Abstracts of talks

A. I. Aptekarev, *Szegő asymptotics for multiple orthogonal polynomials with respect to Angelesco weights*

The system of Angelesco weights: $\{\rho_j(x),\, x \in \Delta_j \subset \mathbb{R}\}_{j=1}^d$, where segments Δ_j: $\Delta_j \cap \Delta_k = \varnothing$, $j \neq k$, is one of the basic systems for multiple orthogonal polynomials (MOPs) $Q_{\vec{n}}$ indexed by $\vec{n} := \{n_j\}_{j=1}^d$:

$$\int Q_{\vec{n}}(x)\, x^k\, \rho_j(x)\, dx = 0, \qquad k = 1, ..., n_j, \quad j = 1, ..., d,$$

where $\deg Q_{\vec{n}} = |\vec{n}| := n_1 + ... + n_d$. It is clear that for $d = 1$ this situation reduces to Q_n – orthogonal polynomials (OPs).

In the talk, we start with Widom's approach to strong (or Szegő type) asymptotics for OPs, then discuss an adaptation of this approach for MOPs with respect to Angelesco system: a known partial result when $d = 2, \vec{n} = (n, n)$, and perspectives for the general case motivated by modern requests from spectral problems for Schrödinger operators on the Cayley tree graph.

R. V. Bessonov, P. V. Gubkin, *Solving discrete nonlinear Shrödinger equation on $\mathbb{Z}$ using Schur's algorithm for analytic functions.*

In 1987, Y. Tsutsumi proved that the nonlinear Shrödinger equation

$$iu_t' = -u_{xx}'' \pm 2|u|^2 u, \quad x, t \in \mathbb{R}, \quad u\big|_{t=0} = u_0$$

admits a unique global weak solution for any initial datum $u_0 \in L^2(\mathbb{R})$. He used a direct method based on a Strichartz estimate. This result is not available by means of the classical inverse scattering theory (IST) due to the following fundamental obstacle: the scattering data for the Dirac equation with $L^2(\mathbb{R})$–potential (the auxiliary problem for NLSE) do not determine the potential uniquely. This fact was discovered in 2002 by A. Volberg and P. Yuditskii on the level of Jacobi matrices.

We discuss how to modify the IST approach to solve the discrete integrable NLS equation (Ablowitz-Ladik equation) with $\ell^2(\mathbb{Z})$ initial data by means of inverse scattering. The argument is based on a new estimate for the classical Schur's algorithm for contractive analytic functions in Szegő class. It also gives a new exponentially fast numerical scheme for solving discrete NLSE. The continuous case remains open.

A. B. Bogatyrev, *Spectral problem of Poincaré and Steklov.*

We consider a spectral problem with a pair of boundary influence operators (transforming the Dirichlet data of a harmonic function into its Neumann data) for a pair of planar domains with a common boundary. Similar problems arise when we justify and optimize the computational methods such as domain decomposition and fictitious domains. The problem reduces to studying a pencil of one-dimensional integral operators with Cauchy and Grunsky kernels. The possibility of finding the eigenvalues and functions of the simplest pencils in a closed analytical form is investigated.

A. L. Delitsyn, *Fast algorithms for solving the nonlinear Schrödinger equation for digital compensation of signal distortions in fiber-optic communication lines.*

Initial value problem for the nonlinear Schrodinger equation

$$i\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial t^2} + |u|^2 u, \quad -\infty < t < \infty, \quad z > 0, \quad u\Big|_{z=0} = u_0(t)$$

is the simplest but realistic model for describing signal propagation in a fiber-optic transmission line. When passing through the information transmission line, the signal is completely distorted and requires restoration. The main problem is the need to solve fast this problem. Initial value problem for the linear Schrodinger equation requires only $O(N \ln N)$ operations (complex multiplications). By fast we mean algorithms that require less than $O(N^2)$ actions.

At the present time, algorithms associated with three different approaches can be considered to solve this problem. The first method that can be considered is the inverse scattering problem method. Formally, it requires $O(N \ln^2 N)$ actions. Its numerical instability is the main obstacle to practical application. If methods for solving the inverse scattering problem are advanced in the field of stability theory, this approach will become extremely relevant in practical terms. The second method is perturbation theory, including the Krylov-Bogolyubov method for a system of ordinary differential equations that approximate the original problem. This approach is the main working method applicable in practice, but it is implemented only for the first correction of perturbation theory. The third possible method is the use of low-rank approximations of Volterra series. Such methods are fast. The main problem is the need to directly calculate Volterra operators and the lack of a mathematical theory that allows one to predict the behaviour of algorithms.

