On the volume of sections of the cube

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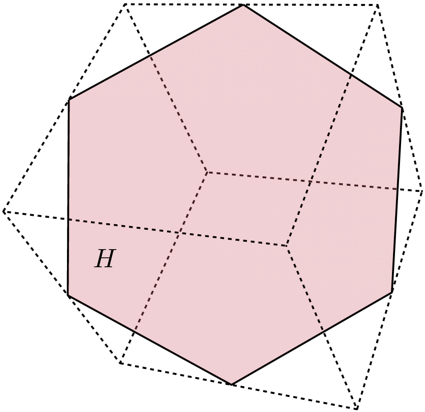
Sections of the cube

- The standard cube $\square^n = [-1, 1]^n$



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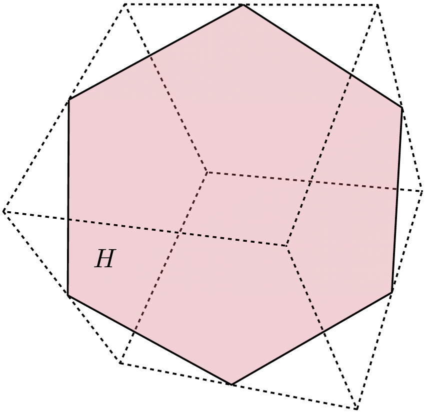

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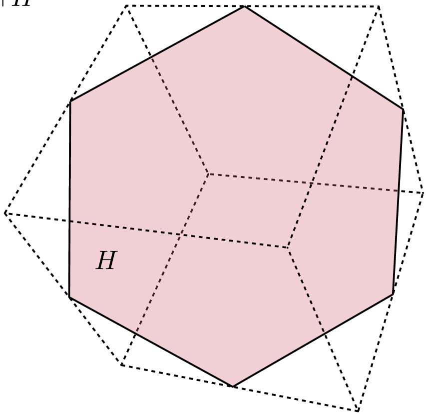
- The standard cube $\square^n = [-1, 1]^n$
- A k-dimensional **linear** subspace H of $\mathbb{R}^n$ $(k \leq n)$
- The intersection $\square^n \cap H$



State of the Art

Vaaler'79

Let H be a k-dimensional linear subspace of $\mathbb{R}^n$. Then

$$\mathrm{vol}_k \square^k \leq \mathrm{vol}_k(\square^n \cap H) \qquad \text{(optimal for all } n \geq k\text{)}.$$

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Instruments:

more peakedness, waists, the Brascamb–Lieb inequality, the Fourier transform, isotropic measures

One ring to rule them all

All known maximizers in Ball's inequalities are affine cubes.

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Conjecture 1.

The maximal volume section of the cube $\square^n$ by a k-dimensional subspace H is attained on subspaces such that the section is an affine cube, i.e.

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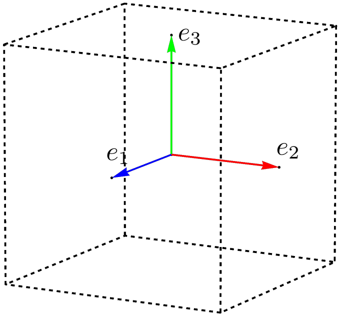
$$C_{\square}^2(n,k) = \left\lceil \frac{n}{k} \right\rceil^{n-k\lfloor n/k \rfloor} \left\lfloor \frac{n}{k} \right\rfloor^{k-(n-k\lfloor n/k \rfloor)}$$

What is a section of the cube?

