Some notes on integration by parts formulae in infinite dimensional spaces

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Introduction. Let H be a separable Hilbert space endowed with a non-degenerate centered Gaussian measure $\mu = \mathcal{N}_Q$. With no loss of generality, we can think of $H = L^2(0,T)$.

In the first part, we extend results about surface measures on level sets $\{g = r\}$ of functions $g : H \to \mathbb{R}$ that satisfy a suitable condition (that is called *local Malliavin condition*; this same hypothesis was already introduced in the literature in different contexts.

Under such conditions, we may construct surface measures on the level sets of the function g. The main result can be stated as follows: in our framework, for each $r \in I$ there exists a Borel measure σ_r in H, concentrated on the level surface $\{g = r\}$ of g (when g is continuous) such that for any $\varphi \in UC_b(H)$ we have

$$\int_{\{g=r\}} \varphi \, \mathrm{d}\sigma_r = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\{r-\epsilon \le g \le r+\epsilon\}} \varphi \, \mathrm{d}\mu, \quad r \in I. \tag{1}$$

We shall call σ_r the *surface measure* related to μ on $\{g = r\}$.

Then, we specialise to the case where μ is the law of some special Gaussian process (like for instance the Brownian motion). We let $H = L^2(0,1)$ and $X = (X(t),\ t \in (0,1))$ be a Gaussian process defined on it; the law of X is a Gaussian measure $\mu \sim \mathcal{N}_Q$ that is concentrated on $E = C([0,1])$. We set $g = \inf X$. Then we are concerned with integrals defined on the convex sets of trajectories $K_r = \{x : (0,1) \to [r,\infty)\} = \{g \ge r\}$ and we shall consider the relevant integration by parts formula

$$\mathbb{E}\big[z(\tau_x)\varphi \big| g = r\big]\rho(r) = -\int_{\{g \ge r\}} [D\varphi \cdot z - \langle Q^{-1/2}x, Q^{-1/2}z\rangle \, \varphi] \, \mathrm{d}\mu, \ \forall\, r < 0.$$

Since, clearly, g is not a continuous function on H, but it is continuous on E, to obtain precise results, we shall be forced to work on E rather than H. We also study the limit case $r = 0$ and apply the obtained results to the cases when X is respectively: (i) a Brownian motion, (ii) a distorted Brownian motion, (iii) a Brownian Bridge. In case (i) and (iii) we recover by a different method the integration by parts formulae from [5] and [2].

References

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