIL}$ and $\Delta \subseteq \mathsf{atomb}$ then $\mathsf{V}(\Delta)$ is sound.*
- *if $\Gamma \subseteq \mathcal{L}_0(\mathsf{parb})$ and T is closed under outer substitutions, then A is also sound.*

General soundness: admissibility

$\succsim^{\mathsf{iGL}}_{\mathsf{N}(\Box)}$ $\succsim^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$

Theorem (Soundness)

[T] is sound for admissibility interpretations, i.e. $[\mathsf{T}] \vdash A \rhd B$ *implies* $A \mathrel{|\!\!\sim^{\mathsf{T}}_{\Gamma}} B$ *for every set* Γ *of propositions and every logic* T *which is closed under outer substitutions. Moreover*

- *if* Γ *is* T*-complete, then* Le^{-} *is sound,*
- *if* Γ *is* T*-prime, then Disj is also sound.*
- *if* Γ *is closed under outer substitutions of* Δ*-conjunctions, i.e.* $A \in \Gamma$ *and* $B \in \Delta$ *implies* $A \wedge \hat{\theta}(B) \in \Gamma$ *(up to* T*-provable equivalence relation), then* Mont(Δ) *is sound.*
- *if* T *has intuitionistic submodel property and* $\Gamma \subseteq \mathsf{NNIL}$ *and* $\Delta \subseteq \mathsf{parb}$ *then* $\mathsf{V}(\Delta)$ *is sound.*

Greatest lower bound (glb)

- B is a (Γ, T)-lb for A if:
  1. $B \in \Gamma$,
  2. $\mathsf{T} \vdash B \to A$.
- B is the (Γ, T)-glb for A, if for every (Γ, T)-lb B' for A we have $\mathsf{T} \vdash B' \to B$.
- Up to T-provable equivalence relation, such glb is unique and we annotate it as $\lfloor A \rfloor_\Gamma^\mathsf{T}$.
- (Γ, T) is downward compact, if every $A \in \mathcal{L}_\Box$ has a (Γ, T)-glb $\lfloor A \rfloor_\Gamma^\mathsf{T}$.
- If $\lfloor A \rfloor_\Gamma^\mathsf{T}$ can be effectively computed, we say that (Γ, T) is recursively downward compact.

Theorem (Visser [2002])

(NNIL, IPC) *is recursively downward compact.*

$\lfloor A \rfloor^{\mathsf{IPC}}_{\mathsf{NNIL}}$ is named A^* in [Visser, 2002], the so called Visser's NNIL algorithm.

Question

One may similarly define the notion of least upper bounds and upward compactness. Does downward compactness imply upward compactness?

(Γ, T)-glb and $\overset{\mathsf{T}}{\underset{\Gamma}{\mid\!\approx}}$

Theorem

B is the (Γ, T)-glb for A iff

- $B \in \Gamma$,
- $\mathsf{T} \vdash B \to A$,
- $A \overset{\mathsf{T}}{\underset{\Gamma}{\mid\!\approx}} B$.

Hence we have $A \overset{\mathsf{T}}{\underset{\Gamma}{\mid\!\approx}} \lfloor A \rfloor_\Gamma^\mathsf{T}$.

Corollary

If $\lfloor A \rfloor_\Gamma^\mathsf{T}$ exists, then for every $B \in \mathcal{L}_\Box$ we have

$$\mathsf{T} \vdash \lfloor A \rfloor_\Gamma^\mathsf{T} \to B \quad \textit{iff} \quad A \overset{\mathsf{T}}{\underset{\Gamma}{\mid\!\approx}} B.$$

Question

The glb may be expressed via preservativity relation $\mid\!\approx^{\mathsf{T}}_{\Gamma}$. *One may think of its twin sister which best suites for lub's:*

$$A \;{}^{*}\!\mid\!\approx^{\mathsf{T}}_{\Gamma} B \quad \textit{iff} \quad \forall E \in \Gamma(\mathsf{T} \vdash A \to E \Rightarrow \mathsf{T} \vdash B \to E).$$

We ask for an axiomatization for ${}^{*}\!\mid\!\approx^{\mathsf{T}}_{\Gamma}$ *when we let* $\mathsf{T} = \mathsf{IPC}$ *and* $\Gamma = \mathsf{NNIL}$.

