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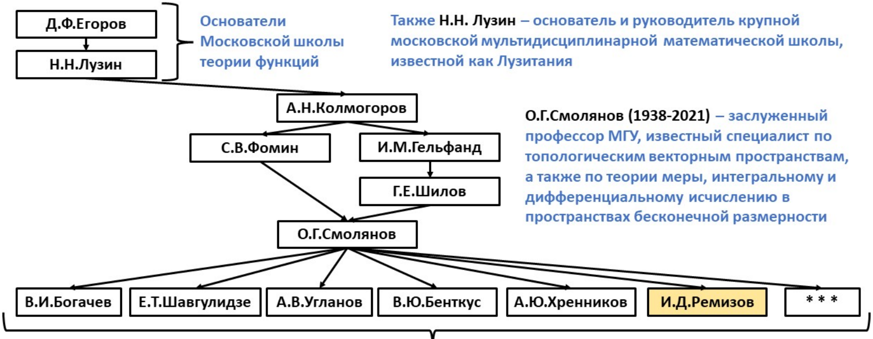
Обзор научных результатов для участия в конкурсе ППС

3 мая 2023

Историческая справка

Математическая генеалогия: И.Д.Ремизов – представитель Московской школы теории функций, ученик О.Г.Смолянова

Стрелками показаны цепочки «Учитель-Ученик» на уровне научного руководства по кандидатским диссертациям:



Среди учеников О.Г.Смолянова более 40 кандидатов наук, более 8 докторов наук

Вклад в науку

И.Д.Ремизов входит в мировой топ-5 специалистов по приближённому вычислению экспонент от дифференциальных операторов с переменными коэффициентами при помощи теоремы Чернова. Всего в мире по этим вопросам насчитывается примерно 20 экспертов. И.Д.Ремизову в этой теме принадлежат:

- Новые концепции (касание по Чернову, квазифейнмановские формулы, аппроксимационное подпространство, быстро и сверхбыстро сходящиеся черновские аппроксимации, построенные на операторе сдвига функции Чернова и др.)
- Фундаментальные результаты (примеры сходящихся с наперёд заданной скоростью черновских аппроксимаций, теорема об оценке сверху на скорость сходимости черновских аппроксимаций - совместно с О.Е.Галкиным, универсальный метод аппроксимации групп унитарных операторов, черновские аппроксимации резольвент)
- Много новых формул, явно выражающих сколь угодно точные аппроксимации к экспоненте от дифференциального оператора через его переменные коэффициенты - в разных пространствах
- Именная формула $R(t) = e^{ia(S(t)-I)}$
- Численные эксперименты (совместно со студентами)

Подробнее о тематике

Иерархия тематик: математика $\supset$ функциональный анализ $\supset$ однопараметрические полугруппы операторов $\supset$ черновские аппроксимации C_0-полугрупп.

Как определить экспоненту? Если $t > 0$, то можно использовать определение с помощью ряда $e^{tL} = \sum_{k=0}^{\infty} \frac{(tL)^k}{k!}$ в случае, если:

- L – вещественное или комплексное число
- L – вещественная или комплексная матрица
- L – ограниченный линейный оператор в вещественном или комплексном банаховом пространстве

В случае, если L – **неограниченный линейный оператор** в банаховом пространстве $\mathcal{F}$, то такая экспонента существует уже не для любого L, и ряд для определения экспоненты использовать уже нельзя. **Под экспонентой e^{tL} в этом случае понимают C_0-полугруппу (C_0-semigroup) с генератором L, т.е. такое отображение** $V\colon [0, +\infty) \to \mathscr{L}(\mathcal{F})$, что при каждом $t \geq 0$ оператор $V(t)$ отображает $\mathcal{F}$ в $\mathcal{F}$ линейно и непрерывно, для каждого $f \in \mathcal{F}$ верно $V(0)f = f$, $V(t_1 + t_2)f = V(t_1)V(t_2)f$ для всех $t_1, t_2 \in [0, +\infty)$, функция $t \mapsto V(t)f$ непрерывна и $V'(0) = L$. **Тогда пишут $V(t) = e^{tL}$.**

Applications of semigroups

This is the contents of the famous book **K.J.Engel, R.Nagel. One-parameter semigroups for linear evolution equations (Springer, 2000):**

