Steklov International Mathematical Center

Logical Perspectives 2021 Workshop

Moscow, Russia, June 17–19, 2021

Abstracts

Logical Perspectives 2021 is supported by:

— the Simons Foundation;

— the Ministry of Science and Higher Education of the Russian Federation (the grant to the Steklov International Mathematical Center, agreement no. 075-15-2019-1614).

Programme Committee

- Matthias Baaz
- Lev D. Beklemishev
- Andre Scedrov
- Stanislav O. Speranski

Organising Committee

- Lev D. Beklemishev
- Stepan L. Kuznetsov
- Stanislav O. Speranski

TABLE OF CONTENTS

J. P. Aguilera, M. Baaz, J. Bydžovský. The number of axioms ... 1

A. Belikov, Y. Petrukhin. Exactly true and non-falsity versions of Deutsch's logic ... 4

Y. E. Gapanyuk, A. N. Nardid, D. A. Zuev. Ideas of metagraph-based types ... 8

O. Grigoriev, Y. Petrukhin. On multilattice counterparts of MNT4, S4, and S5 ... 12

M. Ilić. Gentzenizing R ... 16

S. L. Kuznetsov, S. O. Speranski. Infinitary action logic with exponentiation ... 17

E. La Rosa. A labelled sequent calculus for HYPE ... 21

A. Lolic. On globally sound analytic calculi for quantifier macros ... 23

A. Luxemburg. Set theory in the MathSem program ... 27

G. Olkhovikov. Lindström theorem for predicate intuitionistic logic ... 31

A. A. Onoprienko. A joint logic of problems and propositions ... 33

A. Rodin. Computer-assisted proofs and mathematical understanding: The case of univalent foundations . 35

D. Rogozin. The distributive full Lambek calculus with modal operators: discrete duality and Kripke completeness ... 39

M. Rybakov, D. Shkatov. On relationship between complexity function and complexity of validity in propositional modal logic ... 43

L. Sauras-Altuzarra. Arithmetical applications of Baaz's generalization method ... 47

V. Shehtman. Boxing modal logics ... 48

N. V. Shilov, D. A. Kondratiev. Platform-independent model of fix-point arithmetic for verification of the standard mathematical functions ... 50

D. Skvortsov. On a possibility of finite characterizations for Kripke complete non-recursively axiomatizable superintuitionistic predicate logics ... 53

V. Slyusarev. Definability of graph properties in modal languages ... 56

E. Ternovska. Towards capturing PTIME with no counting construct (but with a version of Hilbert's choice operator epsilon) ... 59

J. Vierling. Unrefutability by clause set cycles ... 60

A. Zapryagaev. Presburger arithmetic and Visser's conjecture ... 65

The Number of Axioms

Juan P. Aguilera
Department of Mathematics
Ghent University
Gent, Belgium
aguilera@logic.at

Matthias Baaz
Institute of Discrete Mathematics and Geometry
Technische Universität Wien
Vienna, Austria
baaz@logic.at

Jan Bydžovský
Institute of Discrete Mathematics and Geometry
Technische Universität Wien
Vienna, Austria
jan.bydz@gmail.com

We explore lower bounds on the number of axioms needed to prove theorems. We deal with first-order logic, formalized as a version of the *sequent calculus* LK introduced by Gentzen [2] (see also Takeuti [6] for additional background): A *sequent* is an expression of the form

$$\Gamma \vdash \Delta \tag{1}$$

where Γ and Δ are finite *multisets* of formulae. The interpretation of (1) is "if all formulae in Γ hold, then some formula among Δ holds". In LK, one starts with axioms and infers other sequents through various *rules of inference*. We measure the length of a proof by the number of sequents that appear in it; we measure the length of a sequent by the number of symbols in it. It is well known that one cannot give a recursive bound on the least possible length of a proof of a provable sequent S in terms of the length of S itself. Below, we prove the following strengthening:

Theorem 1. *There is no recursive bound on the least possible number of distinct axioms in an* LK*-proof of a sequent in terms of its length.*

Here, we do not consider two occurrences of the same axiom $A(a)$ as "distinct," but we do consider as distinct different instances of the same axiom, such as $A(a)$ and $A(b)$. Theorem 1 says that as one considers longer sequents, their minimal proofs not only become "longer," but also "wider," and moreover so in a way that cannot be accounted for by the repetition of axioms.

Intuitionistic logic can be formalized as one of many variants of Gentzen's LJ, which is obtained from LK by adding the restriction that all sequents $\Gamma \vdash \Delta$ contain at most one formula on the right-hand side. All our arguments below apply to intuitionistic logic and to many other related systems. In particular, we have:

Theorem 2. *There is no recursive bound on the least possible number of distinct axioms in an* LJ*-proof of a sequent in terms of its length.*

Among the usual inferences in sequent calculi figures the *cut* rule:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

Gentzen's *Cut-Elimination Theorem* says that the cut rule is redundant, however. Cut-free proofs are useful because they have the *subformula property*: in a proof of a sequent S with no instances of the cut rule, one only finds formulae which are substitution instances of subformulae of formulae in S. This is a desirable property for automated proof search, and other applicatinos. The main tool in the proof of Theorem 1 is the following lower bound on the number of axioms in cut-free proofs:

Theorem 3. *Let S be a provable* LK*-sequent of length s. Denote by m the minimal length of a cut-free* LK*-proof of S and by α the minimal number of* distinct *axioms in a cut-free* LK*-proof of S. Then*

$$\sqrt[s^2]{\frac{1}{s^4}\log_2(m)} \leq \alpha.$$

Finally, we mention another application of Theorem 3. Recall that cut elimination has a high computational cost. A function $f : \mathbb{N} \to \mathbb{N}$ is *elementarily bounded* if it is bounded by a function of the form

$$x \mapsto 2^{2^{\cdot^{\cdot^{\cdot^{2^x}}}}}.$$

An algorithm is *elementary* if it runs in an amount of time which is elementarily bounded. A classical theorem due independently to Orevkov [3] and Statman [5] states that there can be no elementary cut-elimination algorithm for first-order logic. By inspecting Schütte's proof of Gentzen's cut-elimination theorem (see e.g., Schwichtenberg [4]), one sees that this result is optimal, in the sense that the cut-elimination theorem requires computations as simple as possible among non-elementary classes. More precisely, it is easily shown that the cut-elimination theorem is equivalent to the totality of the superexponential function (which maps a natural number n to the result of applying the exponentiation function $x \mapsto 2^x$ n times) over Elementary Arithmetic (EA) (see e.g. Beklemishev [1] for more on relevant subsystems of arithmetic); however, this leaves open the possibility of strengthening the result in other directions; namely, Orevkov and Statman's proofs show that there is a sequence of first-order sequents the nth of which has a proof of length $\mathcal{O}(n)$, but whose shortest cut-free proofs have lengths which cannot be elementarily bounded. Using Theorem 3 we can strengthen this result by showing that those cut-free proofs must necessarily have non-elementarily many distinct axioms.

Theorem 4. *There is no elementary bound on the least possible number of* distinct *axioms of a cut-free* LK*-proof of a sequent in terms of the least possible length of an* LK*-proof of the same sequent.*

References

[1] Beklemishev, L. D. Reflection principles and provability algebras in formal arithmetic. *Russian Math. Surveys 60* (2005), 197–268.

[2] Gentzen, G. K. E. Untersuchungen über das logische Schließen, I. *Math. Z. 39* (1934), 176–210.

[3] Orevkov, V. P. Lower Bounds for Increasing Complexity of Derivations after Cut Elimination (in Russian). *J. Soviet Math.* (1982), 2337–2350.

[4] Schwichtenberg, H. Proof theory: Some applications of cut-elimination. In *Handbook of Mathematical Logic*, J. Barwise, Ed. 1982, pp. 867–896.

[5] Statman, R. Lower Bounds on Herbrand's Theorem. *Proc. Amer. Math. Soc. 75* (1979), 104–107.

[6] Takeuti, G. *Proof Theory (Second ed.).* Dover Publications, 2013.

Exactly True and Non-falsity versions of Deutsch's logic

Alex Belikov[⋆], Yaroslav Petrukhin[∗]

[⋆]Lomonosov Moscow State University, Moscow, Russia
[∗]Lodz University, Lodz, Poland
belerevance@gmail.ru, yaroslav.petrukhin@mail.ru

Abstract

In this report, we study logical systems which represent entailment relations of two kinds. We extend the approach of finding 'exactly true' and 'non-falsity' versions of four-valued logics that emerged in series of recent works on **FDE** to the case of infectious ones, namely to the case of Deutsch's relevant logic introduced in [8, 9].

A lot of interest was paid to so-called infectious logics in recent years. Besides their philosophical significance (see [19, 13]), a number of important results connected with applications of infectious logics in the context of the logical programming and proof theory were also obtained [5, 7, 6, 11, 18]. One interesting four-valued logic can be distinguished among this class of theories, namely Deutsch's $\mathbf{S_{fde}}$ [8, 9]. It can be seen as a rival of well-known Dunn-Belnap's four-valued logic **FDE** [10, 3, 4]. The difference lies in the interpretation of the truth value gaps, as is seen from the matrix below.

We fix a standard propositional language $\mathscr{L}$ with an alphabet $\langle \mathcal{P}, {\sim}, \wedge, \vee, (,) \rangle$, where $\mathcal{P} = \{p, q, r, s, p_1, \ldots\}$ is a set of propositional variables. The set $\mathscr{F}$ of all $\mathscr{L}$-formulas is defined in a standard inductive way. The set $\mathscr{V}_4 = \{\mathtt{T}, \mathtt{B}, \mathtt{N}, \mathtt{F}\}$ contains truth-values which are interpreted as follows: 'true', 'both' (i. e. both true and false), 'none' (i. e. neither true nor false), and 'false', respectively. A valuation is understood as a mapping from $\mathcal{P}$ to $\mathscr{V}_4$. It is extended on the set $\mathscr{F}$ according to the logical matrices which are presented below.

$\mathbf{S_{fde}}$ has the matrix $\langle \mathscr{V}_4, {\sim}, \wedge, \vee, \{\mathtt{T}, \mathtt{B}\} \rangle$, where:

φ	$\sim$
T	F
B	B
N	N
F	T

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	N	F
N	N	N	N	N
F	F	F	N	F

$\vee$	T	B	N	F
T	T	T	N	T
B	T	B	N	B
N	N	N	N	N
F	T	B	N	F

The entailment relation is defined as preserving designated values.

Definition 1. For each $\Gamma \cup \Delta \subseteq \mathscr{F}$, it holds that:

- $\Gamma \models_{\mathbf{S_{fde}}} \Delta$ iff for each valuation v, $v(\gamma) \in \{\mathtt{T}, \mathtt{B}\}$ (for each $\gamma \in \Gamma$) implies $v(\delta) \in \{\mathtt{T}, \mathtt{B}\}$ (for some $\delta \in \Delta$);

In this work we introduce two new logics which differ from $\mathbf{S_{fde}}$ by the definition of the entailment relation. In a manner similar to what has been done by Kapsner[1] and Rivieccio in [14] and Shramko, Zaitsev and Belikov in [16, 17] regarding **FDE**, we consider the corresponding counterparts of $\mathbf{S_{fde}}$. The first one is $\mathbf{S_{etl}}$, the 'exactly true' version of $\mathbf{S_{fde}}$. It differs from $\mathbf{S_{fde}}$ by the set of designated values: it has $\{\mathtt{T}\}$ instead of $\{\mathtt{T}, \mathtt{B}\}$. The second one is $\mathbf{S_{nfl}}$, the 'non-falsity' version of $\mathbf{S_{fde}}$. It has the following set of designated values: $\{\mathtt{T}, \mathtt{B}, \mathtt{N}\}$.

Definition 2. For each $\Gamma \cup \Delta \subseteq \mathscr{F}$, it holds that:

[1]Pietz – before name changing.

- $\Gamma \models_{\mathbf{S_{etl}}} \Delta$ iff for each valuation v, $v(\gamma) = \mathtt{T}$ (for each $\gamma \in \Gamma$) implies $v(\delta) = \mathtt{T}$ (for some $\delta \in \Delta$);
- $\Gamma \models_{\mathbf{S_{nfl}}} \Delta$ iff for each valuation v, $v(\gamma) \in \{\mathtt{T}, \mathtt{B}, \mathtt{N}\}$ (for each $\gamma \in \Gamma$) implies $v(\delta) \in \{\mathtt{T}, \mathtt{B}, \mathtt{N}\}$ (for some $\delta \in \Delta$).

We provide a characterization of $\mathbf{S_{etl}}$ and $\mathbf{S_{nfl}}$ entailment relations with respect to the ones of $\mathbf{K_3}$ [12] and **LP** [15], respectively.

Theorem 1. *Let* $\Gamma \cup \Delta \subseteq \mathscr{F}$.
$\Gamma \models_{\mathbf{S_{etl}}} \Delta$ *iff* $\Gamma \models_{\mathbf{K_3}} \Delta'$ *for some* $\Delta' \subseteq \Delta$ *such that* $var(\Delta') \subseteq var(\Gamma)$.

Theorem 2. *Let* $\Gamma \cup \Delta \subseteq \mathscr{F}$.
$\Gamma \models_{\mathbf{S_{nfl}}} \Delta$ *iff* $\Gamma' \models_{\mathbf{LP}} \Delta$ *for some* $\Gamma' \subseteq \Gamma$ *such that* $var(\Gamma') \subseteq var(\Delta)$.

As to the main result, we introduce sound and complete Gentzen-style calculi (enjoying cut-elimination) for $\mathbf{S_{etl}}$ and $\mathbf{S_{nfl}}$. Consider the following set of axioms and sequent rules:

- Axioms:

  (Ax) $\varphi \Rightarrow \varphi$ (ECQ) $\varphi, {\sim}\varphi \Rightarrow$ (EM) $\Rightarrow \varphi, {\sim}\varphi$

- Structural rules:

  (W$\Rightarrow$) $\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ ($\Rightarrow$W) $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$ (Cut) $\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Theta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Theta, \Pi}$

- Logical rules:

$$(\wedge \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$({\sim}{\sim} \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{{\sim}{\sim}\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}{\sim})\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, {\sim}{\sim}\varphi}$$

$$({\sim}\wedge \Rightarrow)\ \frac{{\sim}\varphi, {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \wedge \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}\wedge)\ \frac{\Gamma \Rightarrow \Delta, {\sim}\varphi \qquad \Gamma \Rightarrow \Delta, {\sim}\psi}{\Gamma \Rightarrow \Delta, {\sim}(\varphi \wedge \psi)}$$

$$({\sim}\vee \Rightarrow)\ \frac{{\sim}\varphi, \Gamma \Rightarrow \Delta \qquad {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \vee \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}\vee)\ \frac{\Gamma \Rightarrow \Delta, {\sim}\varphi, {\sim}\psi}{\Gamma \Rightarrow \Delta, {\sim}(\varphi \vee \psi)}$$

- The restricted versions of the logical rules:

$$(\wedge^H \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \begin{array}{l}\text{provided that}\\ var(\{\varphi, \psi\}) \subseteq var(\Delta)\end{array}$$

$$(\Rightarrow \vee^B)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \quad \begin{array}{l}\text{provided that}\\ var(\{\varphi, \psi\}) \subseteq var(\Gamma)\end{array}$$

Let us make some remarks regarding the rules and sequent calculi already mentioned in the literature. Let us write $\mathfrak{S}_{\mathbf{L}}$ for the sequent calculus for the logic **L**.

1. The axiom (Ax), all the structural rules, and the logical rules ($\wedge\Rightarrow$), ($\Rightarrow\wedge$), ($\vee\Rightarrow$), ($\Rightarrow\vee$), ($\sim\sim\Rightarrow$), ($\Rightarrow\sim\sim$), ($\sim\wedge\Rightarrow$), ($\Rightarrow\sim\wedge$), ($\sim\vee\Rightarrow$), ($\Rightarrow\sim\vee$) form the sequent calculus for **FDE** [1, 2].
2. The extension of $\mathfrak{S}_{\mathbf{FDE}}$ by the axiom (ECQ) is the sequent calculus for $\mathbf{K_3}$ [1].
3. The extension of $\mathfrak{S}_{\mathbf{FDE}}$ by the axiom (EM) is the sequent calculus for **LP** [1].

Let us extend this list by the new results.

4. The axioms (Ax) and (ECQ) as well as all the structural rules and the logical rules ($\sim\sim\Rightarrow$), ($\Rightarrow\sim\sim$), ($\wedge\Rightarrow$), ($\Rightarrow\wedge$), ($\vee\Rightarrow$), ($\Rightarrow\vee^B$), ($\sim\wedge\Rightarrow$), ($\Rightarrow\sim\wedge$), ($\sim\vee\Rightarrow$), ($\Rightarrow\sim\vee$) form the sequent calculus for $\mathbf{S_{etl}}$.
5. The axioms (Ax) and (EM) as well as all the structural rules and the logical rules ($\sim\sim\Rightarrow$), ($\Rightarrow\sim\sim$), ($\wedge^H\Rightarrow$), ($\Rightarrow\wedge$), ($\vee\Rightarrow$), ($\Rightarrow\vee$), ($\sim\wedge\Rightarrow$), ($\Rightarrow\sim\wedge$), ($\sim\vee\Rightarrow$), ($\Rightarrow\sim\vee$) form the sequent calculus for $\mathbf{S_{nfl}}$.

Acknowledgments. This work is supported by the Russian Science Foundation (Project 19-78-00044).

References

[1] Avron, A., "Classical Gentzen-type Methods in Propositional Many-valued Logics", in: Fitting M., Orlowska E. (eds) *Beyond Two: Theory and Applications of Multiple-Valued Logic.* Studies of Fuzziness and Soft Computing, vol. 114. Physica, Heidelberg. (2003): pp. 117–155.

[2] Avron, A., Ben-Naim, J., Konikowska, B. (2007). Cut-free ordinary sequent calculi for logics having generalized finite-valued semantics. Logica Universalis. 1, 41–70.