Normal forms

Define $\Gamma\text{-}\mathsf{NF}_0$ as the set of propositions $B \in \mathcal{L}_\Box$ with either $B \in \Gamma$ or $\boxdot B \in \Gamma$. Then define the set $\Gamma\text{-}\mathsf{NF}$ of propositions in Γ-Normal Form as follows:

$$\Gamma\text{-}\mathsf{NF} := \{A \in \mathcal{L}_\Box : \forall \Box B \in \mathsf{sub}(A)\ B \in \Gamma\text{-}\mathsf{NF}_0\}.$$

Iterating glb's nested inside $\Box$

Completeness

We say that (Γ, T) is (recursively) *strong* downward compact, if it is (recursively) downward compact and for every $\Box B \in \mathsf{sub}(\lfloor A \rfloor_\Gamma^\mathsf{T})$ either we have $\Box B \in \mathsf{sub}(A)$ or $B \in \Gamma\text{-}\mathsf{NF}_0$. We also inductively define $\llbracket A \rrbracket_\Gamma^\mathsf{T}$:

- $\llbracket a \rrbracket_\Gamma^\mathsf{T} = a$ for every atomic a.
- $\llbracket\ \rrbracket_\Gamma^\mathsf{T}$ commutes with $\{\vee, \wedge, \to\}$.
- $\llbracket \Box A \rrbracket_\Gamma^\mathsf{T} := \Box \lfloor \llbracket A \rrbracket_\Gamma^\mathsf{T} \rfloor_\Gamma^\mathsf{T}$. H(Γ, T).

Lemma

If (Γ, T) *is strong downward compact and* $\mathsf{T} \supseteq \mathsf{iK4}$, *then for every* $A \in \mathcal{L}_\Box$ *we have* $\llbracket A \rrbracket_\Gamma^\mathsf{T} \in \Gamma\text{-}\mathsf{NF}$ *and* $\mathsf{H}(\Gamma, \mathsf{T}) \vdash_{\mathsf{T}} A \leftrightarrow \llbracket A \rrbracket_\Gamma^\mathsf{T}$.

Extension property

A class $\mathscr{M}$ of rooted Kripke models is said to has *extension property* if for every finite set $\mathscr{K} \subseteq \mathscr{M}$ there is some finite set of rooted Kripke models $\mathscr{K}'$ such that a variant of $\sum(\mathscr{K}, \mathscr{K}')$ belongs to $\mathscr{M}$.

Prime factorization

Before we continue with the axiomatization and decidability of several preservativities, let us see some preliminaries.

Theorem

Let T *has extension property. Then*

- $\mathsf{N}(\Box) = \mathsf{PN}(\Box)^\vee$ *and* $\mathsf{SN}(\Box) = \mathsf{SPN}(\Box)^\vee$.
- $\mathsf{N} = \mathsf{PN}^\vee$ *and* $\mathsf{SN} = \mathsf{SPN}^\vee$*, whenever* $\mathsf{T} \supseteq \mathsf{iK4C_a}$.

Corollary

$\approx^{\mathsf{T}}_{\mathsf{SN}(\Box)} = \approx^{\mathsf{T}}_{\mathsf{SPN}(\Box)}$ *and* $\vdash^{\mathsf{T}}_{\mathsf{SN}(\Box)} = \vdash^{\mathsf{T}}_{\mathsf{SPN}(\Box)}$ *and* $\vdash^{\mathsf{T}}_{\mathsf{N}(\Box)} = \vdash^{\mathsf{T}}_{\mathsf{PN}(\Box)}$ *and if* $\mathsf{T} \supseteq \mathsf{iK4C_a}$ *then* $\approx^{\mathsf{T}}_{\mathsf{SN}} = \approx^{\mathsf{T}}_{\mathsf{SPN}}$.

Theorem

$A^{\mathsf{h}} := (A^*)^{\Box} = \lfloor A \rfloor_{\mathsf{SN}}^{\mathsf{iGLC_a}}$ *and hence* $(\mathsf{SN}, \mathsf{iGLC_a})$ *is recursively strong downward compact. Moreover* $[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A \rhd A^{\mathsf{h}}$.

Proof.

Derived by ▸ rdc of (NNIL, IPC) from Visser [2002]. ❑

Theorem

$$[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} = \underset{\mathsf{SN}}{\overset{\mathsf{iGLC_a}}{\approx}} = \underset{\mathsf{SN}^\vee}{\overset{\mathsf{iGLC_a}}{\approx}} = \underset{\mathsf{SPN}}{\overset{\mathsf{iGLC_a}}{\approx}} = \underset{\mathsf{SPN}^\vee}{\overset{\mathsf{iGLC_a}}{\approx}} .$$

Moreover all above relations are decidable.

Proof.

