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Characterizations of Geometric Tripotents in Reflexive Complex SFS-Spaces

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Abstract—This paper is devoted to study of the relationship between M-orthogonality and orthogonality in the sense of SFS-spaces in dual space. A geometric characterization of geometric tripotents in reflexive complex SFS-spaces is given.

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1. INTRODUCTION

An important problem of the theory of operator algebras is a geometric characterization of state spaces of operator algebras. In 1989 Friedman and Russo published a paper [4] related to this problem, in which they introduced strongly facially symmetric spaces, largely for the purpose of obtaining a geometric characterization of the predual spaces of JBW*-triples admitting an algebraic structure. Many of the properties required in these characterizations are natural assumptions for state spaces of physical systems. Such spaces are regarded as a geometric model for states of quantum mechanics (see [3–5]).

The principal examples of complex strongly facially symmetric spaces are preduals of complex JBW*-triples, in particular, the preduals of von Neumann algebras (see [5]). In these cases, as shown in [5], geometric tripotents correspond to tripotents in a JBW*-triple and to partial isometries in a von Neumann algebra. In [2], the relationship between M-orthogonality and algebraic orthogonality in JB*-triples was studied. A purely geometric description of the algebraic concept of tripotents was obtained in [6]. Characterizations of real operator algebras with a strongly facially symmetric spaces were obtained in [1, 7, 8].

This paper is devoted to the study of the relationship between M-orthogonality and orthogonality in the sense of SFS-spaces in dual space. A geometric characterization of geometric tripotents in reflexive complex SFS-spaces is given.

2. STRONGLY FACIALLY SYMMETRIC SPACES

Let Z be a real or complex normed space. We say that elements $f, g \in Z$ are orthogonal and write $f \diamond g$ if

$$||f + g|| = ||f - g|| = ||f|| + ||g||.$$

We say subsets $S, T \subset Z$ are orthogonal and write $S \diamond T$, if $f \diamond g$ for all $(f, g) \in S \times T$. For a subset S of Z, we put

$$S^{\diamond} = \{f \in Z : f \diamond g \forall g \in S\};$$

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the set $S^\diamond$ is called the orthogonal complement of S. A convex subset F of the unit ball $Z_1 = \{f \in Z : ||f|| \leq 1\}$ is called a face if the relation $\lambda g + (1-\lambda)h \in F$, where $g, h \in Z_1$, $\lambda \in (0,1)$, implies $g, h \in F$. A face F of the unit ball is said to be norm exposed if $F_u = \{f \in Z_1 : u(f) = 1\}$, for some $u \in Z^*$ with $||u|| = 1$. An element $u \in Z^*$ is called a projective unit if $||u|| = 1$ and $u(g) = 0$ for all $g \in F_u^\diamond$.

A norm exposed face F_u in Z_1 is called a symmetric face if there exists a linear isometry S_u from Z to Z such that $S_u^2 = I$ whose fixed point set coincides with the topological direct sum of the closure $\overline{\mathrm{sp}}F_u$ of the linear hull of the face F_u and its orthogonal complement $F_u^\diamond$, i.e., with $\overline{\mathrm{sp}}F_u \oplus F_u^\diamond$.

A space Z is said weakly facially symmetric (WFS) if each norm exposed face in Z_1 is symmetric.

For each symmetric face F_u, contractive projections $P_k(F_u)$, $k = 0, 1, 2$ on Z are defined as follows. First, $P_1(F_u) = (I - S_u)/2$ is the projection on the eigenspace corresponding to the eigenvalue -1 of the symmetry S_u. Next, $P_2(F_u)$ and $P_0(F_u)$ are defined as projections of Z onto $\overline{\mathrm{sp}}F_u$ and $F_u^\diamond$, respectively; i.e., $P_2(F_u) + P_0(F_u) = (I + S_u)/2$. The projections $P_k(F_u)$ are called geometric Peirce projections.

