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Book of Abstracts

Entropy of a unitary operator on $L^2(\mathbb{T}^n)$

K.A. Afonin[1], D.V. Treschev[1]

[1] *Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia*

Let $\mathcal{X}$ be a nonempty set and let $\mathcal{B}$ be a σ-algebra of subsets $X \subset \mathcal{X}$. Consider the measure space $(\mathcal{X}, \mathcal{B}, \mu)$, where μ is a probability measure: $\mu(\mathcal{X}) = 1$.

Consider the Hilbert space $\mathcal{H} = L^2(\mathcal{X}, \mu)$ with the scalar product and the norm

$$\langle f, g \rangle = \int_{\mathcal{X}} f \overline{g} \, d\mu, \quad \|f\| = \sqrt{\langle f, f \rangle}.$$

Suppose that $F : \mathcal{X} \to \mathcal{X}$ is an endomorphism of the measure space $(\mathcal{X}, \mathcal{B}, \mu)$. Let $\mathrm{End}(\mathcal{X})$ denote the semigroup of all endomorphisms of $(\mathcal{X}, \mathcal{B}, \mu)$. There are two standard constructions associated with any $F \in \mathrm{End}(\mathcal{X})$.

(1) Any such F generates the isometry (a unitary operator if F is an automotphism) U_F on $\mathcal{H}$ (the Koopman operator):

$$L^2(\mathcal{X}, \mu) \ni f \mapsto U_F f = f \circ F, \qquad U_F = \mathrm{Koop}(F).$$

(2) For any $F \in \mathrm{End}(\mathcal{X})$ it is possible to compute the measure entropy (another name is the Kolmogorov-Sinai entropy) $h_\mu(F)$.

Let $\mathrm{Iso}(\mathcal{H})$ denote the set of all isometric linear operators on $\mathcal{H}$. Our question is as follows. Is it possible to determine in some "natural way" a function $\mathfrak{h} \colon \mathrm{Iso}(\mathcal{H}) \to \overline{\mathbb{R}}_+$ so that the diagram

$$\begin{array}{ccc} & \mathrm{End}(\mathcal{X}, \mu) & \\ h_\mu \swarrow & & \searrow \mathrm{Koop} \\ \overline{\mathbb{R}}_+ & \stackrel{\mathfrak{h}}{\longleftarrow} & \mathrm{Iso}(\mathcal{H}) \end{array}$$

is commutative? The quantity $\mathfrak{h}(U)$ is called the entropy of the isometry U.

At the moment, it has been possible to define the entropy $\mathfrak{h}(U)$ of a unitary operator $U \colon \mathcal{H} \to \mathcal{H}$. This definition is based on the concept of the μ-norm of an operator. The μ-norm of an operator was developed by Dmitry Treschev in his paper "μ-norm of an operator". Proc. of Steklov Math. Inst. **310**, 2020. Concerning our definition of $\mathfrak{h}(U)$, many technical questions appear, including the question on the possibility to include into the construction isometries U together with unitary operators, on methods of computation (or at least, on effective lower and upper estimates) of the quantities $\mathfrak{h}(U)$, and many others. Answers to these questions depend on the detailed analysis of the concept of μ-norm. Explicit formulas for calculating the μ-norm of an operator are obtained in the following cases.

- We have computed the μ-norm of a convolution operator on $\mathcal{H} = L^2(\mathbb{T}^n)$, where $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$ is an n-dimensional torus, $n \in \mathbb{N}$.
- We have calculated the μ-norm for a wide class of bounded operators on $L^2(\mathbb{T})$, the so-called operators of diagonal type.
- We have computed the μ-norm for a special class of regular operators.

Bifurcation curves of the Zhukovsky system in Pseudo-Euclidean space

E.S. Agureeva[1]

[1] *Lomonosov Moscow State University, Moscow, Russia*

Topological classification of integrable systems satisfying some conditions was done by A. Fomenko and his coauthors [1]. Generalization of this theory to the case of non-compact Liouville foliations and incomplete Hamiltonian flows is an open and important problem. Such systems are well-known in mathematics, mechanics, and their applications. They appear in the class of super-integrable Bertrand systems on Riemannian manifolds of revolution in a central potential field (e.g., the Kepler problem on the plane) and among the analogues of mechanical systems on various Lie algebras [2].

Recently enlisted by A. Borisov and I. Mamaev [3], "pseudo-Euclidean" analogues of classical systems of rigid body dynamics turn out to be interesting examples of such systems. A pseudo-Euclidean analogue of the Euler top (describing the motion of a "plate" on the Lobachevsky plane) and its generalization via addition of a gyrostat (also called the Zhukovsky integrable case) are among them. Their Liouville foliations on $\mathbf{R}^6(J_1, J_2, J_3, x_1, x_2, x_3)$ are given by four functions: the Casimirs f_1, f_2, the additional first integral K

$$f_1 = x_1^2 + x_2^2 - x_3^2, \qquad f_2 = J_1x_1 + J_2x_2 - J_3x_3, \qquad K = J_1^2 + J_2^2 - J_3^2$$

and the Hamiltonian H with moments of inertia A_i and gyrostatic moment vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, which is equal to zero in the Euler case:

$$H = \frac{(J_1 + \lambda_1)^2}{2A_1} + \frac{(J_2 + \lambda_2)^2}{2A_2} - \frac{(J_3 + \lambda_3)^2}{2A_3}.$$

Regular orbits of coadjoint representation $M^4_{a,b} = \{f_1 = a, f_2 = b\}$ correspond to pairs $(a, b) \neq (0, 0)$. On such $M^4_{a,b}$, both critical points of the momentum map $F = (H, K)$ and noncompact noncritical bifurcations of the Liouville foliation can appear. We describe the bifurcation diagram $\Sigma \subset \mathbf{R}^2(h, k)$ of the map F (i.e., the union of the set of bifurcation values and of the set of critical values of F) in the following way for each regular pair (a, b).

Theorem. *Consider a pseudo-Euclidean analog of the Zhukovsky system and its restriction to the regular orbit $M^4_{a,b}$ for arbitrary $(a, b) \neq (0, 0)$, $\lambda_i \neq 0$, $A_i > 0$, $A_i \neq A_j$ for $i, j = 1, 2, 3$, $i \neq j$. Its bifurcation diagram on $\mathbf{R}^2(h, k)$ is contained in the union of two or three bifurcation curves: the straight line $k = 0$, the straight line $k = b^2/a$ (if $a \cdot b \neq 0$) and the parametrized curve*

$$h(t) = \frac{t^2}{2}\left(\frac{A_1\lambda_1^2}{(1 + A_1t)^2} + \frac{A_2\lambda_2^2}{(1 + A_2t)^2} - \frac{A_3\lambda_3^2}{(1 + A_3t)^2}\right), k(t) = \frac{A_1^2\lambda_1^2}{(1 + A_1t)^2} + \frac{A_2^2\lambda_2^2}{(1 + A_2t)^2} - \frac{A_3^2\lambda_3^2}{(1 + A_3t)^2}.$$

Noncompact noncritical bifurcations correspond to the line $k = 0$. The image of a critical point of F can belong to the parametric curve or (if $a \neq 0$) to the line $k = b^2/a$.

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Examples and singularities of two-dimensional Nijenhuis operators

D. Akpan[1,2]

[1] *Lomonosov Moscow State University, Moscow, Russia*
[2] *Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia*

A Nijenhuis operator L is a $(1,1)$-tensor field on a smooth manifold M with vanishing Nijenhuis torsion $\mathcal{N}_L$. At each point $x \in M$, the algebraic type of $L(x)$ is characterized by (the sizes of blocks in) its Jordan normal form. A point is called *regular* if it has a neighbourhood where the algebraic type of L does not change. In this report we study singularities of a two-dimensional Nijenhuis operator in the case when its trace trL has a non-zero differential at the singular point. A description of such singularities reduces to studying the smoothness of some function which is a fraction depending on partial derivatives of the determinant of L. We completely describe singularities for some special classes of functions trL, $detL$. We also obtained interesting examples of Nijenhuis operators and their singularities.

We will answer the following question in dimension 2: what form can a smooth function $f(x, y)$ have if there exists a Nijenhuis operator L in a neighborhood of the point $(0, 0)$ such that $trL = x$ and $detL = f(x, y)$?

From the paper "Open problems, questions and challenges in finite-dimensional integrable systems" (A. Bolsinov, V. Matveev, E. Miranda, S. Tabachnikov) we know how to restore a Nijenhuis operator in a neighborhood of a differentially non-degenerate point from its invariants (coefficients of the characteristic polynomial).

Theorem 1 [B-M-M-T]. *Let L be a two-dimensional Nijenhuis operator and $trL = x$, $detL = f(x, y)$, with $f_y(0, 0) \neq 0$. Then the point $(0, 0)$ is differentially non-degenerate and L is given in coordinates (x, y) by the following formula:*

$$L = \begin{bmatrix} x - f_x & -f_y \\ \frac{f_x(x-f_x)-f}{-f_y} & f_x \end{bmatrix}. \tag{1}$$

Thus the problem can be divided into 2 cases:
1. $\frac{\partial f}{\partial y} \equiv 0$, i.e., $f = f(x)$;
2. $\frac{\partial f}{\partial y}(0, 0) = 0$ (but $\frac{\partial f}{\partial y} \not\equiv 0$).