- Contents
- Preface
- Prelude
- Linear Dynamical Systems
- Semigroups, Generators, and Resolvents
- Perturbation and Approximation of Semigroups
- Spectral Theory for Semigroups and Generators
- Asymptotics of Semigroups
- Semigroups Everywhere
- A Brief History of the Exponential Function
- Appendix
- Epilogue
- References
- Symbols and Abbreviations
- Index

- Semigroups Everywhere
  - Semigroups for Population Equations
  - Semigroups for the Transport Equation
  - Semigroups for Second-Order Cauchy Problems
  - Semigroups for Ordinary Differential Operators
  - Semigroups for Partial Differential Operators
  - Semigroups for Delay Differential Equations
  - Semigroups for Volterra Equations
  - Semigroups for Control Theory
  - Semigroups for Nonautonomous Cauchy Problems

We will discuss only few of applications of semigroups, and will select only some of those that your lecturer uses or created.

Theorem (summary of well known facts). Suppose that $(A, D(A))$ generates a C_0-semigroup $(e^{tA})_{t\geq 0}$ in Banach space $\mathcal{F}$. Then:
1. For each $u_0 \in D(A)$ Cauchy problem

$$\begin{cases} U'(t) = AU(t), t \geq 0 \\ U(0) = u_0 \end{cases} \tag{1}$$

has a solution $U \in C^1([0, +\infty), \mathcal{F})$ which is unique in $C^1([0, +\infty), \mathcal{F})$ and is given by the formula $U(t) = e^{tA}u_0$.
2. For each $u_0 \in D(A)$, $f \in C([0, +\infty), \mathcal{F})$ Cauchy problem

$$\begin{cases} U'(t) = AU(t) + f(t), t \geq 0 \\ U(0) = u_0 \end{cases}$$

has a solution $U \in C^1([0, +\infty), \mathcal{F})$ which is unique in $C^1([0, +\infty), \mathcal{F})$ and is given as $U(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds$.
3. For each $u_0 \in \mathcal{F}$ Cauchy problem (1) in the integral form $U(t) = u_0 + A\int_0^t U(s)ds$ has a solution $U \in C([0, +\infty), \mathcal{F})$ which is unique in $C([0, +\infty), \mathcal{F})$ and is given as $U(t) = e^{tA}u_0$. This solution is called the *mild solution* of (1) and exists for all $u_0 \in \mathcal{F}$.
4. If $\|e^{tA}\| \leq Me^{wt}$, then for each $\lambda \in \mathbb{C}$ satisfying $Re\lambda > w$ and for each $g \in \mathcal{F}$ equation $\lambda f - Af = g$ has a solution $f \in D(A)$, which is unique in $D(A)$ and is given by the formula $f = R(\lambda, A)g = \int_0^{+\infty} e^{-\lambda t}e^{tA}g dt$.

Example 1. Consider

$$\mathcal{F} = UC_b(\mathbb{R}) = \{f \in \mathbb{R}^{\mathbb{R}} | f \text{ is uniformly continuous and bounded}\}$$

which is a Banach space with the so-called «uniform norm» $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Suppose that functions $a, b, c \in UC_b(\mathbb{R})$ are some known parameters, and define operator A by equality

$$(Af)(x) = a(x)f''(x) + b(x)f'(x) + c(x)f(x) \text{ for all } x \in \mathbb{R}, f \in D(A).$$

Let $(A, D(A))$ be closed linear operator with $D(A)$ satisfying

$$UC_b^2(R) = \{f \in UC_b(R) | f', f'' \in UC_b(R)\} \subset D(A) \subset UC_b(R).$$

Suppose that $(A, D(A))$ generates a C_0-semigroup $(e^{tA})_{t\geq 0}$. Then Cauchy problem for **second order linear parabolic PDE** for $u\colon [0, +\infty) \times \mathbb{R} \to \mathbb{R}$

$$\begin{cases} u_t(t,x) = a(x)u_{xx}(t,x) + b(x)u_x(t,x) + c(x)u(t,x), t \geq 0, x \in \mathbb{R} \\ u(0,x) = u_0(x), x \in \mathbb{R} \end{cases}$$

has solution $u(t,x) = (e^{tA}u_0)(x)$, $U(t) = u(t,\cdot) = [x \mapsto u(t,x)]$. Moreover **second order linear ODE** for $f\colon \mathbb{R} \to \mathbb{R}$

$$a(x)f''(x) + b(x)f'(x) + (c(x) - \lambda)f(x) = -g(x), x \in \mathbb{R}$$

has solution $f(x) = \int_0^{+\infty} e^{-\lambda t}(e^{tA}g)(x)dt$.