A projective unit $u \in Z^*$ is called geometric tripotent if F_u is a symmetric face and $S_u^* u = u$ for the symmetry S_u corresponding to F_u. By $\mathcal{GT}$ and $\mathcal{SF}$ we denote the sets of all geometric tripotents and symmetric faces, respectively; the correspondence $\mathcal{GT} \ni u \mapsto F_u \in \mathcal{SF}$ is one-to-one [4, Proposition 1.6]. For each geometric tripotent u from the dual WFS space Z, we denote the geometric Peirce projections by $P_k(u) = P_k(F_u)$, $k = 0, 1, 2$.

We set

$$U = Z^*, \quad U_1 = Z_1^*, \quad Z_k(u) = Z_k(F_u) = P_k(u)Z, \quad U_k(u) = P_k(F_u) = P_k(u)^* Z^*.$$

The geometric Peirce decomposition

$$Z = Z_2(u) + Z_1(u) + Z_0(u), \quad U = U_2(u) + U_1(u) + U_0(u)$$

holds. Geometric tripotents u and v are said to be orthogonal if $u \in U_0(v)$ (which implies $v \in U_0(u)$) or, equivalently, $u \pm v \in \mathcal{GT}$ (see [3, Lemma 2.5]). More generally, elements a and b of U are said to be orthogonal, denoted $a \diamond b$, if one of them belongs to $U_2(u)$ and the other belongs to $U_0(u)$ for some geometric tripotent u.

A WFS space Z is said to be strongly facially symmetric (SFS) if for each norm exposed face F_u of Z_1 and each $y \in U$ satisfying the conditions $||y|| = 1$ and $F_u \subset F_y$, we have $S_u^* y = y$, where S_u is the symmetry corresponding F_u.

Instructive examples of complex strongly facially symmetric spaces are Hilbert spaces, the preduals of von Neumann algebras or JBW*-algebras, and more generally, the preduals of complex JBW*-triples. Moreover, geometric tripotents correspond to nonzero partial isometries of von Neumann algebras and tripotents in a JBW*-triples (see [5]).

Remark. *Let $u, v \in \mathcal{GT}$ and $u \diamond v$. Then for any $f \in F_{u+v}$ we have*

$$\langle f, u \rangle \geq 0, \quad \langle f, v \rangle \geq 0. \tag{1}$$

Two elements a and b of a normed vector space E are said to be M-orthogonal (see [2]), denoted $a \Box b$, if

$$||a \pm b|| = \max\{||a||, ||b||\}.$$

For a subset H of the normed space E, the M-orthogonal complement (briefly the M-complement) $H^\Box$ of H is defined by

$$H^\Box = \{a \in E : a \Box b, \forall b \in H\}.$$

For a singleton set $\{a\}$ we write $a^\Box$ instead of $\{a\}^\Box$.

For each element $a \in E$ of norm one, the tangent disc S_a at a is defined by

$$S_a = \{b \in E : ||a + \lambda b|| = 1, \forall \lambda \in \mathbb{C}, |\lambda| \leq 1\}.$$

The relations presented in the following lemma will be useful in subsequent considerations. They were proved in [2, Lemma 2.11].

Lemma 1. *Let a be an element of norm one in a complex normed space E. Then*

(i) $a^\Box \cap E_1 = \{b \in E : ||a + tb|| = 1, \ \forall t \in [-1; 1]\}$;

(*ii*) $ia^{\square} \cap a^{\square} \cap E_1 \subseteq \{\sqrt{2}b \in E : ||a + zb|| = 1, \ \forall z \in \mathbb{C}, |z| \leq 1\}$;

(*iii*) $ia^{\square} \cap E_1 = (ia)^{\square} \cap E_1$.

From (i) it is easy to see that $S_a \subseteq ia^{\square} \cap a^{\square} \cap E_1$. From (ii) we have

$$S_a \subseteq ia^{\square} \cap a^{\square} \cap E_1 \subseteq \sqrt{2}S_a. \quad (2)$$

3. CHARACTERIZATIONS OF GEOMETRIC TRIPOTENTS

Let Z be a strongly facially symmetric space and let $U = Z^*$. For a subset G of U we set

$$G^{\diamond} = \{x \in U : x \diamond y, \forall y \in G\}$$

and call $G^{\diamond}$ the orthogonal complement G.