▸ Prime factorization and ▸ $\underset{\mathsf{T}}{\overset{\mathsf{T}}{\approx}} = \underset{\mathsf{T}}{\overset{\mathsf{T}}{\approx}}{}^\vee$ imply $\underset{\mathsf{SN}}{\overset{\mathsf{iGLC_a}}{\approx}} = \underset{\mathsf{SN}^\vee}{\overset{\mathsf{iGLC_a}}{\approx}} = \underset{\mathsf{SPN}}{\overset{\mathsf{iGLC_a}}{\approx}} = \underset{\mathsf{SPN}^\vee}{\overset{\mathsf{iGLC_a}}{\approx}}$.
▸ General soundenss implies $[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \subseteq \underset{\mathsf{SPN}}{\overset{\mathsf{iGLC_a}}{\approx}}$.
To show $\underset{\mathsf{SN}}{\overset{\mathsf{iGLC_a}}{\approx}} \subseteq [\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le}$, let $A \underset{\mathsf{SN}}{\overset{\mathsf{iGLC_a}}{\approx}} B$. Then $A^{\mathsf{h}} \to B$ and hence $[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A^{\mathsf{h}} \rhd B$. Since $[\![\mathsf{iGLC_a}, \mathsf{atomb}]\!]\mathsf{Le} \vdash A \rhd A^{\mathsf{h}}$, Cut implies desired result. ❑

◂ $\downarrow\mathsf{SN}(\Box)^\vee$-rsdc

Theorem

$(\downarrow\mathsf{N}(\Box)^\vee, \mathsf{iGL})$ *is recursively strong downward compact.*
Moreover $[\![\mathsf{iGL}, \mathsf{parb}]\!] \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)^\vee}$.

Proof sketch.

Given A, one must treat outer occurrences of $\Box$'s as parameters, and then $\bigvee \Pi$ in ▸ NNIL(par)-projective approximation will work as $\lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)^\vee}$. ❑

Theorem

All are decidable:

$$[\![\mathsf{iGL},\mathsf{parb}]\!] = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)^{\vee}} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{PN}(\Box)} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{PN}(\Box)^{\vee}} =$$
$$= \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{N}(\Box)} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{N}(\Box)^{\vee}} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{PN}(\Box)} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{PN}(\Box)^{\vee}}.$$

Proof sketch.

▸ Prime factorization and ▸ $\mathrel{|\!\approx}^{\mathsf{T}} = \mathrel{|\!\approx}^{\mathsf{T}}_{\vee}$ imply $\mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)^{\vee}} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{PN}(\Box)} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{PN}(\Box)^{\vee}}$ and $\mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{N}(\Box)} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{N}(\Box)^{\vee}} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{PN}(\Box)} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{PN}(\Box)^{\vee}}$. Moreover ▸ implies $\mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{N}(\Box)} \subseteq \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)}$. ▸ General soundenss implies $[\![\mathsf{iGL},\mathsf{parb}]\!] \subseteq \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{N}(\Box)}$. To show $\mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)^{\vee}} \subseteq [\![\mathsf{iGL},\mathsf{parb}]\!]$, let $A \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)} B$. Hence $\mathsf{iGL} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)^{\vee}} \to B$ and thus $[\![\mathsf{iGL},\mathsf{parb}]\!] \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)^{\vee}} \rhd B$. Since $[\![\mathsf{iGL},\mathsf{parb}]\!] \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{N}(\Box)^{\vee}}$ we are done. ❑

◂ $\mathsf{C}{\downarrow}\mathsf{SN}(\Box)^{\vee}$-rsdc

Theorem

$({\downarrow}\mathsf{SN}(\Box)^{\vee}, \mathsf{iGL})$ *is recursively strong downward compact. Moreover* $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le}^{-} \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{{\downarrow}\mathsf{SN}(\Box)^{\vee}}$.

Proof sketch.

Given A, one first compute $\lfloor A \rfloor^{\mathsf{iGL}}_{{\downarrow}\mathsf{N}(\Box)^{\vee}}$ ▸. Let B is its $\mathsf{N}(\Box)^{\vee}$-projection and define $\lfloor A \rfloor^{\mathsf{iGL}}_{{\downarrow}\mathsf{SN}(\Box)^{\vee}} := \lfloor A \rfloor^{\mathsf{iGL}}_{{\downarrow}\mathsf{N}(\Box)^{\vee}} \wedge B^{\Box}$. ❑

Theorem

$$[\![\mathsf{iGL},\mathsf{parb}]\!]\mathsf{Le}^{-} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)^{\vee}} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SPN}(\Box)} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SPN}(\Box)^{\vee}} =$$
$$= \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SN}(\Box)^{\vee}} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SPN}(\Box)} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SPN}(\Box)^{\vee}}.$$

All are decidable.