[3] Belnap, N.D., "A useful four-valued logic" in *Modern Uses of Multiple-Valued Logic*, ed. by J.M. Dunn, G. Epstein. Boston: Reidel Publishing Company, (1977): 7-37.

[4] Belnap, N.D., "How a computer should think" in *Contemporary Aspects of Philosophy*, ed. by G. Rule. Stocksfield: Oriel Press, (1977): 30-56.

[5] Ciuni R., Ferguson T.M., Szmuc D. Relevant Logics Obeying Component Homogeneity. Australasian Journal of Logic, 15(2), 2018, 301–361.

[6] Ciuni R., Ferguson T.M., Szmuc D. Modeling the interaction of computer errors by four-valued contaminating logics. Springer-Verlag GmbH Germany, part of Springer Nature 2019. R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 119–139, 2019.

[7] Ciuni R., Ferguson T.M., Szmuc D. Logics based on linear orders of contaminating values. Journal of Logic and Computation, 2019, doi:10.1093/logcom/exz009.

[8] Deutsch, H., Relevant analytic entailment, The Relevance Logic Newsletter 2(1) (1977): 26–44.

[9] Deutsch, H., The completeness of S, Studia Logica 38(2) (1979): 137–147.

[10] Dunn, J. M., Intuitive semantics for first-degree entailments and coupled trees, Philosophical Studies 29(3) (1976): 149–168.

[11] Ferguson T.M. Faulty Belnap computers and subsystems of FDE. Journal of Logic and Computation. 26(5) (2016): 1617-1636.

[12] Kleene, S. C., *Introduction to metamathematics*, D. Van Nostrand Company, Inc., New York, Toronto. 1952.

[13] Omori, H., and D. Szmuc, "Conjunction and disjunction in infectious logics", pages 268–283 in A. Baltag, J. Seligman, and T. Yamada (eds.),Logic, Rationality, and Interaction: 6th International Workshop (LORI2017), Springer, Berlin, 2017.

[14] Pietz, A., Rivieccio, U. Nothing but the truth. Journal of Philosophical Logic. 42(1) (2013): 125–135.

[15] Priest, G., "The logic of paradox", Journal of Philosophical Logic, 8 (1979): 219-241.

[16] Shramko, Y., Zaitsev, D., Belikov, A. First degree entailment and its relatives. Studia Logica. 105(6) (2017): 1291–1317.

[17] Shramko, Y., Zaitsev, D., Belikov, A. The FMLA-FMLA Axiomatizations of the exactly true and non-falsity logics and some of their cousins. Journal of Philosophical Logic. 48(5) (2019): 787–808.

[18] Szmuc, D.E.: Defining LFIs and LFUs in extensions of infectious logics. Journal of Applied Non-Classical Logics 26(4) (2016): 286–314.

[19] Szmuc, D.E., Omori, H. A note on Goddard and Routley's significance logic. The Australasian Journal of Logic, [S.I.], v.14, n.2, p. 431-448, july 2018.

Ideas of Metagraph-Based Types

Yuriy E. Gapanyuk, Anatoly N. Nardid, and Dmitry A. Zuev

Bauman Moscow State Technical University, Moscow, Russia
gapyu@bmstu.ru, nazgul109@gmail.com, zuynew@yandex.ru

Abstract

The metagraph model is a kind of "complex networks with emergence" model. To process and transform metagraph data, the metagraph agents are used. The combination of the metagraph data model and the metagraph agent model makes it possible to represent various type systems in the form of a metagraph model.

According to the HOTT book [5]: "the basic concept of type theory, that the term a is of type A, which is written: $a : A$. This expression is traditionally thought of as akin to: 'a is an element of the set A'. However, in homotopy type theory we think of it instead as: 'a is a point of the space A' ."

We propose the basic ideas of an approach in which a is a subgraph in a complex graph A. According to [1]: "a complex network is a graph (network) with non-trivial topological features – features that do not occur in simple networks such as lattices or random graphs but often occur in graphs modeling of real systems." The terms "complex network" and "complex graph" are often used synonymously. According to [2]: "the term 'complex network,' or simply 'network,' often refers to real systems while the term 'graph' is generally considered as the mathematical representation of a network." In this paper, we also consider these terms synonymous.

One of the essential kinds of such complex network models is "complex networks with emergence." The term "emergence" is used in general system theory. The emergent element means a whole that cannot be separated into its component parts. As far as the authors know, currently, there are two "complex networks with emergence" models that exist: hypernetworks and metagraphs.

The hypernetwork model [4] is mature, and it helps to understand many aspects of complex networks with an emergence. However, from the authors' point of view, the metagraph model is more flexible and convenient than a hypernetwork model for use in information systems [3].

According to paper [3], the metagraph approach may be considered as a higher-level structural framework for the representation of dynamical complex graph structures.

The metagraph is described as follows: $MG = \langle V, MV, E \rangle$, where MG – metagraph; V – set of metagraph vertices; MV – set of metagraph metavertices; E – set of metagraph edges.

Metagraph vertex is described by set of attributes: $v_i = \{atr_k\}, v_i \in V$, where atr_k – attribute.

Metagraph edge is described by set of attributes, the source and destination vertices (or metavertices): $e_i = \langle v_S, v_E, \{atr_k\} \rangle, e_i \in E$, where e_i – metagraph edge; v_S – source vertex (metavertex) of the edge; v_E – destination vertex (metavertex) of the edge; atr_k – attribute.

The metagraph fragment is defined as $MG_i = \{ev_j\}, ev_j \in (V \cup E \cup MV)$, where MG_i – metagraph fragment; ev_j – an element that belongs to union of vertices, edges and metavertices.

The metagraph metavertex: $mv_i = \langle \{atr_k\}, MG_f \rangle, mv_i \in MV$, where mv_i – metagraph metavertex; atr_k – attribute, MG_f – metagraph fragment.

From the general system theory point of view, metavertex is a particular case of manifestation of emergence principle, which means that metavertex with its private attributes and connections became whole that cannot be separated into its component parts. The example of metagraph representation is represented in Fig. 1.

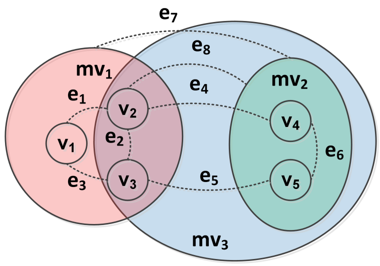

Figure 1: The example of metagraph representation.

The example contains three metavertices: mv_1, mv_2, and mv_3. Metavertex mv_1 contains vertices v_1, v_2, v_3 and connecting them edges e_1, e_2, e_3. Metavertex mv_2 contains vertices v_4, v_5, and connecting them edge e_6. Edges e_4, e_5 are examples of edges connecting vertices v_2–v_4 and v_3-v_5 are contained in different metavertices mv_1 and mv_2. Edge e_7 is an example of the edge connecting metavertices mv_1 and mv_2. Edge e_8 is an example of the edge connecting vertex v_2 and metavertex mv_2. Metavertex mv_3 contains metavertex mv_2, vertices v_2, v_3 and edge e_2 from metavertex mv_1 and also edges e_4, e_5, e_8 showing emergent nature of metagraph structure.

Consider the basics of the object-oriented data structures representation using the metagraph approach. We review only data structures containing data fields in form *name* : *type* : *value* where *type* may be atomic type, complex type or list (collection) type.

The data structure formally may be defined as follows: $DS = \langle ds_T, DS_F \rangle, ds_T \in TP, DS_F = \{fld^i\}$, where DS – data structure; ds_T – data structure type belongs set of types TP; DS_F – set of data structure fields fld^i.

The field is defined as follows: $fld^i = \langle fld_N, fld_T, fld_V \rangle, fld_T \in TP$, where fld_N – field name; fld_T – field type belongs set of types TP, fld_V – field value of type fld_T.

Every type tp belonging to set of types TP must be either atomic type TP_A or complex type TP_C or list (collection) type TP_L. The atomic type TP_A corresponds to the only value. The complex type TP_C contains a set of corresponding field types fld_T. The list type TPL is a collection of elements of any type: $(\forall tp \in TP) tp = TP_A | TP_C = \{fld_T\} | TP_L = [TP]$.

The example showing one of the possible cases of metagraph representation of object–oriented data structure is represented in Fig. 2.

Data structure DS and its corresponding type are represented as a metavertices bound with edge ds_T. The set of data structure fields DS_F (also represented as a metavertex) consists of three fields fld^1, fld^2 and fld^3.

Field fld^1 with the name “field1” corresponds to the atomic type “int” with value “1”. Field fld^1 is represented as a metavertex, field name fld^1_N, and value fld^1_V are represented as inner vertices. The field type is represented as edge fld^1_T bound field metavertex with atomic type TP_A vertex.

Field fld^2 with the name “field2” corresponds to the complex type consists of fields “field2_1” of type “int” with value “2” and “field2_2” of type “string” with value “string2”. Field fld^2 is represented as a metavertex, field name fld^2_N is represented as inner vertex, and value fld^2_V is represented as inner metavertex containing metavertices fld^{2_1} and fld^{2_2} correspondings to subfields “field2_1” and “field2_2” with their values. Field fld^2 type is represented as edge fld^2_T bound field metavertex with complex type TP_C metavertex. The TP_C metavertex contains

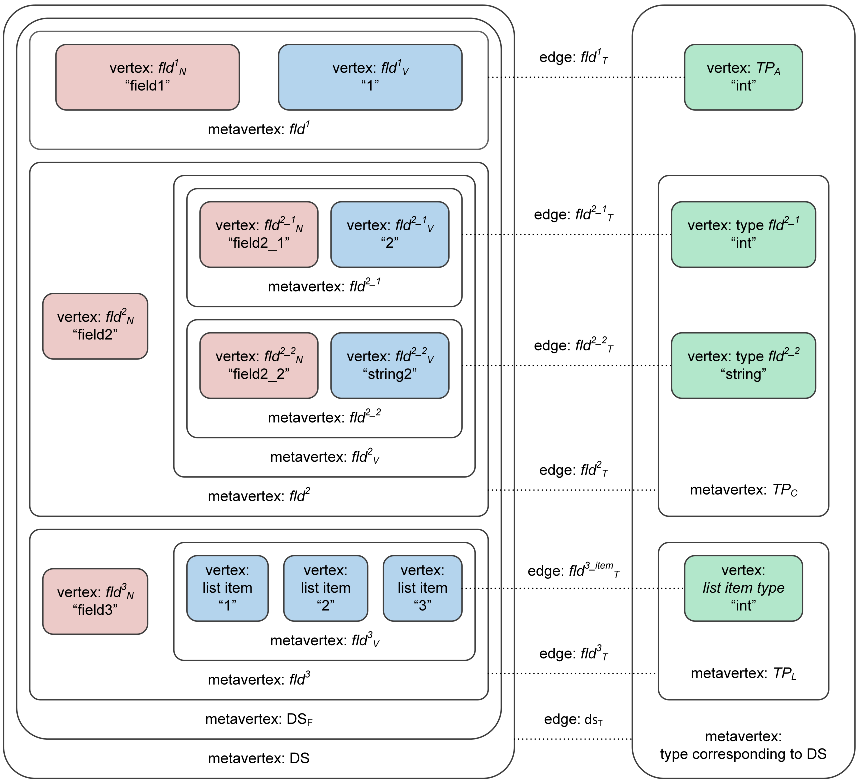

Figure 2: The metagraph representation of object-oriented data structure.

inner vertices corresponding to subfields $fld^{2\text{-}1}$ and $fld^{2\text{-}2}$ types. The edges $fld^{2\text{-}1}_T$ and $fld^{2\text{-}2}_T$ bound subfields $fld^{2\text{-}1}$ and $fld^{2\text{-}2}$ metavertices with corresponding subtypes vertices.

Field fld^3 with the name "field3" corresponds to the list (collection) type "list of int" with value "1, 2, 3". Field fld^3 is represented as a metavertex, field name fld^3_N is represented as inner vertex and value fld^3_V is represented as inner metavertex corresponding to the list containing vertices corresponding to the list items. The field type is represented as edge fld^3_T bound field metavertex with list (collection) type TP_L metavertex. The TP_L metavertex contains inner vertex corresponds to the list item type. List items bound with list item type with $fld^{3\text{-}item}_T$ edge (shown only for list item "3" in order not to clutter the figure).

The example shows that the object–oriented data structure may be represented using the metagraph approach without losing detailed information. In conclusion, we note other features of the metagraph model related to the description of types:

- Types are considered as fragments of a complex graph, in which not only the values of the vertices are important, but also the relationships between them. This makes the metagraph model related to the ontological knowledge model.
- To process and transform metagraph data, the metagraph agents are used. The metagraph agent may be represented as a set of metagraph fragments. The distinguishing feature of

the metagraph agent is its homoiconicity, which means that it can be data structure for itself.

- The combination of the metagraph data model and the metagraph agent model makes it possible to represent various type systems in the form of a metagraph model. In this case, the relationships between the elements of the model are represented explicitly.

References

[1] Manoj B.S., Abhishek Chakraborty, and Rahul Singh. *Complex Networks: A Networking and Signal Processing Perspective*. Prentice Hall, 2018.

[2] Victor Chapela, Regino Criado, Santiago Moral, and Miguel Romance. *Intentional Risk Management through Complex Networks Analysis*. SpringerBriefs in Optimization. Springer International Publishing, first edition, 2015.

[3] Valeriy M. Chernenkiy, Yuriy E. Gapanyuk, Anatoly N. Nardid, Anton V. Gushcha, and Yuriy S. Fedorenko. The hybrid multidimensional-ontological data model based on metagraph approach. In Alexander K. Petrenko and Andrei Voronkov, editors, *Perspectives of System Informatics*, volume 10742 of *LNCS*, pages 72–87, Cham, 2018. Springer International Publishing.

[4] Jeffrey Johnson. *Hypernetworks in the Science of Complex Systems*. Series on Complexity Science. Imperial College Press, 2013.

[5] The Univalent Foundatiosn Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. http://homotopytypetheory.org/book/, 2013.

On multilattice counterparts of **MNT4**, **S4**, and **S5**

Oleg Grigoriev[⋆], Yaroslav Petrukhin[∗]

[⋆]Lomonosov Moscow State University, Moscow, Russia
[∗]Lodz University, Lodz, Poland
grig@philos.msu.ru, yaroslav.petrukhin@mail.ru

Abstract

In this report, we are going to introduce three recently developed modal multilattice logics based on **MNT4**, **S4**, and **S5** in the form of cut-free sequent and hypersequent calculi as well as in the form of algebraic semantics.

Multilattice logic $\mathbf{ML_n}$ was designed by Shramko [9] in order to generalize frameworks of Arieli and Avron's bilattice logic [1], Shramko and Wansing's trilattice logic [10], and Zaitsev's tetralattice logic [11]. Modal multilattice logic $\mathbf{MML_n}$ was developed by Kamide and Shramko [7]. They expected that this logic will be a multilattice version of **S4**. However, as argued in [5, 4], it is not really the case. **S4** proves the interdefinability of necessity and possibility modal operators, while, as follows from the embedding theorem of **S4** into $\mathbf{MML_n}$ [7], the latter logic does not have the interdefinability axioms. Moreover, the algebraic structure suggested by Kamide and Shramko is too weak to be an adequaete semantics for $\mathbf{MML_n}$ (see [5] for the details). The closure and interior operators introduced by Kamide and Shramko are rather multilattice versions of Tarski's operators (which are suitable for **MNT4**), than Kuratowski ones (which are needed for **S4**). It has motivated us to present a genuine multilattice version of **S4** based on Kuratowski's closure and interior operators (we call this logic $\mathbf{MML_n^{S4}}$) and a multilattice version of **MNT4** based on Tarski's operator (we call this logic $\mathbf{MML_n^{MNT4}}$).

Moreover, we consider one more logic: $\mathbf{MML_n^{S5}}$ which is a multilattice version of **S5**. Its algebraic semantics is based on Halmos closure and interior operators. What is important in the case of **S5** (since we a interested not only in algebraic, but proof-theoretical aspects of multilattice logics), **S5** has an impressive amount of various proof systems. In particular, it has at least eight various cut-free hypersequent calculi (see [6] for the latest one and [2] for a survey of the others). This feature of **S5** makes it a good candidate for the development on its base of non-standard modal logics (for example, multilattice modal logics).

Let us introduce the notion of multilattice.

Definition 1. [7, p. 319, Definitions 2.1 and 2.2] A *multilattice* is a structure $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n\rangle$, where $n > 1$, $S \neq \emptyset$, $\leqslant_1, \ldots, \leqslant_n$ are partial orders such that $\langle S, \leqslant_1\rangle, \ldots, \langle S, \leqslant_n\rangle$ are lattices with the corresponding pairs of meet and join operators $\langle \cap_1, \cup_1\rangle, \ldots, \langle \cap_n, \cup_n\rangle$ as well as the corresponding j-inversion operators $-_1, \ldots, -_n$ which satisfy the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

$$a \leqslant_j b \text{ implies } -_j b \leqslant_j -_j a; \qquad a \leqslant_k b \text{ implies } -_j a \leqslant_k -_j b; \qquad -_j -_j a = a.$$

Definition 2 (Ultralogical multilattice)**.** [7, p. 319, Definitions 2.3 and 2.4] A pair $\langle \mathcal{M}_n, \mathcal{U}_n\rangle$ is called an *ultralogical multilattice* iff $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n\rangle$ is a multilattice and $\mathcal{U}_n \subsetneq S$ satisfies the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

- $a \cap_j b \in \mathcal{U}_n$ iff $a \in \mathcal{U}_n$ and $b \in \mathcal{U}_n$ ($\mathcal{U}_n$ is a multifilter on $\mathcal{M}_n$);
- $a \cup_j b \in \mathcal{U}_n$ iff $a \in \mathcal{U}_n$ or $b \in \mathcal{U}_n$ ($\mathcal{U}_n$ is a prime multifilter on $\mathcal{M}_n$);
- $a \in \mathcal{U}_n$ iff $-_j -_k a \notin \mathcal{U}_n$ ($\mathcal{U}_n$ is an ultramultifilter on $\mathcal{M}_n$).