A. V. Domrin, *Properties of real-analytic solutions of the nonlinear Schrödinger equation and related equations.*

We prove that every local real-analytic solution of the focusing nonlinear Schrödinger equation in dimension 1+1 can be extended analytically to a strip (which can sometimes be enlarged to a half-plane or the whole plane) parallel to the axis of the spatial variable, and this also holds in the defocusing case if we admit singularities of pole type. The question of the maximal domain of analyticity of solutions and the possible types of singularities will be discussed for the vector and matrix versions of NLS as well as for the equations in their hierarchies and, moreover, for the Heisenberg magnetic model and the Landau–Lifshits equation.

L. L. Frumin, A. E. Chernyavsky, *Inverse spectral scattering transform algorithm for the Manakov system.*

A numerical algorithm for solving the inverse spectral scattering problem associated with Manakov's model of the vector nonlinear Schrödinger equation is described. This model of wave processes simultaneously takes into account dispersion, nonlinear and polarization effects. It is in demand in nonlinear problems of theoretical physics and physical optics, and is especially promising for describing the propagation of optical radiation along fiber communication lines. In the presented algorithm, the solution to the inverse scattering problem is based on inverting a set of nested matrices of a discretized system of Gelfand-Levitan-Marchenko integral equations using a block version of the Toeplitz algorithm of the

Levinson type. Numerical tests carried out by comparing the calculations with known exact analytical solutions confirm the stability and second-order accuracy of the proposed algorithm. An example is given of the use of the algorithm to simulate the collision of a pair of differently polarized vector Manakov solitons.

A. D. Mednykh, O. A. Danilov, *Discrete analytic functions and Taylor series.*

History. The notion of discrete analytic function on the Gaussian lattice $\mathbb{G} = \mathbb{Z} + i\,\mathbb{Z}$ was given by R. F. Isaacs [1]. He classified these functions into functions of first and second kind and investigated those of first kind. Further J. Ferrand [2] and R. J. Duffin [3] created the theory of discrete analytic functions of second kind (from now on: discrete analytic functions). Important results which are connected with a behaviour of discrete analytic and harmonic functions at infinity was obtained by S. L. Sobolev [4]. New combinatorial and analytical ideas to the theory were input by D. Zeilberger [5]. They were generalized by A. D. Mednykh [6]. An advance of nonlinear theory of discrete analytic functions based on usage of circle patterns began by W. Thurston [7] and his students [8], [9]. In that way an approximation with rapid convergence was obtained in the theory of conformal maps of Riemann surfaces.

Definition. From now on, $\mathbb{G} = \{x + i\,y : x, y \in \mathbb{Z}\}$ is the Gaussian integer lattice and $\mathbb{G}^+ = \{x + i\,y \in \mathbb{G} : x \geq 0, y \geq 0\}$ is a part of Gaussian plane contained in the first quadrant. A complex function f defined on some subset $E \subset \mathbb{G}$ is called discrete analytic on E if for any square $\{z, z + 1, z + 1 + i, z + i\} \subset E$ there holds:

$$\frac{f(z+1+i) - f(z)}{i+1} = \frac{f(z+i) - f(z+1)}{i-1}$$

or equivalently

$$\bar{\partial} f(z) = f(z) + i f(z+1) + i^2 f(z+1+i) + i^3 f(z+i) = 0.$$

A discrete analytic function on all $\mathbb{G}^+$ is called entire discrete. Let us denote the set of all discrete analytic functions on E and on $\mathbb{G}^+$ by $\mathcal{D}(E)$ and $\mathcal{D}(\mathbb{G}^+)$ correspondingly.