Example 2. Consider dimension $d \in \mathbb{N}$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ and

$$\mathcal{F} = UC_b(\mathbb{R}^d) = \{f \in \mathbb{R}^{\mathbb{R}^d} | f \text{ is uniformly continuous and bounded}\}$$

which is a Banach space with the so-called «uniform norm» $\|f\| = \sup_{x \in \mathbb{R}^d} |f(x)|$. Suppose that functions $a_{ij}, b_i, c \in UC_b(\mathbb{R}^d)$ are some known parameters, and define operator A by equality

$$(Af)(x) = \sum_{i,j=1}^{d} a_{ij}(x) f_{x_i x_j}(x) + \sum_{i=1}^{d} b_i(x) f_{x_i}(x) + c(x) f(x) \text{ for all } x \in \mathbb{R}^d.$$

Suppose that $(A, D(A))$ generates a C_0-semigroup $(e^{tA})_{t \geq 0}$. Then Cauchy problem for **second order linear parabolic PDE**, $u \colon [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$

$$\begin{cases} u_t(t,x) = \sum\limits_{i,j=1}^{d} a_{ij}(x) u_{x_i x_j}(t,x) + \sum\limits_{i=1}^{d} b_i(x) u_{x_i}(t,x) + c(x) u(t,x), t \geq 0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x), x \in \mathbb{R}^d \end{cases}$$

has solution $u(t,x) = (e^{tA} u_0)(x)$, $U(t) = u(t, \cdot) = [x \mapsto u(t,x)]$.
Moreover **second order linear elliptic PDE** for $f \colon \mathbb{R}^d \to \mathbb{R}$

$$\sum_{i,j=1}^{d} a_{ij}(x) f_{x_i x_j}(x) + \sum_{i=1}^{d} b_i(x) f_{x_i}(x) + (c(x) - \lambda) f(x) = -g(x), x \in \mathbb{R}^d$$

has solution $f(x) = \int_0^{+\infty} e^{-\lambda t} (e^{tA} g)(x) dt$.

Chernoff approximations of C_0-semigroups

Definition.[1] Operator-valued function G is called *Chernoff-tangent* to the operator L iff all conditions are met:
(CT0)Let us use symbol $\mathscr{L}(\mathcal{F})$ to denote the set of all linear bounded operators in a Banach space $\mathcal{F}$. Let the operator $L\colon \mathcal{F} \supset D(L) \to \mathcal{F}$ be linear and closed.
(CT1) G is defined on $[0, +\infty)$, takes values in $\mathscr{L}(\mathcal{F})$, and the function $t \longmapsto G(t)f$ is continuous for each $f \in \mathcal{F}$.
(CT2) $G(0) = I$, i.e. $G(0)f = f$ for each $f \in \mathcal{F}$.
(CT3) There exists such a dense subspace $\mathcal{D} \subset \mathcal{F}$ that for each $f \in \mathcal{D}$ there exists a limit

$$G'(0)f = \lim_{t\to 0} \frac{G(t)f - f}{t}.$$

(CT4) The closure of the operator $(G'(0), \mathcal{D})$ is equal to $(L, D(L))$.
Remark. Informal meaning: $G(t) = I + tL + o(t)$ as $t \to 0$.

[1] I.D. Remizov. Quasi-Feynman formulas – a method of obtaining the evolution operator for the Schrödinger equation// Journal of Functional Analysis, 270:12 (2016)

Chernoff approximations of C_0-semigroups

Remark. In the definition of the Chernoff tangency the family $(G(t))_{t\geq 0}$ usually does not have a semigroup composition property, which in fact is a reason why we often can find a simple formula for $G(t)$. Each C_0-semigroup $\left(e^{tL}\right)_{t\geq 0}$ is Chernoff-tangent to its generator L, but if L is a differential operator with variable coefficients then usually we do not have a simple formula for e^{tL}. We should not expect to have such a formula beacause the Cauchy problem for parabolic equation $[u'_t(t) = Lu(t), u(0) = u_0]$ has the solution $u(t) = e^{tL}u_0$, so finding a formula for e^{tL} is equivalent to finding a formula that solves this Cauchy problem for each $u_0 \in \mathcal{F}$, which is usually not an easy task. However, we can obtain approximations to $e^{tL}u_0$ via the Chernoff theorem.