Theorem 2. *Let L be a two-dimensional Nijenhuis operator and $trL = x$, $detL = f(x)$. Then either $L = \begin{bmatrix} x-\alpha & 0 \\ c(x,y) & \alpha \end{bmatrix}$ or $L = \begin{bmatrix} \frac{x}{2} & 0 \\ c(x,y) & \frac{x}{2} \end{bmatrix}$, where $\alpha \in \mathbf{R}$, $c(x, y)$ is an arbitrary smooth function. In particular, $f(x) = \alpha x - \alpha^2$ or $f(x) = \frac{x^2}{4}$.*

Definition. *A function $g(x, y)$ is called* an admissible discriminant *if there exists a smooth Nijenhuis operator L such that $trL = x$ and $(\frac{trL}{2})^2 - detL = g(x, y)$.*

Theorem 3. *Let $g(x, y)$ be a smooth function such that $g(0, 0) = g_y(0, 0) = 0$, $g_{yy}(0, 0) \neq 0$. Then $g(x, y)$ is an admissible discriminant if and only if there exists a transformation of the second coordinate $\tilde{y} = \tilde{y}(x, y)$, $\frac{\partial \tilde{y}}{\partial y}(0, 0) \neq 0$, that reduces g to one of the following two forms: either $g = \pm\tilde{y}^2 + \frac{x^2}{4}$ or $g = \pm\tilde{y}^2$. In coordinates $(x, \tilde{y})$, the corresponding Nijenhuis operators are respectively $L = \begin{bmatrix} x & \pm 2\tilde{y} \\ \frac{\tilde{y}}{2} & 0 \end{bmatrix}$ and $L = \begin{bmatrix} \frac{x}{2} & \pm 2\tilde{y} \\ \frac{\tilde{y}}{2} & \frac{x}{2} \end{bmatrix}$.*

Early theorems in dynamics and abstraction from Euclidean geometry

A. Albouy[1]

[1]*Institut de Mécanique Céleste et de Calcul des Éphémérides, Paris, France*

The deduction of Kepler's laws from Newton's laws necessarily involves the Euclidean metric. But part of it may be presented on an affine space, while the remaining part is successful when replacing the Euclidean metric by the simplest kind of Finsler metric (Jacobi-Darboux problem). This analysis of the geometrical structure is surprisingly relevant when discussing the history of dynamics. For example, the version of the second law of dynamics given by Hooke in 1674 appears as both inaccurate and Euclidean. His version of 1679 is accurate and no longer Euclidean. In Newton's Principia, Proposition VI, which translates the second law, is inaccurate and Euclidean in the first edition (1687). Newton changed it in the second edition (1713) in another statement which is accurate and no longer Euclidean. We will present similar facts, as the classical analogy between the gravitational attraction inside a sphere and inside an ellipsoid. The next level of abstraction, the locally projective space, is also relevant. We will recall results of projective dynamics about the divergence of the gravitational force field. We will present the relation between classical identities due to Halphen and the property of the Jacobi-Darboux problem.

Parts of these results are from collaborations with Antonio Ureña and with Zhao Lei.

Nonlinear stability of regular vortex N-gons in a Bose – Einstein condensate

E. Artemova[1], A. Kilin[1]

[1] *Ural Mathematical Center, Udmurt State University, Department of Theoretical Physics, Universitetskaya 1, Izhevsk*

The problem of stability of rotating regular vortex N-gons (Thomson's configurations) in a Bose – Einstein condensate in a harmonic trap is considered. A reduction procedure on the level set of the momentum integral is proposed. The dependence of the velocity of rotation ω of vortex polygon about the center of the trap is obtained as a function of the number of vortices N and the radius of the configuration, R. The analysis of the orbital linear and nonlinear stability of the motion of such configurations is carried out. For $N \leqslant 6$, regions of orbital stability of configurations in the parameter space are constructed. It is shown that vortex N-gons for $N > 6$ are unstable for any parameters of the system.

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Nonlinear resonances in Hamiltonian systems and kinetic equations

A. Artemyev[1,2], A. Neishtadt[2,3] and A. Vasiliev[2]

[1]*EPSS UCLA, Los Angeles, CA, USA*
[2]*IKI RAS, Moscow, Russia*
[3]*Loughborough University, Loughborough, UK*

Nonlinear resonant interaction of charged particles and electromagnetic waves is the common phenomenon in various space plasma systems. The most interesting question here is how to describe a long term dynamics of a large particle ensemble where each particle experiences a multiple nonlinear resonances. This problem of plasma kinetics can be resolved within the probabilistic approach based on analysis of Hamiltonian equations of nonlinear resonances. In this presentation we discuss such characteristics of nonlinear resonances as a probability of phase trapping and rate of drift due to scattering. We show that these characteristics being derived from the Hamiltonian equations allow to construct a kinetic equation describing dynamics of the entire particle ensemble.

On Local Variables in a Neighborhood of Periodic Solutions of an Autonomous Hamiltonian System

B.S. Bardin[1,2,3]

[1]*Moscow Aviation Institute (National Research University) Moscow, Russia*
[2]*Mechanical Engineering Research Institute of the Russian Academy of Sciences Moscow, Russia*
[3]*Moscow Automobile and Road Construction State Technical University (MADI) Moscow, Russia*

Autonomous Hamiltonian systems possess natural families of periodic solutions. This solutions are unstable in the sense of Lyapunov. However, in classical and celestial mechanics the problem of orbital stability of motions described by the above periodic solutions is of great interest. The most general approach for studying the orbital stability bases of introduction of the so-called local variables in a neighborhood of the unperturbed periodic orbit. This approach allows to reduce the problem of orbital stability to the problem of stability in the sense of Lyapunov of an equilibrium position of the periodic Hamiltonian system, which describes the motion on energy level corresponding to the unperturbed periodic orbit.

It is well known that local variables can be always introduced. In spite of this theoretical fact it is usually not easy to choose the local variables by a proper way. In particular, a singularity can appear in the Hamiltonian system, when it is written in local variables. Moreover, introduction of the local variables becomes especially complicate when the periodic orbit is not given analytically, but we have only its numerical presentation.

In this work, we propose a new method for introducing local variables. A constructive algorithm for constructing canonical transformation in the form of series in powers of local variables is described. This method makes it possible to avoid the singularity when introducing local variables and can be used both for analytical presentation of periodic orbit and for numerical one. As an application, the following problems of classical and celestial mechanics are considered: the problem of orbital stability of periodic motions of a heavy rigid body with a fixed point in the Bobylev-Steklov case of and the problem of orbital stability of periodic motions of a dynamically symmetric satellite in a circular orbit. The results of the study of orbital stability, obtained on the basis of the proposed method, are in full agreement and significantly supplement the results obtained earlier in the above problems by other methods [1,2].

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Statistics of distribution of families of periodic solutions to the Hill problem

A.B. Batkhin[1], N.V. Batkhina[2]

[1]*Keldysh Institute of Applied Mathematics of RAS, Moscow, Russia*
[2]*MAEF Secondary School No. 17, Shchyolkovo, Russia*

All generating solutions of families of periodic orbits of the second kind according to Poincare of the planar circular Hill problem can be described in terms of limiting arc-solutions of the integrable Henon problem. Each generating solution is a finite sequence composed according to certain rules from a countable number of arcs of two types, joined at the origin of coordinates (the singular point of equations of motion) by parts of hyperbolic trajectories. The equations of the Hill problem are invariant under the discrete group of automorphisms of the phase space with two generators, which leads to the division of all solutions into groups of three types of symmetry: nonsymmetric, singly symmetric, and doubly symmetric. Each generating sequence uniquely determines the symmetry type, the global multiplicity of the orbit and other characteristics of the corresponding periodic solutions of the generated family near its limit values.

On a finite subset of arc-solutions a set of generating sequences is constructed on which the distributions of the generated families by symmetry types and by global multiplicities of the corresponding periodic orbits are computed. It is shown that among all generating sequences consisting of no more than six arcs, the vast majority of periodic solutions have some symmetry. For generating sequences of length eight or more, most of the corresponding families consist of nonsymmetric periodic solutions. Similar results were obtained for sampling over global multiples of periodic orbits.

On the Stability of the Equilibrium Positions of a Rigid Body with a Suspension Point Vibrating in Three-dimensional Space

M.V. Belichenko[1]

[1]*Moscow Aviation Institute (National Research University) Moscow, Russia*

We consider the motion of a rigid body with a suspension point performing the specified high frequency periodic motion with small amplitude in three-dimensional space. The body mass center is located in the main plane of inertia. A wide case of suspension point vibrations, including motion along vertical and horizontal ellipses, arbitrary motion in the horizontal plane, as well as certain motions in three-dimensional space was studied. The approximate autonomous system of equations of motion written in the form of canonical Hamiltonian equations is investigated. In this form, the effect of the suspension point vibrations is represented as an additional potential field — a vibration potential [1]. The problem of the existence, number and stability of the body equilibrium positions is solved.

It is shown that the number and stability of the equilibrium positions depend on two vibration parameters, which are differences in the intensity of the horizontal and vertical components of vibrations. Four equilibrium positions, in which the center of mass is in the vertical upper and vertical lower positions, exist at any values of the vibration intensities. The main plane of inertia with mass center can hold two orthogonal positions.

At strong vibration intensities, equilibrium positions with an inclined mass center radius-vector arise. The body's mass center radius-vector may also be located in two vertical perpendicular planes. In the case of strong vertical vibrations in lateral equilibria, the body mass center lies above the suspension point, and in the case of strong horizontal vibrations, below it.

The stability of the obtained equilibria was studied. Sufficient equilibrium stability conditions are the conditions that the quadratic part of the Hamiltonian of perturbed motion is positive determined. The necessary conditions for stability of equilibria require that the roots of the characteristic equation of the linearized equations of perturbed motion be imaginary. The study showed that all necessary stability conditions were the same as the sufficient conditions.

It is obtained that at low vibration intensities only one of the lower equilibrium positions of the body is stable. In the case when vertical vibrations are much stronger than horizontal vibrations, one of the upper vertical positions will also become stable. In the case of strong horizontal vibrations, the lower equilibrium position becomes unstable, but a lateral stable equilibrium position appears.

The study, which was carried out at the Moscow Aviation Institute (National Research University), was financially supported by the Russian Science Foundation (project 19-11-00116).