Proof sketch.

▸ Prime factorization and ▸ $\mathrel{|\!\approx}^{\mathsf{T}}_{\mathsf{T}} = \mathrel{|\!\approx}^{\mathsf{T}}_{\mathsf{T}^{\vee}}$ imply $\mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)^{\vee}} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SPN}(\Box)} = \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SPN}(\Box)^{\vee}}$ and $\mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SN}(\Box)^{\vee}} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SPN}(\Box)} = \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SN}(\Box)^{\vee}}$. Moreover ▸ implies $\mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} \subseteq \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)}$. ▸ General soundenss implies $[\![\mathsf{iGL},\mathsf{parb}]\!]\mathsf{Le}^{-} \subseteq \mathrel{|\!\sim}^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$. To show $\mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)^{\vee}} \subseteq [\![\mathsf{iGL},\mathsf{parb}]\!]\mathsf{Le}^{-}$, let $A \mathrel{|\!\approx}^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)} B$. Hence $\mathsf{iGL} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)^{\vee}} \to B$ and thus $[\![\mathsf{iGL},\mathsf{parb}]\!]\mathsf{Le}^{-} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\downarrow\mathsf{SN}(\Box)^{\vee}} \rhd B$.

Theorem

$(\mathsf{C}{\downarrow}\mathsf{SN}(\Box)^{\vee}, \mathsf{iGL})$ *is recursively strong downward compact.*
Moreover $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{C}{\downarrow}\mathsf{SN}(\Box)^{\vee}}$.

Proof sketch.

Given A, one first compute $\lfloor A \rfloor^{\mathsf{iGL}}_{{\downarrow}\mathsf{SN}(\Box)^{\vee}}$. Then define

$$\lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{C}{\downarrow}\mathsf{SN}(\Box)^{\vee}} := \boxdot \lfloor A \rfloor^{\mathsf{iGL}}_{{\downarrow}\mathsf{SN}(\Box)^{\vee}}$$

❑

Theorem

$$[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} = \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)} = \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee} = \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SPN}(\Box)} = \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SPN}(\Box)^\vee}$$

Moreover all mentioned relations are decidable.

Proof.

▸ Prime factorization and ▸ $\mathrel{|\!\!\approx}^{\mathsf{T}}_{\Gamma} = \mathrel{|\!\!\approx}^{\mathsf{T}}_{\Gamma^\vee}$ imply $\mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)} = \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee} = \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SPN}(\Box)} = \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SPN}(\Box)^\vee}$. ▸ General soundenss implies $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} \subseteq \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SPN}(\Box)}$. To show $\mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee} \subseteq [\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le}$, let $A \mathrel{|\!\!\approx}^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)} B$. Hence $\mathsf{iGL} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee} \to B$ and thus $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee} \rhd B$. Since $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)^\vee}$ we are done. ❑

Theorem

$(\mathsf{SN}(\Box), \mathsf{iGL})$ *is recursively strong downward compact. Moreover* $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{LeA} \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$. ▸ *Axioms*

A flavour of proof.

The computation of $\lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$ is not just an add on for the Visser's NNIL-algorithm. One must go inside that algorithm and make some additional instruction.

$x \to B$ is approximated by $\lfloor B[\hat{x} : \top] \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$.

$B \to x$ is approximated by $\lfloor \neg B \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$. ❑

Examples

back to e1

- $\lfloor x \rfloor = \bot$
- $\lfloor A \rfloor = \bot$ if $A \in \mathcal{L}_0(\mathsf{var})$ and A is not a theorem of IPC.
- $\lfloor p \to x \rfloor = \neg p$
- $\lfloor \Box x \to x \rfloor = \neg \Box x$
- $\lfloor Boxdotx \to y \rfloor = \neg \Box x.$

Theorem

$$[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{LeA} = |\!\!\approx^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} = |\!\!\approx^{\mathsf{iGL}}_{\mathsf{SPN}(\Box)} = |\!\!\approx^{\mathsf{iGL}}_{\mathsf{SPN}(\Box)^\vee}$$

Moreover all mentioned relations are decidable.

Proof.

