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Applications of stochastic limit of quantum theory (part I)

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Main idea of stochastic limit

Basic physical intuition:
– In a large class of dynamical systems one can naturally distinguish **two time scales**:
a **slow** and a **fast** one.

In the stochastic limit the two time scales are separated and the intuition:
fast degrees of freedom $\sim$ *pure noise*
becomes **literally true**.

Separation of the two time scales $\iff$:
– look at the fast degrees of freedom with the clock of the slow degrees freedom.
Consequence:
the fast degrees of freedom seem to be infinitely fast, i.e.
they look like **pure noise** (**white noise**)

The separation of time scales is achieved by **combining the two fundamental asymptotic techniques of physics**
– perturbation theory (*small parameter*),
– scattering theory (*long times*),
into **a new asymptotic technique**,
– stochastic limit (a 'parameter is **small with respect to times**')

$$\lambda \to 0 \quad , \quad \tau \to \infty \quad \textbf{but} \quad \lambda^2 \tau \to t \quad \text{(finite)}$$

Equivalently

$$\tau \mapsto t/\lambda^2 \quad , \quad \lambda \to 0$$

Not surprising emergence of wave and scattering operators in more sophisticated models.

Framework in this talk: **standard open system scheme**:

system + **environment**.

Moreover the **environment** will be of **Boson** type.

Basic ingredients of stochastic limit:

System

System free Hamiltonian.

Stochastic limit, simplest models: discrete spectrum

$$H_S = \sum_k \varepsilon_k P_k \text{ (\textbf{free system–Hamiltonian})}$$

More complex models: continuous spectrum.

ρ_S, **initial state** of the system, is **arbitrary**.

$$\mathcal{H}_S \text{ –\textbf{system Hilbert space}}$$

Observables of the environment.

To fix the ideas we take the observables of the environment to be given by the hermitean elements of the CCR algebra CCR($\mathcal{H}_1$) over a fixed environment 1–particle Hilbert space

$$\mathcal{H}_1 = L^2(\mathbb{R}^d) \qquad ; \qquad (d \geq 3)$$

$$[a_k, a^+_{k'}] = \delta(k - k') \quad \text{momentum representation}$$

$$[a_f, a^+_g] = \int\int dk dk' \delta(k - k') \bar{f}(k) g(k')$$

$f, g \in L^2(\mathbb{R}^d) \equiv \mathcal{H}_1$.

A **scalar boson field** is defined by a **state** φ on this algebra.
φ, the initial state of the environment,
determines the state space of the environment (cyclic representation).

φ Gaussian states

gauge invariant

$$\varphi(a^+_{f_j} \cdot a^+_{f_k}) = \varphi(a^-_{f_j} \cdot a^-_{f_k}) = 0$$

covariance

$$\varphi(a^+_f a_g) = \langle g, Nf \rangle$$

our covariances are diagonal in momentum space:

$$Nf(k) = N_k f(k) \quad ; \quad n \geq 0$$

Meaning of N (in distribution sense)

$$\langle a^+_k a_h \rangle := \varphi(a^+_k a_h) = \delta(k-h) N_k \langle a^+_k a_k \rangle$$

mean density of quanta at momentum k.

Summing up the covariance

$$\begin{pmatrix} \langle a_k^+ a_h \rangle & 0 \\ 0 & \langle a_k a_h^+ \rangle \end{pmatrix} = \begin{pmatrix} N_k & 0 \\ 0 & 1 + N_k \end{pmatrix} \delta(k-h)$$

$$=: \begin{pmatrix} \text{non–Fock term} & 0 \\ 0 & \text{Fock term} \end{pmatrix}$$

Equilibrium and non–equilibrium states.
$N(k)$ is a positive function $\Rightarrow$ one can always write $N(k)$ in the form

$$N(k) =: \frac{1}{e^{\beta(k)} - 1} \ (\textbf{non–linear Planck factor})$$

where

$$\beta(k) := \lg \frac{N(k) + 1}{N(k)} > 0$$

Momentum (frequency) dependent inverse temperature.