Lemma 2. *Let Z be a SFS-space and x be an element in U. Then the orthogonal complement $x^{\diamond}$ of x is contained in the M-orthogonal complement $x^{\square}$ of x, i.e.*

$$x^{\diamond} \subset x^{\square}. \quad (3)$$

The proof of Lemma 2 follows from [3, Lemma 2.1(i)].

Lemma 3. *Let Z be a SFS-space and $u \in \mathcal{GT}$. Then $u^{\square} \cap U_1 = u^{\diamond} \cap U_1$.*

Proof. Let $u \in \mathcal{GT}$ and $y \in u^{\diamond} \cap U_1$. By lemma 2, $y \in u^{\square} \cap U_1$.

Let us suppose that $y \in u^{\square} \cap U_1$. By the definition of the set $u^{\square}$ it follows that

$$||u \pm y|| = \max\{||u||, ||y||\} = 1.$$

Then for every $f \in F_u$ we have

$$|1 \pm \langle f, y\rangle| = |\langle f, u\rangle \pm \langle f, y\rangle| = |\langle f, u \pm y\rangle| \leq ||u \pm y|| = 1.$$

But this inequality is valid only for $\langle f, y\rangle = 0$. Therefore, $F_u \subset F_{u \pm y}$. Then by [3, Lemma 2.8] we obtain that

$$u \pm y = u + P_0(u)^*(u \pm y) = u + P_0(u)^*(u)u \pm P_0(u)^*(y) = u \pm P_0(u)^* y.$$

Therefore, $y \in U_0(u)$. Hence, $y \in u^{\diamond} \cap U_1$. Thus, if u is a geometric tripotent, then $u^{\square} \cap U_1 = u^{\diamond} \cap U_1$. The proof is complete. $\square$

The next result is directly follows from Lemmas 2 and 3.

Theorem 1. *Let Z be a SFS-space and let G be non-empty subset of $\mathcal{GT}$. Then the sets $G^{\square} \cap U_1$ and $G^{\diamond} \cap U_1$ coincide.*

Theorem 2. *Let Z be a reflexive complex SFS-space and let u be an element in U of norm one. Then the following conditions are equivalent:* (a) $u \in \mathcal{GT}$; (b) $u^{\square} \cap U_1 = u^{\diamond} \cap U_1$; (c) $u^{\square} \cap U_1 = iu^{\square} \cap U_1$; (d) $S_u = u^{\diamond} \cap U_1$.

Proof. An implication $(a) \Longrightarrow (b)$ have already proved in Lemma 3.

$(a) \Longrightarrow (c)$. Take an arbitrary tripotent u. Since $u^{\diamond}$ is a complex subspace of U, then $u^{\diamond} = (iu)^{\diamond}$. Therefore, by Lemma 1 (iii), and it follows from (b) that

$$iu^{\square} \cap U_1 = (iu)^{\square} \cap U_1 = (iu)^{\diamond} \cap U_1 = u^{\diamond} \cap U_1 = u^{\square} \cap U_1.$$

$(a) \Longrightarrow (d)$. Let u be a tripotent and y be any element of $u^{\diamond} \cap U_1$. Since $u^{\diamond}$ is a complex subspace of U, it follows from (b) that, for all $z \in \mathbb{C}$, $|z| \leq 1$, $zy \in u^{\diamond} \cap U_1 = u^{\square} \cap U_1$. Therefore, $||u + zy|| = \max\{||u||, ||zy||\} = 1$, that is, y lies in S_u.

