Mordukhovich subdifferential and well-posedness

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In the present paper we continue the research started in [1] and [2]. Let X and P be Banach spaces and $h\colon X\times P\to\mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function. Consider the problem

$$\mathcal{P}_h\colon\quad \text{minimize } h(x,p) \text{ over } x\in X$$

with parameter $p\in P$. The *optimal value* of $\mathcal{P}_h$ at $p\in P$ is $h_{\inf}(p)=\inf_{x\in X}h(x,p)$ and $x\in X$ is called a *solution* of $\mathcal{P}_h$ at $p\in P$ if $h(x,p)=h_{\inf}(p)\in\mathbb{R}$. A sequence $\{x_k\}\subset X$ is called *minimizing* for $\mathcal{P}_h$ at $p\in P$ if $\lim_{k\to\infty}h(x_k,p)=h_{\inf}(p)$.

Let $\mathcal{P}_h$ admit at $p_0\in P$ a unique solution x_0. Define the function $\Delta_{h,p_0}\colon P\to[0,+\infty)$ as

$$\Delta_{h,p_0}(p)=\inf_{\{x_k\}\text{ is a minimizing sequence for }\mathcal{P}_h\text{ at }p}\ \liminf_{k\to\infty}\|x_k-x_0\|,\qquad p\in P.$$

The problem $\mathcal{P}_h$ is called *approximately well-posed* (AWP) at $p_0\in P$ if it admits a unique solution at p_0, $h_{\inf}(p)$ is finite for p in some neighborhood of p_0 and

$$\lim_{p\to p_0}\Delta_{h,p_0}(p)=0.$$

If, in addition, there exists a constant $L>0$ such that $\Delta_{h,p_0}(p)\le L\|p-p_0\|$ for all p in some neighborhood of p_0, then the problem $\mathcal{P}_h$ is called *Lipschitz approximately well-posed* (LAWP) at p_0 with constant L.

We elaborate some subdifferential calculus of the optimal value (marginal) function $h_{\inf}(\cdot)$ provided that $\mathcal{P}_h$ is AWP or LAWP. Then we state some sufficient conditions for $\mathcal{P}_h$ to be AWP and LAWP.

Given a function $f\colon X\to\mathbb{R}\cup\{+\infty\}$ and $\varepsilon\ge 0$, the *Fréchet ε-subdifferential* of f at $x_0\in\operatorname{dom}f:=\{x\in X\colon f(x)\in\mathbb{R}\}$ is

$$\partial^{F,\varepsilon}f(x_0)=\big\{x^*\in X^*\colon\ \forall\eta>0\ \exists\delta>0\colon\ \forall x\in B_\delta(x_0)\\ \langle x^*,x-x_0\rangle\le f(x)-f(x_0)+(\varepsilon+\eta)\|x-x_0\|\big\},$$

where $B_\delta(x_0) = \{x \in X\colon \|x - x_0\| < \delta\}$. If $\varepsilon = 0$, then $\partial^F f := \partial^{F,\varepsilon} f$ is called the *Fréchet subdifferential*.

The *Mordukhovich limiting subdifferential* $\partial^L f(x_0)$ at $x_0 \in \operatorname{dom} f$ is the set of $x^* \in X^*$ such that there exist $\varepsilon_k \downarrow 0$, $x_k \to x_0$ with $f(x_k) \to f(x_0)$, and $x_k^* \to x^*$ weakly star and $x_k^* \in \partial^{F,\varepsilon_k} f(x_k)$ for all $k \in \mathbb{N}$.

We use $\partial^{F,\varepsilon} h(x,p)$ and $\partial^L h(x,p)$ to denote respectively the Fréchet ε-subdifferential and the limiting subdifferential of h at $(x,p) \in \operatorname{dom} h$ with respect to the norm $\|(x,p)\| = \|x\| + \|p\|$ in $X \times P$. We denote by $\partial_x^{F,\varepsilon} h(x,p)$ the Fréchet ε-subdifferential of the function $h(\cdot,p)$ at the point x. We shall use $\partial_x^L h(x,p)$ to denote the set of $x^* \in X^*$ such that there exist $\varepsilon_k \downarrow 0$, $(x_k,p_k) \to (x_0,p_0)$ with $h(x_k,p_k) \to h(x,p)$, and $x_k^* \to x^*$ weakly star and $x_k^* \in \partial_x^{F,\varepsilon_k} h(x_k,p_k)$ for all $k \in \mathbb{N}$. Similarly we define $\partial_p^{F,\varepsilon} h(x,p)$ and $\partial_p^L h(x,p)$.

A function $f\colon X \to \mathbb{R}\cup\{+\infty\}$ is called *lower regular* at a point $x \in \operatorname{dom} f$ (see [3]) whenever $\partial^L f(x) = \partial^F f(x)$.