Suppose that
the function $N(k)$ depends only on $\omega(k)$:

$$N(k) \ \rightarrow \ N(\omega(k))$$

Then the same will be true for $\beta(k)$, i.e.

$$\beta(\omega(k)) := \lg \frac{N(\omega(k)) + 1}{N(\omega(k))} > 0$$

and that the **function** β **is linear**

$$\tilde{\beta}(\omega(k)) = \beta \cdot \omega(k) \qquad (\beta \text{ a constant}) \; > 0$$

Then non–linear Planck factor is reduced to the **usual Planck factor**:

$$N(\omega(k)) = \frac{1}{e^{\beta(\omega(k))} - 1} = \frac{1}{e^{\beta \cdot \omega(k)} - 1}$$

Fundamental open problem for physics: **can non–equilibrium states be experimentally realized?** (At least for some special choices of the function $N(k)$ or, more likely $N(\omega(k))$).

If the answer to the above question is 'yes', then several quite non–trivial experimental predictions of the stochastic limit can be experimentally checked.

In the following I will describe one of these predictions which leads to a natural **local equilibrium** extension of a fundamental principle of equilibrium statistical physics:
the principle of detailed balance.

Dynamics

$H_E := \Gamma_{Bos}(H_1)$ (**environment free Hamiltonian**)

is the 2–d quantization of

$H_1 := \omega_k$ (1–**particle environment free Hamiltonian**)

in momentum representation ($k \in \mathbb{R}^d$).

$$(H_1 f)(k) = \omega_k f(k)$$

multiplication operator, in momentum space, by the **dispersion function** ω_k.
Typical examples

$$\omega_k = |k| \text{ (non–relativistic QED)}$$

$$\omega_k = |k|^2/2m$$

(non–relativistic gas of mass–m particles).
In solid state physics, many more examples.

Initial state of the compound system

$$\rho = \rho_S \otimes \varphi$$

$\mathcal{H}_S \otimes \Gamma_\varphi(\mathcal{H}_1)$ – **space of the compound system**

Interaction Hamiltonian, weak coupling case

$$H_I^{(\lambda)} := \lambda H_I$$

λ is a small parameter.

$$\varphi_\lambda := \varphi \text{ independent of } \lambda$$

Total Hamiltonian

$$H_{\text{tot}}^{(\lambda)} := H_S \otimes 1 + 1 \otimes H_0 + \lambda H_I$$

Evolutions

– Schrödinger evolution operator at time t **in interaction representation**:

$$U^{(\lambda)}(t) := e^{itH_0} \cdot e^{-itH_{\text{tot}}^{(\lambda)}} \qquad (1)$$

Schrödinger equation in interaction representation

$$\partial_t U_t^{(\lambda)} = -iH_I^{(\lambda)}(t) U_t^{(\lambda)} \qquad ; \qquad t \geq 0$$

with initial condition

$$U_0^{(\lambda)} = 1$$

– **Separation of time scales**

$$t \to t/\lambda^2 \qquad ; \qquad \lambda \to 0$$

$$\partial_t U^{(\lambda)}(t/\lambda^2) = -iH_I^{(\lambda)}(t/\lambda^2) U_{t/\lambda^2}^{(\lambda)}$$

Simplest example:

$$H_I^{(\lambda)}(t/\lambda^2) = \sum_\omega \left(E_\omega \otimes a_{t/\lambda^2,k,\omega} + \text{h.c.} \right)$$

$$a_{t/\lambda^2,k,\omega} := \frac{1}{\lambda} e^{-i\frac{t}{\lambda^2}(\omega(k)-\omega)} a_k$$

Results of the stochastic limit. As $\lambda \to 0$:
1) **Evolved rescaled fields $\to$ white noise**:

$$a_{t/\lambda^2,k,\omega} := \frac{1}{\lambda} e^{-i\frac{t}{\lambda^2}(\omega(k)-\omega)} a_k \longrightarrow b_\omega(t,k)$$

satisfying the commutation relations

$$[b_\omega(t,k), b^\dagger_{\omega'}(t',k')] = \delta_{\omega,\omega'} 2\pi\delta(t-t')\delta(k-k')\delta(\omega(k)-\omega)$$