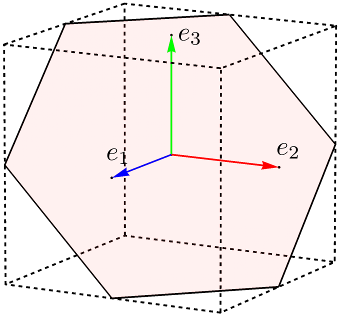
$$\square^n = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n : \ |\langle x, e_i \rangle| \leq 1\}$$



$$\square^n = [-1, 1]^n$$

What is a section of the cube?

$$\square^n \cap H = \left(\bigcap_{i=1}^{n} \{ x \in \mathbb{R}^n : \; |\langle x, e_i \rangle| \leq 1 \} \right) \cap H$$



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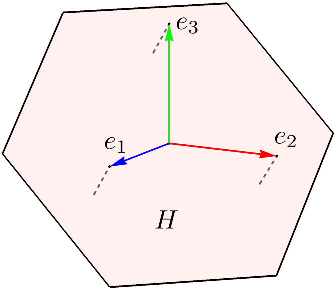

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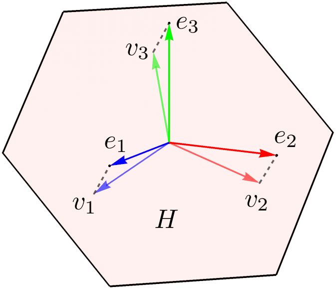

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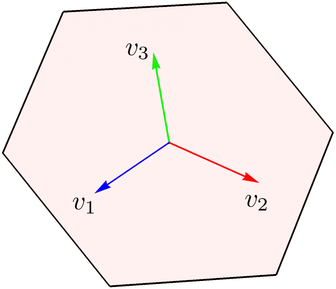
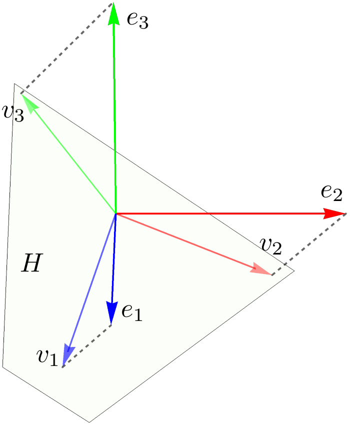
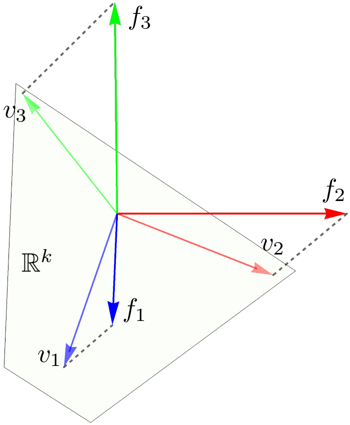


Tight frames



H is a subspace of $\mathbb{R}^n$



$\{f_1, f_2, f_3\}$ is an onb of $\mathbb{R}^n$

Tight frames

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An ordered n-tuple of vectors $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^k$ is *a tight frame* if the vectors of S are the projections of an orthonormal basis of $\mathbb{R}^n$.

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Recall

$$\square^n \cap H = \bigcap_{i=1}^{n} \{x \in H : \ |\langle x, v_i \rangle| \leq 1\}$$

Two 'equivalent' problems

Problem 1.

Find

$$\max \operatorname{vol}_k \left(\square^n \cap H\right)$$

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How to make out a tight frame?

More freedom!

How to make out a tight frame?

Folklore

TFAE:

1. $\{v_1, \dots, v_n\}$ is a tight frame in $\mathbb{R}^k$;
2. $\sum_1^n v_i \otimes v_i = \mathrm{Id}_{\mathbb{R}^k}$.

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$$A_S > 0 \quad \Rightarrow \quad \underbrace{A_S^{-1/2} S = \{A_S^{-1/2} v_1, \dots, A_S^{-1/2} v_n\}}_{\text{is a tight frame}}$$

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Definition 4.

An n-tuple of vectors in a linear space $\mathbb{R}^k$ that spans $\mathbb{R}^k$ is called a *frame*.