Remark. Chernoff's theorem says that if e^{tL} exists, G is Chernoff-taangent to L, and $\|G(t)\|$ behaves similar to $\|e^{tL}\|$ then $G(t/n)^n \to e^{tL}$ as $n \to \infty$. It is a natural fact because for the trivial case $\mathbb{R} = \mathcal{F} = \mathscr{L}(\mathcal{F})$ we have $G\colon [0, +\infty) \to \mathbb{R}$, and condition

$$G(t/n)^n = (1 + (tL/n) + o(1/n))^n \to e^{tL} \text{ as } n \to \infty$$

follows from the "second remarkable limit theorem".

Chernoff approximations of C_0-semigroups

Theorem (P. R. CHERNOFF, 1968). Let $\mathcal{F}$ and $\mathscr{L}(\mathcal{F})$ be as before. Suppose that the operator $L\colon \mathcal{F} \supset Dom(L) \to \mathcal{F}$ is linear and closed, and function G takes values in $\mathscr{L}(\mathcal{F})$. Suppose that these assumptions are fulfilled:
(E) There exists a C_0-semigroup $(e^{tL})_{t\geq 0}$ with the generator $(L, D(L))$.
(CT) G is Chernoff-tangent to $(L, D(L))$.
(N) There exists such $\omega \in \mathbb{R}$, that $\|G(t)\| \leq e^{\omega t}$ for all $t \geq 0$.
Then for each $f \in \mathcal{F}$ we have $(G(t/n))^n f \to e^{tL} f$ as $n \to \infty$ with respect to norm in $\mathcal{F}$ locally uniformly in t, i.e. for each $T > 0$

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| e^{tL} f - (G(t/n))^n f \right\| = 0. \qquad (C)$$

Remark. Expressions $(G(t/n))^n$ are called Chernoff approximations to the semigroup e^{tL}. If condition (C) holds then G is called: a Chernoff function for operator L and (sometimes) a Chernoff function for the semigroup $(e^{tL})_{t\geq 0}$, also in that case family $(G(t))_{t\geq 0}$ is called Chernoff-equivalent to the semigroup $(e^{tL})_{t\geq 0}$.

Concrete example of Chernoff approximation

Theorem.[2] Suppose that $d \in \mathbb{N}$ is an arbitrary number, and index j runs from 1 to d. Let $e_j \in \mathbb{R}^d$ be a constant d-dimensional vector with 1 at position j and 0 at other $d-1$ positions. Suppose that functions $a_j, b_j, c\colon \mathbb{R}^d \to \mathbb{R}$ are uniformly continuous and bounded, moreover a_j are positive and bounded from zero. For each $x \in \mathbb{R}^d$, $t \geq 0$, $f \in UC_b(\mathbb{R}^d)$ and $\varphi \in C_b^\infty(\mathbb{R}^d)$ define

$$(S(t)f)(x) = \frac{1}{4d}\sum_{j=1}^{d}\left(f\left(x + 2\sqrt{a_j(x)td}e_j\right) + f\left(x - 2\sqrt{a_j(x)td}e_j\right)\right) + \\ + \frac{1}{2}f(x + 2tb(x)) + tc(x)f(x),$$

$$(H\varphi)(x) = \sum_{j=1}^{d} a_j(x)\varphi_{x_j x_j}(x) + \sum_{j=1}^{d} b_j(x)\varphi_{x_j}(x) + c(x)\varphi(x).$$

Then e^{tH} exists, and for each $f \in UC_b(\mathbb{R}^d)$ we have a convergence

$$\lim_{n\to\infty}\left(\Big(S(t/n)\Big)^n f\right)(x) = \left(e^{tH}f\right)(x) \in \mathbb{R}$$

uniformly in $x \in \mathbb{R}^d$ and locally uniformly in $t \geq 0$.

[2] I.D.Remizov. Solution-giving formula to Cauchy problem for multidimensional parabolic equation with variable coefficients// Journal of Mathematical Physics, vol 60 (2019)

Дальнейшая информация

Профиль И.Д.Ремизова на Общероссийском математическом портале
https://www.mathnet.ru/php/person.phtml?option_lang=rus&personid=76353
был недавно обновлён и содержит следующие разделы:

- Основные темы научной работы
- Полученные научные результаты (по состоянию на март 2023)
- Темы в работе (по состоянию на март 2023)
- Образование
- Работа
- Организационная работа
- Преподавание
- Награды и премии
- Personalia
- Список публикаций
- Доклады и лекции в базе данных Math-Net.Ru