Theorem 1. *Let $x_0 \in X$ be a solution of $\mathcal{P}_h$ at $p_0 \in P$. Then for all $\varepsilon \geq 0$*

$$\{0\} \times \partial^{F,\varepsilon} h_{\inf}(p_0) \subset \partial^{F,\varepsilon} h(x_0,p_0) \subset \partial_x^{F,\varepsilon} h(x_0,p_0) \times \partial_p^{F,\varepsilon} h(x_0,p_0).$$

If, in addition, $\mathcal{P}_h$ is AWP at p_0, then

$$\{0\} \times \partial^L h_{\inf}(p_0) \subset \partial^L h(x_0,p_0) \subset \partial_x^L h(x_0,p_0) \times \partial_p^L h(x_0,p_0).$$

Theorem 1 correlates with the results of Thibault [5, Proposition 3.1] and those of Ngai, Luc and Théra [6, Theorem 2.5].

The next theorem provides sufficient conditions for $\mathcal{P}_h$ to be LAWP (and consequently AWP).

Theorem 2. *Let $x_0 \in X$, $p_0 \in P$ and $\lambda > 0$, $\mu, \gamma \in \mathbb{R}$ be such that for all p in some neighborhood of p_0 and for all $x \in X$*

$$h(x,p) \geq h(x_0,p_0) + \lambda\|x - x_0\| - \mu\|p - p_0\|,$$

$$h(x_0,p) \leq h(x_0,p_0) + \gamma\|p - p_0\|.$$

Then $\mathcal{P}_h$ admits a unique solution x_0 at p_0 and $\mathcal{P}_h$ is LAWP at p_0 with constant $L = (\mu + \gamma)/\lambda$.

The rest of the paper is devoted to the infimal convolution problem.

The Moreau-type *infimal convolution* of two functions $f, g\colon X \to \mathbb{R} \cup \{+\infty\}$ is

$$(f \boxplus g)(p) = \inf_{x\in X}\big(f(x) + g(p - x)\big), \qquad p \in X.$$

The *infimal convolution problem* $\mathcal{P}_{f,g}$ at a point $p \in X$ is the problem $\mathcal{P}_h$ with

$$h(x,p) = f(x) + g(p-x), \qquad x,p \in X.$$

From now on we assume that functions f and g are lower semicontinuous, which guarantees lower semicontinuity of $h(x,p) = f(x) + g(p-x)$.

Theorem 3. *Let $x_0 \in X$ be a solution of $\mathcal{P}_{f,g}$ at $p_0 \in X$. Then for all $\varepsilon \geq 0$*

$$\partial^{F,\varepsilon}(f \boxplus g)(p_0) \subset \big(\partial^{F,\varepsilon} f(x_0)\big) \cap \big(\partial^{F,\varepsilon} g(p_0 - x_0)\big).$$

If, in addition, $\mathcal{P}_{f,g}$ is AWP at p_0, then

$$\partial^{L}(f \boxplus g)(p_0) \subset \big(\partial^{L} f(x_0)\big) \cap \big(\partial^{L} g(p_0 - x_0)\big).$$

If, in addition, f is continuously differentiable in some neighborhood of x_0 or g is continuously differentiable in some neighborhood of $p_0 - x_0$, then the latter two inclusions are equalities.

Theorem 4. *Suppose that $x_0 \in X$ is the solution of $\mathcal{P}_{f,g}$ at $p_0 \in X$ and the problem $\mathcal{P}_{f,g}$ is LAWP at p_0 with constant L. Then for all $\varepsilon \geq 0$*

$$\big(\partial^{F,\varepsilon} f(x_0)\big) \cap \big(\partial^{F,\varepsilon} g(p_0 - x_0)\big) \subset \partial^{F,(2L+1)\varepsilon}(f \boxplus g)(p_0).$$

If, in addition, f and g are lower regular at points x_0 and $p_0 - x_0$ respectively, then $f \boxplus g$ is lower regular at p_0.

Theorem 5. *Suppose that $x_0, z_0 \in X$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha + \beta > 0$, $f\colon X \to \mathbb{R} \cup \{+\infty\}$ and $g\colon X \to \mathbb{R}$ are such that for all $x, z \in X$*

$$f(x) - f(x_0) \geq \alpha \|x - x_0\|, \tag{1}$$

$$g(z) - g(z_0) \geq \beta \|z - z_0\|, \tag{2}$$

$$g(z) - g(z_0) \leq \gamma \|z - z_0\|. \tag{3}$$

Then $\mathcal{P}_{f,g}$ admits a unique solution x_0 at $p_0 = x_0 + z_0$ and $\mathcal{P}_{f,g}$ is LAWP at p_0 with constant $L = (|\beta| + \gamma)/(\alpha + \beta)$.

If, in addition, f and g are lower regular at points x_0 and z_0 respectively, then $f \boxplus g$ is lower regular at p_0.

Theorems 3–5 improve Theorem 5.5 in [4], Theorems 3.1, 3.2 in [1] and Theorems 3.1, 4.2 in [2].

References

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Построение множества разрешимости в дифференциальных играх с простыми движениями (Construction of the solvability set for differential games with simple motion)*

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Пусть движение управляемой системы на плоскости описывается динамикой простых движений [1]:

$$\dot{x} = u + v, \qquad u \in P, \quad v \in Q, \quad t \in [0, \vartheta], \quad \vartheta > 0.$$

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