▸ Prime factorization and ▸ $|\!\!\approx^{\mathsf{T}}_{\mathsf{T}} = |\!\!\approx^{\mathsf{T}}_{\mathsf{T}^\vee}$ imply $|\!\!\approx^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} = |\!\!\approx^{\mathsf{iGL}}_{\mathsf{SPN}(\Box)} = |\!\!\approx^{\mathsf{iGL}}_{\mathsf{SPN}(\Box)^\vee}$. ▸ General soundenss implies $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{LeA} \subseteq |\!\!\approx^{\mathsf{iGL}}_{\mathsf{SPN}(\Box)}$. To show $|\!\!\approx^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} \subseteq [\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{LeA}$, let $A \mathrel{|\!\!\approx^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}} B$. Hence $\mathsf{iGL} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} \to B$ and thus $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{LeA} \vdash \lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} \rhd B$. Since $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{LeA} \vdash A \rhd \lfloor A \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$ we are done. ❑

- $\mathsf{H}(\Gamma, \mathsf{T}) := \{\Box A \to \Box B : A \mathrel{|\!\!\approx^{\mathsf{T}}_{\Gamma}} B\}$.
- Hence $\mathsf{iGLH}(\Gamma, \mathsf{T})$ is iGL plus the axiom $\mathsf{H}(\Gamma, \mathsf{T})$.
- $\mathsf{iGLH} := \mathsf{iGLH}(\mathsf{C}{\downarrow}\mathsf{SN}(\Box), \mathsf{iGL})$. (provability logic of HA)
- $\mathsf{iGLH}^{\Box} := \mathsf{iGLH}(\mathsf{SN}(\Box), \mathsf{iGL})$. (complete but not sound)
- $\mathsf{iGLC_aH}_\sigma := \mathsf{iGLC_aH}(\mathsf{SN}, \mathsf{iGLC_a})$.($\Sigma_1$-provability logic of HA)

$\{\mathsf{T}, \Delta\}$

T: All theorems of T.

V(Δ) : $B \to C \rhd \bigvee_{i=1}^{n+m} \{B\}_\Delta(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$.

Mont(Δ): $A \rhd B \to (C \to A) \rhd (C \to B)$ for every $C \in \Delta$.

Le: $A \rhd \Box A$ for every A.

Disj: $(B \rhd A \wedge C \rhd A) \to (B \vee C) \rhd A$.

Conj: $[(A \rhd B) \wedge (A \rhd C)] \to (A \rhd (B \wedge C))$.

Cut: $[(A \rhd B) \wedge (B \rhd C)] \to (A \rhd C)$.

MP: $A,\ A \to B\ /\ B$.

PNec: $A \to B\ /\ A \rhd B$.

- $\mathsf{iPH} := \{\!\{\mathsf{iGL}, \mathsf{parb}\}\!\}$.
- Iemhoff [2003] introduced iPH and proved (de Jongh & Visser) that iPH is sound for arithmetical interpretations in HA, i.e. $\mathsf{iPH} \vdash A$ implies $\mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A)$ for every α.
- Iemhoff [2003] conjectures that iPH is also complete for the arithmetical interpretations.
- $\mathsf{iPH}_\sigma := \{\!\{\mathsf{iGLC_a}, \mathsf{atomb}\}\!\}$.
- The same proof implies that iPH_σ is sound for Σ_1-interpretations in HA.
- It is quite natural to expect that iPH_σ is also complete for such interpretations.

iPH includes iGLH

Lemma

$\mathsf{iGLH} \vdash A$ *implies* $\mathsf{iPH} \vdash A$.

Proof.

By induction on the proof complexity of $\mathsf{iGLH} \vdash A$. All cases are trivial except for when A is an axiom instance of $\mathsf{H}(\mathsf{C}{\downarrow}\mathsf{SN}(\Box), \mathsf{iGL})$, i.e. $A = \Box B \to \Box C$ with $B \mathrel{|\!\!\!\approx^{\mathsf{iGL}}_{\mathsf{C}\downarrow\mathsf{SN}(\Box)}} C$. implies that $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} \vdash B \rhd C$. Then use induction on the proof $[\![\mathsf{iGL}, \mathsf{parb}]\!]\mathsf{Le} \vdash B \rhd C$. ❑

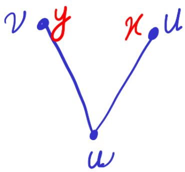
- Iemhoff [2003] provides soundness and completeness of iPH for some class of intuitionistic modal Kripke models.
- Iemhoff [2001a] also proved some partial completeness results corresponding to some fragments of iGLH.
- Mentioned Kripke models are mainly infinite. This makes them difficult to work with.
- Here we provide Kripke-style semantic for provability and preservativity which enjoys finite-model property.
- The main idea is that we assign a proposition φ_w to each node w and

$$\mathcal{K}, w \Vdash \Box B \quad \text{iff} \quad \forall\, u \sqsupset w \;(\; \varphi_w \vdash B)$$