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On the satellite formation flying mission to display graphics from space

Sh. Biktimirov[1]

[1] *Skolkovo Institute of Science and Technology, Moscow, Russia*

A concept of deploying satellite formations to display graphics from space is proposed. The formation satellites are equipped with sunlight reflectors, which under certain lighting conditions and given the right attitude can be observed from the Earth as bright stars in the sky. When deployed into appropriately selected orbits, the spacecraft can be grouped into a pixel image.

The first part of the work is devoted to mission design, where orbit selection, solar reflector sizing, and formation's orbital configuration design are considered. The target orbit for the mission is derived based on the graphics demonstration requirements. It states that the formation satellites should be lit by the Sun and be above the horizon during the graphics demonstration, while the point of the observer should be at the darker side of the Earth with negative Sun declination at the moment. We consider circular Sun-synchronous orbits oriented close to the Earth terminator plane. For the sake of repeating demonstrations at the densely populated Earth regions, the orbit is also designed in a way to have a repeating ground track. The solar reflector size is chosen based on individual pixel brightness requirements. A parametric model is designed, yielding an intensity of the reflected sunlight at the observer's location depending on the solar reflector size, reflectivity coefficient, satellite elevation and its distance to the observer, and sunlight incident angle to the reflector. The minimum required reflector size is calculated for the worst-case geometrical condition corresponding to the smallest intensity of the reflected sunlight at the observer's location during a demonstration. Formation's orbital configuration is a set of closed relative trajectories built with respect to the leader satellite. The trajectories are obtained with the aid of analytical solutions to the Hill-Clohessy-Wiltshire equations describing relative motion dynamics between two satellites at close circular orbits.

The second part of the work is devoted to control algorithms. The formation operates in two regimes: reconfiguration and maintenance. The former is used to deploy the formation of satellites after a cluster launch and to reconfigure the formation's orbital configuration for displaying different graphics. The reconfiguration is performed with the aid of impulsive maneuvers that allows correcting state error within a relatively short time (2-3 orbital periods). In order to minimize the fuel consumption within formation satellites during reconfiguration, the assignment problem is solved for each reconfiguration. It is made with the aid of combinatorial optimization aiming to find a reconfiguration scenario yielding minimum total fuel consumption of all formation satellites while maximizing minimum remaining fuel among the formation satellites. The maintenance regime aims to keep the relative trajectories for all formation satellites within a pre-defined tolerance. The maintenance is performed with the aid of continuous control based on a linear-quadratic regulator. A procedure for linear-quadratic regulator tuning based on a genetic algorithm is proposed. Extensive numerical simulations have been performed for different graphics demonstration missions to demonstrate the performance of the proposed formation flying control algorithms.

Dynamics near the homoclinic set in slow-fast Hamiltonian systems

Sergey Bolotin[1]

[1] *Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia*

In slow-fast systems, fast variables change with the rate of order one, and slow variables with the rate of order $\epsilon \ll 1$. The system obtained for $\epsilon = 0$ is called frozen. If the frozen Hamiltonian system has one DOF, then in the region where the level curves of the frozen Hamiltonian are closed, there is an adiabatic invariant. A.Neishtadt showed that when the fast variable crosses a separatrix of the frozen system, the adiabatic invariant performs quasirandom jumps of order ϵ. We partially extend Neishtadt's result to slow-fast Hamiltonian systems with many DOF. If the frozen system has a hyperbolic equilibrium possessing transverse homoclinics, for small ϵ there are local analogs of adiabatic invariants for trajectories in a neighborhood of the homoclinic set. The slow variables evolve in a quasirandom way, shadowing trajectories of systems whose Hamiltonians are these adiabatic invariants. This extends the work of V.Gelfreich and D.Turaev who considered similar phenomena away from critical points of the frozen Hamiltonian.

Normal forms: a powerful technique to study the Earth's space debris dynamics

A. Celletti[1]

[1] *University of Rome Tor Vergata, Italy*

A real threat for operative satellites and space missions is represented by space debris, which amount to millions of small objects orbiting our planet. The study of their dynamics is of paramount importance, since it allows to understand the location of regular and chaotic regions. These studies might help to develop mitigation, maintenance and control strategies based on mathematical investigations.

In this context, the following perturbative methods have been proven very successful: (i) a multiscale normalization procedure to compute the proper elements, which are quasi-invariants of the dynamics; (ii) perturbation theory to get the long-term behaviour of the debris orbits; (iii) Nekhoroshev's theorem to obtain exponential stability times.

In particular, the proper elements in (i) allow us to identify groups of fragments associated to the same break-up event and to back-trace the fragments to a parent body. The results are corroborated by statistical data analysis.

This talk refers to works in collaboration with: I. De Blasi, C. Efthymiopoulos, G. Pucacco, T. Vartolomei.

ENTROPY OF UNITARY OPERATOR ON $\mathbb{C}^J$

A. Chernyshev[1]

[1]*Lomonosov Moscow State University, Moscow, Russia*

The paper considers the problem of constructing a quantum analogue of the metric entropy of a unitary operator in a Hilbert space. In this paper, an explicit formula for the entropy of the unitary operator is obtained. It turns out that this entropy is determined through the corresponding doubly stochastic operator.

Estimation of instability times in Hamiltonian system: a Shannon entropy approach

P. Cincotta[1]

[1] *University of La Plata, Argentina*

In this talk it is shown that the Shannon entropy is an efficient dynamical indicator that provides a direct measure of the diffusion rate and thus a time-scale for the instabilities arising when dealing with chaotic motion. After a review of the theory behind this approach, two particular applications are presented; a 4D symplectic map that models the dynamics around a double resonance and an exoplanetary system approximated by the Three Body Problem. Successful results are obtained for instability time-scales when compared with direct long range integrations (N-body or just iterations).

Classification of three-dimensional linear Nijenhuis operators with functionally independent invariants

S. Degtyareva[1,2]

[1] *Lomonosov Moscow State University, Moscow, Russia*
[2] *Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia*

Let M be a smooth n-manifold and L be a smooth $(1,1)$-tensor field on M. L is called a *Nijenhuis operator field* (*Nijenhuis operator*) if for any two vector fields u, v we have:

$$N_L(u, v) \equiv 0,$$

where

$$N_L(u, v) = L^2[u, v] + [Lu, Lv] - L[Lu, v] - L[u, Lv]$$

is the *Nijenhuis tensor* (*torsion*).

From the paper "Nijenhuis Geometry" (A. Bolsinov, V. Matveev, A. Konyaev) we know the canonical form for all Nijenhuis operators.

Theorem 1 [B-M-K]. *Let $P(t) = det(tE - L) = t^n + \sigma_1(x)t^{n-1} + \ldots + \sigma_n(x)$ be the characteristic polynomial for the Nijenhuis operator L. Then in a neighborhood of a differentally non-degenerate point (where all $\sigma_i(x)$ are functionally independent) the Nijenhuis operator has the following form:*

$$L = J^{-1} \begin{bmatrix} -\sigma_1 & 1 & \ldots & \ldots & 0 \\ -\sigma_2 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \\ -\sigma_{n-1} & 0 & \ldots & \ldots & 1 \\ -\sigma_n & 0 & \ldots & \ldots & 0 \end{bmatrix} J,$$

where $J = (\frac{\partial \sigma_i(x)}{\partial x_j})$.

The report is related to the study of Nijenhuis operators L, i.e., operator fields with zero Nijenhuis tensor N_L. More precisely, the paper deals with "linear" Nijenhuis operators, i.e., those that are given on a linear space and whose components depend linearly on the coordinates. It is known that there is a natural one-to-one correspondence between linear Nijenhuis operators and left-symmetric algebras, i.e., the questions about the classification of these objects are equivalent.

The report describes the problem of classifying linear Nijenhuis operators in the three-dimensional case (or, what is the same, three-dimensional left-symmetric algebras) with an additional condition that all $\sigma_i(x)$ (the basic symmetric polynomials of the eigenvalues of the operator) are functionally independent. The latter condition can also be interpreted as the functional independence of the eigenvalues of the operator fields under consideration, although these functions (eigenvalues) are not uniquely defined.

The main result of the report is the following: obtaining a list of all possible linear Nijenhuis operators satisfying the additional condition from above, and proving their pairwise non-equivalence.

The question of linearization of Nijenhuis operators is very important, for example, in the work of A. Konyaev "Nijenhuis geometry II: Left-symmetric algebras and linearization problem for Nijenhuis operators" there is a classification of left-symmetric algebras in some cases. There are also results by D. Akpan and A. Oshemkov on the linearization of Nijenhuis operators in the two-dimensional case when the determinant $detL$, restricted to a regular level set of the trace trL, has a non-degenerate singularity.

ON THE LINEARIZABILITY PROBLEM IN A HAMILTONIAN SYSTEM WITH ONE DEGREE OF FREEDOM

N. Elfimov[1]

[1]*Lomonosov Moscow State University, Moscow, Russia*

The Hamiltonians represented as the product of two real-analytic functions are considered. The conditions under which the corresponding phase flow of the system will be diffeomorphic to the phase flow of a system with a quadratic Hamiltonian and the conditions under which this system is isochronous are found.

Integrable solutions for binary star systems with planet

Fionov Andrey[1]

[1] *OCRV (Russian Railway Company subsidiary), Sirius, Russia*

General task is the following: define positions and velocities in time for a system of binary stars and a planet. In view of binary-planetary systems kinematics there are several main cases.

- P-type binary-planetary systems when a planet is orbiting two stars
- S-type binary-planetary systems when a planet is orbiting one of the star
- Planet is placed in the Lagrange point of a binary system

In this report conditions for all of these cases are considered: initial star positions, masses, velocities. It is important to notice that masses of stars are much more than a mass of a planet. Analytically, integrable solutions exist for two body problems and in some cases for three body problem, for example, when a body is located in one of the Lagrange points.