Variational principle

$$\underbrace{S}_{\text{tight frame}} \xrightarrow{\quad\mathbf{T}\quad} \tilde{S} \xrightarrow{\left(A_{\tilde{S}}\right)^{-1/2}} \underbrace{S'}_{\text{tight frame}},$$

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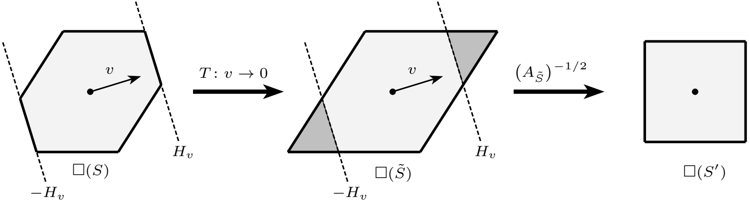

Figure: Here we map one vector to zero. $H_v = \{x \in \mathbb{R}^k : \langle x, v\rangle = 1\}$

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NS condition (Tsiutsiurupa and I.'20).

The maximum is attained at a tight frame $S = \{v_1, \ldots, v_n\}$ iff for an arbitrary frame $\tilde{S}$ inequality

$$\frac{\operatorname{vol}_k \square(\tilde{S})}{\operatorname{vol}_k \square(S)} \leq \frac{1}{\sqrt{\det A_{\tilde{S}}}}$$

holds.

Discrete operations

Theorem 1 (Tsiutsiurupa and I.'20).

Let S be a global maximizer and $v \in S$. Then

$$\frac{k}{n+k} \le |v|^2 \le \frac{k}{n-k}.$$

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Hint.

Map a short vector to a long one and vice versa.

First order approximation

Perturbation

$$v_i \mapsto v_i + \tau x \qquad \text{for every} \quad i \in J \subset [n] \quad (\tau \in \mathbb{R}, x \in \mathbb{R}^k)$$

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What do we want?

$$1 + \Delta Vol \cdot \tau + o(\tau) \leq 1 + C \cdot \tau + o(\tau)$$

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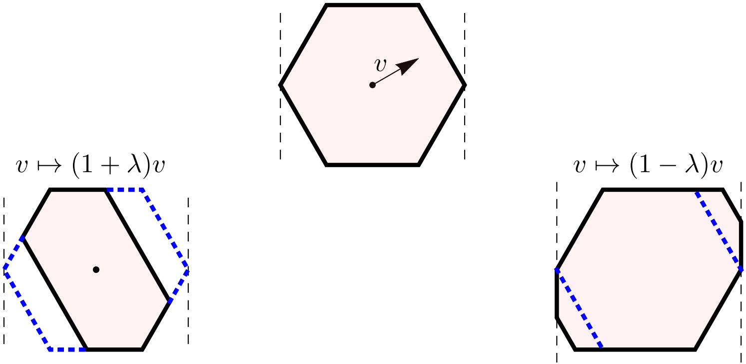
Theorem 2 (I.'18).

Let $S = \{v_1, \ldots, v_n\}$ be a tight frame and $\tilde{S} = \{v_1 + \tau x_1, \ldots, v_n + \tau x_n\}$, where $\tau \in \mathbb{R}$, $x_i \in \mathbb{R}^k$, $i \in [n]$. Then,

$$\frac{1}{\sqrt{\det A_{\tilde{S}}}} = 1 - \tau \sum_{i=1}^{n} \langle x_i, v_i \rangle + o(\tau).$$

Hidden obstruction



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Lemma 1 (Tsiutsiurupa and I.'20).

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Hint

Move v into direction of a vertex of $\operatorname{conv}\{\pm S\}$.

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Definition 5.

For a given frame S in $\mathbb{R}^k$ and $u \in \mathbb{R}^k$, a facet F of $\square(S)$ and a vector $u \in \mathbb{R}^k$, we define an F-substitution in the direction u as follows:

- each vector v of S such that $F \subset H_v$ is substituted by $v + u$;
- each vector v of S such that $-F \subset H_v$ is substituted by $v - u$;
- all other vectors of S remain the same.

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Everything is smooth for an F-substitution!