An example

back to e1

$$\nVdash \Box(x \vee y) \to (\Box x \vee \Box y)$$



$v \Vdash \boxdot(x \vee y)$

$u \Vdash \boxdot(x \vee y)$

$\varphi_u = \boxdot x$

$\varphi_v = \boxdot y$

Definition

$\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \varphi)$ is called a $(\Delta, \Gamma, \mathsf{T})$-semantic if $\tilde{\mathcal{K}} := (W, \preccurlyeq, \sqsubset, V)$ is a transitive conversely well-founded Kripke model for the intuitionistic modal logic and

- φ is a function on $\sqsubset$-accessible nodes of W.
- $\mathcal{K}, w \Vdash \varphi_w$.

$$\mathcal{K}, w \Vdash \Box A \quad \Leftrightarrow \quad \forall u \sqsupset w \; \varphi_u, \Delta_u \vdash_{\mathsf{T}} A$$

in which Δ_u is defined in the following line. Given a set of modal propositions Y, define

$$Y_w := \{E \in Y : \mathcal{K}, w \Vdash E\}.$$

Definition

- $\mathcal{K}$ is called Y-*full* if $\varphi_u, \Delta_u \vdash_{\mathsf{T}} Y_u$ for every $u \in W\sqsubset$. $\mathcal{K}$ is called *full* if it is Γ-full. We say that $\mathcal{K}$ is A-full if it is Y-full for $Y := \{B : \Box B \in \mathsf{sub}(A)\}$.
- We say that $\mathcal{K}$ has a property of intuitionistic modal Kripke models (like transitive) if $\tilde{\mathcal{K}}$ is so.
- Whenever $\Gamma = \Delta$ we simply say that $\mathcal{K}$ is a (Γ, T)-semantic. In this case it doesn't matter what φ_w.

Theorem

Forcing relationship for finite $(\Delta, \Gamma, \mathsf{T})$-semantic is decidable whenever (Δ, T) is recursively downward compact and T is sound.

Proof.

Let $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, X)$ be a $(\Delta, \Gamma, \mathsf{T})$-semantic. We show decidability of $\mathcal{K}, w \Vdash A$ by induction.

- $A = \Box B$. It is enough to decide $\Delta_u \vdash_{\mathsf{T}} \varphi_u \to B$ for every $u \sqsupset w$. Since (Δ, T) is recursively downward compact, one may effectively compute $\lfloor \varphi_u \to B \rfloor_{\Delta}^{\mathsf{T}}$. By definition of $\lfloor . \rfloor_{\Gamma}^{\mathsf{T}}$ it is enough to decide $\Delta_u \vdash_{\mathsf{T}} \lfloor \varphi_u \to B \rfloor_{\Delta}^{\mathsf{T}}$ which is equivalent to $\mathcal{K}, u \Vdash \lfloor \varphi_u \to B \rfloor_{\Delta}^{\mathsf{T}}$. Then induction hypothesis implies decidability of $\mathcal{K}, u \Vdash \lfloor \varphi_u \to B \rfloor_{\Delta}^{\mathsf{T}}$. ❑

Definition of Preservativity Semantic

We extend $\mathcal{K}, w \Vdash A$ to the language $\mathcal{L}_\rhd$ as follows:

$$\mathcal{K}, w \Vdash B \rhd C \quad \Leftrightarrow$$
$$\forall u \sqsupset w\ \forall E \in \Delta\ (\Delta_u, \varphi_u \vdash_{\scriptscriptstyle\mathsf{T}} E \to B \text{ implies } \Delta_u, \varphi_u \vdash_{\scriptscriptstyle\mathsf{T}} E \to C),$$

Relation to Preservativity

Soundness

Theorem

$\underset{\Gamma}{\overset{\mathsf{T}}{\approx}}$ is sound for $(\Delta, \Gamma, \mathsf{T})$-semantics, i.e. given such preservativity semantics $\mathcal{K}$, we have $\mathcal{K} \Vdash A \rhd B$ whenever $A \underset{\Gamma}{\overset{\mathsf{T}}{\approx}} B$.

Proof.