The formulas of $\mathbf{ML_n}$ are built from the set $\mathcal{P} = \{p_n \mid n \in \mathbb{N}\}$ of propositional variables, negations $\neg_1, \ldots, \neg_n$, conjunctions $\wedge_1, \ldots, \wedge_n$, and disjunctions $\vee_1, \ldots, \vee_n$. A valuation v is defined as a mapping from $\mathcal{P}$ to S. It is extended into complex formulas as follows: $v(\neg_j \phi) = -_j v(\phi)$, $v(\phi \wedge_j \psi) = v(\phi) \cap_j v(\psi)$, and $v(\phi \vee_j \psi) = v(\phi) \cup_j v(\psi)$. The entailment relation is defined as follows:

$\Gamma \models_{\mathbf{ML}_n} \Delta$ iff for each De Morgan ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(\gamma) \in \mathcal{U}_n$ (for each $\gamma \in \Gamma$), then $v(\delta) \in \mathcal{U}_n$ (for some $\delta \in \Delta$).

In the next definition we adopt the notions of Tarski, Kuratowski, and Halmos closure and interior operators for the multilattice case (we follow Cattaneo and Ciucci [3]).

Definition 3. We say that a multilattice $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ have *Tarski* operators iff for each $j \leqslant n$ the unary operators of *interior* I_j and *closure* C_j can be defined on S and satisfy the subsequent conditions ($a, b, c \in S$, $1 := c \cup_j \neg_j \neg_k c$, $0 := c \cap_j \neg_j \neg_k c$, $k \neq j$):

$I_j(a) \leqslant_j a$;	$C_j(a) \cup_j C_j(b) \leqslant_j C_j(a \cup_j b)$;	$-_k I_j(a) = I_j(-_k a)$;
$I_j(a) = I_j I_j(a)$;	$I_j(1) = 1$;	$-_k C_j(a) = C_j(-_k a)$;
$I_j(a \cap_j b) \leqslant_j I_j(a) \cap_j I_j(b)$;	$C_j(0) = 0$;	$I_j(a) = -_j -_k C_j(-_j -_k a)$;
$a \leqslant_j C_j(a)$;	$-_j I_j(a) = C_j(-_j a)$;	$C_j(a) = -_j -_k I_j(-_j -_k a)$.
$C_j(a) = C_j C_j(a)$;	$-_j C_j(a) = I_j(-_j a)$;	

Tarski operators are said to be *Kuratowski* ones iff the subsequent conditions are fulfilled: $I_j(a \cap_j b) = I_j(a) \cap_j I_j(b)$ and $C_j(a) \cup_j C_j(b) = C_j(a \cup_j b)$. Kuratowski operators are said to be *Halmos* ones iff the subsequent conditions are fulfilled: $I_j(-_j I_j(a)) = -_j I_j(a)$ and $C_j(-_j C_j(a)) = -_j C_j(a)$.

The formulas of modal multilattice logics are built not only from propositional connectives, but necessity operators $\Box_1, \ldots, \Box_n$ and possibility operators $\Diamond_1, \ldots, \Diamond_n$. The entailment relation in modal multilattice logics is understood in the following way:

$\Gamma \models \Delta$ in $\mathbf{MML}_n^{\mathbf{MNT4}}$ (resp., $\mathbf{MML}_n^{\mathbf{S4}}$, $\mathbf{MML}_n^{\mathbf{S5}}$) iff for each ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ with Tarski (resp., Kuratowski, Halmos) operators and each valuation v, it holds that if it holds that if $v(\gamma) \in \mathcal{U}_n$ (for each $\gamma \in \Gamma$), then $v(\delta) \in \mathcal{U}_n$ (for some $\delta \in \Delta$).

Let us introduce a sequent calculus for the logic $\mathbf{MML_n^{MNT4}}$. By a sequent we understood a pair written as $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of formulas. In what follows, the letter π denotes a set which is either empty or consists of exactly one formula from the list $\Box_j \psi, \neg_j \Diamond_j \psi, \neg_k \Box_j \psi$, where $k \neq j$; the letter δ denotes a set which is either empty or consists of exactly one formula from the list $\Diamond_j \psi, \neg_j \Box_j \psi, \neg_k \Diamond_j \psi$, where $k \neq j$. The axioms are as follows:

$$\text{(A) } \phi \Rightarrow \phi \qquad (\text{A}_\neg)\ \neg_j \phi \Rightarrow \neg_j \phi$$

The structural rules are cut (which is admissible) and weakening. The non-negated logical rules are as follows:

$$(\wedge_j \Rightarrow)\ \frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge_j \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge_j \psi}$$

$$(\vee_j \Rightarrow)\ \frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee_j \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee_j)\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee_j \psi}$$

The jj-negated logical rules are as follows:

$$(\neg_j\wedge_j \Rightarrow)\ \frac{\neg_j\phi, \Gamma \Rightarrow \Delta \qquad \neg_j\psi, \Gamma \Rightarrow \Delta}{\neg_j(\phi \wedge_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_j\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\phi, \neg_j\psi}{\Gamma \Rightarrow \Delta, \neg_j(\phi \wedge_j \psi)}$$

$$(\neg_j\vee_j \Rightarrow)\ \frac{\neg_j\phi, \neg_j\psi, \Gamma \Rightarrow \Delta}{\neg_j(\phi \vee_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_j\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\phi \qquad \Gamma \Rightarrow \Delta, \neg_j\psi}{\Gamma \Rightarrow \Delta, \neg_j(\phi \vee_j \psi)}$$

$$(\neg_j\neg_j \Rightarrow)\ \frac{\phi, \Gamma \Rightarrow \Delta}{\neg_j\neg_j\phi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_j\neg_j)\ \frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \neg_j\neg_j\phi}$$

The kj-negated logical rules as follows:

$$(\neg_k\wedge_j \Rightarrow)\ \frac{\neg_k\phi, \neg_k\psi, \Gamma \Rightarrow \Delta}{\neg_k(\phi \wedge_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_k\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\phi \qquad \Gamma \Rightarrow \Delta, \neg_k\psi}{\Gamma \Rightarrow \Delta, \neg_k(\phi \wedge_j \psi)}$$

$$(\neg_k\vee_j \Rightarrow)\ \frac{\neg_k\phi, \Gamma \Rightarrow \Delta \qquad \neg_k\psi, \Gamma \Rightarrow \Delta}{\neg_k(\phi \vee_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_k\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\phi, \neg_k\psi}{\Gamma \Rightarrow \Delta, \neg_k(\phi \vee_j \psi)}$$

$$(\neg_k\neg_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \phi}{\neg_k\neg_j\phi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_k\neg_j)\ \frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_k\neg_j\phi}$$

The non-negated modal rules are as follows:

$$(\Box_j \Rightarrow)\ \frac{\phi, \Gamma \Rightarrow \Delta}{\Box_j\phi, \Gamma \Rightarrow \Delta} \quad (\Rightarrow \Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \Diamond_j\phi} \quad (\Rightarrow \Box_j)\ \frac{\pi \Rightarrow \Diamond_j\Lambda, \phi}{\pi \Rightarrow \Diamond_j\Lambda, \Box_j\phi} \quad (\Diamond_j \Rightarrow)\ \frac{\phi, \Box_j\Lambda \Rightarrow \delta}{\Diamond_j\phi, \Box_j\Lambda \Rightarrow \delta}$$

The jj-negated modal logical rules:

$$(\Rightarrow \neg_j\Box_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\phi}{\Gamma \Rightarrow \Delta, \neg_j\Box_j\phi} \qquad (\neg_j\Diamond_j \Rightarrow)\ \frac{\neg_j\phi, \Gamma \Rightarrow \Delta}{\neg_j\Diamond_j\phi, \Gamma \Rightarrow \Delta}$$

$$(\neg_j\Box_j \Rightarrow)\ \frac{\neg_j\phi, \Box_j\Lambda \Rightarrow \delta}{\neg_j\Box_j\phi, \Box_j\Lambda \Rightarrow \delta} \qquad (\Rightarrow \neg_j\Diamond_j)\ \frac{\pi \Rightarrow \Diamond_j\Lambda, \neg_j\phi}{\pi \Rightarrow \Diamond_j\Lambda, \neg_j\Diamond_j\phi}$$

The kj-negated modal logical rules:

$$(\neg_k\Box_j \Rightarrow)\ \frac{\neg_k\phi, \Gamma \Rightarrow \Delta}{\neg_k\Box_j\phi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_k\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\phi}{\Gamma \Rightarrow \Delta, \neg_k\Diamond_j\phi}$$

$$(\Rightarrow \neg_k\Box_j)\ \frac{\pi \Rightarrow \Diamond_j\Lambda, \neg_k\phi}{\pi \Rightarrow \Diamond_j\Lambda, \neg_k\Box_j\phi} \qquad (\neg_k\Diamond_j \Rightarrow)\ \frac{\neg_k\phi, \Box_j\Lambda \Rightarrow \delta}{\neg_k\Diamond_j\phi, \Box_j\Lambda \Rightarrow \delta}$$

A sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ [4] is obtained from the one for $\mathbf{MML}_n^{\mathbf{MNT4}}$ by the replacement in each of modal rule the letters δ and π, respectively, with the sets $\{\Box_j\Gamma_1, \neg_j\Diamond_j\Gamma_2, \neg_k\Box_j\Gamma_3\}$ and $\{\Diamond_j\Delta_1, \neg_j\Box_j\Delta_2, \neg_k\Diamond_j\Delta_3\}$ (where $k \neq j$) as well as $\Box_j\Lambda$ and $\Diamond_j\Lambda$, respectively, with $\{\Box_j\Lambda_1, \neg_j\Diamond_j\Lambda_2, \neg_k\Box_j\Lambda_3\}$ and $\{\Diamond_j\Lambda_1, \neg_j\Box_j\Lambda_2, \neg_k\Diamond_j\Lambda_3\}$. Because of the lack of space, we are not able to present here a hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ based on Restall's hypersequent calculus for **S5** [8], but the reader may find it in [4]. All the calculi for modal multilattice logics are show to be sound, complete, and cut-free.

Acknowledgments. The report of Yaroslav Petrukhin supported by the grant from the National Science Centre, Poland, grant number DEC-2017/25/B/HS1/01268.

References

[1] Arieli, O., Avron, A., “Reasoning with logical bilattices”, *Journal of Logic, Language and Information* 5 (1996): 25–63.

[2] Bednarska, K., Indrzejczak, A., “Hypersequent calculi for S5: the methods of cut elimination”, *Logic and Logical Philosophy*, 24, 3 (2015): 277–311.

[3] Cattaneo, G., Ciucci, D. Lattices with interior and closure operators and abstract approximation spaces. In Transactions on rough sets X. Peters, J. F.; Skowron, A.; Wolski, M.; Chakraborty, M. K., Wu, W.-Z. (Eds.) Lecture notes in computer sciences (Vol. 5656, pp. 67–116). Berlin: Springer. 2009.

[4] Grigoriev, O., Petrukhin, Y., “On a multilattice analogue of a hypersequent S5 calculus”, *Logic and Logical Philosophy*, 28, 4 (2019): 683–730.

[5] Grigoriev, O., Petrukhin, Y. “Two proofs of the algebraic completeness theorem for multilattice logic”, Journal of Applied Non-Classical Logics 29, 4 (2019): 358–381.

[6] Indrzejczak A. “Two Is Enough – Bisequent Calculus for S5”. In: Herzig A., Popescu A. (eds) Frontiers of Combining Systems. FroCoS 2019. Lecture Notes in Computer Science, vol 11715. Springer, Cham. 2019.

[7] Kamide, N., Shramko, Y. “Modal Multilattice Logic”, *Logica Universalis* 11, 3 (2017): 317–343.

[8] Restall, G., “Proofnets for S5: Sequents and circuits for modal logic”, pages 151172 in *Logic Colloquium 2005*, series “Lecture Notes in Logic”, no. 28, Cambridge University Press, 2007.

[9] Shramko, Y., “Truth, falsehood, information and beyond: the American plan generalized”, In: Bimbo, K. (ed.) J. Michael Dunn on Information Based Logics, Outstanding Contributions to Logic. Springer, Dordrecht, (2016): 191–212.

[10] Shramko, Y., Wansing, H., “Some useful sixteen-valued logics: how a computer network should think”, *Journal of Philosophical Logic*, 34 (2005), 121–153.

[11] Zaitsev, D., “A few more useful 8-valued logics for reasoning with tetralattice EIGHT4”, *Studia Logica*, 92, 2 (2009): 265–280.

Gentzenizing R

Mirjana Ilić*

University of Belgrade, Faculty of Economics, Belgrade, Serbia
mirjanailic@ekof.bg.ac.rs

Abstract

It is well–known that extending a sequent calculus of positive relevant logic, for example Dunn's LK_+ [4], so as to handle a negation is not trivial. Belnap [1], solved this problem, using a concept of 'Display logic', but by going outside the standard vocabulary for R. Namely, to the standard $\{\rightarrow, \wedge, \vee, \sim\}$ he added not only, t and $\circ$, which are also needed in LK_+, but T and $\sim_b$, where T is the disjunction of all propositions and $\sim_b$ is Boolean negation. Another solution of this problem was presented by Brady [3], who in addition to t and $\circ$, used also the classical negation, denoted by $-$, and additional structural connective $\star$, corresponding to $\otimes$, defined by $\alpha \otimes \beta = \alpha \wedge - \sim \beta$, in order to set up the left–handed sequent system with signed formulae, for R. Significantly simpler sequent calculus was presented by Bimbó and Dunn [2], but only for the fragment $R^t_{\rightarrow}$ of R.

We have tried to set up a sequent system for R, less entangled than Brady's or Belnap's. Bearing in mind that RW allows a simple gentzenization on the standard vocabulary, GRW [6], we formulate the system GR by adding the intensional contraction rule

$$\frac{\vdash \Gamma[\Pi;\Pi]}{\Gamma[\Pi]}\ (\mathrm{WI})$$

to GRW. We prove that GR presents the sequent calculus for R. Unfortunately, the rule of cut cannot be eliminated in GR [7].

References

[1] N. D. Belnap, Jr., *Display Logic,* Journal of Philosophical Logic, **11** (1982), pp. 375–417.

[2] K. Bimbó, J, M. Dunn, *New consecution calculi for* $R^t_{\rightarrow}$*,* Notre Dame Journal of Formal Logic, **53** (2012), pp. 491–501.

[3] R. T. Brady, *Gentzenizations of relevant logics with distribution*, The Journal of Symbolic Logic, **61(2)** (1996), pp. 402–420.

[4] J. M. Dunn, *A 'Gentzen system' for positive relevant implication*, The Journal of Symbolic Logic, **38** (1973), pp. 356–357.

[5] G. Gentzen, *Investigations into logical deduction*, The Collected Papers of Gerhard Gentzen, Szabo, M. E. (ed.) North–Holland (1969), pp. 68–131.

[6] M. Ilić, B. Boričić, *A cut-free sequent calculus for relevant logic RW*, Logic Journal of IGPL, **22(4)** (2014), pp. 673-695.

[7] M. Ilić, B. Boričić, *A note on the system GRW with the intensional contraction rule*, to appear in Logic Journal of IGPL, DOI:10,1093/jigpal/jzaa002.

*This work is supported by the Ministry of Science and Technology of Serbia, grant number ON174026.

Infinitary Action Logic with Exponentiation

Stepan L. Kuznetsov[1] and Stanislav O. Speranski[1,2]

[1] Steklov Mathematical Institute of RAS
[2] St. Petersburg State University

The Lambek calculus [9] was introduced as a logical framework for describing natural language syntax. In order to be useful for such applications, the Lambek calculus is highly substructural, including neither contraction, nor weakening, nor permutation structural rules. The only structural rule kept is implicit associativity. From a modern point of view [1], the Lambek calculus can be considered as a non-commutative intuitionstic version of Girard's linear logic [3]. Thus, the Lambek can be further extended by linear logic connectives, such as additives and (sub)exponentials.

The derivability problem for the basic Lambek calculus is NP-complete [13]. The multiplicative-additive Lambek calculus (viz., the Lambek calculus extended with additive conjunction and disjunction, denoted by MALC) is PSPACE-hard [4, 6]. Extending the Lambek calculus with an exponential modality yields an undecidable (Σ_1^0-complete) system [10]. A more fine-grained system can be obtained by extending MALC with a family of structural modalities, called subexponentials, cf. [11] Such a non-commutative version of the subexponential extension of linear logic was studied by Kanovich et al. [5]. The Lambek calculus with subexponentials is also undecidable, provided that at least one of the subexponentials allows the rule of non-local contraction.

Action logic, or the Lambek calculus with additives further extended with iteration (Kleene star), originates in the works of Pratt [14] and Kozen [7]. Buszkowski and Palka [2, 12] considered a stronger version of action logic, where iteration is governed by an ω-rule instead of inductive-style axioms. This system is called *infinitary action logic*. Buszkowski and Palka proved that it is Π_1^0-complete (thus, in particular, not computably enumerable).

We study an extension of MALC with *both* Kleene star and a family subexponentials. This extension is called *infinitary action logic with exponentiation* and denoted by $!\mathrm{ACT}_\omega$.

Formulae of $!\mathrm{ACT}_\omega$ are built from propositional *variables* ($\mathrm{Var} = \{p_1, p_2, p_3, \ldots\}$) and the *multiplicative unit (truth)* constant $\mathbf{1}$ using the following binary connectives:

- multiplicative connectives: *left implication* $\multimap$, *right implication* $\multimapinv$, and *product (multiplicative conjunction)* $\otimes$;
- additive connectives: *conjunction* $\&$ and *disjunction* $\oplus$

and the following unary connectives:

- *iteration (Kleene star)* *;
- *subexponentials:* we fix a partially ordered set $\langle \mathcal{I}, \preceq \rangle$ of subexponential labels, and three subsets of $\mathcal{I}$, called $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{E}$, upwardly closed w.r.t. $\preceq$

  For each $s \in \mathcal{I}$ we introduce a unary connective $!^s$.

