Schwarzian Integrals and Wiener Measure

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We derive the explicit form of the polar decomposition of the Wiener measure, and obtain the equation connecting functional integrals in conformal quantum mechanics to those in the Schwarzian theory. Using this connection we evaluate some nontrivial functional integrals in the Schwarzian theory and also find the fundamental solution of the Schroedinger equation in imaginary time in the model of conformal quantum mechanics.

We find that the two measures are connected with each other by the equation

$$w(dx) = e^{-\frac{1}{4r^2}}\,(\varphi'(0)\varphi'(1))^{\frac{3}{4}}\mu_{\frac{2}{r}}(d\varphi)\;dr, \quad x(t) = \frac{r}{\sqrt{(\varphi^{-1}(t))'}}$$

$$\mu_{\frac{2}{r}}(d\varphi) = \frac{1}{\sqrt{\varphi'(0)\varphi'(1)}}\exp\left\{\frac{r^2}{4}\left[\frac{\varphi''(0)}{\varphi'(0)} - \frac{\varphi''(1)}{\varphi'(1)} + \int\limits_0^1 \mathcal{S}_\varphi(t)dt\right]\right\}d\varphi$$

$$x \in C_+([0,\,1])\,, \qquad \varphi \in Diff_+^1([0,\,1])\,, 0 < r < +\infty\,,$$

which we name "Polar decomposition of the Wiener measure". $\mathcal{S}_\varphi(t) = \left(\frac{\varphi''(t)}{\varphi'(t)}\right)' - \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2$ is known as the Schwarzian derivative.

Here $C_+([0,\,1])$ is the space of positive continuous function of the interval $[0,\,1]$, $Diff_+^1([0,\,1])$ is the group of diffeomorphism of the interval $[0,\,1]$.

We obtain that functional integrals in the Schwarzian theory can be written

$$\int\!\!\int F(x_{r,\varphi})\,e^{-\frac{1}{4r^2}}\exp\left\{\frac{r^2}{4}\int\limits_{S^1}\left[\mathcal{S}_\varphi(t) + 2\pi^2\left(\varphi'(t)\right)^2\right]dt\right\}d\varphi\,dr =$$

$$\int F(x)\exp\left\{-\frac{1}{2}\int\limits_{S^1}\left[(x'(t))^2 - \pi^2(x(t))^2\right]dt\right\}dx$$

where $\varphi \in Diff_+^1(S^1)$, $x \in C_+(S^1)$, $x_{r,\varphi}(t) = \frac{r}{\sqrt{(\varphi^{-1}(t))'}}$. Here $C_+(S^1)$ is the space of positive continuous function of the circle S^1, $Diff_+^1(S^1)$ is the group of diffeomorphism of the circle S^1. Talk is based on V. V. Belokurov and E. T. Shavgulidze, *Unusual view of the Schwarzian theory, Mod. Phys. Lett. A* **33** (2018) 1850221, [arXiv:1806.05605]

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