Let $A \underset{\Gamma}{\overset{\mathsf{T}}{\approx}} B$ and $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \varphi)$ be a $(\Delta, \Gamma, \mathsf{T})$-semantics and $w \sqsubset u \in W$ and $E \in \Delta$ such that $\varphi_u, \Delta_u, E \vdash_{\mathsf{T}} A$. Hence there is a finite set $\Phi_u \subseteq \Delta_u$ such that $\Phi_u, E, \varphi_u \vdash A$. By conjunctive closure condition, we have $\bigwedge \Phi_u \wedge E \wedge \varphi_u \in \Gamma$ and thus by $A \underset{\Gamma}{\overset{\mathsf{T}}{\approx}} B$ we get $\Phi_u, E, \varphi_u \vdash_{\mathsf{T}} B$. Hence we have $\varphi_u, \Delta_u, E \vdash_{\mathsf{T}} B$. ❑

Soundness

Theorem

$\mathsf{iGLH}(\Gamma, \mathsf{T})$ *is sound for* $(\Delta, \Gamma, \mathsf{T})$*-semantics whenever* $\mathsf{IPC} \subseteq \mathsf{T}$ *and* $\mathsf{SN}(\Box) \subseteq \Delta$.

- The proof is by induction on $A \in \mathcal{L}_\Box$ and W ordered by $\sqsubset$.
- One may use ▶ to show soundness of $\mathsf{H}(\Gamma, \mathsf{T})$.
- $\mathsf{SN}(\Box) \subseteq \Delta$ is needed for soundness of iGL and necessitation.
- The proof is straightforward.

Completeness

Theorem

$\mathsf{iGLH}(\Gamma, \mathsf{T})$ *is complete for good* (Γ, T)*-semantics, if* (Γ, T) *is sdc and* $\Gamma \supseteq \mathsf{SN}(\Box)$ *is closed under conjunctions and* $\mathsf{T} \supseteq \mathsf{IPC}$.

Proof.

Let $\mathsf{iGLH}(\Gamma, \mathsf{T}) \nvdash A$. Then by ▸ we also have $\mathsf{iGLH}(\Gamma, \mathsf{T}) \nvdash \llbracket A \rrbracket_{\Gamma}^{\mathsf{T}}$ and hence $\mathsf{iGL} \nvdash \llbracket A \rrbracket_{\Gamma}^{\mathsf{T}}$. ▸ implies that there is some good Kripke model $\tilde{\mathcal{K}} := (W, \preccurlyeq, \sqsubset, V)$ such that $\tilde{\mathcal{K}}, w_0 \nVdash \llbracket A \rrbracket_{\Gamma}^{\mathsf{T}}$. Define φ arbitrary for $\mathcal{K} := (W, \preccurlyeq, \sqsubset, V, \varphi)$. We have $\mathcal{K} \nVdash A$. ❑

Corollary

iGLH_σ ($\mathsf{iGLC_aH}_\sigma$) *is sound and complete for* ($\mathsf{C_a}$) *good* $(\mathsf{SN}, \mathsf{iGLC_a})$*-semantics.*

Corollary

$\mathsf{iGLH}^\Box$ *is sound and complete for good* $(\mathsf{SN}(\Box), \mathsf{iGL})$*-semantics.*

Corollary

iGLH *is sound for* $(\mathsf{SN}(\Box), \mathsf{C}{\downarrow}\mathsf{SN}(\Box), \mathsf{iGL})$*-semantics.*

1st-red

- Since $\mathsf{C{\downarrow}SN}(\Box)$ is not closed under conjunctions we do not have completeness of iGLH for $(\mathsf{C{\downarrow}SN}(\Box), \mathsf{iGL})$-semantics.
- Like most useful results, the proof of following theorem is not easy!
- Its proof needs its own saturation and truth lemmas.
- See the manuscript for details.

Theorem

iGLH *is complete for good* $(\mathsf{SN}(\Box), \mathsf{C{\downarrow}SN}(\Box), \mathsf{iGL})$*-semantics.*

Corollary

iGLH *is decidable.*

Back to example $\nVdash \Box(x \vee y) \to (\Box x \vee \Box y)$

- That model is $(\mathsf{SN}(\Box), \mathsf{C}{\downarrow}\mathsf{SN}(\Box), \mathsf{iGL})$-semantic.
- More details on why $\mathcal{K}, w \nVdash \Box y$:

Proof.

Enough to show $\mathsf{SN}(\Box)_u, \boxdot x \nvdash y$. If $\mathsf{SN}(\Box)_u \vdash \boxdot x \to y$, then $\mathsf{SN}(\Box)_u \vdash \lfloor \boxdot x \to y \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)}$. As we saw earlier, $\lfloor \boxdot x \to y \rfloor^{\mathsf{iGL}}_{\mathsf{SN}(\Box)} = \neg\Box x$. Thus $\mathsf{SN}(\Box)_u \vdash \neg\Box x$. Hence by soundness of iGL we have $\mathcal{K}, u \Vdash \neg\Box x$, a contradiction. ❑

- This shows that $\mathsf{iGLH} \nvdash \Box(x \vee y) \to (\Box x \vee \Box y)$.