  Intuitively, $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{E}$ mean the sets of subexponentials for which we allow weakening, contraction, and permutation (exchange) rules respectively.

The axioms and rules of $!\mathrm{ACT}_\omega$ are as follows:

$$\overline{A \vdash A}\ \text{(id)}$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \vdash C}{\Gamma, \Pi, A \multimap B, \Delta \vdash C}\ (\multimap\, \vdash) \qquad \frac{A, \Pi \vdash B}{\Pi \vdash A \multimap B}\ (\vdash\, \multimap)$$

$$\frac{\Pi \vdash A \quad \Gamma, B, \Delta \vdash C}{\Gamma, B \multimapinv A, \Pi, \Delta \vdash C}\ (\multimapinv\, \vdash) \qquad \frac{\Pi, A \vdash B}{\Pi \vdash B \multimapinv A}\ (\vdash\, \multimapinv)$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \otimes B, \Delta \vdash C}\ (\otimes \vdash) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\vdash \otimes)$$

$$\frac{\Gamma, \Delta \to C}{\Gamma, \mathbf{1}, \Delta \to C}\ (\mathbf{1} \vdash) \qquad \frac{}{\vdash \mathbf{1}}\ (\vdash \mathbf{1})$$

$$\frac{\Gamma, A_1, \Delta \vdash C \quad \Gamma, A_2, \Delta \vdash C}{\Gamma, A_1 \oplus A_2, \Delta \vdash C}\ (\oplus \vdash) \qquad \frac{\Pi \to A_i}{\Pi \to A_1 \oplus A_2}\ (\vdash \oplus)_i,\ i = 1, 2$$

$$\frac{\Gamma, A_i, \Delta \to C}{\Gamma, A_1 \,\&\, A_2, \Delta \to C}\ (\& \vdash)_i,\ i = 1, 2 \qquad \frac{\Pi \to A_1 \quad \Pi \to A_2}{\Pi \to A_1 \,\&\, A_2}\ (\vdash \&)$$

$$\frac{(\Gamma, A^n, \Delta \vdash C)_{n \in \mathbb{N}}}{\Gamma, A^*, \Delta \vdash C}\ ({}^* \vdash)_\omega \qquad \frac{\Pi_1 \to A \quad \ldots \quad \Pi_n \vdash A}{\Pi_1, \ldots, \Pi_n \vdash A^*}\ (\vdash {}^*)_n,\ n \geqslant 0$$

$$\frac{\Gamma, A, \Delta \vdash C}{\Gamma, !^s A, \Delta \vdash C}\ (! \vdash) \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \vdash B}{!^{s_1} A_1, \ldots, !^{s_n} A_n \vdash !^s B}\ (\vdash !),\ s_i \succeq s$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^w A, \Delta \to C}\ (\mathrm{weak}),\ w \in \mathcal{W}$$

$$\frac{\Gamma, \Phi, !^e A, \Delta \vdash C}{\Gamma, !^e A, \Phi, \Delta \vdash C}\ (\mathrm{perm})_1,\ e \in \mathcal{E} \qquad \frac{\Gamma, !^e A, \Phi, \Delta \vdash C}{\Gamma, \Phi, !^e A, \Delta \vdash C}\ (\mathrm{perm})_2,\ e \in \mathcal{E}$$

$$\frac{\Gamma, !^c A, \Phi, !^c A, \Delta \vdash C}{\Gamma, !^c A, \Phi, \Delta \vdash C}\ (\mathrm{ncontr})_1,\ c \in \mathcal{C} \qquad \frac{\Gamma, !^c A, \Phi, !^c A, \Delta \vdash C}{\Gamma, \Phi, !^c A, \Delta \vdash C}\ (\mathrm{ncontr})_2,\ c \in \mathcal{C}$$

$$\frac{\Pi \vdash A \quad \Gamma, A, \Delta \vdash C}{\Gamma, \Pi, \Delta \vdash C}\ (\mathrm{cut})$$

Since (ncontr) and (weak) derive (perm), we explicitly postulate $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

Derivations in $!\mathrm{ACT}_\omega$ are trees which can be infinitely branching, but should be well-founded (that is, infinite paths are not allowed).

The cut rule is eliminable, which is established by a juxtaposition of two arguments. The first one is cut elimination in infinitary action logic, performed by Palka [12] using transfinite induction. The second one is cut elimination is the subexponential extension of MALC by Kanovich et al. [5], using a version of Gentzen's mix rule.

Our main result is that a combination of exponential and Kleene star yields a system of hyperarithmetical complexity:

Theorem 1. *If $\mathcal{C} \neq \varnothing$, then the derivability problem in* $!\mathrm{ACT}_\omega$ *is* Π^1_1*-complete.*

The proof of the lower bound, Π^1_1-hardness, is based on encoding Kozen's result on the complexity of Horn theories for *-continuous Kleene algebras [8]. The upper bound is established by quite a general argument, based on the form of the rules and derivations in the calculus.

Another measure of complexity of $!\mathrm{ACT}_\omega$ is its *closure ordinal.* The closure ordinal is defined as follows. Let $\mathscr{D}$ be the *immediate derivability operator.* The $\mathscr{D}$ operator is a mapping of sets of sequents into sets of sequents. For a set of sequents S and a sequent s we have $s \in \mathscr{D}(S)$ if and only if either $s \in S$, or s is an axiom, or s is obtained by one of the inference rules from sequents belonging to S.

By $\mathscr{D}^\alpha$, for an ordinal α, we denote the α-th transfinite iteration of $\mathscr{D}$. The closure ordinal is the smallest ordinal α such that $\mathscr{D}^\alpha(\varnothing) = \mathscr{D}^{\alpha+1}(\varnothing)$. The existence of such α follows from the Knaster–Tarski theorem.

We compute the closure ordinal for $!\mathrm{ACT}_\omega$:

Theorem 2. *If $\mathcal{C} \neq \varnothing$, the closure ordinal for* $!\mathrm{ACT}_\omega$ *(for the $\mathscr{D}$ operator defined above using axioms and rules of* $!\mathrm{ACT}_\omega$*) is* ω_1^{CK}*, that is, the smallest non-computable ordinal, known as the Church–Kleene ordinal.*

Thus, we have established exact complexity bounds for $!\mathrm{ACT}_\omega$, both in terms of the complexity class for the derivability problem and in terms of the closure ordinal of the immediate derivability operator. Complexity of naturally arising fragments of $!\mathrm{ACT}_\omega$, with $\mathcal{C} = \varnothing$ (that is, where no subexponential allows contraction) or where $!^c$, $c \in \mathcal{C}$, cannot be applied to formulae containing the Kleene star, is left for future research.

References

[1] V. M. Abrusci (1990). A comparison between Lambek syntactic calculus and intuitionistic linear logic. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 36:11–15.

[2] W. Buszkowski (2007). On action logic: equational theories of action algebras. *Journal of Logic and Computation*, 17(1):199–217.

[3] J.-Y. Girard (1987). Linear logic. *Theoretical Computer Science,* 50(1):1–102.

[4] M. Kanovich (1994). Horn fragments of non-commutative logics with additives are PSPACE-complete. In: *Proceedings of 1994 Annual Conference of the EACSL*, Kazimierz, Poland.

[5] M. Kanovich, S. Kuznetsov, V. Nigam, A. Scedrov (2019). Subexponentials in non-commutative linear logic. *Mathematical Structures in Computer Science,* 29(8):1217–1249.

[6] M. Kanovich, S. Kuznetsov, A. Scedrov (2019). The complexity of multiplicative-additive Lambek calculus: 25 years later. In: *Logic, Language, Information, and Computation, WoLLIC 2019*, vol. 11541 of *LNCS*, Springer, pp. 356–372.

[7] D. Kozen (1994). On action algebras. In: J. van Eijck and A. Visser, editors, *Logic and Information Flow,* MIT Press, pp. 78–88.

[8] D. Kozen (2002). On the complexity of reasoning in Kleene algebra. *Information and Computation,* 179:152–162.

[9] J. Lambek (1958). The mathematics of sentence structure. *The American Mathematical Monthly,* 65:154–170.

[10] P. Lincoln, J. Mitchell, A. Scedrov, N. Shankar (1992). Decision problems for propositional linear logic. *Annals of Pure and Applied Logic,* 56(1–3):239–311.

[11] V. Nigam, D. Miller (2009). Algorithmic specifications in linear logic with subexponentials. In: *Proc. PPDP 2009*, pp. 129–140.

[12] E. Palka (2007). An infinitary sequent system for the equational theory of *-continuous action lattices. *Fundamenta Informaticae,* 78:295–309.

[13] M. Pentus (2006). Lambek calculus is NP-complete. *Theoretical Computer Science,* 357(1):186–201.

[14] V. Pratt (1991). Action logic and pure induction, in: *JELIA 1990: Logics in AI*, vol. 478 of *LNCS (LNAI)*, Springer, pp. 97–120.

Elio La Rosa

A Labelled Sequent Calculus for HYPE

Abstract

In this work, we will present and discuss a calculus for HYPE's system as developed in [Lei18]. G3HYPE is a labelled calculus built to provide a proof system for HYPE's model theory, as it allows us to express in syntactic terms statements concerning not only formulas in the language, but possible intercurrent relations between different states as well. Structural rule admissibility will be shown to hold for the system, along with some of the most important metatheorems. Possible recovery of different logics in the system will be finally shown.

This work is devoted to finding a proper sequent-style calculus reflecting most of the attractive features that HYPE semantics display by the use of modern proof theoretic techniques.

HYPE is a system of non-classical logic developed in [Lei18]. Philosophical and technical motivation for the system are manifold and will not be discussed in length here. They include, and are not limited to, the study of an easily extendible semantic framework useful for specifying different logics, for applications in the field of semantic paradoxes, and for a possible background system for hyperintensional operators. While some of this research has already been hinted at or partially developed in [Lei18], other works are, at the time of writing, in development.

Our aim however will not only consist in offering a completion of [Lei18] from a proof theoretical side. We will in fact try to provide and argue for new syntactic grounds on which a study of HYPE's characteristic features can be brought on.

Taken as a logic per se, HYPE's official one,[1] is not difficult to spell out starting from the axioms provided in [Lei18]. This system, while possessing the advantage of being simple to define, seems to be not expressive enough to represent the possibility provided by such a rich semantics, which includes relations typical of mereological systems (fusion), combined with star states and (in)compatibility relations. Moreover, the strategy adopted for proving (strong) completeness through the construction of a canonical model is particularly involving. As we shall see, it seems to lead to an inexact result, namely the fact that the axiomatic system presented in [Lei18] is complete for the variable domain variant of HYPE.[2] For these reasons, we will introduce a different kind of framework, namely a linguistic extension of a G3-style classical sequent calculus. With such system we aim at providing a Classical logic base calculus for HYPE's characteristic semantic clauses and model-theoretic relations by representing them in the language of the derivation. To this end, a system of labels in the style of [Neg05] and [DN12] is employed. Similarly to labelled calculi in Modal and Intuitionistic logics, variable and constant labels will be used to label formulas.

This permits us to obtain a calculus G3HYPE with an algorithmic Cut admissibility procedure. In this case, the metatheorems will be proved in a simple and direct way in order to show such closeness to the actual semantics from one side and the benefit of the employed proof theoretic machinery from the other. The completeness proof will indeed be shown by the simple construction of a proof-search reduction tree, by generalising the method of Schütte-Takeuti [Tak13]. By the internalisation of HYPE's first-order semantics, however, we can actually achieve more, namely, we will obtain a system that enables us to reason with HYPE's model

[1] There are many other ways in which HYPE semantics is sharpened. We are going to consider the ones specified in [Lei18] for the propositional fragment

[2] In order to obtain the constant domain one, substitution of logical equivalents must be assumed in the axiomatic system, as it is not a derivable property of it.

theory, and in which many semantic observations made in [Lei18] can be derived in the system without ad-hoc additions.

Finally, we would like to remark that the logics recaptured by extensions of HYPE by imposing restriction on the models can therefore be recaptured in the proof system by formalising such restriction as rules over the relations between the labels. This logics include Classical logic, Intuitionistic one, First Degree Entailment (FDE) [Bel77], Strong Kleene logic (K3) [Kle38], Logic of Paradox (LP) [Pri79] and therefore, as we will show, Strict-Tolerant logic (ST), that is, Classical logic minus Cut [CERvR12].

References

[Bel77] Nuel D Belnap. A useful four-valued logic. In *Modern uses of multiple-valued logic*, pages 5–37. Springer, 1977.

[CERvR12] Pablo Cobreros, Paul Egré, David Ripley, and Robert van Rooij. Tolerant, classical, strict. *Journal of Philosophical Logic*, 41(2):347–385, 2012.

[DN12] Roy Dyckhoff and Sara Negri. Proof analysis in intermediate logics. *Archive for Mathematical Logic*, 51(1-2):71–92, 2012.

[Kle38] Stephen Cole Kleene. On notation for ordinal numbers. *The Journal of Symbolic Logic*, 3(4):150–155, 1938.

[Lei18] Hannes Leitgeb. HYPE: A system of hyperintensional logic (with an application to semantic paradoxes). *Journal of Philosophical Logic*, 2018.

[Neg05] Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5-6):507, 2005.

[Pri79] Graham Priest. The logic of paradox. *Journal of Philosophical logic*, 8(1):219–241, 1979.

[Tak13] Gaisi Takeuti. *Proof theory*. Dover, 2013.

On Globally Sound Analytic Calculi for Quantifier Macros

Anela Lolic[1]
(joint work with Matthias Baaz)

Institute of Discrete Mathematics and Geometry, TU Wien, Vienna, Austria,
anela@logic.at

Abstract

In this work we present a methodology to construct globally sound but possibly locally unsound analytic calculi for partial theories of Henkin quantifiers. It is demonstrated that locally sound analytic calculi do not exist for any reasonable fragment of the full theory of Henkin quantifiers.

Henkin introduced the general idea of dependent quantifiers extending classical first-order logic [4], cf. [5] for an overview. This leads to the notion of a partially ordered quantifier with m universal quantifiers and n existential quantifiers, where F is a function that determines for each existential quantifier on which universal quantifiers it depends (m and n may be any finite number). The simplest Henkin quantifier that is not definable in ordinary first-order logic is the quantifier Q_H binding four variables in a formula. A formula A using Q_H can be written as $A_H = \begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v)$. This is to be read "For every x there is a u and for every y there is a v (depending only on y)" s.t. $A(x, y, u, v)$. If the semantical meaning of this formula is given in second-order notation, the above formula is semantically equivalent to the second-order formula $\exists f \exists g \forall x \forall y A(x, y, f(x), g(y))$, where f and g are function variables. Systems of partially ordered quantification are intermediate in strength between first-order logic and second-order logic. Similar to second-order logic, first-order logic extended by Q_H is incomplete [7]. In proof theory incomplete logics are represented by partial proof systems, c.f. the wealth of approaches dealing with partial proof systems for second-order logic. However, in contrast to second-order logic only a few results deal with the proof theoretic aspect of the use of branching quantifiers in partial systems.[1]

The first step in this work is to establish an analytic function calculus with a suitable partial Henkin semantics. We choose a multiplicative function calculus based on pairs of multisets as sequents corresponding to term models and refer to this calculus as **LF**. Besides the usual propositional inference rules of **LK** the quantifier inference rules of **LF** are

- $\forall$-introduction for second-order function variables

$$\frac{A(t(t_1^*, \ldots, t_n^*))\Gamma \to \Delta}{\forall f^* A(f^*(t_1^*, \ldots, t_n^*)), \Gamma \to \Delta}\ \forall_l^n$$

  t is a term and $t_1^*, \ldots, t_n^*$ are semi-terms.

$$\frac{\Gamma \to \Delta, A(f(t_1^*, \ldots, t_n^*))}{\Gamma \to \Delta, \forall f^* A(f^*(t_1^*, \ldots, t_n^*))}\ \forall_r^n$$

[1] The most relevant paper is the work of Lopez-Escobar [6], describing a natural deduction system for Q_H. The setting is of course intuitionistic logic. The formulation of the introduction rule for Q_H corresponds to the introduction rule right in the sequent calculus developed in this paper. The system lacks an elimination rule.

f is a free function variable (eigenvariable) of arity n which does not occur in the lower sequent and $t_1^*, \ldots, t_n^*$ are semi-terms.

- $\exists$-introduction for second-order function variables

$$\frac{A(f(t_1^*, \ldots, t_n^*)), \Gamma \to \Delta}{\exists f^* A(f^*(t_1^*, \ldots, t_n^*)), \Gamma \to \Delta} \exists_l^n$$

f is a free function variable (eigenvariable) of arity n which does not occur in the lower sequent and $t_1^*, \ldots, t_n^*$ are semi-terms.

$$\frac{\Gamma \to \Delta, A(t(t_1^*, \ldots, t_n^*))}{\Gamma \to \Delta, \exists f^* A(f^*(t_1^*, \ldots, t_n^*))} \exists_r^n$$

t is a term and $t_1^*, \ldots, t_n^*$ are semi-terms.

LF is obviously cut-free complete w.r.t. term models by the usual Schütte argument and admits effective cut-elimination. The question arises why not to be content with the second-order representation of Henkin quantifiers. The answer is twofold: First of all, a lot of information can be extracted from cut-free proofs but only on first-order level. This includes (i) suitable variants of Herbrand's theorem with or without Skolemization, (ii) the construction of term-minimal cut-free proofs and (iii) the development of suitable tableaux provers. (i) fails due to the failure of second-order Skolemization, (ii) and (iii) fail because of the undecidability of second-order unification and the impossibility to obtain most general solutions.