In case of S-type binary-planetary systems to define kinematics of stars and a planet in some conditions one may divide solution in 2 levels:

1. Rotation of stars around barycenter of a binary star system, neglecting the mass of a planet.
2. Rotation of a planet and a star around its barycenter inside the binary star system.

After definition radius vectors of stars in the system coordinates of the center of star masses one may define velocity components in time range. This is a first level solution. The same type solution will be used for a second level: Star - Planet.

Analytical solution for radius, velocities considered in this report with different levels as well as conditions are considered. Having velocities, we obtain radial and tangent velocity in system of coordinates of a star system related to observer.

Two and three body problem for modeling stars movements with planet might be important for exoplanets discovery research. Indeed, the most efficient methods to find exoplanets candidates are: Transit Photometry, Radial Velocity with Doppler, Pulsar Timing. All of the methods above are related to movements analysis of a star due to planet move around a common barycenter. Thus, modelling of the process is important. Peculiarities of different inclinations of a planet plane in respect to a star is important and the solutions will be reported aswell. For different masses of a planet will be shown star dynamics for Sirius binary star and Alpha Centauri triple star.

Attitude control of gyrostat without energy supply

A.P.Ivanov[1]

[1]*Moscow Institute of Physics and Technology, Russia*

In the classical Euler case of the rigid body dynamics, each pair of physically permissible values of the moment and energy integrals corresponds to two symmetric polodia on the ellipsoid of inertia.

The transition from one such trajectory to its twin does not violate conservation laws, but is not feasible in this configuration. To solve this problem, it is proposed to temporarily expand the dimension of the system by adding to the main body a certain rotor having a common axis with the body. The rotor can be in two states: 1) fixed, i.e. its the relative angular velocity $\mathbf{\Omega}$ is zero and 2) free, i.e. the rotor rotates around the axis by inertia. Some kind of "latch" is used to change these states. The situation when the rotor is a body of rotation relative to this axis is known as the integrable Volterra-Zhukovsky case.

The proposed control algorithm is reduced to two switches: 1)$\rightarrow$ 2) and back (at a moment when $\mathbf{\Omega} = \mathbf{0}$), where the control variables are the switching moments. It is shown that the transition between trajectories I and II is always possible for sufficiently large values of the moment of inertia of the rotor.

More control possibilities are provided by the use of an asymmetric rotor, which from a formal point of view leads to a non-integrable system with chaotic dynamics. The main difference from the integrable case is that after stopping the "rotor" we get a body with a new mass distribution. In addition, there is an unlimited choice of stopping points where $\mathbf{\Omega} = \mathbf{0}$, leading to different phase portraits of the classical problem. Numerical examples are given.

Some mathematical aspects of the theory of dynamic tides

P. Ivanov[1]

[1] *P.N. Lebedev Physical Institute, Moscow, Russia*

In my talk I am going to review certain development in theoretical techniques used to describe tidal interactions of stars and giant planets. The whole approach is based on the decomposition of stellar/planetary perturbations induced by tides in series over normal modes corresponding to solutions of the problem of free stellar/planetary pulsations. I'll show how to bring this problem to a standard self-adjoint form, for a uniformly rotating gaseous self-gravitating object and discuss a non-trivilal application of the self-adjoint approach to description of inertial waves in fully convective planets and stars. Later, the formalism will be used for calculation of energy and angular momentum transferred to a fully convective rotating object moving on a highly elliptical (technically, parabolic) orbit after a periastron flyby. In the end, I am going to show that when many periastron passages are considered a chaotic evolution of the object semi-major axis is possible under certain conditions.

A flux-based statistical theory for the three-body problem

B. Kol[1]

[1]*The Hebrew University of Jerusalem, Israel*

The generic, non-hierarhical, three-body system is known to be chaotic. In fact, it is so chaotic that one expects that a statistical solution is the optimal solution. Yet, despite considerable progress, all extant statistical approaches display two flaws. First, probability was equated with phase space volume, thereby ignoring the fact that significant regions of phase space describe regular motion, including post-decay motion. Secondly and relatedly, an adjustable parameter, the strong interaction region, which is a sort of cutoff, was a central ingredient of the theory. The talk will describe a theory that is based on phase-space flux, rather than phase-space volume, which remedies these flaws. Statistical predictions for the identity of the escaper, and other measurable quantities, will be shown to agree with computerized simulations considerably better than previous theories.

Moreover, the flux-based theory enables to predict the distribution of decay times. This prediction relies on the definition and determination of a regularized phase-volume for the system, and the latter led us to a decomposed formulation of the problem. Basically, this decomposition separates the motion of the instantanoues plane defined by the three bodies, from the motion of the bodies within the plane.

On the exo-planet precession under torqes due to three celestial bodies with the evolution of the satellite's orbit

P.S. Krasilnikov[1]

[1]*Moscow Aviation Institute (National Research University), Moscow, Russian Federation*

We investigate the non-resonant evolution of the axial tilt of hypothetical exo-Earth in the gravitational field of a star, planet's satellite (exo-Moon) and outer planet (exo-Jupiter). The exo-Earth is assumed to be rigid, axially symmetric ($A = B$) and almost spherical. We assume the orbits of the both exo-planets to be Keplerian ellipses with focus in the star, the orbit of exo-Moon to be evolving Keplerian ellipse with slowly changing of ascending node longitude Ω and periapsis argument ϖ

Assuming the frequencies of the unperturbed orbital elliptical motion to be order one, we obtain the canonical averaged equations decribing the perturbed oscillations of the exo-Moon spin axis. These equations contain parameters changing slowly over time. Using the smallness of the planets masses relative to the mass of the star, we have obtained simplified equations of oscillations of exo-Earth spin axis by the small parameter method. Time integration of simplified equations get the axial tilt of exo-Moon as the function of time:

$$\delta_1(t) = \delta_{11} + \frac{\sin\delta_{11}}{C\omega_r}\left[\frac{\Lambda\Xi}{4(\omega_0-\varepsilon_1)}\cos 2\alpha(1-\cos 2i_2) + \left(\frac{D_1^{(3)}-D_2^{(3)}}{2K}\cos 2\varphi_3 + \frac{D_4^{(3)}}{K}\sin 2\varphi_3\right)\right] +$$

$$+\frac{\cos\delta_{11}}{C\omega_r}\left[\frac{\Lambda\Xi}{(\omega_0-\varepsilon_1)}\cos\alpha\sin 2i_2 - 2D_5^{(3)}\frac{\sin\varphi_3}{K} + 2D_6^{(3)}\frac{\cos\varphi_3}{K}\right]$$

Here

$$\varphi_3 = \omega_0 t + \varphi_{30}, \quad \alpha = (\omega_0-\varepsilon_1)t - \Omega_{20} + \varphi_{30}, \quad \Xi = \frac{\mu_2}{2a_2^3(1-e_2^2)^{3/2}}$$

δ_{11} is the unperturbed value of the axial tilt, ω_0 is the precession frequency of the exo-Earth spin axis, ε_1 is the ascending node longitude frequency of the exo-Moon orbit. The first terms in square brackets are due to the exo-Moon, the rest of the terms are due to the exo-Jupiter. Hence it follows that the torques from the exo-Jupiter create a secular, long-period oscillation mode in axial tilt with a frequency equals to frequency of unperturbed spin axis precession of the exo-Earth. The impact of a Moon on the evolution of a exo-Earth spin axis is that short-period harmonics ($\omega_0 \ll \varepsilon_1$) appear in the oscillations of the axial tilt. The frequency of such oscillations close coincides with the evolution frequency of the ascending node longitude of the exo-Moon orbit.

We have calculated the evolution of exo-Earth axial tilt for two exo-planetary systems, i.e. for a system similar to the solar system, and for a planetary exo-system 7 Canis Majoris. The effect of destabilization (stabilization) of the exo-Earth tilt oscillations due to torques exerted by exo-Moon and exo-Jupiter is described.

This work was performed at the Moscow Aviation Institute and supported by the Russian Science Foundation, project no. 22-21-00560

Superintegrable Bertrand mechanical systems

E. Kudryavtseva[1]

[1] *Moscow State University, Moscow, Russia*

The problem of description of superintegrable systems (i.e., systems with closed trajectories in a certain domain) in the class of spherically symmetric natural mechanical systems goes back to Bertrand and Darboux. Dynamical and geometric properties of such systems have been studied by many mathematicians (Killing, Besse, Perlick, Kozlov, Borisov, Mamaev, Santoprete, Ballesteros, Enciso, Herranz, Ragnisco). However in full generality, the problem of description of all such systems remained open because of the so-called "problem of equators" (by an *equator*, we mean a parallel which is a geodesic).

Bertrand proved (1873) that, in Newtonian mechanics, the Kepler potential $V_K(r) = -a/r + b$ and the harmonic oscillator potential $V_H(r) = ar^2 + b$ on the Euclidean plane ($a, b \in \mathbf{R}$, $a > 0$) are distinguished by the property that: (i) all the bounded trajectories are closed and (ii) there exist non-circular closed orbits. Natural mechanical systems possessing the above property will be called *Bertrand systems*. Darboux (1877) and Perlick (1992) extended the result of Bertrand by obtaining a complete description of all spherically symmetric Bertrand systems, whose underlying Riemannian manifolds of revolution have no equators [**?**]. Other well-known examples of Bertrand systems (which have equators and, therefore, are not in the Darboux–Perlick list) are

- the systems with the Kepler potential on the round spheres (or their "rational coverings"),
- the systems with the harmonic oscillator potential on pear-shaped surfaces whose equators split them into two subsurfaces from the Darboux–Perlick list, and
- the systems with zero potential on (pear-shaped or spherical) Tannery surfaces classified by Besse (1978).

We describe all spherically-symmetric Bertrand systems [**?**, **?**], including those with equators. We prove, in particular, that the restriction of any Bertrand system to the union of its non-circular closed orbits is essentially the same as one of the systems mentioned above.