2) **states of the field** $\to$ **states of the white noise**:

if the initial state of the field is a mean zero gauge invariant Gaussian state with correlations:

$$\langle a_k^\dagger a_{k'} \rangle = N(k)\delta(k-k')$$

then the state of the limit white noise will be of the **same type** with correlations

$$\langle b_\omega^\dagger(t,k) b_{\omega'}(t',k') \rangle = \delta_{\omega,\omega'} 2\pi\delta(t-t')\delta(k-k')\delta(\omega(k)-\omega)N(k)$$

$$\langle b_\omega(t,k) b_{\omega'}^\dagger(t',k') \rangle =$$

$$= \delta_{\omega,\omega'} 2\pi\delta(t-t')\delta(k-k')\delta(\omega(k)-\omega)(N(k)+1)$$

Notice that the states live on different algebras, but they are **both gaussian** and both **with the same density of quanta** $N(k)$.

3a) Schrödinger evolution $\to$ **White noise evolution**

$$U^{(\lambda)}(t/\lambda^2) \to U_t$$

3b) Schrödinger equation $\to$ **White noise Hamiltonian equation**

$$\partial_t U(t) = -iH(t)U_t \text{ (WN Schödinger eq.)}$$

3c) Heisenberg evolution

$$(U^{(\lambda)}(t/\lambda^2))^* X U^{(\lambda)}(t/\lambda^2) \to U_t^* X U_t =: j_t(X)$$

– **White noise Heisenberg equation**:

$$\partial_t U(t)^* X U(t) = -i[H(t), j_t(X)]$$

for the evolution of observables X of the test particle.

The new equations are:
– Singular equations:

$$H(t) \;'' ='' \;\; H(\delta(t))$$

– But you can read in these equations a lot of **new physical phenomena**:
this was **impossible** with the original equations.

Usual operator methods can say very little on these singular equations.

Theorem (Accardi, Lu, Volovich)
White noise **Hamiltonian** equations are equivalent to **stochastic differential equations** (classical or quantum).
More specifically:
White noise Hamiltonian equation $\equiv$ Stochastic Schrödinger equation;
White noise Heisenberg equation $\equiv$ Stochastic Schrödinger equation;

The proof of the equivalence is quite non–trivial.

This is true (and new) also in the classical case.
From the **mathematical** point of view, this is **one of the basic results** of SL theory.
It proves the equivalence of the physicist approach to stochastic calculus (which is based on distribution theory) with the mathematical approach (which is based on semi–martingales).
It has opend the way to a **multi–dimensional stochastic calculus**.

Connection with Markov semi–groups:
From SDE, via **quantum Feynman–Kac formula**,

$$E\left(U(t)^* X U(t)\right) = \mathsf{Tr}_{envir}(U(t)^* X U(t))$$

$$= P^t(X) = e^{t\mathcal{L}}(X)$$

you get Markov semi–groups .
Differential description:

$$\partial_t P^t(X) = \mathcal{L}(P^t(X))$$

master equation.

From where do the re–scaled fields come?

$$a_{t/\lambda^2,k,\omega} := \frac{1}{\lambda} e^{-i\frac{t}{\lambda^2}(\omega(k)-\omega)} a_k$$

Answer: from the Heisenberg evolution of a_k

$$(U^{(\lambda)}(t/\lambda^2))^* a_k U^{(\lambda)}(t/\lambda^2) = \frac{1}{\lambda} e^{-i\frac{t}{\lambda^2}(\omega(k)-\omega)} a_k$$

Notice that the maps

$$X \mapsto u_{t/\lambda^2}(X) := (U^{(\lambda)}(t/\lambda^2))^* a_k U^{(\lambda)}(t/\lambda^2)$$

are $*$–automorphisms, i.e.

$$u_{t/\lambda^2}(XY) = u_{t/\lambda^2}(X) u_{t/\lambda^2}(Y) \quad ; \; u_{t/\lambda^2}(X^*) = u_{t/\lambda^2}(X)^* \tag{2}$$

and that the convergence result

$$\frac{1}{\lambda} e^{-i\frac{t}{\lambda^2}(\omega(k)-\omega)} a_k \longrightarrow b_\omega(t,k)$$

can be written as

$$u_{t/\lambda^2}(a_k) \longrightarrow b_\omega(t,k)$$

Therefore it is natural to expect that

$$u_{t/\lambda^2}(a_k^+ a_k) = u_{t/\lambda^2}(a_k^+) u_{t/\lambda^2}(a_k) \longrightarrow b_\omega^+(t,k) b_\omega(t,k)$$

