The Dynamics of Birkhoff's Sums

A. A. Shukur [1]

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Introduction. Let X be a compact topological space and $\alpha : X \to X$ be a continuous invertable map. This kind of maps generate a dynamical systems (cascades) such as: $\alpha^k(x) = \alpha(\alpha^{k-1}(x)),\ k \in \mathbb{Z}$. For $f : X \to \mathbb{C}$ and $n \in \mathbb{Z}$, the Birkhoff sums $f(n, x)$ is represented by

$$f(n,x) = f(n,x,\alpha) = \begin{cases} \sum_{k=0}^{n-1} f(\alpha^k(x)) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -\sum_{k=n}^{-1} f(\alpha^k(x)) = -f(-n; \alpha^n(x)) & \text{for } n < 0. \end{cases} \tag{1}$$

Particularly, the behavior of the Birkhoff sums is related to ergodic theorem, this fact is shown in the next discussion:

Let $PM_\alpha(X)$ be a set of probability Borel measures in X, which invariant relatively to α. The Birkhoff's ergodic theorem says, if $\mu \in PM_\alpha(X)$ and $f \in L_1(X, \mu)$, then the limit of the Birkhoff average exist, μ-almost everywhere (see [3]). In case of continuous functions, the following result presented:

Theorem 1. [3] If X be a compact topological space, $\alpha : X \to X$ be a continuous map and $f \in C(X)$, then

$$\lim_{n\to\infty} \max_X \frac{1}{n} f(n, x, \alpha) = \max\{\int_X f(x)d\mu : \mu \in PM_\alpha(X)\},$$

$$\lim_{n\to\infty} \min_X \frac{1}{n} f(n, x, \alpha) = \min\{\int_X f(x)d\mu : \mu \in PM_\alpha(X)\}.$$

Moreover, the map α is called *strictly ergodic,* if there exist only one invariant probability measure μ. From Theorem 1, follows that the following convergent where $f \in C(X)$ holds:

$$\lim_{n\to\infty} \max_X \frac{1}{n} f(n; x) = \int_X f(x)d\mu. \tag{2}$$

$$\lim_{n\to\infty} \min_X \frac{1}{n} f(n; x) = \int_X f(x)d\mu. \tag{3}$$

In the present work, we will provide a detailed description about the convergence of (2) and (3). The estimates of powers of operators generated by irrational are given.

Main results.

Let $T = \mathbb{R}/\mathbb{Z}$ be the unit circle and the map $x \to x + h$ generates the rotation such that $\alpha_h : T \to T$ with angle $2\pi h$ where h is irrational number. For a function $f \in C(T)$ the Birkhoff sums $f(n, x, h)$ is represented by

[1]Belarusian State University, Department of Functional Analysis, Belarus, Minsk. University of Kufa, Department of Mathematics, Iraq, AlNajaf. Email: shukur.math@gmail.com

$$f(n,x;h) = \begin{cases} f(x) + f(x+h) + \cdots + f(x+(n-1)h) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -[f(x-h) + f(x-2h) + \cdots + f(x-nh)] & \text{for } n < 0 \end{cases}$$

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Theorem 2. [1] Let h be irrational number. For any sequence of numbers σ_n, which monotonic converge to zero, there exist a continuous function φ with zero average such that Birkhoff sums $f(n,h,\varphi)$ is growing such as faster than

$$f(n,h,\varphi) \geq n\sigma_n.$$

Theorem 3. [2] Let φ be a continuous function with zero average, which is not trigonometrical polynomial. For any monotonic converge to zero σ_n, there exist an irrational number h, such that $f(n_k,h,\varphi)$ is growing such as faster than

$$f(n_k,h,\varphi) \geq n_k\sigma_{n_k}.$$

If φ is smooth, then $f(q_k,h,\varphi)$ is bounded.

The proof was based on some facts of number theory and ergodic theory.

References

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