We also describe all rotationally-symmetric superintegrable (in a domain of slow motions) magnetic geodesic flows [**?**]. We show that all sufficiently slow motions in a central magnetic field on a two-dimensional manifold of revolution are periodic if and only if the metric has a constant scalar curvature and the magnetic field is homogeneous, i.e., proportional to the area form.

This work was supported by the Russian Science Foundation (project No. 22-11-00164).

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The Problem of Motion of a Rigid Body with a Fixed Point in a Flow of Particles

A.S. Kuleshov[1] and M.M. Gadzhiev[1]

[1]*Lomonosov Moscow State University, Moscow, Russia*

The problem of motion of a rigid body with a fixed point in a free molecular flow of particles is considered. We assume that the flow consists of identical non-interacting particles, moving with constant velocity in a fixed direction in the absolute space. We assume also that the collision of particles with a rigid body is perfectly inelastic. Under these assumptions we derive the equations of motion of the body. We show that the obtained equations generalize the classical Euler – Poisson equations of motion of a heavy rigid body with a fixed point and they are represented in the form of the classical Euler – Poisson equations in the case, when the surface of the body in a flow of particles is a sphere. Problems of the existence of first integrals in the considered system are discussed. The integrable cases for the obtained equations of motion of the body (generalized Euler case, generalized Lagrange case and generalized Hess case) are investigated.

Modeling of degenerate singularities of integrable Hamiltonian systems by billiard books

A.A. Kuznetsova[1]

[1]*Lomonosov Moscow State University, Moscow, Russia*

In one of his works, A.T. Fomenko formulated a conjecture about modeling all non-degenerate integrable systems with two degrees of freedom by suitable billiards as "configuration manifolds". V.V. Vedyushkina and I.S. Kharcheva proved that an arbitrary non-degenerate singularity of the Liouville foliation on a regular isoenergy surface Q^3, also called a *Bott 3-atom*, and any base of the Liouville foliation on Q^3 are realized by an algorithmically specified billiard book.

It turns out that the Fomenko conjecture is also valid for some Hamiltonian systems whose additional first integral is not Bott on the energy level Q^3, i.e., it has degenerate singularities of rank 1. In the work of I.M. Nikonov, formulas were found expressing the number of degenerate atoms with one singular point. It turned out that there are exactly five such atoms of multiplicity 3, three of which are orientable, and two atoms are non-orientable.

This talk shows how to model, by using billiard books, bifurcations of Liouville foliations, whose 2D bases contain arcs corresponding to orientable non-Morse multi saddles.

The work is supported by the Russian Science Foundation (project no. 21-11-00355).

Dynamical constraints on extrasolar systems

Anne-Sophie Libert[1]

[1] *University of Namur, Department of Mathematics, Namur, Belgium*

The diversity of the hundreds of discovered extrasolar systems puzzles our understanding of the formation and long-term evolution of planetary systems. The detected planetary systems generally suffer from large observational uncertainties. In this talk, I will discuss recent results showing how dynamical studies can be useful to constrain the orbital parameters of tightly packed planetary systems which harbor two-body resonances and/or chains of resonances involving three or more planets. More precisely, I will show how i) periodic orbits can serve as dynamical clues to validate the parametrization of detected systems, ii) TTVs keep track of the migration history of planetary systems, and iii) TTVs provide signatures of three-body resonances accessible by future monitoring of the systems. Applications to K2-21, K2-24, Kepler-9, Kepler-108, and TRAPPIST-1 will be discussed. This talk aims at illustrating how the interplay between formation, dynamics, and stability can contribute to bridge the gap between observations and theoretical studies.

Complete sets in bi-involution construction for singular points of Lie algebras

F.I. Lobzin[1,2]

[1]*Lomonosov Moscow State University, Moscow, Russia*
[2]*Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia*

Let g be a Lie algebra, and let g^* be the dual space. Consider the following Poisson structures on g^*:

$$\mathcal{A}_x(x) = (c_{ij}^k x_k), \quad \mathcal{A}_a(x) = (c_{ij}^k a_k), \quad a, x \in g^*.$$

A Lie algebra is called *completely integrable* if over this Lie algebra there is a complete set of functions that are in involution. The greatest practical interest is sets of polynomials.

- Mishchenko-Fomenko's conjecture (**proved**): on the dual space g^* of any Lie algebra g, there is a complete set of polynomials that are in involution.
- Generalized Mishchenko-Fomenko's conjecture: on the dual space g^* of any Lie algebra g, there is a complete set of polynomials in bi-involution, i.e., a set that is simultaneously in involution with respect to $\mathcal{A}_x$ and $\mathcal{A}_a$.

The first conjecture was proved by Sadetov in 2004 [1], but the sets obtained by him did not always turn out to be in involution with respect to the bracket $\mathcal{A}_a$ with a frozen argument. It is worth noting that when applying the general method for constructing sets in bi-involution (Mischenko-Fomenko's argument shift method), firstly, the resulting sets are not complete for some Lie algebras (for example, for some non-semisimple Lie algebras), secondly, functions of these sets are not functionally independent for some values of the parameter a (for example, for some singular a).

In my report I will talk about method for constructing sets in bi-involution and sufficient conditions for completeness of such sets, obtained by this method:

Theorem 1 (Construction of complete sets with respect to singular elements a)**.**
Let an element $a \in g^*$ be such that

1) *there is $x \in g^*$ such that the plane spanned by x and a, without the line λa consists only of regular elements,*
2) *$ind(Ann(a)) = ind\, g$.*

Then the set of polynomials, generated by shifting by the element a, can be complemented to a complete set in bi-involution by a set of polynomials in involution, such that their differentials are in $Ann(a)$.

We recall the notation $Ann(a) = \{\xi \in g \mid \forall \eta \in g\ \langle a, [\xi, \eta]\rangle = 0\}$, the annihilator of the element $a \in g^*$, i.e., the stationary subalgebra of a in the sense of the coadjoint representation ($Ann(a) = \{\xi \in g \mid \ ad_\xi^* a = 0\}$). If $Ann(a)$ has the minimal dimension over all elements of g^*, then $dim\ Ann(a)$ is denoted by $ind\, g$, and a is called *regular* (otherwise it is called *singular*).

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Non-integrability of the three body problem

A.J. Maciejewski[1], M. Przybylska[1] and T. Combot[2]

[1] *University of Zielona Gra, Zielona Gra, Poland*
[2] *Université de Burgogne, Dijon, France*

In 2001 Alexei Tsygvintsev proved the non-integrability of the classical planar three body problem. His proof is based on the analysis of the monodromy group of the variational equations along the parabolic homographic solution. Later, in 2003, Delphine Boucher and Jacques-Arthur Weil, using the Morales–Ramis obtained the same result. They investigated the differential Galois group of the variational equations along the parabolic homographic solution. In 2011 the first two authors proved non-integrability of a certain generalization of the three body problem analysing the differential Galois group of the variational equations along the homothetic solution.

In this paper we investigate integrability of the problem on fixed common levels of the energy and the angular momentum first integrals. As in the above-mentioned studies we analyse the differential Galois group of the variational equations. However, as a particular solution we take a homographic solution with arbitrary eccentricity and energy. We prove that except the zero eccentricity the differential Galois group is not virtually Abelian and, as a consequence, the problem is not integrable on the respective common levels of first integrals.

Analysis of the Orbital Stability of Periodic Pendulum Oscillations of a Heavy Rigid Body with a Fixed Point under the Goryachev-Chaplygin Condition

B. Maksimov[1]

[1]*Moscow Aviation Institute (National Research University), Moscow, Russia*

The motion of a rigid body with a fixed point in a uniform gravity field is considered. It is assumed that the main moments of inertia of the body are in the ratio 1:1:4 and there are no restrictions imposed on the position of the center of mass of the body.

The problem of the orbital stability of periodic pendulum oscillations of a body is investigated. By introducing the chosen local coordinates in a convenient way [1] and performing an isoenergetic reduction, it was possible to obtain the equations of perturbed motion in a simpler canonical Hamiltonian form for studying. It allowed to reduce the problem of orbital stability to the stability problem of the equilibrium position of a second-order system with periodic coefficients, the right-hand sides of which depend on two parameters (energy integral constant and the angle between the radius vector of the center of mass and the equatorial plane of the ellipsoid of inertia).

Based on the analysis of the linearized system, a diagram of stability of pendulum oscillations has been constructed, on which the regions of orbital instability (parametric resonance) and regions of orbital stability in the linear approximation are indicated.

For parameters' values from regions of orbital stability in linear approximation an additional nonlinear analysis based on theorems of the KAM theory has been performed for non-resonant and resonant cases. To this end, Hamiltonian function has been normalized up to fourth order terms. At small values of oscillations amplitudes, the normalization has been performed analytically. For arbitrary parameters' values it was performed by constructing symplectic mapping generated by phase flow of the Hamiltonian system [2]. By means of an analysis of the coefficients of the normalized Hamiltonian function on the basis of the theorems of the KAM theory, rigorous conclusions on the orbital stability of pendulum oscillations were obtained.

This work was financially supported by the Russian Foundation for Basic Research, project No. 20-01-00637.