Main improvement of SL over previous techniques

In the SL one obtains the **full unitary (reversible) evolution**,
not only the **reduced (irreversible) evolution**.
SL can solve problems that in previous theories **cannot even be formulated**.
For example:

– decay of transition probabilities

$$|\langle \Phi_{S,E}, U_t \Phi_{S,E} \rangle|^2 \tag{3}$$

Even in the simplest cases, when

$$\Phi_{S,E} = \Phi_S \otimes \Phi_E$$

this becomes

$$\mathsf{Tr}_{S,E}\left(U_t^* \left(\Phi_S \Phi_S^* \otimes \Phi_E \Phi_E^* U_t\right)\right) \tag{4}$$

and this cannot be calculated if you only have information on the master equation, because in this case you are limited to expectation values of the form:

$$\mathsf{Tr}_{S,E}\left(U_t^* \left(\Phi_S \Phi_S^* \otimes 1_E U_t\right)\right) \tag{5}$$

$$= \mathsf{Tr}_S \left(P^t \left(\Phi_S \Phi_S^*\right)\right)$$

Most important: one can compute the time evolution of **the slow observables of the environment** not only the **observables of the system**.
This opens new possibilities for physics.
Let me illustrate this with an example having to do with non–equilibrium statistical mechanics.

Time evolution of the slow observables of the environment: Currents and micro–currents

Let

$$n_k = a_k^\dagger a_k \qquad ; \qquad k \in \mathbb{R}^d$$

be the number operator density.

The stochastic limit allows to calculate its time evolution $U_t^\dagger n_k U_t$.

The expectation value of $U_t^\dagger n_k U_t$ with respect to the initial state of the system and noise,

$$\langle U_t^\dagger n_k U_t \rangle := (\mathrm{Tr}(\rho \,\cdot\,) \otimes \varphi)\,(U_t^\dagger n_k U_t)$$

gives the mean number of quanta at time t.

Its time derivative defines

the current density of quanta

$$J(t,k) := \frac{d}{dt}\langle U_t^\dagger n_k U_t \rangle$$

Using quantum Ito formula one finds

$$J(t,k) = \frac{d}{dt}\langle U_t^\dagger n_k U_t\rangle = 2\sum_{\epsilon_m > \epsilon_n} \delta(\omega(k) - (\epsilon_m - \epsilon_n))$$

$$\pi|g_\omega(k)|^2((N(k)+1)\rho_{mm}(t) - N(k)\rho_{nn}(t))$$

where $\rho_{mm}(t)$ are the (diagonal) matrix elements, in the eigen–vectors of the free system Hamiltonian of the initial state ρ of the system time–evolved with the reduced evolution (Markov semi–group). Thus the number current density is a sum of **microscopic quanta currents densities** $J_{\omega_{mn}}(k,t)$ defined by

$$J_{\omega_{mn}}(k,t) := 2\delta(\omega(k) - (\epsilon_m - \epsilon_n))$$

$$\pi|g_\omega(k)|^2((N(k)+1)\rho_{mm}(t) - N(k)\rho_{nn}(t))$$

Notice that one has one micro–current density for each strictly positive Bohr frequency $\omega_{mn} := \epsilon_m - \epsilon_n > 0$.