For the converse inclusion, combine (b) and (c) with the relations (2) to obtain

$$S_u \subseteq u^{\square} \cap iu^{\square} \cap U_1 = u^{\square} \cap u^{\square} \cap U_1 = u^{\square} \cap U_1 = u^{\diamond} \cap U_1.$$

Suppose, that $u \in U$ is not a geometric tripotent. By the spectral theorem for reflexive SFS-spaces (see [3, Theorem 1]), every element $u \in U$ is uniquely represented in the next form

$$u = \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n,$$

where $\lambda_1 > \lambda_2 > ... > \lambda_n > 0$, $u_k \in \mathcal{GT}$ and $u_k \diamond u_m$, $(k \neq m, k, m = 1, 2, ..., n, n \in \mathbb{N})$. Then, by [3, Lemma 2.1 (i)], it follows that

$$1 = ||u|| = ||\lambda_1 u_1 + \lambda_2 u_2 + ... + \lambda_n u_n|| = \max\{||\lambda_1 u_1||, ||\lambda_2 u_2||, ..., ||\lambda_n u_n||\}$$
$$= \max\{|\lambda_1|||u_1||, |\lambda_2|||u_2||, ..., |\lambda_n|||u_n||\} = \max\{\lambda_1, \lambda_2, ..., \lambda_n\}.$$

Therefore, $\lambda_1 = 1$.

In order to disprove (c), we need to find an element y which lies in $u^\square \cap U_1$ but not in $iu^\square$. Similarly, we show hat there exists an element h which lies in $u^\square \cap U_1$ and S_u but not in $u^\diamond$.

$(c) \Longrightarrow (a)$. Let $u^\square \cap U_1 = iu^\square \cap U_1$ and $y = \sqrt{1-\lambda_2^2} i u_2 + ... + \sqrt{1-\lambda_n^2} i u_n$. Again, by [3, Lemma 2.1 (i)], we have

$$||y|| = ||\sqrt{1-\lambda_2^2} i u_2 + ... + \sqrt{1-\lambda_n^2} i u_n|| = \max\{\sqrt{1-\lambda_2^2}, ..., \sqrt{1-\lambda_n^2}\} = \sqrt{1-\lambda_n^2} < 1,$$
$$||u \pm y|| = ||u_1 + (\lambda_2 \pm \sqrt{1-\lambda_2^2} i) u_2 + ... + (\lambda_n \pm \sqrt{1-\lambda_n^2} i) u_n||$$
$$= \max\{1, |\lambda_2 \pm \sqrt{1-\lambda_2^2} i|, ..., |\lambda_n \pm \sqrt{1-\lambda_n^2} i|\} = 1.$$

Therefore,

$$\max\{||u||, ||y||\} = \max\{1, \sqrt{1-\lambda_n^2}\} = 1 = ||u \pm y||.$$

This shows that u and y are M-orthogonal.

On the other hand, by [3, Lemma 2.1 (i)], we have

$$\max\{||u||, ||iy||\} = \max\{||u||, ||y||\} = \max\{1, \sqrt{1-\lambda_n^2}\} = 1,$$
$$||u - iy|| = ||u_1 + (\lambda_2 + \sqrt{1-\lambda_2^2}) u_2 + ... + (\lambda_n + \sqrt{1-\lambda_n^2}) u_n||$$
$$= \max\{1, \lambda_2 + \sqrt{1-\lambda_2^2}, ..., \lambda_n + \sqrt{1-\lambda_n^2}\} > 1.$$

Hence, u and $\pm iy$ are not M-orthogonal, and y is not contained in $iu^\square$.

$(b) \Longrightarrow (a)$. Let $u^\square \cap U_1 = u^\diamond \cap U_1$ and $h = (1-\lambda_2)u_2 + ... + (1-\lambda_n)u_n$. Again, by [3, Lemma 2.1 (i)], we have

$$||h|| = ||(1-\lambda_2)u_2 + ... + (1-\lambda_n)u_n|| = \max\{1-\lambda_2, ..., 1-\lambda_n\} = 1-\lambda_n < 1,$$
$$||u + h|| = ||u_1 + u_2 + ... + u_n|| = \max\{||u_1||, ..., ||u_n||\} = 1,$$
$$||u - h|| = ||u_1 + (2\lambda_2 - 1)u_2 + ... + (2\lambda_n - 1)u_n|| = \max\{1, |2\lambda_2 - 1|, ..., |2\lambda_n - 1|\} = 1.$$

Therefore,

$$\max\{||u||, ||h||\} = \max\{1, 1-\lambda_n\} = 1 = ||u \pm h||.$$

This shows that u and h are M-orthogonal, i.e. $h \in u^\square \cap U_1$.