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2. A. P. Markeev: Stability of equilibrium states of hamiltonian systems: a method of investigation (Mech. Solids, 2004, 39 (6), pp. 1-8)

On the metric stability and the Nekhoroshev estimate of the velocity of Arnold diffusion in a special case of three-body problem

A.P. Markeev[1,2]

[1]*Ishlinskii Institute for Problems in Mechanics RAS, Russia, Moscow*
[2]*Moscow Aviation Institute (National Research University),Russia, Moscow*

A study is made of the stability of triangular libration points in the nearly-circular restricted three-body problem in the spatial case. The problem of stability for most (in the sense of Lebesgue measure) initial conditions in the planar case has been investigated earlier. In the spatial case, an identical resonance takes place: for all values of the parameters of the problem the period of Keplerian motion of the two main attracting bodies is equal to the period of small linear oscillations of the third body of negligible mass along the axis perpendicular to the plane of the orbit of the main bodies. In this paper it is assumed that there are no resonances of the planar problem through order six. Using classical perturbation theory, KAM theory and algorithms of computer calculations, stability is proved for most initial conditions and the Nekhoroshev estimate of the time of stability is given for trajectories starting in an addition to the above-mentioned set of most initial conditions.

This research was carried out within the framework of the state assignment (registration No. AAAA-A20-120011690138-6) at the Ishlinskii Institute for Problems in Mechanics, RAS, and at the Moscow Aviation Institute (National Research University)

Long-term dynamics of the inner planets in the Solar System: chaos and stability

F. Mogavero[1]

[1]*Observatoire de Paris, France*

I will present the line of research on the long-term motion of the inner planets in the Solar System that I have been following for several years, in collaboration with J. Laskar and N. H. Hoang [1-3].

I will describe the new dynamical model of a forced secular inner Solar System that we introduced last year, and explain how we revealed the origin of chaotic behaviour of the orbits by computer algebra.

I will finally present the main ideas that we are following to address the statistical stability of the orbits of the inner planets over the lifetime of the Solar System.

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Typical rank-1 singularities of integrable systems

M.V. Onufrienko[1,2]

[1]*Moscow State University, Moscow, Russia*
[2]*Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia*

It is known that the classification of typical rank-one orbits of integrable Hamiltonian systems with n degrees of freedom reduces to the description of critical points which appear in typical parametric families of smooth $\mathbf{Z}_s$-invariant functions in two variables with $n-1$ parameters, where the group $G \cong \mathbf{Z}_s$ acts by rotations. In this talk, we solve the latter problem for $n \leq 3$.

Fix a natural number s and consider the cyclic group $G \subset SO(2)$ of order s consisting of rotations $z \mapsto e^{2\pi \ell i/s} z$ of the plane $\mathbf{C} \cong \mathbf{R}^2$, where $z = x + iy \in \mathbf{C}$, $0 \leq \ell < s$.

Consider the Morse functions $F_{s,0} = F_{s,0}^{\pm,\pm}(x,y) = \pm|z|^2 = \pm(x^2+y^2)$ for $s \geq 1$, and $F_{s,0} = F_{s,0}^{+,-}(x,y) = x^2 - y^2$ for $s = 1,2$, and two families of G-invariant functions $F_{s,k}$, $k = 1,2$:

$$F_{s,1} = F_{s,1}(x,y,a,\lambda) = \begin{cases} \pm x^2 \pm y^{s+2} + \lambda y^s, & s = 1,2, \\ Re(z^3) + \lambda |z|^2, & s = 3, \\ Re(z^s) \pm a|z|^4 + \lambda|z|^2, & s \geq 4, \end{cases}$$

$$F_{s,2} = F_{s,2}(x,y,a,\lambda) = \begin{cases} \pm x^2 \pm y^{2s+2} + \lambda_2 y^{2s} + \lambda_1 y^s, & s = 1,2, \\ Re(z^4) \pm (1+\lambda_2)|z|^4 \pm a|z|^6 + \lambda_1|z|^2, & s = 4, \\ Re(z^5) \pm a|z|^6 + \lambda_2|z|^4 + \lambda_1|z|^2, & s = 5, \\ Re(z^s) + a_1|z|^6 \pm a_2|z|^8 + \lambda_2|z|^4 + \lambda_1|z|^2, & s \geq 6. \end{cases}$$

Here $\lambda = (\lambda_i) \in \mathbf{R}^k$ are small parameters, $a = (a_j) \in \mathbf{R}^m$ are *"moduli"*, $m \in \{0,1,2\}$ is the *modality*; $a^2 \neq 1$ if $(s,k) = (4,1)$; $a > 0$ if $s \neq k+3$; $a_1^2 \neq 1$ if $(s,k) = (6,2)$; $a_1 > 0$ otherwise.

One shows that any smooth G-invariant function $f(x,y)$ has a Taylor series at $(0,0)$ of the form

$$\mathrm{Re} \sum_{p,q \geq 0} c_{pq} |z|^{2p} z^{qs} = c_0 + c_1|z|^2 + \mathrm{Re}(c_2 z^s + c_3|z|^2 z^s) + c_4|z|^4 + c_5|z|^6 + c_6|z|^8 + \dots \tag{1}$$

Theorem 1. *Let $f(x,y)$ be a smooth G-invariant function, where $s \geq 3$. Then there exists a smooth G-equivariant change of variables reducing the function $f(x,y)$ to the form $F_{s,k}(x,y,a,0) + const$, $k \leq 2$ (with some value $a \in \mathbf{R}^m$) if and only if the coefficients c_i of the Taylor series* (1) *satisfy the following system of k equations and several inequalities:*
(a) $c_1 \neq 0$ *for* $k = 0$,
(b) $c_1 = 0$ *and* $c_2 \neq 0$, *with* $a = |c_4|/|c_2|^{4/s}$ *for* $k = 1$,
(c) $c_1 = 0$, $|c_2| = |c_4| > 0$ *with* $a = |c_5 - c_4 \mathrm{Re}(\frac{c_3}{c_2})|/|c_2|^{3/2}$ *for* $k = 2$ *and* $s = 4$;
$c_1 = c_4 = 0$ *and* $c_2 \neq 0$ *with* $a = |c_5|/|c_2|^{6/5}$ *for* $k = 2$ *and* $s = 5$;
$c_1 = c_4 = 0$ *and* $c_2 \neq 0$ *with* $a_1 = \frac{|c_5|}{|c_2|^{6/s}}$, $a_2 = (c_6 - c_5 \mathrm{Re}(\frac{6c_3}{sc_2}))/|c_2|^{8/s}$ *for* $k = 2$ *and* $s \geq 6$.

Thus, by Theorem 1, in "typical" parametric families of smooth G-invariant functions with at most two parameters, only singularities of the form $F_{s,k}(x,y,a,0) + const$ appear, $k \leq 2$.

The corresponding rank-one orbit of an integrable Hamiltonian system with $n \leq 3$ degrees of freedom has a neighbourhood in which the system is fibrewise diffeomorphic to the *standard model*

$$M_{\frac{\ell}{s}} = (V/G) \times W, \quad \mathcal{F}_{\frac{\ell}{s},k,a} : M_{\frac{\ell}{s}} \to \mathbf{R}^n, \quad \mathcal{F}_{\frac{\ell}{s},k,a}(x,y,\varphi_1,\lambda,\varphi') = (\lambda, F_{s,k}(x,y,a(\lambda),\lambda')),$$

where $0 \leq \ell < s$, $(\ell,s) = 1$, $0 \leq k < n$, $\lambda' = (\lambda_1,\dots,\lambda_k)$, $a(\lambda) = (a_j(\lambda))_{j=1}^m$, $a_j(\lambda)$ are smooth functions, $a_1(\lambda) \equiv 1$ for $(s,k) \notin \{(4,1),(6,2)\}$; $a_2(\lambda) \equiv 1$ for $(s,k) = (6,2)$. Here we consider the action of the group G on the solid torus $V = D^2 \times S^1$ of the form $(z,\varphi_1) \mapsto (e^{2\pi i \frac{\ell}{s}} z, \varphi_1 + \frac{2\pi}{s})$. Furthermore $W = D^{n-1} \times (S^1)^{n-2}$ is a cylinder with coordinates $\lambda = (\lambda_i)_{i=1}^{n-1}$, $\varphi' = (\varphi_j)_{j=2}^{n-1}$.

Bifurcation of two Liouville tori in one problem of vortex dynamics

G. Palshin[1]

[1]*Moscow Institute of Physics and Technology, Dolgoprudny, Russia*

This report is a continuation of the study of *the constrained problem of three magnetic vortices* [**?**, **?**], where one of the vortices is fixed at the point of origin. The model is described in Hamiltonian form:

$$H = \frac{\Gamma_1}{\lambda_1} \ln|r_1| + \frac{\Gamma_2}{\lambda_2} \ln|r_2| + \frac{\Gamma_1}{\lambda_1}\frac{\Gamma_2}{\lambda_2} \ln|r_1 - r_2|, \quad \Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \quad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \quad \alpha = 1, 2,$$

where $r_\alpha = (x_\alpha, y_\alpha)$ is a vortex coordinate, Γ_α is a constant vortex intensity and λ_α is a vortex polarity, which takes values ± 1 depending on a direction of magnetization. Moreover, the system admits additional first integral $F = \Gamma_1 r_1^2 + \Gamma_2 r_2^2$, so it is *completely Liouville integrable*.

In previous works a bifurcation diagram of the integral map $\mathcal{F}(\mathbf{r}) = (F(\mathbf{r}), H(\mathbf{r}))$ was found and a reduction to a Hamiltonian system with one degree of freedom was performed [**?**]. Numerical research has shown that for $\Gamma_1 \approx -13.552767614968$, $\Gamma_2 = -1$, $\lambda_1 = 1$, and $\lambda_2 = -1$, the bifurcation diagram contains *two-into-two* Liouville tori bifurcation, that corresponds to the *rare atom-bifurcation* D_2 [**?**]. Previously, this atom was numerically found by Moskvin in the Dullin-Matveev integrable case of rigid body dynamics [**?**].