Two F-substitutions

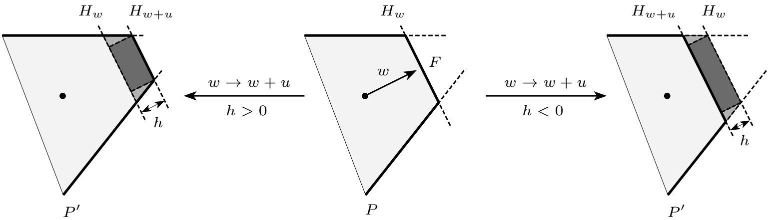

Figure: Parallel shift of a facet by vector $u = hw/|w|$

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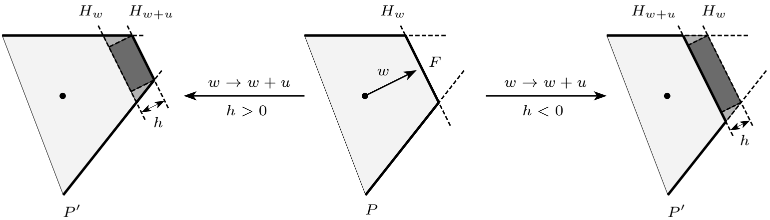

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$$\mathrm{vol}_k P' - \mathrm{vol}_k P = h \, \mathrm{vol}_{k-1} F + o(h).$$

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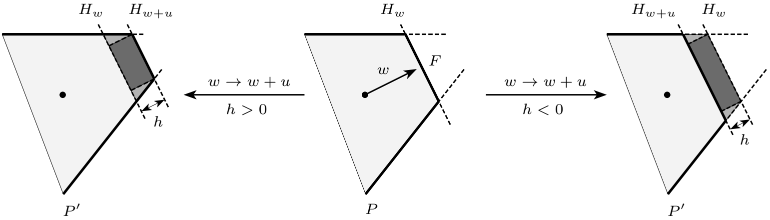

Figure: Parallel shift of a facet by vector $u = hw/|w|$

Lemma 2 (Tsiutsiurupa and I.'20).

Let S be a local maximizer and F be a facet of $\square(S)$. Denote $P_F = \operatorname{conv}\{0, F\}$. Then

$$\frac{\operatorname{vol}_{k-1} F}{\operatorname{vol}_k \square(S)} = \frac{1}{2} \sum_{\star} |v|^2,$$

where the summation is over all vectors $v \in S$ such that either $F \subset H_v$ or $-F \subset H_v$.

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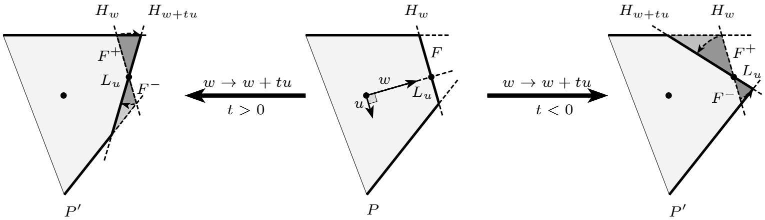

Figure: A rotation of the facet F around L_u.

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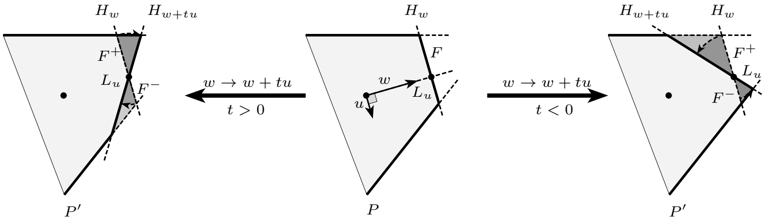

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Lemma 3 (Tsiutsiurupa and I.'20).

Let S be a local maximizer and $v \in S$. Then the line span$\{v\}$ intersects the boundary of $\square(S)$ in the centroid of the facet $\square(S) \cap H_v$.