Let us suppose that $u \diamond h$, $h = (1-\lambda_2)u_2 + ... + (1-\lambda_n)u_n$. By [3, Lemma 2.1(ii)], $F_u \subset F_{u+h}$. Then for any $f \in F_u$ we have $1 + \langle f, h\rangle = \langle f, u+h\rangle = 1$. Hence, $\langle f, h\rangle = 0$. On the other hand

$$1 = \langle f, u+h\rangle = \langle f, u_1 + u_2 + ... + u_n\rangle,$$

i.e. $F_u \subset F_{u_1+u_2+...+u_n}$. By (1), for each $f \in F_u$ we have

$$\langle f, h\rangle = (1-\lambda_2)\langle f, u_2\rangle + ... + (1-\lambda_n)\langle f, u_n\rangle \neq 0.$$

Therefore h does not contained in $u^\diamond$.

$(d) \Longrightarrow (a)$. Let $S_u = u^\diamond \cap U_1$ and $h = (1-\lambda_2)u_2 + ... + (1-\lambda_n)u_n$. For any $z \in \mathbb{C}$, $|z| \leq 1$, by [3, Lemma 2.1 (i)], we have

$$||u + zh|| = ||u_1 + (\lambda_2 + (1-\lambda_2)z)u_2 + ... + (\lambda_n + (1-\lambda_n)z)u_n||$$
$$= \max\{1, |\lambda_2 + (1-\lambda_2)z|, ..., |\lambda_n + (1-\lambda_n)z|\} = 1.$$

Therefore $h \in S_u$. But h does not contained in $u^\diamond$. The proof is complete. □

For a norm-one element u in U we consider the sets $X_1(u)$ and $X_2(u)$ defined by

$$X_1(u) = \{y \in U : \exists t > 0, ||u \pm ty|| = 1\},$$
$$X_2(u) = \{y \in U : \forall \lambda \in \mathbb{C}, ||u \pm \lambda y|| = \max\{1, ||\lambda y||\}\}.$$

Theorem 3. *Let Z be a reflexive complex SFS-space and let u be an element in U of norm one. Then u is a geometric tripotent if and only if $X_1(u)$ and $X_2(u)$ coincide.*

Proof. Suppose that $u \in \mathcal{GT}$. Observe that the inclusion $X_2(u) \subset X_1(u)$ is immediate from the definition of these sets. Hence we need only to show that $X_1(u) \subset X_2(u)$. Consider an element y in $X_1(u)$, i.e. there exists $t > 0$ with $||u \pm ty|| = 1$. Then

$$2||ty|| = ||u + ty - (u - ty)|| \leq ||u + ty|| + ||u - ty|| = 2.$$

Hence ty lies in U_1. Therefore $\max\{||u||, ||ty||\} = 1$ and ty lies in $u^{\square}$. From this and theorem 2 (b), it follows that $ty \in u^{\square} \cap U_1 = u^{\diamond} \cap U_1$. The relation (3) implies that, for all $\lambda \in \mathbb{C}$,

$$\lambda y = \frac{\lambda}{t} ty \in sp(u^{\diamond} \cap U_1) = u^{\diamond} \subset u^{\square}.$$

Therefore, y lies in $X_2(u)$, as required.

Suppose that $X_1(u) = X_2(u)$, and consider an element y in $u^{\square} \cap U_1$. From Lemma 1 (i) we see that $u^{\square} \cap U_1 \subset X_1(u)$, and we have that y lies in $X_2(u)$. In particular,

$$||u \pm iy|| = \max\{1, ||iy||\} \leq 1.$$

This shows that iy and $-iy$ are elements of $u^{\square} \cap U_1$. Hence y lies in $iu^{\square} \cap U_1$. We conclude that $u^{\square} \cap U_1 \subset iu^{\square} \cap U_1$. The reverse inclusion is obtained from similar arguments. By Theorem 2 (c), u is a geometric tripotent. The proof is complete. □

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