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The three-body problem and its implications: from secular to chaotic evolution

H. Perets[1]

[1] *Technion - Israel Institute of Technology, Israel*

The three-body problem, dealing with the dynamics and evolution of three gravitating bodies is one of the oldest, generally still open problems in modern science, from the days of Newton to these days. Its study frustrated Newton, explored by Euler and Lagrange, and lead Poincare to introduce the field of chaos. Indeed, it is an active and extensively studied field to this day both in physics and mathematics. The evolution of three-body systems plays a key role in the evolution of stars and planets in the universe and serves as a major player in the production of various cosmic explosions and the production of gravitational wave sources. It thereby, indirectly, also affects the production of (mostly) heavy elements in the universe, produced in such explosions. I will briefly introduce the history of this problem and its importance, and then explore the dynamics of three-body systems in different regimes. I will first describe the evolution of secular and quasi-secular hierarchical triple systems, which have some similarities to coupled oscillators, and pinpoint various important implications for such evolution studied in my group from the scales of Solar system asteroids and moons, through mergers of compact objects such as white dwarfs and neutron stars, to the production of gravitational-wave sources near massive black holes. In the second part of the talk, I will focus on the solution to the chaotic, non-hierarchical regime of the three-body problem considered a major challenge and the holy grail in this field. It could effectively be analyzed only through brute-force few-body numerical simulations available with sufficient computational power only over the last few decades. Due to the chaotic nature of the problem, predicting the evolution of a single chaotic three-body system is effectively impossible, however, as I'll show, a solution to the distribution of the final outcomes, i.e. the cross-sections and branching ratios for the various outcomes can be tackled using detailed-balance approach and coupled with the use of a random-walk approach. Our novel solution provides, for the first time, a statistical-analytical description of the chaotic three-body problem throughout its dynamical evolution and can directly reproduce the results of the decades of numerical studies, without even requiring any fitting parameter. Furthermore, our approach allows a robust method to include additional dissipative processes to the problem and/or additional external potentials and perturbations, aspects that are critical for realistic astrophysical systems.

Model of formation of non-spherical satellites of the planet

A. Pestrikov[1]

[1]*Lomonosov Moscow State University, Moscow, Russia*

- Considered the formation problem of satellite systems of the planets of Solar system, and using such a model to describe the possible areas of accumulation of space debris in near-Earth space.
- Described the possible capture of irregular satellites and the possible capture of cosmic masses in the framework of the plane hyperbolic three-body problem, as well as the plane parabolic three-body problem.
- Constructed a numerical example of such capture, and calculated it's probability for some types of orbits

Dynamics of a Chaplygin Sphere on a Vibrating Plane

E. Pivovarova[1] and A. Kilin[1]

[1]*Ural Mathematical Center, Udmurt State University, Izhevsk, Russia*

This work addresses the problem of a dynamically asymmetric balanced sphere rolling on a plane performing horizontal periodic oscillations. We show that in the system under consideration the projections of the angular momentum onto the axes of the fixed coordinate system remain unchanged. The investigation of the reduced system on a fixed level set of first integrals reduces to analyzing a three-dimensional period advance map on $SO(3)$. The analysis of this map suggests that in the general case the problem considered is nonintegrable. We find partial solutions to the system which are a generalization of permanent rotations and correspond to nonuniform rotations about a body- and space-fixed axis. We also find a particular integrable case which, after time is rescaled, reduces to the classical Chaplygin sphere rolling problem on the zero level set of the area integral.

Also we consider the controlled motion of this system. The motion of the sphere is controlled by the controlled rotation of the internal gyrostats. The paper addresses two control problems concerning the construction of controls which generate motion along a trajectory given either on a moving plane or in a fixed frame of reference. It is shown that, using a control torque constant in the fixed frame of reference, the general problem can be reduced to the problem of control on the zero level set of the angular momentum integral. It is proved that, on the zero level set of the angular momentum integral, the system under consideration is completely controllable according to the Rashevsky – Chow theorem. Control algorithms for the motion of the sphere along an arbitrary prescribed trajectory are constructed. Examples are given of controls for the sphere rolling in a straight line in an arbitrary direction and in a circle, and for the sphere turning so that the position of the center of mass, both relative to the moving plane and relative to the fixed frame of reference, remains unchanged.

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On the covering of a Hill's region by solutions in systems with gyroscopic forces

I. Polekhin[1]

[1] *Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia*

Consider a Lagrangian system with the Lagrangian containing terms linear in velocity. By analogy with the systems in celestial mechanics, we call a bounded connected component of the possible motion area of such a system a Hills region. Suppose that the energy level is fixed and the corresponding Hills region is compact. We present sufficient conditions under which any point in the Hills region can be connected with its boundary by a solution with the given energy. We consider two classical celestial-mechanical systems: the planar restricted circular three-body problem and its simplification, the Hills problem. Numerical and analytical analyses of the covering of a Hills region by solutions starting with zero velocity at its boundary are presented. We show that, in all considered cases, there always exists an area inside a Hills region that is uncovered by the solutions.

Dynamics of two vortices in a Bose-Einstein condensate

G. Pruss[1]

[1] *Moscow Institute of Physics and Technology (National Research University), Moscow, Russia*

We study completely Liouville integrable Hamiltonian system with two degrees of freedom describing the dynamics of two vortex filaments in a Bose-Einstein condensate which is enclosed in a harmonic trap with axial symmetry. The vortex filaments are parallel to the axis of symmetry. As a result, the problem becomes two-dimensional. Motivation of our interest for the system is the analysis of phase topology. The main role in such a study is played by the bifurcation diagram of the momentum map. Our presentation includes results for the both cases the vortices of the same signs and opposite one. For each cases, the bifurcation diagram is constructed. The types of special periodic motions and their stability are determined for various ratios of intensities.

Dynamics of dipole in a stationary non-homogeneous electromagnetic field

M. Przybylska[1], A.J. Maciejewski[1]

[1] *University of Zielona Gra, Zielona Gra, Poland*

The non-relativistic equations of motion for a dipole in a stationary non-homogeneous electromagnetic field are derived and analysed. It is shown that they are Hamiltonian with respect to a certain degenerated Poisson structure. The dynamics of the system is complex because the motion of the centre of mass of the dipole is coupled with its rotational motion. The problem of the existence of linear in momenta first integrals is discussed. The presence of such first integral appears to be related with a linear symmetry of electric and magnetic fields. These integrals are used for the separation of rotational motion from the motion of the centre of mass. Also results of search of quadratic in momenta first integrals for uniform and stationary electromagnetic fields are reported. Deriving equations of motion of a dipole in arbitrary stationary electromagnetic fields and analysis of described by them dynamics is important for the construction of electromagnetic traps for polar particles. Certain novel design for a trap of this type for confinement of neutral particles having a permanent electric dipole moment is presented.

Existence of extremals of the action functional in celestial mechanics

T. Salnikova[1], E. Kugushev[1]

[1] *Lomonosov Moscow State University, Moscow, Russia*

Consider the problem of the existence of geodesics in the Jacobi metric when the configuration space is non-compact. Let us consider N material points under conservative forces. We study the question of the "mobility" in this N body problem i.e., of the existence of a motion in which the points of the system will move from a given initial position to any given final position, with an appropriate choice of the initial velocity. Using the principle of least action in the form of Hamilton for the case when the potential energy of the system is continuously differentiable and bounded from above, and using the principle of least action in the Jacobi form for the case of a gravitational potential, or Lennard-Jones type potential, the mobility property is proved.

The question of mobility in the three-body problem, can be applied to the problems of the evolution of space debris clouds. Property of particle mobility makes it possible for cloud particles to migrate from the area of influence of one large body (Earth) to the area of gravitational influence of another large body flying past (an asteroid). And vice versa. The mobility property also can be used in the conjecture of formation of the irregular (non-spherical) satellites of the Solar system planets (like Phobos and Deimos). Let a double asteroid, consisting of a large body and rotating in its vicinity of a small body, fly with parabolic (hyperbolic) velocity near the planet. During the approach, it is possible to intercept a small body, which will remain in the field of the gravitational influence of the planet, and a large body will move away indefinitely.

The possible capture of irregular satellites and the possible capture of cosmic masses in the framework of the plane hyperbolic three-body problem, as well as the plane parabolic three-body problem is justified through numerical simulation.

Analysis of the Orbital Stability of Plane Periodic Motions of a Heavy Rigid Body with a Fixed Point in the Hess Case

A.A. Savin[1]

[1]*Moscow Aviation Institute (National Research University) Moscow, Russia*

We consider a motion of a rigid body with a fixed point in the uniform gravity field. The geometry of the masses of the body corresponds to the Hess case, in which the center of mass of the body places in the plane of the largest and smallest principal axes of inertia. In this case, plane pendulum motions around the middle principal axis keeping unchanged horizontal position in the space are possible.

In this work, the problem on the orbital stability of pendulum periodic motions (oscillations or rotations) is investigated. The analysis of orbital stability is performed with respect to perturbations for which the projection of the angular momentum to the vertical takes the value zero. There are three parameters in the problem: two parameters describing the geometry of masses, and a parameter specifying the family of unperturbed motions. In the neighborhood of the unperturbed periodic orbit, local variables that are action-angle variables on unperturbed motion were introduced, and equations of perturbed motion were obtained. An isoenergetic reduction at the energy level corresponding to the unperturbed motion was performed. It allows to reduce the study of the orbital stability of the unperturbed motion of the original autonomous system with two degrees of freedom to the study of the stability of the equilibrium position of a nonautonomous periodic system with one degree of freedom. An explicit form of the monodromy matrix of the linearized reduced system was obtained.

In the case of rotations, this matrix has a lower triangular shape with unequal reciprocal elements on the main diagonal. This allowed us to show the orbital instability of the pendulum rotations. In the case of oscillations, it was possible to show that the so-called identical resonance takes place, when for all values of the parameters the linear system has a multiple multiplier equal to unity. It is shown that, for most values of parameters, the monodromy matrix has a lower-triangular shape. It leads to the orbital instability in the linear approximation. Nonlinear stability analysis based on the method developed in [1] has been also performed. It was shown that the transcendental case takes place and pendulum osculations are also orbitally unstable in the original nonlinear system.