Integrating over $\mathbb{R}^d$ the micro–current number density associated to the frequency $\omega_{mn}(k,t)$, one obtains the micro–current number current associated to the Bohr frequencey $\omega_{mn}(k,t)$:

$$J_{\omega_{mn}}(t) =$$

$$\left(\int_{\mathbb{R}^d} dk\ \delta(\omega(k) - (\epsilon_m - \epsilon_n))2\pi|g_\omega(k)|^2(N(k)+1)\right)\rho_{mm}(t)$$

$$-\left(\int_{\mathbb{R}^d} dk\ \delta(\omega(k) - (\epsilon_m - \epsilon_n))2\pi|g_\omega(k)|^2 N(k)\right)\rho_{nn}(t)$$

Recalling the form of the (real part of the) **generalized transport coefficients (or susceptibilities)**:

$$\Gamma_{-,\omega} = \pi \int dk\, |g(k)|^2 (N(k) + 1)\delta(\omega(k) - \omega)$$

$$\Gamma_{+,\omega} = \pi \int dk\, |g(k)|^2 N(k)\delta(\omega(k) - \omega)$$

we see that the quanta micro–currents are

$$J_{\omega_{mn}}(t) = \Gamma_{-,\omega}\ \rho_{mm}(t) - \Gamma_{+,\omega}\ \rho_{nn}(t)$$

A corollary of this result is:

Theorem. In any state ρ **of the system** which is stationary for the reduced (Markov) evolution, i.e.

$$\rho(t) = \rho\, (\Leftrightarrow\ \ \rho_{mm}(t) = \rho_{mm})$$

each quanta micro–current is constant:

$$J_{\omega_{mn}} = \Gamma_{-,\omega}\rho_{mm} - \Gamma_{+,\omega}\rho_{nn}$$

hence, a fortiori, the total number current J is constant.

This seems to be a new principle in non–equilibrium statistical mechanics.
It was called: **dynamical detailed balance**.
Dynamical because there are currents.
Detailed because it concerns with currents associated with single Bohr frequencies.
See the paper:
L. Accardi, K. Imafuku,
Dynamical detailed balance and local KMS condition for non-equilibrium states,
Int. J. Mod. Phys. B, **18** (4) & (5) (2004) 435–467
`quant-ph/0209088`
Recall that:

$$\Gamma_{-,\omega} := \mathrm{Re}((g|g)^{-}_{\omega}) = \pi \int dk\, |g(k)|^2 (N(k)+1)\delta(\omega(k)-\omega)$$

$$\Gamma_{+,\omega} := \mathrm{Re}((g|g)^{+}_{\omega}) = \pi \int dk\, |g(k)|^2 N(k)\delta(\omega(k)-\omega)$$

Detailed Balance

In equilibrium one does not expect macro–currents, i.e.

$$J = 0$$

Definition
A stationary state ρ is said to satisfy the **detailed balance** condition if all micro–currents are zero, i.e.

$$J_{\omega_{mn}}(t) = \Gamma_{-,\omega}\rho_{mm} - \Gamma_{+,\omega}\rho_{nn} \qquad ; \qquad \forall\omega_{mn}$$

Thus, **if we suppose** that

$$\rho_{mm} > 0 \qquad ; \qquad \forall m$$

this is equivalent to

$$\frac{\Gamma_{-,\omega}}{\Gamma_{+,\omega}} = \frac{\rho_{nn}}{\rho_{mm}} \qquad ; \qquad \forall\omega_{mn}$$

The quotient

$$\frac{\Gamma_{-,\omega}}{\Gamma_{+,\omega}}$$

has an important physical interpretation.
Suppose that
the function $N(k)$ **depends only on** $\omega(k)$:

$$N(k) \;\to\; N(\omega(k))$$

then the generalized transport coefficients become respectively:

$$\Gamma_{-,\omega_{mn}} = \pi \int dk\,|g(k)|^2 (N(\omega(k)) + 1)\delta(\omega(k) - \omega_{mn})$$

$$= (N(\omega_{mn}) + 1)\pi \int dk\,|g(k)|^2 \delta(\omega(k) - \omega_{mn})$$

$$=: (N(\omega_{mn}) + 1)c_{\omega_{mn}}$$

$$\Gamma_{+,\omega_{mn}} = \pi \int dk\,|g(k)|^2 N(\omega(k))\delta(\omega(k) - \omega_{mn})$$

$$= \pi \int dk\,|g(k)|^2 N(\omega_{mn})\delta(\omega(k) - \omega_{mn})$$

$$= N(\omega_{mn})c_{\omega_{mn}} \text{ (\textbf{the same} } c_{\omega_{mn}})$$

Therefore

$$\frac{\Gamma_{-,\omega_{mn}}}{\Gamma_{+,\omega_{mn}}} = \frac{(N(\omega_{mn})+1)c_{\omega_{mn}}}{N(\omega_{mn})c_{\omega_{mn}}} = \frac{N(\omega_{mn})+1}{N(\omega_{mn})}$$