Theorem

The provability logic of HA *is* iGLH *and hence is decidable.*

Proof.

Soundness: $\mathsf{iGLH} \vdash A$ implies $\mathsf{HA} \vdash \alpha_{\mathsf{HA}}(A)$ for every α.
$\mathsf{iGLH} \subseteq \mathsf{iPH}$ and soundness of iPH.

Completeness: $\mathsf{iGLH} \nvdash A$ implies $\mathsf{HA} \nvdash \alpha_{\mathsf{HA}}(A)$ for some α.
$\mathsf{iGLH} \nvdash A$ implies $\mathsf{iGLC_aH_\sigma} \nvdash \theta(A)$ for some propositional substitution θ (we will see later). Then arithmetical completeness of $\mathsf{iGLC_aH_\sigma}$ implies desired result. ❑

What happens without H and H_σ

2nd-red

Theorem

$\mathsf{iGL} \nvdash A$ *implies* $\mathsf{iGLC_a} \nvdash \beta(A)$ *for some propositional substitution* β.

Proof.

Since $\mathsf{iGL} \nvdash A$, there is some $\mathcal{K} := (W, \preccurlyeq, \sqsubset, V)$ with $\mathcal{K} \nVdash A$.
Define $\mathcal{K}'$: for every $w \in W$ add a fresh atomic p_w and let it be forced (satisfied) at w and its successor/above nodes. No other atomics are forced at w.
Define $\beta(x) := \bigvee_{\mathcal{K}, w \Vdash x} Q_w$ and $Q_w := q_w \wedge \bigwedge_{w \sqsubset u} \neg q_u$.
Claim. $\mathcal{K}, w \Vdash A$ iff $\mathcal{K}', w \Vdash \beta(A)$. ❑

$\mathsf{iGLH} \nvdash A$ implies $\mathsf{iGLC_aH}_\sigma \nvdash \theta(A)$

The proof is broken in two steps:

1. $\mathsf{iGLH} \nvdash A$ implies $\mathsf{iGLH}^\Box \nvdash \gamma(A)$ for some γ.
2. $\mathsf{iGLH}^\Box \nvdash A$ implies $\mathsf{iGLC_aH}_\sigma \nvdash \beta(A)$ for some β.

Both are proved via provability semantics.

$\mathsf{iGLH}^\Box \nvdash A$ implies $\mathsf{iGLC_aH}_\sigma \nvdash \beta(A)$

Sketch of the proof

- Since $\mathsf{iGLH}^\Box \nvdash A$, by completeness of $(\mathsf{SN}(\Box), \mathsf{iGL})$-semantics, $\mathcal{K} \nVdash A$ for some $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \top)$.
- On the other hand, $\mathsf{iGLC_aH}_\sigma$ is sound for $\mathsf{CP_a}$ $(\mathsf{SN}(\Box), \mathsf{SN}, \mathsf{iGLC_a})$-semantics.
- One must transform $\mathcal{K}$ to a $(\mathsf{SN}(\Box), \mathsf{SN}, \mathsf{iGLC_a})$-semantic.
- The transformation is a uniform collection of transformations for iGL and $\mathsf{iGLC_a}$.

iGLH $\nvdash A$ implies iGLH$^\Box \nvdash \gamma(A)$

- The first step reduction is not as elementary as the second one.
- It uses features of relative projectivity and simultaneous fixed point theorem.

$\mathsf{iGLH} \nvdash A$ implies $\mathsf{iGLH}^\Box \nvdash \gamma(A)$

Sketch of the proof.

- $\mathsf{iGLH} \nvdash A$ implies $\mathcal{K} \nVdash A$ for some good $(\mathsf{SN}(\Box), \mathsf{C{\downarrow}SN}(\Box), \mathsf{iGL})$-semantic $\mathcal{K} = (W, \preccurlyeq, \sqsubset, V, \varphi)$.
- Since $\varphi_w \in \mathsf{C{\downarrow}SN}(\Box)$ there is a $(\mathsf{SN}(\Box), \mathsf{iGL})$-projective substitution θ_w such that $\hat{\theta}_w(\varphi_w) \in \mathsf{SN}(\Box)$.
- The main idea is that one-bye-one we must kill φ_w's and send them in to the set $\mathsf{SN}(\Box)$.
- Good news: when some φ_w goes in to $\mathsf{SN}(\Box)$ it remains there since $\mathsf{SN}(\Box)$ is closed under substitutions.
- Bad news: These $\hat{\theta}_w$'s are not even a substitution.
- Solution: use the simultaneous fixed point theorem in iGL .

❑

Thanks For Your Attention

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