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Asymptotics for Gaussian Beams of the Schrödinger equation with a delta potential

O. Shchegortsova[1]

[1] *Lomonosov Moscow State University, Moscow, Russia*

We consider the Cauchy problem for the Schrödinger equation with a delta potential localized on a surface M of codimension 1. The initial condition represents a rapidly oscillating wave packet localized in a neighborhood of a point x_0. Such solutions are called narrow Gaussian beams and constructed with the help of Maslov complex germ theory [1,2].

In this problem Schrödinger operator with a delta potential is defined as a self-adjoint extension of the Schrödinger operator with a smooth potential. The domain of a such an operator consists of functions satisfying the following boundary conditions on M:

$$\begin{cases} \psi(r(y)-0,t) = \psi(r(y)+0,t), \\ h(\frac{\partial\psi}{\partial m}(r(y)-0,t) - \frac{\partial\psi}{\partial m}(r(y)+0,t)) = q(y)\psi(r(y),t). \end{cases}$$

Here $x = r(y) \in M$, $\psi(r(y) \pm 0)$ are the limits of the function ψ on the positive and negative sides of the surface M, and m is an orienting unit normal to M.

We provide the solution in terms of the Maslov complex germ theory. The effect of the delta potential is evident in the fact that the corresponding Lagrangian manifold turns out to be a pair of manifolds with complex germs that describes the reflection and scattering of the beam. The form of the delta potential defines a mapping between these manifolds.

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Resonant exoplanet dynamics and planetary chaotic zones

I.I. Shevchenko[1,2]

[1] *Saint Petersburg State University, 7/9 Universitetskaya nab., 199034 Saint Petersburg, Russia*
[2] *Institute of Applied Astronomy, Russian Academy of Sciences, 191187 Saint Petersburg, Russia*

The report considers the conditions for emergence and observational manifestations of the planetary chaotic zone (the Wisdom gap), as well as other chaotic and resonant structures in the planetesimal disks of exoplanet systems. Methods for analytical description of the chaotic planetary zones, estimation of their radial sizes and the basic timescales of chaotic dynamics in these zones, namely, the Lyapunov times and chaotic diffusion times, are discussed. The dependences of these quantities on the mass parameter of the star–planet system are considered and compared with results of numerical experiments. A relatively new concept of the planetary "broad chaotic zone" phenomenon and the conditions for its manifestation are discussed.

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Adiabatic approximation in dynamical studies of exoplanetary systems in mean-motion resonance

V.V. Sidorenko[1]

[1]*Keldysh Institute of Applied Mathematics RAS, Moscow, Russia*

The available observational information indicates that in exoplanetary systems mean motion resonances (MMR) often occur. An effective approach to the study of secular effects caused by MMR was proposed by J. Wisdom (1985), but so far it has been used only in the framework of the restricted three-body problem (R3BP). We demonstrate how Wisdom's approach should be modified so that it can be applied to the general three-body problem. As an example, we consider the dynamics of a system consisting of a star and two planets in co-orbital motion, interpreting this situation as a 1:1 MMR.

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Singularities of a Lagrange Top with a Vibrating Suspension Point

S.V. Sokolov[1] and P.E. Ryabov[2]

[1]*Moscow Institute of Physics and Technology, Moscow, Russia*
[2]*Financial University, Moscow, Russia*

We consider a completely integrable Hamiltonian system that describes the dynamics of a Lagrange top with a vibrating suspension point. Initially system has a two and a half degrees of freedom and after averaging of fast oscillations of suspension point can be describe in the terms of vibration potential as an autonomous Hamiltonian system with two degrees of freedom. The results of a stability analysis of equilibrium positions are clearly presented. It turns out that, in the case of a vibrating suspension point,both equilibrium positions can be unstable, which corresponds to the existence of focus singularities in the considered model.

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On Numerical Continuation, Stability and Bifurcation Analysis of Periodic Motions of Autonomous Hamiltonian Systems

E.A. Sukhov[1]

[1]*Moscow Aviation Institute (National Research University) Moscow, Russia*

Natural families of periodic motions play an important role in investigating dynamical behaviour of autonomous Hamiltonian systems. Said families can be constructed in form of power series with respect to small parameter using approach based on Lyapunov's method. Their orbital stability can be established by using general Theory of Stability and methods of KAM theory. There is a large body of work by many authors dedicated to investigating periodic motions analytically. However, analytical computation is only possible in a limited array of particular cases such as small neighborhoods of known equilibria or stationary motions. To construct and analyze a natural family for all admissible values of its parameters it is necessary to use a numerical approach.

In this work we present a numerical approach for constructing natural families of periodic motions of autonomous Hamiltonian systems and investigating their bifurcations and orbital stability. The computation is carried out in three stages. The first stage involves obtaining the so-called base motions which is done either analytically employing an approach developed in works [1,2] or numerically using Poincare maps. On the second stage base motions are continued to the bounds of their existence domains using a numerical method developed by Sokolskiy and Karimov with modifications proposed in [3]. The second stage also involves linear orbital stability analysis which is carried out alongside the numerical continuation. On the third stage bifurcation diagrams of the obtained families are mapped in the problem's parameter space and bifurcation manifolds are identified. Finally, Poincare maps are computed in the neighborhood of the bifurcation points and new families of periodic motions are identified and continued.

To illustrate and verify the proposed approach we investigated two cases both of which are described by autonomous Hamiltonian systems with two degrees of freedom. The first case is a dynamically symmetric satellite moving about its center of mass on a circular orbit. The second case is restricted four-body problem in which three massive bodies form a stable Lagrangian configuration and the fourth body of a negligible mass moves about one of the relative equilibria points. For both cases natural families of periodic motions have been obtained analytically and their existence and linear orbital stability domains have been constructed numerically. Bifurcation diagrams have also been constructed for the families of periodic motions emanating from Regular precessions of a dynamically symmetric satellite.

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An example of billiard in Celestial Mechanics

Susanna Terracini

University of Turin
irene.deblasi@unito.it

A new type of dynamical model, describing the motion of a point-mass particle in an elliptic galaxy with a massive central core (such as, for example, a Black Hole), is studied.
This kind of model belongs to the more general class of the refraction billiards, which are particularly useful as a way to describe the dynamics of particles under the action of discontinuous potentials. In our case, a refraction interface (a regular closed curve) separates a Keplerian potential with positive energy from a two-dimensional homogeneous harmonic potential.
The dynamical properties of the system depend crucially on the geometric features of the interface. In particular, this talk focuses on three main aspects: the linear stability of particular (homothetic) equilibrium trajectories, the existence of orbits with prescribed rotation number for close-to-circle interfaces and the existence of a symbolic dynamics under more general assumptions.

Work in collaboration with V. Barutello and Irene De Blasi.

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Linearization by means of a functional parameter

D. Treschev[1]

[1] *Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia*

Let us consider a Hamiltonian system near an equilibrium position or a symplectic mapping around a fixed point in a $2n$-dimensional phase space. Assume that the system depends on a functional parameter that is a function of n variables. We study the possibility of using a functional parameter to obtain a system conjugate to a linear system on an open set.

Lie algebras with bases generating two-dimensional subalgebras

E.S. Zhikhareva[1]

[1]*Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Moscow, Russia*

It is known that the classification of left-invariant Nijenhuis operators on Lie groups reduces (in the case when all eigenvalues are real and simple) to the description of bases in Lie algebras having the property that any two basis vectors generate a two-dimensional subalgebra.

This task has not been studied enough yet. In my work, the first important step has been taken in solving the problem of classification of left-invariant Nijenhuis operators.

- **Problem statement:** Describe all Lie algebras g^n which in some basis $\mathtt{e}_1, \dots, \mathtt{e}_n$ satisfy the following condition: $[\mathtt{e}_i, \mathtt{e}_j] = \alpha_{ij}\mathtt{e}_i + \beta_{ij}\mathtt{e}_j$ for all $i, j \in \overline{1, n}$, where $\alpha_{ij}, \beta_{ij} \in \mathbf{R}$.
- **Result:** A complete description of the bases with the specified property for three-dimensional Lie algebras is obtained. As it turned out, such bases exist for most of them.

We note that the corresponding left-invariant Nijenhuis operators on the corresponding Lie group G are given by diagonal matrices with respect to the basis $\mathtt{e}_1, \dots, \mathtt{e}_n$ with constant elements on the diagonal, where each vector $\mathtt{e}_i$ is regarded as a left-invariant vector field on G.

Stability loss delay and shimmy for accelerated motion of a Castor wheel

A.A. Zobova[1]

[1] *Lomonosov Moscow State University, Moscow, Russia*

We study the acceleration of a Castor wheel along a horizontal plane with friction. The driving force T, constant in the inertial frame, acts on a lightweight Castor wheel's fork. The friction force and torque act in the contact between the wheel and the plane. We study the linear stability of rectilinear motions with a constant velocity V of the mass center C. Under quite general assumptions on the dependence of the friction force and torque on phase variables, the matrix of the linearized system is block-diagonal, so the equations separate into "transverse" and "longitudinal" subsystems. "Transverse" subsystem governs the dynamics of the course angle θ and lateral slip; "longitudinal" subsystem governs vertical oscillations, longitudinal slip, and the wheel's angular velocity about its axis. For a particular friction model, the "longitudinal" subsystem is always stable. In contrast, the "transverse" subsystem becomes unstable when the velocity V exceeds a critical value V_{cr}, which depends on the wheel offset, and the Andronov-Hopf bifurcation occurs. Instability is associated with the shimmy phenomenon and results in the oscillation of the course angle θ.

Numerical simulation of the wheel acceleration in the nonlinear statement shows the stability loss delay — oscillations of the course angle occur at $V > V_{cr}$. The breakdown to periodic motion occurs the later, the longer the phase trajectory stays in the vicinity of stable stationary motion.