depending only on N (universality).
Recalling that $N(\omega)$ is the density of environment quanta (photon, phonons,gas particles, . . .) at frequency ω,
we see that the identity

$$\frac{\Gamma_{-,\omega_{mn}}}{\Gamma_{+,\omega_{mn}}} = \frac{N(\omega_{mn})+1}{N(\omega_{mn})}$$

generalizes the well known **Einstein formula** of radiation theory (see Heitler's book):

$$\frac{W_{emission}}{W_{absorption}} = \frac{\overline{n}_\omega + 1}{\overline{n}_\omega} \tag{6}$$

giving the ratio of the probability rate of emission and absorption of a light quantum by an atom.

The quotient of the transport coefficients (generalized susceptivities) provide a **non–equilibrium generalization of Einstein formula**.

To prove this fact consider the detailed balance condition:

$$\Gamma_{-,\omega}\rho_{mm} = \Gamma_{+,\omega}\rho_{nn} \qquad ; \qquad \forall\omega_{mn}$$

If $c_{\omega_{mn}} = 0$, the the ω_{mn}–micro–current is absent.

If $c_{\omega_{mn}} \neq 0$ then solutions the DB condition is equivalent to

$$\frac{\rho_{mm}}{\rho_{nn}} = \frac{\Gamma_{+,\omega_{mn}}}{\Gamma_{-,\omega_{mn}}} < 1$$

This suggests to define a **non–linear frequency dependent temperature function**:

$$\frac{\Gamma_{+,\omega_{mn}}}{\Gamma_{-,\omega_{mn}}} =: e^{-\beta(\omega_{mn})}$$

We call $e^{-\beta(k)}$ the **non–linear Gibbs factor**.

If the detailed balance principle is satisfied, then:

$$e^{-\beta(\omega_{mn})} = \frac{\rho_{mm}}{\rho_{nn}} = \frac{\Gamma_{+,\omega_{mn}}}{\Gamma_{-,\omega_{mn}}} = e^{-\beta(\omega_{mn})} = e^{-\beta(\varepsilon_m - \varepsilon_n)}$$

we deduce

$$e^{-\beta(\varepsilon_m - \varepsilon_n)} = \frac{\rho_{mm}}{\rho_{nn}} = \frac{\rho_{mm}}{\rho_{kk}} \frac{\rho_{kk}}{\rho_{nn}}$$

$$= e^{-\beta(\varepsilon_m - \varepsilon_k)} e^{-\beta(\varepsilon_k - \varepsilon_k)} \Leftrightarrow$$

$$\Leftrightarrow \beta(\varepsilon_m - \varepsilon_k) = \beta(\varepsilon_m - \varepsilon_k) + \beta(\varepsilon_k - \varepsilon_k)$$

Thus, fixing ε_0 to be the minimum energy level (assumed to exist) and defining

$$\tilde{\beta}(\varepsilon_m) := \beta(\varepsilon_m - \varepsilon_0)$$

one finds

$$\tilde{\beta}(\varepsilon_m) = \tilde{\beta}(\varepsilon_m) + \tilde{\beta}(\varepsilon_k)$$

Therefore the function β must be linear

$$\tilde{\beta}(x) = \beta \cdot x \qquad (\beta \text{ a constant}) \; > 0$$

Thus we find the Gibbs state at inverse temperature β and we recover the original formulation of Einstein formula:

$$\frac{W_{\text{emission}}(\omega)}{W_{\text{absorption}}(\omega)} = e^{\beta\omega} \qquad (7)$$

Philip Anderson
More is different,
Science, New Series, Vol. 177, No. 4047. (Aug. 4, 1972), pp. 393-396.

Reductionism: if we know the fundamental laws, we know everything.

This point of view turned out to be too naive.

A more realistic point of view is constructionism:

... at each level of complexity entirely new properties arise ...

Psychology is not applied biology, nor biology is applied chemistry ...