Therefore, we construct the analytic calculus **LH** by deriving first-order rules from second-order rule macros. The language $\mathcal{L}_H$ of **LH** is based on the usual language of first-order logic with exception that the quantifiers are replaced by the quantifier Q_H. With exception of the quantifier-rules, **LH** corresponds to the calculus **LK** in a multiplicative setting. The idea is to abstract the eigenvariable conditions from the premises of the inference macros in **LF**. To obtain **LH**, we replace the quantifier rules of **LK** by

$$\frac{\Gamma \to \Delta, A(a, b, t_1, t_2)}{\Gamma \to \Delta, \begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v)} Q_{H_r}$$

a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1 and t_2 are terms s.t. t_1 must not contain b and t_2 must not contain a.[2]

$$\frac{A(t_1', t_2', a, b), \Pi \to \Gamma}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v), \Pi \to \Gamma} Q_{H_{l_1}}$$

where a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1', t_2' are terms s.t. b does not occur in t_2' and a and b do not occur in t_1'.

$$\frac{A(t_1', t_2', a, b), \Pi \to \Gamma}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v), \Pi \to \Gamma} Q_{H_{l_2}}$$

[2]Note that such a rule was already used by Lopez-Escobar in [6].

where a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1', t_2' are terms s.t. a does not occur in t_1' and a and b do not occur in t_2'. [3]

Cuts in **LH** can be eliminated following Gentzen's procedure and we obtain a midsequent theorem. However, **LH** is incomplete: Assume towards a contradiction the sequent $\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x,y,u,v) \to \begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} (A(x,y,u,v) \vee C)$ is provable. Then it is provable without cuts. A cut-free derivation after deletion of weakenings and contractions has the form:

$$\cfrac{\cfrac{A(a,b,c,d) \to A(a,b,c,d)}{A(a,b,c,d) \to A(a,b,c,d) \vee C}}{\vdots}$$

Due to the mixture of strong and weak positions in Q_H none of Q_{H_r}, $Q_{H_{l_1}}$, $Q_{H_{l_2}}$ can be applied.

The inherent incompleteness of **LH** even for trivial statements is a consequence of the fact that Q_H represents a quantifier inference macro combining quantifiers in a strong and a weak position. This phenomenon occurs already on the level of usual first-order logic when quantifiers defined by macros of quantifiers such as $\forall x \exists y$ are considered [2].

The solution is to consider sequent calculi with concepts of proof which are globally but not locally sound, similar to [1]. This means that all derived statements are true but that not every sub-derivation is meaningful. We obtain for **LF** and **LH** globally, but possibly locally unsound calculi $\mathbf{LF}^{++}$ and $\mathbf{LH}^{++}$ by weakening the eigenvariable conditions and show soundness, completeness and cut-elimination for the novel calculus $\mathbf{LH}^{++}$ [3]. The main results are[4]:

Lemma 1. *An $\mathbf{LH}^{++}$-derivation φ with cuts can be immediately transformed into an $\mathbf{LF}^{++}$-derivation φ' with cuts.*

Lemma 2. *An $\mathbf{LF}^{++}$-derivation φ where the end-sequent contains only quantifiers in blocked distinct sequences $\exists f \exists g \forall x \forall y$ can be transformed into a cut-free $\mathbf{LF}^{++}$-derivation φ' where the quantifiers in the sequence $\exists f \exists g \forall x \forall y$ belonging to a block in the end-sequent are inferred immediately one after the other.*

Lemma 3. *A cut-free $\mathbf{LF}^{++}$-proof φ with blocked quantifier inferences $\exists f \exists g \forall x\ \forall y$ from atomic axioms and only such blocks of quantifiers in the end-sequent can be transformed into a cut-free $\mathbf{LH}^{++}$-proof φ' from atomic axioms.*

Theorem 1. $\mathbf{LH}^{++}$ *is sound, cut-free complete w.r.t. the intended semantics and admits an effective cut-elimination.*

It is obvious that the methodology developed in this work can be extended to arbitrary Henkin quantifiers, however not to arbitrary macros of quantifiers, where repeated alternations between strong and weak quantifiers are allowed.

References

[1] Juan P. Aguilera and Matthias Baaz. Unsound inferences make proofs shorter. *J. Symb. Log.*, 84(1):102–122, 2019.

[3] The usual quantifier rules of **LK** ($\forall_l$,$\forall_r$ and $\exists_l$, $\exists_r$) can be obtained by partial dummy applications of Q_H.

[4] All proofs can be found in [3].

[2] Matthias Baaz and Anela Lolic. Note on globally sound analytic calculi for quantifier macros. In *WoLLIC*, volume 11541 of *Lecture Notes in Computer Science*, pages 486–497. Springer, 2019.

[3] Matthias Baaz and Anela Lolic. A globally sound analytic calculus for henkin quantifiers. In *LFCS*, volume 11972 of *Lecture Notes in Computer Science*, pages 128–143. Springer, 2020.

[4] L. Henkin. Some remarks on infinitely long formulas. In *Journal of Symbolic Logic*, volume 30, pages 167–183. Pergamon Press, 1961.

[5] Michał Krynicki and Marcin Mostowski. Henkin quantifiers. In *Quantifiers: logics, models and computation*, pages 193–262. Springer, 1995.

[6] E.G.K. Lopez-Escobar. Formalizing a non-linear Henkin quantifier. *Fundamenta Mathematicae*, 138(2):93–101, 1991.

[7] Marcin Mostowski and Konrad Zdanowski. Degrees of logics with Henkin quantifiers in poor vocabularies. *Archive for Mathematical Logic*, 43(5):691–702, 2004.

Set Theory in the MathSem Program

Luxemburg Andrey

UMCA Technologies, Moscow, Russia

andr_lux@mail.ru

Abstract.
Knowledge representation is a popular research field in IT. As mathematical knowledge is most formalized, its representation is important and interesting. Mathematical knowledge consists of various mathematical theories. In this paper we present a deductive system that derives mathematical notions, axioms and theorems of elementary set theory. All these notions, axioms and theorems can be considered a small mathematical theory.

Keywords: Semantic network · semantic net· mathematical logic · set theory · axiomatic systems · formal systems · semantic web · prover · ontology · knowledge representation · knowledge engineering · automated reasoning

1 Introduction

The term "knowledge representation" usually means representations of knowledge aimed to enable automatic processing of the knowledge base on modern computers, in particular, representations that consist of explicit objects and assertions or statements about them. We are particularly interested in the following formalisms for knowledge representation:

1. First order predicate logic [1, 4].
2. Deductive (production) systems. In such a system there is a set of initial objects, rules of inference to build new objects from initial ones or ones that are already build, and the whole of initial and constructed objects [5].

In this paper we describe a part of the project and a part of the interactive computer application for automated building of mathematical theories.

Studies in this area are mainly connected writing programs for automatic theorem proving, the development of the semantic Internet, ontologies.

First-order theorem proving is one of the most mature subfields of automated theorem proving. On the other hand, it is still semi-decidable, and a number of sound and complete calculi have been developed, enabling fully automated systems. More expressive logics, such as higher order logics, allow the convenient expression of a wider range of problems than first order logic, but theorem proving for these logics is less well developed.

In our project, unlike other provers, where it is necessary to translate the theorem and the axioms needed for its proof in a formal language and directly to the internal language of the system itself, on the contrary, a formal written axioms and theorems are generated automatically by a computer program. Using the language of set theory and axiomatic set theory can be constructed a significant part of mathematics. That is why as the original object taken membership predicate. In this paper, therefore, is considered as an example provide you with the basic concepts of set theory (empty set, subset, membership, inclusion, intersection, union, powerset, Cartesian product). With the help of program MathSem one can build the axioms and theorems of set theory. In the future, a deductive system is expected to bring and represent in the form of a semantic net framework of set theory, Euclidean geometry, group theory and graph theory.

2 Description of the Project

We define a formal language (close to first-order predicate logic), and a deductive (production) system that builds expressions in this language. There are rules for building new objects from initial (atomic) ones and the ones already built. Objects can be either statement (predicates), or definitions (these could be predicates or truth sets of predicates). The membership predicate is taken as the atomic formula. Rules for building new objects include logical operations (conjunction, disjunction, negation, implication), adding a universal or existential quantifier, and one more rule: building the truth set of a predicate. One can consider symbols denoting predicates and sets, and also the predicates and sets themselves (when an interpretation or model is fixed). One more rule allows substitution of an individual variable or a term for a variable. Further, when we have built a new formula, we can simplify it using term-rewriting rules and logical laws (methods of automated reasoning).

In order to prove theorems one can apply well-known methods of automated reasoning (resolution method, method of analytic tableaux, natural deduction, inverse method), as well as new methods based on the knowledge of «atomic» structure of the formula (statement) that we are trying to prove. For a new formula written in the formal language a human expert (mathematician) can translate it into «natural» language (Russian, English etc.), thus we obtain a glossary of basic notions of the system. More complicated formulae are translated into natural language using an algorithm and the glossary. The deductive system constructed here is based on classical first-order predicate logic. The initial object is the membership predicate, and the derivations result into mathematical notions and theorems. The computer program (algorithm) builds formulae from atomic ones (makes the semantic net of the derivation).

3 Software Description

The MathSem program is being written by Vitaliy Tatarinsev.

In this program, complicated formulae are built from atomic ones «manually». The formulae built can be saved in a Word file along with their descriptions. One can also upload formulae from a Word file. Below one can find an example of building circa 30 formulae. Notably, all the signature of set theory is built from formulae with length (number of atomic formulae) not greater than two.

N	Formula	Notation	Symbol	Natural language
1	$[\ (x_0 \in A_0)\ ]$	$P_0(x_0, A_0)$		
2	$[\ (x_0 \in A_1)\ ]$	$P_0(x_0, A_1)$		
3	$[\ (x_1 \in A_0)\ ]$	$P_0(x_1, A_0)$		
4	$[\ (x_1 \in A_1)\ ]$	$P_0(x_1, A_1)$		
5	$[\neg(x_0 \in A_0)\ ]$	$P_1(x_0, A_0)$		
6	$\forall(x_0)[\ (x_0 \in A_0)\ ]$	$P_2(A_0)$	$A_0 = I$	A_0 -universe
7	$\forall(A_0)[\ (x_0 \in A_0)\ ]$	$P_3(x_0)$		
8	$\exists(x_0)[\ (x_0 \in A_0)\ ]$	$P_4(A_0)$	$A_0 \neq \varnothing$	A_0 not empty set
9	$\exists(A_0)[\ (x_0 \in A_0)\ ]$	$P_5(x_0)$		
10	$\{x_0 \mid P_0(x_0, A_0)\}$	$M_0(A_0)$	A_0	A_0
11	$\{A_0 \mid P_0(x_0, A_0)\}$	$R_0(x_0)$		Ri are sets consisting of sets (comments)
12	$[((x_0 \in A_0) \& (x_0 \in A_1))]$	$P_6(x_0, A_0, A_1)$		
13	$[((x_0 \in A_0) \vee (x_0 \in A_1))]$	$P_7(x_0, A_0, A_1)$		
14	$[\neg(x_1 \in A_0)\ ]$	$P_8(x_1, A_0)$		
15	$[\neg(x_1 \in A_1)\ ]$	$P_9(x_1, A_1)$		
16	$[\neg(x_0 \in A_1)\ ]$	$P_{10}(x_0, A_1)$		
17	$[\ ((x_0 \in A_0) \& (x_1 \in A_0))\ ]$	$P_{11}(x_0, A_0, x_1)$		
18	$[\ ((x_0 \in A_0) \& \neg(x_0 \in A_1))\ ]$	$P_{12}(x_0, A_0, A_1)$		
19	$\{x_0 \mid P_{12}(x_0, A_0, A_1)\}$	$M_1(A_0, A_1)$	$A_0 \setminus A_1$	difference of A_0 and A_1
20	$\{x_0 \mid P_6(x_0, A_0, A_1)\}$	$M_2(A_0, A_1)$	$A_0 \cap A_1$	intersection of A_0 and A_1
21	$\{x_0 \mid P_1(x_0, A_0)\}$	$M_3(A_0)$		the complement to A_0
22	$\{x_0 \mid P_7(x_0, A_0, A_1)\}$	$M_4(A_0, A_1)$	$A_0 \cup A_1$	union of A_0 and A_1
23	$[\ ((x_0 \in A_0) \vee \neg(x_0 \in A_1))\ ]$	$P_{13}(x_0, A_0, A_1)$		
24	$\{x_0 \mid P_{13}(x_0, A_0, A_1)\}$	$M_5(A_0, A_1)$		
25	$[\ ((x_0 \in A_0) \& (x_1 \in A_1))\ ]$	$P_{14}(x_0, A_0, x_1, A_1)$		
26	$\{<x_0, x_1> \mid P_{14}(x_0, A_0, x_1, A_1)\}$	$M_6(A_0, A_1)$	A_0 x A_1	Cartesian product of A_0 and A_1
27	$\forall(x_0)[((x_0 \in A_0) \vee \neg(x_0 \in A_1))]$	$P_{15}(A_0, A_1)$	$A_1 \subset A_0$	A_1 subset A_0
28	$\{A_1 \mid P_{15}(A_0, A_1)\}$	$R_1(A_0)$		the powerset of A_0
29	$\exists(x_0)\ \exists(A_0)[\ (x_0 \in A_0)]$	$P_{16}()$		TRUE $A_0 = \{x_0\}$

Table 1.

References

1. Vereshchagin N. K., Shen A.: Languages and Calculi (in Russian), MCCME, Moscow (2008).

2. Vagin V. N., Golovina E. Yu., Zagoryanskaya A. A., Fomina M. V.: Exact and Plausible Reasoning in Intellectual Systems (in Russian). Fizmatlit, Moscow, (2004)..
3. Luxemburg A. A.: Automated Construction of Mathematical Theories (in Russian). Editorial URSS, Moscow (2005)
4. Robinson J.A., Voronkov A. (Eds.): Handbook of Automated Reasoning (in 2 volumes). Elsevier and MIT Press (2001)
5. Maslov S. Yu.: Theory of Deductive Systems and its Applications (in Russian). Radio i svyaz, Moscow (1986)
6. VUE http://vue.tufts.edu/about/index.cfm

LINDSTRÖM THEOREM FOR PREDICATE INTUITIONISTIC LOGIC

GRIGORY OLKHOVIKOV

The talk will be devoted to the explanation of the main result of [4] (joint work with G. Badia and R. Zoghifard). The paper extends the main result of [1] to several variants of first-order intuitionistic logic. More precisely, we consider the following family of six logics which we call standard intuitionistic logics, $StIL$ for short:

- Intuitionistic first-order logic without equality;
- Intuitionistic first-order logic with extensional equality;
- Intuitionistic first-order logic with intensional equality;
- Intuitionistic logic of constant domains without equality;
- Intuitionistic logic of constant domains with extensional equality;
- Intuitionistic logic of constant domains with intensional equality.

We supply these logics with the semantics in style of 'modified Kripke semantics' of [2, Sect 5.3] and define, for every system in $StIL$, an appropriate intuitionistic variant of first-order bisimulation (initially introduced in [3] for the intuitionistic first-order logic without equality under the name of first-order asimulation).

We then define the notion of abstract intuitionistic logic $\mathcal{L}$ as a quadruple of the form $(Str_{\mathcal{L}}, L, \models_{\mathcal{L}}, \boxplus_{\mathcal{L}})$, where $Str_{\mathcal{L}}$ is a function returning, for every signature Θ, the class of $\mathcal{L}$-admissible pointed Θ-models $Str_{\mathcal{L}}(\Theta)$ and L is a function returning the set $L(\Theta)$ of Θ-sentences in $\mathcal{L}$; next, $\models_{\mathcal{L}}$ is a class-relation such that, if $\alpha \models_{\mathcal{L}} \beta$, then there exists a signature Θ such that $\alpha \in Str_{\mathcal{L}}(\Theta)$, and $\beta \in L(\Theta)$; informally this is to mean that β holds in α. The relation $\models_{\mathcal{L}}$ is only assumed to be defined (i.e. to either hold or fail) for the elements of the class $\bigcup\{(Str_{\mathcal{L}}(\Theta), L(\Theta)) \mid \Theta \text{ is a signature}\}$ and to be undefined otherwise.

The fourth element in our quadruple, $\boxplus_{\mathcal{L}}$, is then a function, returning, for every $(\mathcal{M}, w) \in Str_{\mathcal{L}}(\Theta)$, for every tuple $\bar{c}$ of pairwise distinct constants outside Θ, and for every tuple $\bar{a}$ of objects in $(\mathcal{M}, w)$ a non-empty set of admissible constant expansions of a certain family of submodels of $\mathcal{M}$ by this new tuple of constants in such a way that $\bar{a}$ ends up being exactly the tuple of their values at $(\mathcal{M}, w)$.

In order for such a quadruple $\mathcal{L}$ to count as an abstract intuitionistic logic, several groups of additional requirements need to be satisfied, and we will offer a detailed formulation and motivation of these requirements in the talk.

We further define that, given a pair of abstract intuitionistic logics $\mathcal{L}$ and $\mathcal{L}'$, we say that $\mathcal{L}'$ *expressively extends* $\mathcal{L}$ and write $\mathcal{L} \sqsubseteq \mathcal{L}'$ iff all of the following holds:

- $Str_{\mathcal{L}} = Str_{\mathcal{L}'}$;

(a) Department of Philosophy I, Ruhr University Bochum, Bochum, Germany
(a) Department of Philosophy, Ural Federal University, Ekaterinburg, Russia
E-mail address: (a) grigory.olkhovikov@rub.de, grigory.olkhovikov@gmail.com.