Theorem 1. *Every discrete analytic function $f \in \mathcal{D}(\mathbb{G}^+)$ has a Taylor expansion in terms of $\pi_k(z)$:*

$$f(z) = \sum_{0}^{\infty} a_k \pi_k(z), \quad z \in \mathbb{G}^+.$$

Theorem 2. *Above mentioned expansion is not unique. More precisely,*

$$f(z) = \sum_{0}^{\infty} a_k \pi_k(z) \equiv 0, \quad z \in \mathbb{G}^+ \Leftrightarrow F(s) = 0, \quad s \in \mathbb{Z}.$$

Theorem 3. *A homomorphism $\Theta : \mathcal{A}(U_R) \to \mathcal{D}(Q_R)$ is "onto" and $\Theta(F) \equiv 0 \Leftrightarrow F(s) = 0$, $s \in \mathbb{Z}$, $|s| < R$. In this case*

$$Ker\,\Theta = \langle F_N(\xi) \rangle = F_N \cdot (U_R)$$

is a principal ideal in $\mathcal{A}(U_R)$ generated by $F_N(\xi) = \xi \prod_{k=1}^{N}(\xi^2 - k^2)$, where $N = [R]$, if R is non-integer and $R - 1$ otherwise.

Theorem 4. *Let $f \in \mathcal{D}(\mathbb{G}^+)$. Then there exists a function $F(\xi) = \sum_{|k|=0}^{\infty} a_k \frac{\xi^k}{(1+i)^{|k|}} \in \mathcal{A}(\mathbb{C}^n)$ such that $f(z) = \sum_{|k|=0}^{\infty} a_k \pi_k(z)$ and this expansion converges absolutely for all $z \in \mathbb{G}^+$. In addition, $\Theta F = 0 \Leftrightarrow F(s) = 0$ for all $s \in \mathbb{Z}^n$.*

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V. Yu. Novokshenov, *Discrete Riemann-Hilbert problem and interpolation of entire functions.*

We consider two problems in complex analysis which were developed in Ufa in 1970s years. These are a Riemann-Hilbert problem about jump of a piecewise-analytic function on a contour and a problem of interpolation of entire functions on a countable set in the complex plane. A progress in recent years led to comprehension that they have much common in subject. The first problem arrives as an equivalent of the inverse scattering problem applied for integrating nonlinear differential equations of mathematical physics. The second problem is a natural generalization of Lagrange formula for polynomial with given values on a finite set of points. It is shown that both problems can be united by generalization of the Riemann-Hilbert problem on a case of "discrete contour", where a "jump" of analytic function takes place. This formulation of the discrete matrix Riemann problem can be applied now for various problems of exactly solvable difference equations as well as estimates of spectrum of random matrices. In the paper we show how the discrete matrix Riemann problem provides a way to integrate nonlinear difference equations such as a discrete Painlevé equation. On the other hand, it is shown how assignment of residues to meromorphic matrix functions is effectively reduced to an interpolation problem of entire functions on a countable set in $\mathbb{C}$ with the only accumulation point at infinity. Other application of discrete matrix Riemann problem includes calculation of Fredholm determinants emerging in combinatorics and group representation theory.

R. V. Romanov, *Large time behavior of lossy systems governed by Dirac and Schrödinger operators.*

We discuss the behaviour at large times of systems with losses. Typical examples involve Schrödinger or Dirac evolution governed by complex potentials and the linear transport operator. When the losses are weak the evolution resembles the situation in the selfadjoint scattering theory. At the opposite end, when the losses get strong the evolution reduces to the exponential decay. In the talk we are going to discuss an intermediate regime where the losses are strong enough to prevent scattering but too weak to move the spectrum off the real line and cause the exponential decay.

A. O. Smirnov, *Spectral curves and vector integrable nonlinear equations.*

Studies of the two-component vector nonlinear Schrödinger equation, the Kundu-Eckhaus vector equation and the Gerdjikov-Ivanov vector equation have shown that the spectral curves of multiphase solutions of these equations have unusual properties. In particular,

- These equations are invariant with respect to orthogonal transformations of solutions. And the spectral curves of multiphase solutions are also invariant with respect to orthogonal transformations of solutions. I.e., it is impossible to know the direction of the wave vector from the spectral curve.
- The procedure for constructing the simplest nontrivial solutions of these equations showed that the equation for the length of the vector appears first. Then, from the additional relations, an equation follows that determines the dependence of the direction of the vector on its length. I.e., the solution of the equation is determined not so much by the dynamics of its components as by the dynamics of the length of the vector and its direction.
- For all vector equations, there are parameter values at which the direction of the vector is fixed. I.e., the evolution of the vector is reduced to the evolution of the length of the vector. In these cases, the spectral curve splits into separate components and the evolution of the vector is determined by a curve of a smaller kind than in the case when the direction of the vector is not fixed.

We have studied examples illustrating these provisions.

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