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3. Denote $F = \square(S) \cap H_v$ and $P_F = \operatorname{conv}\{0, F\}$. Then

$$\frac{\operatorname{vol}_k P_F}{\operatorname{vol}_k \square(S)} = \frac{1}{2} \frac{\sum_{\star} |w|^2}{k},$$

where the summation is over all vectors $w \in S$ such that either $w = v$ or $w = -v$.

Planar case $k = 2$

Theorem 4 (Tsiutsiurupa and I.'20).

Conjecture 1 is true for $n > k = 2$. For any two-dimensional subspace $H \subset \mathbb{R}^n$ the following inequality holds

$$\operatorname{Area}(\square^n \cap H) \leq C_{\square}(n, 2) \operatorname{vol}_2 \square^2 = 4\sqrt{\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor}.$$

This bound is optimal and is attained if and only if $\square^n \cap H$ is a rectangle with the sides of lengths $2\sqrt{\left\lceil \frac{n}{2} \right\rceil}$ and $2\sqrt{\left\lfloor \frac{n}{2} \right\rfloor}$.

Planar case $k = 2$

Sketch.

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1. $\square(S)$ is cyclic.

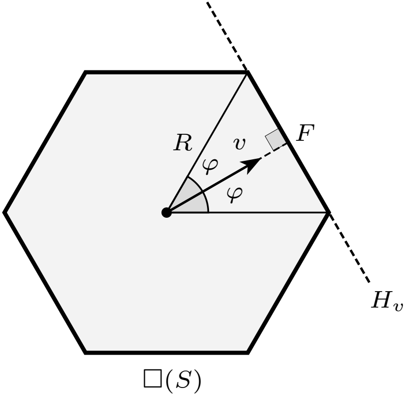
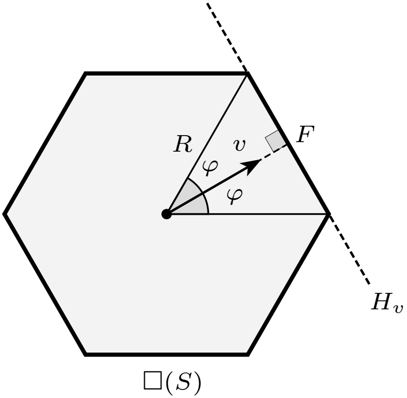


Figure: R – the circumradius;
$2f$ – the number of edges

Planar case $k = 2$

Sketch.

1. $\square(S)$ is cyclic.
2. $$R^2 = \frac{1}{|v|^2 \cos^2 \varphi} \leq \frac{n+1}{2} \frac{1}{\cos^2 \frac{\pi}{2f}}$$



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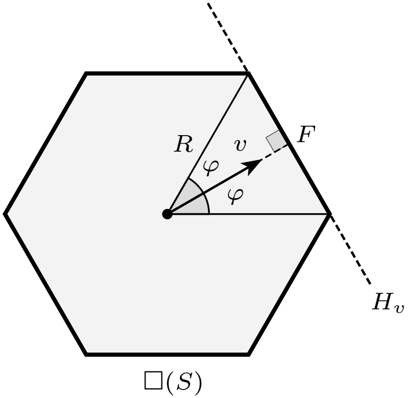

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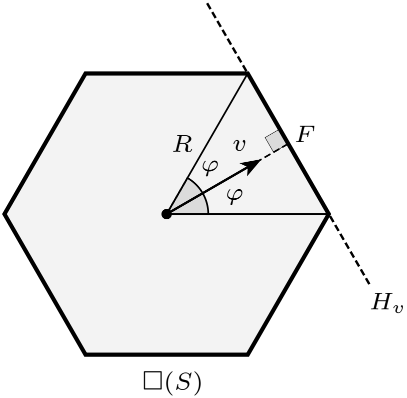
4. Profit! Equality for $f = 3$ and $n = 7$.



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Thank you for your attention!