- If $(\mathcal{M}, w) \in Str_{\mathcal{L}}(\Theta)$, $\bar{c}$ is a tuple of pairwise distinct constants outside Θ, and $\bar{a}$ is an appropriate tuple of objects, then we have $(\mathcal{M}, w)\boxplus_{\mathcal{L}'}(\bar{c}_n/\bar{a}_n) \subseteq (\mathcal{M}, w)\boxplus_{\mathcal{L}}(\bar{c}_n/\bar{a}_n)$
- For every $\phi \in L(\Theta)$ there exists a $\psi \in L'(\Theta)$ such that for every $(\mathcal{M}, w) \in Str_{\mathcal{L}}(\Theta)$ we have:

$$\mathcal{M}, w \models_{\mathcal{L}} \phi \Leftrightarrow \mathcal{M}, w \models_{\mathcal{L}'} \psi.$$

If both $\mathcal{L} \sqsubseteq \mathcal{L}'$ and $\mathcal{L}' \sqsubseteq \mathcal{L}$ hold, then we say that the logics $\mathcal{L}$ and $\mathcal{L}'$ are *expressively equivalent* and write $\mathcal{L} \bowtie \mathcal{L}'$.

We show how the logics in $StIL$ can be represented as abstract intuitionistic logics in the form outlined above and consider abstract extensions of these logics. Some of these extensions enjoy useful model-theoretic properties among which the following three are of particular interest: (1) Compactness, (2) Tarski Union Property, and (3) preservation under $\mathcal{L}$-asimulations for some $\mathcal{L} \in StIL$.

Our main result is then that no standard intuitionistic logic has proper extensions that display the combination of all the three useful properties mentioned above. In other words, we establish the following:

Theorem 1. *Let $\mathcal{L}$ be an abstract intuitionistic logic and let $\mathcal{L}' \in StIL$. If $\mathcal{L}' \sqsubseteq \mathcal{L}$ and $\mathcal{L}$ is preserved under $\mathcal{L}'$-asimulations, compact, and has Tarski Union Property, then $\mathcal{L}' \bowtie \mathcal{L}$.*

Acknowledgments. Grigory Olkhovikov is supported by Deutsche Forschungsgemeinschaft (DFG), Project OL 589/1-1.

References

[1] G. Badia and G. Olkhovikov. A Lindström theorem for intuitionistic propositional logic. *Notre Dame Journal of Formal Logic*, 61 (1): 11–30 (2020).

[2] D. van Dalen. *Logic and Structure*. 4 ed. Springer, 2008.

[3] G. K. Olkhovikov. Model-theoretic characterization of intuitionistic predicate formulas. *Journal of Logic and Computation*, 24: 809–829 (2014).

[4] G. Olkhovikov, G. Badia and R. Zoghifard. A Lindström theorem for intuitionistic first-order logic. (submitted), preprint available at http://arxiv.org/abs/2103.17024.

A joint logic of problems and propositions

Onoprienko A. A.

Abstract

We consider the joint logic of problems and propositions suggested by S. A. Melikhov. We prove that its propositional part is complete with respect to models constructed by S. Artemov and T. Protopopesku for intuitionistic epistemic logic. We also show that this logic conservatively extends the intuitionistic epistemic logic IEL$^+$.

In a commentary to his collected works [2], Kolmogorov remarked that his paper [3] «was written in hope that with time, the logic of solution of problems [i.e., intuitionistic logic] will become a permanent part of a [standard] course of logic. A unified logical apparatus was intended to be created, which would deal with objects of two types — propositions and problems.» Melikhov [5] construct a formal system QHC (a joint logic of problems and propositions) which contain two types of variables: problem and proposition. Formulas of QHC are built from variables by using standart classical and intuitionistic connectives $\vee, \wedge, \neg, \to$, modalities ! and ? and quantifiers $\forall, \exists$.
All propopositions (all problems) satisfy all inference rules and axioms schemes of classical (intuitionistic) predicate logic. Formulas of this two types are interconnected of two modalities. The modality ! inputs a proposition p and outputs a problem $!p$ «Find a proof of p». The modality ? inputs a problem α and outputs a proposition $?\alpha$ «There exists a solution of α». There are following axioms schemes and inference rules for modalities:

$$1)\ !(p \to q) \to (!p \to !q);\quad 2)\ ?(\alpha \to \beta) \to (?\alpha \to ?\beta);$$
$$3)\ \frac{p}{!p};\quad 4)\ \frac{\alpha}{?\alpha};$$
$$5)\ ?!p \to p;\quad 6)\ \alpha \to !?\alpha;\quad 7)\ \neg !0.$$

Melikhov examined several types of models for logic QHC [6], but completeness theorem failed even for propositional part HC of this logic. Author [1] considered algebraical models and Kripke-type semantic for logic HC. Compleness theorem and finite model property are proved for this types of models.
Artemov and Protopopesku [4] considered audit set models for intuitionistic epistemic logic IEL$^+$.

Definition 1. *Audit set scale is a triplet $(W, \preccurlyeq, Aud)$, where $(W, \preccurlyeq)$ is a standard intuitionistic scale (W is a set, $\preccurlyeq$ is a partial order), $Aud \subseteq W$ is a subset of audit states such that*

$$\forall a \in W\ \exists b \in W (a \preccurlyeq b \wedge b \in Aud).$$

Let us define the evaluation $\models$ for intuitionistic formulas by standard way, for classical formulas by naturally way only in audit states, and for modalities this way:

$$a \models ?\alpha \Leftrightarrow a \models \alpha \text{ (for } a \in \text{Aud)}$$
$$a \models !p \Leftrightarrow \forall b \in \text{Aud}(a \preccurlyeq b \Rightarrow b \models p) \text{ (for } a \in W\text{)},$$

We obtain a audit set model of logic HC.

Theorem 1. *Logic HC is complete with respect to audit set models. Moreover, the finite model property holds.*

Corollary 1. *Logic IEL$^+$ is complete with respect to finite audit set models.*

Corollary 2. *Logics HC and IEL$^+$ are decidable.*

Logic QHC is conservative extension of classical logic, intuitionistic logic and modal logic S4. This fact was proved by Melikhov. There was an open questions whether QHC is conservative extension of modal intuitionistic logic QH4 (let us denote $\nabla = !?$) which propositional part coincide which logic IEL$^+$. It is possible to extend audit set model of logic HC to models of logic QHC by attaching a set of «available objects» to each element of W. Author proved following theorem by using this models.

Theorem 2. *Logic QHC is conservative extension of logic QH4.*

References

[1] Onopienko A.A. Kripke type semantics for a logic of problems and propositions. Graduate work, protected 25.05.18. MSU, Faculty of Mechanics and Mathematics, Department of Mathematical Logic and Theory of Algorithms.

[2] A. N. Kolmogorov, On the papers on intuitionistic logic, Колмогоров А. Н. Избранные труды. Математика и механика, Наука, М., 1985, pp. 393; English transl., Selected Works of A. N. Kolmogorov. Vol. I, Mathematics and its Applications (Soviet Series), vol. 25, Kluwer, Dordrecht, 1991, pp. 451

[3] Kolmogoroff, Zur Deutung der intuitionistischen Logik, Math. Z. 35 (1932), no. 1, 58–65 (German). Russian transl. in Колмогоров А. Н. Избранные труды. Математика и механика, Наука, М. (1985), pp. 142–148.

[4] Artemov S., Protopopescu T. Intuitionistic Epistemic Logic https://arxiv.org/abs/1406.1582v2

[5] Melikhov S. A. A Galois connection between classical and intuitionistic logics. I: Syntax https://arxiv.org/abs/1312.2575

[6] Melikhov S. A. A Galois connection between classical and intuitionistic logics. II: Semantics https://arxiv.org/abs/1504.03379

Computer-Assisted Proofs and Mathematical Understanding
the case of Univalent Foundations

Andrei Rodin (andrei@philomatica.org)

Institute of Philosophy, Russian Academy of Sciences

The computer-assisted proof of Four Colour Map theorem (4CT) published by Kenneth Appel, Wolfgang Haken and John Koch back in 1977 [1] prompted a continued philosophical discussion on the epistemic value of computer-assisted mathematical proofs [10],[9],[3],[2],[7],[8]. We briefly overview this discussion and then show how the Univalent Foundations of Mathematics (UF) meets some earlier stressed epistemological concerns about computer-assisted proofs and thus offers a new possibility to fill the gap between computer-assisted and traditional mathematical proofs. We demonstrate the argument with a proof of basic theorem in Algebraic Topology formalised in UF and implemented in AGDA [6].

1 Overview

In their proof of 4CT Appel and his co-authors used a low-level computer code written specifically for this purpose in order to check one by one 1482 different cases (configurations), which was not feasible by hand. More recently a fully formalised version of Appel&Haken&Koch's proof has been implemented with Coq [4]. A philosophical discussion on this proof has been started by Thomas Tymoczko [10] who argues that the computer-assisted proof of 4CT does not qualify as mathematical proof in anything like the usual sense of of the word because the computer part of this proof cannot be surveyed and verified in detail by human mathematician or even a group of human mathematicians. On this ground Tymoczko suggests that the computer-assisted proof of 4CT represents a wholly new kind of *experimental* mathematics akin to experimental natural sciences, where the computer plays the role of experimental equipment.

Paul Teller in his response to Tymoczko [9] argues that Tymoczko misconceives of the concept of mathematical proof by confusing the epistemic notion of verification that something is a proof of a given statement with this proof itself, which under Teller's general conception of mathematical proof has no intrinsic epistemic content in it. Assuming that the published proof of 4CT is indeed a proof, Teller argues that it is unusual only in how one gets an epistemic access (if any) to it but that, contra Tymoczko, there is nothing unusual in the involved concept of mathematical proof itself.

Commenting on Teller's analysis in 2008 Dag Prawitz [7] approves on Teller's distinction between a proof and its verification. However since Prawitz's conception of proof unlike Teller's is essentially epistemic, Prawitz comes to a different conclusion. Contra Teller and in accordance with Tymoczko Prawitz argues that *if* Appel&Haken&Koch's alleged proof is indeed a proof then it comprises a crucial empirical evidence provided by computer and thus is not deductive.

Mic Detlefsen and Mark Luker in their response to Tymoczko [3] quite convincingly show that the difference between the computer-assisted proof of 4CT and traditional mathematical proofs is less dramatic than Tymoczko says. For traditional mathematical proofs quite often, and perhaps even typically, comprise some "blind" symbolic calculations like one that is needed in order to compute the product $50 \times 101 = 5050$. How much a given symbolic calculation is epistemically transparent or blind, is, according to Detlefsen&Luker, a matter of degree rather than a matter of principle.

2 Local and Global Surveyability of Mathematical Proofs

O. Bradley Bassler [2] suggests to distinguish between *local* and *global* surveyability of mathematical proofs. By local surveyability of proof p Bassler understands the property of p that makes it possible for a human to follow each elementary step of p. Bassler argues that local serveyability of p does not, by itself, make p epistemically transparent or surveyable in the usual intended sense because on the top of local surveyability it requires at least a minimal *global* surveyability, which allows one to see that all steps of p taken together provide p with a sufficient epistemic force that warrants its conclusion on the basis of its premises. In the historical part of his paper Bassler shows that there is an unfortunate tendency to neglect the global surveability in proofs by assuming that it reduces to the local one.

When one applies the distinction between local and global surveyability in the analysis of Appel&Haken&Koch's proof of 4CT the resulting picture is more complex than one suggested by Tymoczko [10]. The computer part of the proof is fully locally surveyable in the sense that each piece of the computer code can be checked and interpreted by human (since it is written by human). Arguments explaining why the computation so encoded, if performed correctly, completes the proof of the theorem, which Appel&Haken&Koch present in the form of traditional mathematical prose, provide a global survey of this proof and of this computation in particular. What this proof still lacks is rather an expected surveyability and traceability at the intermediate scale between the general understanding of what the given computation computes and the low-level computational steps expressed with the program code.

3 Univalent Foundations and Spatial Intuition

Homotopy Type theory (HoTT), which is the mathematical core of UF [5], allows one to think of formal derivations in Martin-Löf Type theory (MLTT) as homotopical spatial constructions . When this base calculus or its fragment is implemented in the form of programming code then the same homotopical interpretation along with the associated spatial intuition applies to the code. This spatial (homotopical) intuition makes formal symbolic derivations and the corresponding programming code humanly surveyable in a new way: on the top of the *local* surveyability that allows one to control elementary steps of the process, and in addition to the high-scale *global* surveyability that provides one with a general understanding of the resulting construction, the homotopical spatial intuition provides an epistemic access to the intermediate mesoscopic level of this construction, which allows one to follow and control all significant steps of formal reasoning ignoring its minute details. Such an intuitive reading of the formalism bridges the usual gap between the rigour formal representation of mathematical reasoning with logical calculi, on the one hand, and the conventional representations of mathematical reasoning, which typically heavily use various symbolic means of expression without strict syntactic rules, on the other hand. Thus HoTT supports a representation of mathematical reasoning in general and mathematical proof in particular, which is:

- fully formal in the sense that it uses a symbolic calculus with an explicit rigorous syntax;
- computer-checkable;
- supported by a spatial (homotopical) intuition that balances local and global aspects of mathematical intuition in the usual way.

A simple (but not trivial) example of mathematical proof represented in this way is found in [6]. It is a proof of basic theorem in Algebraic Topology according to which the fundamental group $\pi_1(S^1)$ of (topological) circle is S^1 (isomorphic to) the infinite cyclic group $\mathbb{Z}$, which is canonically represented as the additive group of integers.

Let *base* be a point of given circle S^1 (the base point). This judgement is formally reproduced with the MLTT syntax as formula

$$b : S^1$$

Then loops associated with this base point are terms of form:

$$loop : b =_{S^1} b$$

The resulting formal proof and its implementation in a programming code are interpretable in terms of such intuitive spatial (homotopical) constructions all the way through.

4 Conclusion

The UF-based approach in computer-assisted theorem proving allows the user to follow mathematical arguments at the crucial mesoscopic level of the proof structure, which is necessary for human understanding of mathematical proofs in anything like the usual sense of the word. In this case a computer-assisted proof does no longer appear as a "black box proof" where significant parts of the argument remain epistemically opaque and are replaced by non-deductive empirical evidences. This feature makes UF-based formal computer-assisted proofs quite like traditional mathematical proofs in accordance with the general line of Detlefsen&Luker's argument [3].

References

[1] K. Appel and W. Haken. Every planar map is four colorable. *Illinois Journal of Mathematics*, 21(3):429–567, 1977.

[2] O. Bradley Bassler. The surveyability of mathematical proof: A historical perspective. *Synthese*, 148(1):99–133, 2006.

[3] M. Detlefsen and M. Luker. The four-color theorem and mathematical proof. *Journal of Philosophy*, 77(12):803–820, 1980.

[4] G. Gonthier. A computer-checked proof of the four colour theorem, 2005. http://research.microsoft.com/ gonthier/4colproof.pdf.

[5] Univalent Foundations Group. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study (Princeton); available at http://homotopytypetheory.org/book/, 2013.

[6] D.R. Licata and M. Shulman. *Calculating the Fundamental Group of the Circle in Homotopy Type Theory*. https://https://arxiv.org/abs/1301.3443, 2013.

[7] D. Prawitz. Proofs verifying programs and programs producing proofs: A conceptual analysis. In R. Lupacchini and G. Corsi, editors, *Deduction, Computation, Experiment : Exploring the Effectiveness of Proof*, pages 81–94. Springer, 2008.

[8] G. D. Secco and L.C. Pereira. Proofs versus experiments: Wittgensteinian themes surrounding the four-color theorem. In Marcos Silva, editor, *How Colours Matter to Philosophy*, pages 289–307. Springer, 2017.

[9] P. Teller. Computer proof. *The Journal of Philosophy*, 77(12):797–803, 1980.

[10] Th. Tymoczko. The four-color problem and its philosophical significance. *The Journal of Philosophy*, 76(2):57–83, 1979.

The Distributive Full Lambek Calculus with Modal Operators: Discete duality and Kripke completeness

Daniel Rogozin[1*]

Lomonosov Moscow State University, Moscow, Russia
daniel.rogozin@serokell.io

In this talk, we study bounded distributive residuated lattices with modal operators $\Box$ and $\Diamond$ and their logics. We show that any canonical logic is Kripke complete via discrete duality and canonical extensions. We show that a given canonical modal extension of the distributive full Lambek calculus is the logic of its frames if its variety is closed under canonical extensions.

By the distributive full Lambek calculus with modal operators we mean the logic of the following kind:

Definition 1. A residual normal distributive modal logic is the set of sequents Λ that contains axioms (1)-(14) and closed under inference rules below:

$$p \Rightarrow p$$

- $\bot \Rightarrow p$
- $p \Rightarrow \top$
- $p_i \Rightarrow p_1 \vee p_2, i = 1, 2$
- $p_1 \wedge p_2 \Rightarrow p_i, i = 1, 2$
- $p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee (p \wedge r)$
- $p \bullet (q \bullet r) \Leftrightarrow p \bullet (q \bullet r)$
- From $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \theta$ infer $\varphi \Rightarrow \theta$
- From $\varphi \Rightarrow \psi$ and $\theta \Rightarrow \psi$ infer $\varphi \vee \theta \Rightarrow \psi$
- From $\varphi \bullet \theta \Rightarrow \psi$ infer $\theta \Rightarrow \varphi \setminus \psi$ and vice versa
- $p \bullet \mathbf{1} \Leftrightarrow \mathbf{1} \bullet p \Leftrightarrow p$
- $\Diamond(p \vee q) \Leftrightarrow \Diamond p \vee \Diamond q$
- $\Diamond\bot \Rightarrow \bot$
- $\Box p \wedge \Box q \Leftrightarrow \Box(p \wedge q)$
- $\top \Rightarrow \Box\top$
- $\Box p \bullet \Box q \Rightarrow \Box(p \bullet q)$
- From $\varphi(p) \Rightarrow \psi(p)$ infer $\varphi[p := \psi] \Rightarrow \psi[p := \gamma]$
- From $\varphi \Rightarrow \psi$ and $\varphi \Rightarrow \theta$ infer $\varphi \Rightarrow \psi \wedge \theta$
- From $\theta \bullet \varphi \Rightarrow \psi$ infer $\theta \Rightarrow \psi / \varphi$, and vice versa

In fact, a residual normal distributive modal logic extends normal distributive normal modal logic, the logic of bounded distributive lattices with modal operators introduced in [6]. To define relational semantics we introduce ternary Kripke frames with the additional binary modal relations. Such a ternary frame might be considered as a noncommutative generalisation of a modal relevant Kripke frame described, e.g., here [10]. As it is usual in the relational semantics of substructural logic, product and residuals have the ternary semantics as in, e.g., [1].

Definition 2. A Kripke frame is a tuple $\mathcal{F} = \langle W, R, R_\Box, R_\Diamond, \mathcal{O}\rangle$, where $R \subseteq W^3$, $R_\Box, R_\Diamond \subseteq W^2$, $\mathcal{O} \subseteq W$.

Note that R, $R_\Box$, and $R_\Diamond$ have certain conditions that we define in more detail during our talk. A Kripke model is a Kripke frame with an equipped valuation function that maps

*The research is supported by the Presidential Council, research grant MK-430.2019.1.

each propositional variable to $\leq$-upwardly closed subset of worlds. Let $\mathcal{F} = \langle W, R, R_\Box, R_\Diamond, \mathcal{O}\rangle$ be a Kripke frame, a Kripke model is a pair $\mathcal{M} = \langle \mathcal{F}, \vartheta\rangle$, where $\vartheta : \mathrm{PV} \to \mathrm{Up}(W, \leq)$. Here, $\mathrm{Up}(W, \leq)$ is the collection of all upwardly closed sets. Variables, $\wedge$, $\vee$, $\bot$, and $\top$ are understood standardly. The truth conditions for product, residuals, and $\mathbf{1}$ are understood with a ternary relation and the distinguished subset $\mathcal{O}$. $\Box$ and $\Diamond$ are understood as usual Kripkean necessity and possibility defined in terms of $R_\Box$ and $R_\Diamond$ relations.

The soundness theorem is formulated and proved in stadardly.

Theorem 1. *Let $\mathbb{F}$ be a class of Kripke frames, then* $\mathrm{Log}(\mathbb{F}) = \{\varphi \Rightarrow \psi \mid \mathbb{F} \models \varphi \Rightarrow \psi\}$ *is a residual distributive normal modal logic.*

Now we define algebraic semantics for such logics. The underlying algebraic structure for us is a residuated lattice [7]. A residuated lattice is called bounded distributive if its lattice reduct is bounded distributive. A residuated lattice morphism is a map $f : \mathcal{L}_1 \to \mathcal{L}_2$ that commutes with all operations. A residuated distributive modal algebra is a distributive bounded residuated lattice extended with normal modal operators $\Box$ and $\Diamond$ that distribute over finite infima and suprema correspondingly. One may also consider such algebras as full Lambek algebras [8] [9] lattice reducts of which are bounded distributive lattices. Note that we also require that $\Box$ is also "normal" with respect to a product. Such a "normality" corresponds to the promotion principle widely used in linear logic. This "normality" requirement is introduced as the additional inequation, more precisely:

Definition 3. A residuated distributive modal algebra (RDMA) is an algebra $\mathcal{M} = \langle \mathcal{R}, \Box, \Diamond\rangle$ with the following conditions for each $a, b \in \mathcal{R}$:

1. $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box\top = \top$
2. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$, $\Diamond\bot = \bot$
3. $\Box a \cdot \Box b \leq \Box(a \cdot b)$

A RDMA-morphism is a residuated lattice morphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ such that $f(\Box a) = \Box(f(a))$ and $f(\Diamond a) = \Diamond(f(a))$.

One may associate with an arbitrary residual normal modal logic its variety as follows:

Definition 4. Let $\mathcal{L}$ be a residual normal modal logic, then $\mathcal{V}_\mathcal{L}$ is a variety defined by the set unequations $\{\varphi \leq \psi \mid \mathcal{L} \vdash \varphi \Rightarrow \psi\}$

The usual Lindenbaum-Tarski construction provides us algebraic completeness for each residual distributive normal modal logic.

Theorem 2. *Let $\mathcal{L}$ be a residual normal modal logic, then there exists an RDMA $\mathcal{M}_L$ such that $\mathcal{L} \vdash \varphi \Rightarrow \psi$ iff $\mathcal{R}_L \models \varphi \leq \psi$*

Now we define completely distributive residuated perfect lattice as a distributive version of residuated perfect one defined in [2].

Definition 5. A distributive residuated lattice $\mathcal{L} = \langle L, \bigvee, \bigwedge, \cdot, \backslash, /, \varepsilon\rangle$ is called perfect distributive residuated lattice, if:

- $\mathcal{L}$ is a perfect distributive lattice

- $\cdot$, $\backslash$, and $/$ are binary operations on L such that $/$ and $\backslash$ right and left residuals of $\cdot$, repsectively. $\cdot$ is a complete operator on L, and $/ : L \times L^{\delta} \to L$, $\backslash : L^{\delta} \times L \to L$ are complete dual operators.

Here we formulate canonical extensions for bounded distributive lattices with a residuated family in the fashion of [3]. Note that one may provide canonical extensions for Heyting algebras similarly as it is described here [4]. See this paper to get acquainted with canonical extensions for bounded distributive lattices with operators closely [5].

Lemma 1. *Let $\mathcal{L} = \langle L, \cdot, \backslash, /, \varepsilon \rangle$ be a bounded distributive residuated lattice, then so $\mathcal{L}^{\sigma} = \langle L^{\sigma}, \cdot^{\sigma}, \backslash^{\pi}, /^{\pi} \rangle$ is. Moreover, $\mathcal{L}^{\sigma}$ is a perfect residuated distributive lattice.*

Definition 6. Let $\mathcal{L}$ be a perfect distributive residuated lattice and $\Box, \Diamond$ unary operators on $\mathcal{L}$, then $\mathcal{M} = \langle \mathcal{L}, \Box, \Diamond \rangle$ is called a perfect distributive residuated modal algebra, if

- $\Box \bigwedge A = \bigwedge \{\Box a \mid a \in A\}$
- $\Diamond \bigvee A = \bigvee \{\Diamond a \mid a \in A\}$
- $\Box a \cdot \Box b \leq \Box (a \cdot b)$

where $A \subseteq \mathcal{L}$

Given $\mathcal{M}$, $\mathcal{N}$ perfect distributive modal algebras, a map $\mathcal{M} \to \mathcal{N}$ is a homomorphism, if f is a complete lattice homomorphism that preserves product, residuals, modal operators, and the multiplicative identity.

Let us show that the variety of all RDMA is closed under canonical extensions.

Lemma 2. *Let $\mathcal{M}$ be a RDMA, then $\mathcal{M}^{\sigma}$ is a perfect DRMA, where $\mathcal{M}^{\sigma}$ is a canonical extension of the undelying bounded distributive residuated lattice with extended $\Box$ and $\Diamond$.*

Definition 7. A residual normal modal logic $\mathcal{L}$ is called canonical, if $\mathcal{V}_{\mathcal{L}}$ is closed under canonical extensions

Given a Kripke frame $\mathcal{F} = \langle W, R, R_{\Box}, R_{\Diamond}, \mathcal{O} \rangle$, we construct a complex algebra $\mathcal{F}^{+}$ as defining operations and constants $\mathrm{Up}(W, \leq)$. It is clear that $\bot = \emptyset$, $\top = W$, $\mathbb{1} = \mathcal{O}$, $A \wedge B = A \cap B$, $A \vee B = A \cup B$. Residuals, product, and modal operators are obtained via ternary and binary modal relations.

Let us define $\mathcal{M}_{+}$, the dual Kripke frame a perfect RDMA $\mathcal{M}$. Let us define the following relations on $\mathcal{J}^{\infty}(\mathcal{M})$: $a R_{\Diamond} b \Leftrightarrow b \leq \Diamond a$, $a R_{\Box} b \Leftrightarrow \Box \kappa(a) \leq \kappa(b)$, and $Rabc \Leftrightarrow a \cdot b \leq c$. The structure $\mathcal{M}_{+} = \langle \mathcal{J}^{\infty}(\mathcal{M}), \leq, R, R_{\Diamond}, R_{\Box}, \mathcal{O} \rangle$ is the dual frame of a perfect RDMA $\mathcal{M}$, where $\mathcal{O} = \uparrow \{\varepsilon\}$.

Here, $\mathcal{J}^{\infty}(\mathcal{M})$ is the set of all completely join irreducible elements of a perfect RDMA $\mathcal{R}$ and κ is the isomorphism between the set of all completely join irreducible elements and the set of all completely meet irreducible elements defined as $a \mapsto \bigvee(- \uparrow a)$.

Lemma 3.

1. *A complex algebra of a Kripke frame $\mathcal{F}$ defined as $\mathcal{F}^{+} = \langle \mathrm{Up}(W, \leq), \wedge, \vee, \bot, \top, \backslash, /, \cdot, \mathcal{O}, [R_{\Box}], \langle R_{\Diamond} \rangle, \rangle$ is a perfect DRMA.*
2. *Let $\mathcal{M}$ be a perfect DRMA, then $\mathcal{M}_{+}$ is a Kripke frame*

Theorem 3.

1. *Let* $\mathcal{F} = \langle W, R, R_\Box, R_\Diamond, \mathcal{O} \rangle$ *be a Kripke frame, then* $\mathcal{F} \cong (\mathcal{F}^+)_+$
2. *Let* $\mathcal{M} = \langle M, \bigvee, \bigwedge, \Box, \Diamond, \varepsilon \rangle$ *be a perfect DRMA, then* $\mathcal{R} \cong (\mathcal{M}_+)^+$
3. *Functors* $(.)_+ : \mathrm{pDRMA} \rightleftarrows \mathrm{KF} : (.)^+$ *establish a dual equivalence between the category of Kripke frames and the category of perfect DRMAs.*

The discrete duality established above together with canonical extensions of residuated distributive modal algebras provides the following consequence:

Theorem 4. *Let* $\mathcal{L}$ *be a canonical residual distributive modal logic, then* $\mathcal{L}$ *is Kripke complete.*

References

[1] Hajnal Andréka and Szabolcs Mikulás. Lambek calculus and its relational semantics: Completeness and incompleteness. *Journal of Logic, Language and Information*, 3(1):1–37, Jan 1994.

[2] J. Michael Dunn, Mai Gehrke, and Alessandra Palmigiano. Canonical extensions and relational completeness of some substructural logics. *J. Symbolic Logic*, 70(3):713–740, 09 2005.

[3] Mai Gehrke. Topological duality and algebraic completions.

[4] Mai Gehrke. *Canonical Extensions, Esakia Spaces, and Universal Models*, pages 9–41. Springer Netherlands, Dordrecht, 2014.

[5] Mai Gehrke and Bjarni Jónsson. Bounded distributive lattice expansions. *Mathematica Scandinavica*, 94(1):13–45, Mar. 2004.

[6] Mai Gehrke, Hideo Nagahashi, and Yde Venema. A sahlqvist theorem for distributive modal logic. *Annals of Pure and Applied Logic*, 131(1):65 – 102, 2005.

[7] Peter Jipsen and Constantine Tsinakis. A survey of residuated lattices. In *Ordered algebraic structures*, pages 19–56. Springer, 2002.

[8] Hiroakira Ono. Semantics for substructural logics. *Substructural logics*, 1993.

[9] Hiroakira Ono. *Modal and Substructural Logics*, pages 47–60. Springer Singapore, Singapore, 2019.

[10] Takahiro Seki. A sahlqvist theorem for relevant modal logics. *Studia Logica*, 73(3):383–411, Apr 2003.

On relationship between complexity function and complexity of validity in propositional modal logic

Mikhail Rybakov[123] and Dmitry Shkatov[4]

[1] Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia
[2] National Research University Higher School of Economics, Moscow, Russia
[3] Tver State University Tver, Russia
m_rybakov@mail.ru
[4] School of Computer Science and Applied Mathematics,
University of the Witwatersrand, Johannesburg, South Africa
shkatov@gmail.com

Abstract

We prove the existence, for any complexity class or degree of unsolvability C, of a linearly approximable extension of the unimodal propositional logic **K** whose variable-free fragment is C-hard. A similar result is proven for extensions of the unimodal propositional logic **KTB**.

1 Introduction

The study of computational properties of propositional modal, and related, logics has been historically concerned with estimating the size of smallest Kripke frames separating formulas from logics. The function f_L estimating, for a logic L, the size of smallest L-frames refuting L-inconsistent formulas is called (see e.g. [3, Chapter 18]) the *complexity function of* L. The interest in the complexity function of a logic L is largely due to it giving us an estimate of the running time of a decision algorithm for L-validity. In particular, provided the Kripke semantics for L is reasonably "natural," polynomiality of f_L implies that L can be shown to be polynomially equivalent to the classical propositional logic **Cl**—in the sense that the complexity of L-validity is the same as the complexity of **Cl**-validity modulo a polynomial—using a natural construction originally proposed by A. Kuznetsov [6] for the propositional intuitionistic logic **Int**, but subsequently adapted to propositional modal logics whose semantics can be described using classical propositional formulas [3, §18.1].[1]

The existence or otherwise of an inherent link between the nature of the complexity function of L and the complexity of L-validity is a natural question that, as far as we know, has not been explicitly considered in the literature. The existence, or at least plausibility, of such a link seems to be an underlying assumption in Kuznetsov's work [6]. The impression that such a link is plausible might well arise from similarities in the constructions used in establishing PSPACE-hardness of **Int** [15, 18] and the minimal normal modal logic **K** [7], on the one hand, and those used in proving the exponentiality of the complexity function of **Int** [19], [3, §18.2], [18] and **K** [1, §6.7], on the other.

A. Urquhart [16] has shown that, in propositional modal logic, enjoyment of the finite model property is compatible with undecidability. Building on Urquhart's work, E. Spaan [14,

[1] Perhaps the most widely known example of a "natural" propositional modal logic—i.e., one not purposefully constructed to exhibit a logic with a sought property—whose complexity function is polynomial but that is not polynomially equivalent to **Cl** unless NP = PSPACE is the linear-time temporal logic **LTL** [13]; the semantics of **LTL**, however, involves evaluating formulas with respect to paths rather than worlds, which precludes a straightforward application of Kuznetsov's construction.

Theorem 2.1.1] has shown that even the polynomial-size model property is compatible with undecidability; in fact, Spaan has shown that this is so even for single-variable fragments of extensions of **K**.

We further extend Spaan's results, in three respects: we show that similar connections hold, first, for arbitrary complexity classes or degrees of unsolvability, second, for even variable-free fragments, and third, for the logics higher up in the lattice of the normal modal logics (namely, extensions of **KTB**).

2 Preliminaries

We consider a propositional modal language containing a single unary modal operator $\Box$. We use the standard terminology and notation related to modal logic, as can be found in [3] and [1]. We use $\mathit{NExt}\,L$ to denote the lattice of normal extensions of the propositional modal logic L.

Recall that a propositional modal logic L is said to have the finite model property (fmp) if every L-consistent formula is satisfiable in a finite model based on an L-frame (equivalently, every formula not in L is refuted in a finite model based on an L-frame).

Given a logic L that has the fmp, the *complexity function of* L (see, e.g., [3, §18.1]) is defined by

$$f_L(n) \quad = \quad \max\big\{ \min\{|\mathfrak{F}| : \mathfrak{F} \models L,\, \mathfrak{F} \not\models \varphi\} : |\varphi| \leqslant n \text{ and } \varphi \notin L\big\},$$

where $|\mathfrak{F}|$ is the cardinality of a frame $\mathfrak{F}$ and $|\varphi|$ is the size of a formula φ. A logic L is *polynomially* (respectively, *linearly*) *approximable*, if there exists a positive constant c such that $f_L(n) \leqslant n^c$ (respectively, $f_L(n) \leqslant c \cdot n$), for sufficiently large n.

3 Main results

Given $n \geqslant 2$, let $\mathfrak{F}_n = \langle W_n, R_n\rangle$ be a Kripke frame where $W_n = \{w_0, \ldots, w_n, w^*\}$ and $R_n = \{\langle w_k, w_{k+1}\rangle : 0 \leqslant k < n\} \cup \{\langle w_0, w^*\rangle\}$. Given $n \geqslant 1$, define $\alpha_n = \Diamond\Box\bot \wedge \Diamond^n\Box\bot$.

Lemma 3.1. *Let $m, k > 2$. Then, $\mathfrak{F}_m, x \models \alpha_k$ if, and only if, $k = m$ and $x = w_0$.*

Let $\mathbb{A} = \mathbb{N} \setminus \{0, 1\}$. Given a set $I \in 2^{\mathbb{A}}$, define $2 \cdot I = \{2n : n \in I\}$, $\mathfrak{C}_I = \{\mathfrak{F}_n : n \in \mathbb{A} \setminus 2 \cdot I\}$, and $L_I = L(\mathfrak{C}_I)$. Lemma 3.1 immediately gives us the following:

Lemma 3.2. *For every $n \in \mathbb{A}$,*

$$\neg\alpha_{2n} \in L_I \iff \mathfrak{F}_{2n} \notin \mathfrak{C}_I \iff n \in I.$$

Thus, L_I-validity is as hard as the decision problem for the set I. Therefore, given any complexity class or degree of unsolvability C, with an appropriate choice of I, we obtain a C-hard normal modal logic L_I—in fact, since formulas α_n contain no propositional variables, even the variable-free fragment of L_I is C-hard.[2]

Lemma 3.3. *For every $I \subseteq \mathbb{A}$, the logic L_I is linearly approximable.*

Lemmas 3.3 and 3.2 give us the following:

Theorem 3.4. *Let C be a complexity class or a degree of unsolvability. Then, there exists a linearly approximable logic $L \in \mathit{NExt}\,\mathbf{K}$ whose constant fragment is C-hard.*

[2]For many "natural" modal logics, the complexity of their variable-free fragments coincides with the complexity of the full logic [2, 9].

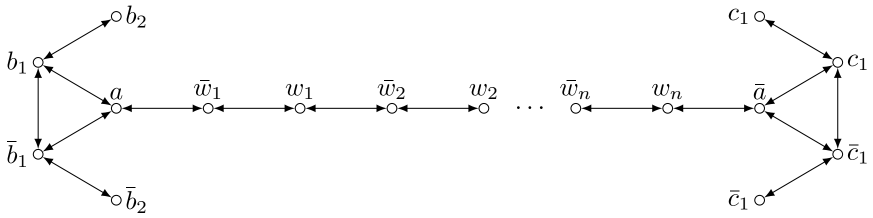

Figure 1: Frame $\mathfrak{F}_n^{rs}$

Similar examples can be constructed higher up in the lattice of the normal propositional modal logics. We illustrate this claim with extensions of the logic of reflexive and symmetric frames **KTB**.

For every $n \geqslant 1$, let $\mathfrak{F}_n^{rs} = \langle W_n, R_n \rangle$ be a Kripke frame (see Figure 1) where

$$W_n \quad = \quad \{w_1, \dots, w_n\} \cup \{\overline{w}_1, \dots, \overline{w}_n\} \cup \{a, \overline{a}, b_1, \overline{b}_1, b_2, \overline{b}_2, c_1, \overline{c}_1, c_2, \overline{c}_2\}$$

and R_n is the reflexive and symmetric closure of the relation

$$\{\langle \overline{w}_k, w_k \rangle : 1 \leqslant k \leqslant n\} \ \cup \{\langle a, \overline{w}_1, \langle \overline{a}, w_n \rangle\} \ \cup$$
$$\{\langle a, b_1 \rangle, \langle a, \overline{b}_1 \rangle, \langle b_1, b_2 \rangle, \langle \overline{b}_1, \overline{b}_2 \rangle, \langle b_1, \overline{b}_1 \rangle\} \ \cup \{\langle \overline{a}, c_1 \rangle, \langle \overline{a}, \overline{c}_1 \rangle, \langle c_1, c_2 \rangle, \langle \overline{c}_1, \overline{c}_2 \rangle, \langle c_1, \overline{c}_1 \rangle\}.$$

Recursively define the sequence of formulas

$$\begin{array}{lcl} \zeta_0 & = & \neg p \wedge \Diamond^{=2}\Box p \wedge \Diamond^{=2}\Box\neg p; \\ \zeta_{k+1} & = & \neg p \wedge \Diamond(p \wedge \Diamond\zeta_k), \end{array}$$

and define, for every $n \geqslant 1$ (letting $\Diamond^{=2}\varphi = \Diamond\Diamond\varphi \wedge \neg\Diamond\varphi$),

$$\gamma_n \quad = \quad p \wedge \Diamond^{=2}\Box p \wedge \Diamond^{=2}\Box\neg p \wedge \Diamond\zeta_n \wedge \bigwedge_{k=0}^{n-1} \neg\Diamond\zeta_k.$$

Lemma 3.5. *Let $m, k > 2$ and let x be a world in $\mathfrak{F}_m^{rs}$. Then, γ_k is satisfiable at x if, and only if, $k = m$ and $x \in \{a, \overline{a}\}$.*

Let $\mathfrak{C}_I^{rs} = \{\mathfrak{F}_n^{rs} : n \in \mathbb{A} \setminus 2 \cdot I\}$ and $L_I^{rs} = L(\mathfrak{C}_I^{rs})$. Lemma 3.5 immediately gives us the following:

Lemma 3.6. *For every $n \in \mathbb{A}$,*

$$\neg\gamma_{2n} \in L_I^{rs} \quad \Longleftrightarrow \quad \mathfrak{F}_{2n}^{rs} \notin \mathfrak{C}_I^{rs} \quad \Longleftrightarrow \quad n \in I.$$

Therefore, L_I^{rs}-validity is as hard as the decision problem for the set I. Consequently, given any complexity class or degree of unsolvability C, with an appropriate choice of I, we obtain a C-hard normal extension L_I^{rs} of **KTB**—in fact, since formulas γ_n contain only one propositional variable, even the single-variable fragment of L_I^{rs} is C-hard.[3]

Lemma 3.7. *For every $I \subseteq \mathbb{N}^+$, the logic L_I^{rs} is linearly approximable.*

Lemmas 3.7 and 3.6 give us the following:

Theorem 3.8. *Let C be a complexity class or a degree of unsolvability. Then, there exists a linearly approximable logic $L \in NExt$ **KTB** whose single-variable fragment is C-hard.*

[3] For many "natural" modal logics, the complexity of their single-variable fragments coincides with the complexity of the full logic [5, 4, 2, 17, 8, 10, 11, 12].

References

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2001.

[2] Alexander Chagrov and Mikhail Rybakov. How many variables does one need to prove PSPACE-hardness of modal logics? In *Advances in Modal Logic*, volume 4, pages 71–82, 2003.

[3] Alexander Chagrov and Michael Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.

[4] Stéphane Demri and Philippe Schnoebelen. The complexity of propositional linear temporal logics in simple cases. *Information and Computation*, 174:84–103, 2002.

[5] Joseph Y. Halpern. The effect of bounding the number of primitive propositions and the depth of nesting on the complexity of modal logic. *Aftificial Intelligence*, 75(2):361–372, 1995.

[6] A. V. Kuznetsov. Tools for detecting non-derivability or non-expressibility. In V. A. Smirnov, editor, *Logical Inference. Proceedings of the USSR Symposium on the Theory of Logical Inference*, pages 5–23. Nauka, 1979. In Russian.

[7] Richard E. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM Journal on Computing*, 6(3):467–480, 1977.

[8] Mikhail Rybakov and Dmitry Shkatov. Complexity and expressivity of branching- and alternating-time temporal logics with finitely many variables. In Bernd Fischer and Tarmo Uustalu, editors, *Theoretical Aspects of Computing, ICTAC2018*, volume 11187 of *Lecture Notes in Computer Science*, pages 396–414. Springer, 2018.

[9] Mikhail Rybakov and Dmitry Shkatov. Complexity and expressivity of propositional dynamic logics with finitely many variables. *Logic Journal of the IGPL*, 26(5):539–547, 2018.

[10] Mikhail Rybakov and Dmitry Shkatov. On complexity of propositional linear-time temporal logic with finitely many variables. In J. van Niekerk and B. Haskins, editors, *Proceedings of SAICSIT2018*, pages 313–316. ACM, 2018.

[11] Mikhail Rybakov and Dmitry Shkatov. Complexity of finite-variable fragments of propositional modal logics of symmetric frames. *Logic Journal of the IGPL*, 27(1):60–68, 2019.

[12] Mikhail Rybakov and Dmitry Shkatov. Complexity of finite-variable fragments of products with **K**. *Journal of Logic and Computation*, 31(2):426–443, 2021.

[13] A. Prasad Sistla and Edmund M. Clarke. The complexity of propositional linear temporal logics. *Journal of ACM*, 32(3):733–749, 1985.

[14] Edith Spaan. *Complexity of Modal Logics*. PhD thesis, Universiteit van Amsterdam, 1993.

[15] Richard Statman. Intuitionistic propositional logic is polynomial-space complete. *Theoretical Computer Science*, 9:67–72, 1979.

[16] Alasdair Urquhart. Decidability and the finite model property. *Journal of Philosophical Logic*, 10(3):367–370, 1981.

[17] Vítězslav Švejdar. The decision problem of provability logic with only one atom. *Archive for Mathematical Logic*, 42(8):763–768, 2003.

[18] Vítězslav Švejdar. On the polynomial-space completeness of intuitionistic propositional logic. *Archive for Mathematical Logic*, 42(7):711–716, 2003.

[19] Michael Zakharyaschev and S. V. Popov. On the complexity of Kripke countermodels in intuitionistic propositional calculus. In *Proceedings of the 2nd Soviet-Finland Logic Colloquium*, pages 32–36, 1979. In Russian.

Arithmetical applications of Baaz's generalization method

Lorenzo Sauras-Altuzarra *

Extended abstract

In mathematics, examples are very important, but not all of them are equally good. Intuitively, the more an example reflects the potential of a theorem, the better it is. In fact, if an example sufficiently represents the essence of a theorem, then it can be almost as instructive as the proof itself. Mathematical teaching via examples has many endorsements, notably Babylonian mathematics, which mainly consisted of collections of examples (see Van der Waerden [3, Chapter 3]).

Baaz's generalization method formalizes a way of measuring the quality of an example. Indeed, from a concrete example E of a certain universal theorem T, it generates another universal theorem $t(E)$, with its corresponding proof. A subsequent comparison between T and $t(E)$ may show how well E approximates T.

But there is yet another possible use of this procedure: if applied to an answer to a concrete case of an open problem (such as a proof that 641 divides the fifth Fermat number), it will output a result that can be particularized to a partial answer to the question (such as a sufficient condition for a number to be a divisor of an arbitrary Fermat number).

This talk will explain in detail how to apply this method in elementary number theory. A general explanation of the algorithm is given by Baaz [1].

In addition, if time permits, some arithmetical results (and derived open problems) will be commented, for example:

1. $k \cdot 2^s + 1 \mid F_n$, for every $k, n, r, s \in \mathbb{N}^+$ such that $r \cdot s \leq 2^{n-1}$ and $k \cdot 2^s + 1 \mid k^{2 \cdot r} + 2^{2^n - 2 \cdot r \cdot s}$;
2. $i \mid F_n$, for every $c, i, n \in \mathbb{N}^+$ such that $i \mid (2^{2^{n-1}} - i \cdot c)^2 + 1$; and
3. $2^{2^n - 4 \cdot (n+2)} + i^4 \mid F_n$, for every $i, n \in \mathbb{N}^+$ such that $n > 4$ and $i \cdot 2^{n+2} + 1 = 2^{2^n - 4 \cdot (n+2)} + i^4$

(F_n denotes the n^{th} Fermat number). For proofs of these theorems and some other associated questions, see Sauras-Altuzarra [2].

References

[1] M. Baaz, Note on the generalization of calculations, *Theoret. Comput. Sci.* **224** (1999), 1–2.

[2] L. Sauras-Altuzarra, Some arithmetical problems that are obtained by analyzing proofs and infinite graphs, arXiv:2002.03075 (2020).

[3] B. L. Van der Waerden, *Science Awakening*, Oxford University Press, 1961.

*Institut für Diskrete Mathematik und Geometrie, TU Wien, Wiedner Hauptstrasse 8-10/104, A-1040, Vienna, Austria. E-mail: lorenzo.sauras@tuwien.ac.at.

Boxing modal logics

Valentin Shehtman[123]

[1] Institute for Information Transmssion Problems, RAS
shehtman@netscape.net
[2] National Research University Higher School of Economics
[3] Moscow State University

We consider normal 1-modal logics, propositional and predicate. For the basic definitions cf. [1], [2].

1 Propositional logics

For a set of modal formulas Γ, put

$$\Box\Gamma := \{\Box A \mid A \in \Gamma\}.$$

For a modal propositional logic L put

$$\Box{\cdot}L := \mathbf{K} + \Box L.$$

Lemma 1.1. $\Box{\cdot}(\mathbf{K} + \Gamma) = \mathbf{K} + \Box\Gamma$.

It turns out that $\Box{\cdot}L$ inherits many properties of L.

Theorem 1.2.

- *If L is Kripke complete, then $\Box{\cdot}L$ is Kripke complete.*
- *If L is strongly Kripke complete, then $\Box{\cdot}L$ is strongly Kripke complete.*
- *If L is canonical, then $\Box{\cdot}L$ is canonical.*
- *If L has the FMP, then $\Box{\cdot}L$ has the FMP.*
- *If L is locally tabular, then $\Box{\cdot}L$ is locally tabular.*
- *If L is has a finite modal depth, then $\Box{\cdot}L$ has a finite modal depth:*

$$md(\Box{\cdot}L) \leq md(L) + 1.$$

Hence, in particular, we obtain many new examples of locally tabular logics.

Corollary 1.3. *The logics $\mathbf{K} + \Box^n(p \to \Box p)$ (and all their extensions) are locally tabular.*

Another consequence is the FMP for some logics of trees. Recall that a *tree* (irreflexive and intransitive) is a rooted frame, in which every point (but the root) has a unique predecessor. A *reflexive tree* is a reflexive closure of a tree.

Theorem 1.4. *The logic of every serial tree has the FMP.*

Theorem 1.5. *The logic of every reflexive tree has the FMP.*

Theorem 1.6. *The logic of every tree validating*

$$\Diamond\top \to \Diamond^2\top \wedge \Diamond\Box\bot$$

has the FMP.

2 Predicate logics

Recall that $\mathbf{Q\Lambda}$ is the minimal predicate extension of a propositional logic $\mathbf{\Lambda}$; $\mathbf{T} = \mathbf{K} + \Box p \to p$.

For a predicate modal logic L we also define boxing:

$$\Box{\cdot}L := \mathbf{QK} + \Box L.$$

For modal predicate logics a direct analogue of Lemma 1.1 does not hold. It is replaced by the following

Lemma 2.1. *$\Box{\cdot}(\mathbf{QT} + \Gamma) = \mathbf{QT} + \Box\Gamma + \Box\forall ref$, where*

$$\Box\forall ref := \Box\forall x(\Box P(x) \to P(x)).$$

Axiomatization of boxing in other cases remains an open problem.

Definition 2.2. *A* predicate modal theory *Γ is a set of closed predicate modal formulas with constants.*

A predicate modal theory Γ is satisfiable *in a predicate Kripke frame $\mathbf{F}$ if there exists a Kripke model M over $\mathbf{F}$, a world w in M and a map δ from constants of Γ to the domain of w such that $M, w \vDash \delta \cdot \Gamma$.*

A predicate modal logic L is strongly Kripke complete *if every L-consistent countable theory Γ is satisfiable in a Kripke frame validating L.*

Theorem 2.3. *Let $\mathbf{\Lambda}$ be a modal propositional logic containing $\mathbf{T}$. If $\mathbf{Q\Lambda}$ is strongly Kripke complete, then $\Box{\cdot}\mathbf{Q\Lambda}$ is strongly Kripke complete.*

There are several well-known examples of logics $\mathbf{\Lambda}$ above $\mathbf{T}$, for which $\mathbf{Q\Lambda}$ is strongly Kripke complete: $\mathbf{T}$, $\mathbf{S4}$, $\mathbf{S5}$, $\mathbf{S4.2}$, $\mathbf{S4.3}$, $\mathbf{Triv}$. So in these cases boxing preserves strong Kripke completeness.

The definition of strong completeness can be extended to Kripke sheaf semantics. Then we can prove a better result:

Theorem 2.4. *If a predicate modal logic L is strongly Kripke sheaf complete, then $\Box \cdot L$ is strongly Kripke sheaf complete.*

On the other hand, quite often logics of the form $\mathbf{QK} + \Box\Gamma$ are Kripke (and Kripke sheaf) incomplete. In particular, we have

Theorem 2.5. *If $\mathbf{\Lambda}$ is any consistent modal propositional logic containing $\mathbf{T}$, then $\mathbf{Q}(\Box{\cdot}\mathbf{\Lambda})$ is Kripke incomplete, and $\Box{\cdot}(\mathbf{Q\Lambda}) = \mathbf{Q}(\Box{\cdot}\mathbf{\Lambda}) + \Box\forall ref$ is its Kripke completion.*

References

[1] A. Chagrov, M. Zakharyaschev. *Modal Logic.* Oxford University Press, 1997.

[2] D. Gabbay, V. Shehtman, D. Skvortsov. *Quantification in nonclassical logic*, Vol. 1. Elsevier, 2009.

[3] V. Shehtman. *Bisimulation games and locally tabular logics.* Russian Mathematical Surveys, 71(5), pp. 979-981, 2016.

[4] V. Shehtman. *On Kripke completeness of modal predicate logics around quantified* **K5**. Submitted.

Platform-independent model of fix-point arithmetic for verification of the standard mathematical functions

Nikolay V. Shilov[1] and Dmitry A. Kondratyev[2]

[1] Innopolis University, Innopolis, Russia
shiloviis@mail.ru
[2] A.P. Ershov Institute of Informatics Systems, Novosibirsk, Russia
apple-66@mail.ru

Abstract

In the talk we present axiomatic of fix-point computer arithmetics that we use in our platform-independent incremental combined approach to specification and verification of the standard functions `sqrt`, `cos` and `sin` that implement mathematical functions $\sqrt{}$, cos and sin.

1 Introduction

One who has a look at verification research and practice may observe that there exist *verification in large (scale)* and *verification in small (scale)*: verification in large deals (usually) behavioral properties of large-scale complex critical systems like the *Curiosity* Mars mission [4], while verification in small addresses (usually) functional properties of small programs like computing the standard trigonometry functions [3, 2].

Our research "Platform-independent approach to formal specification and verification of standard mathematical functions" deals with *verification in small.* It may look like that it is about the same topic as [3, 2] i.e. formal verification of the standard computer functions that implement mathematical functions. But there are serious differences between [3, 2] and our research project.

Our research project is aimed onto a development of an incremental combined approach to the specification and verification of the standard mathematical functions. Platform-independence means that we attempt to design a relatively simple axiomatization of the computer arithmetic in terms of real, rational, and integer arithmetic (i.e. the fields $\mathbb{R}$ and $\mathbb{Q}$ of real and rational numbers, the ring $\mathbb{Z}$ of integers) but don't specify neither base of the computer arithmetic, nor a format of numbers' representation. Incrementality means that we start with the most straightforward specification of the simplest easy to verify algorithm in real numbers and finish with a realistic specification and a verification of an algorithm in computer arithmetic. We call our approach combined because we start with a manual (pen-and-paper) verification of some selected algorithm in real numbers, then use these algorithm and verification as a draft and proof-outlines for the algorithm in computer arithmetic and its manual verification, and finish with a computer-aided validation of our manual proofs with some proof-assistant system (to avoid appeals to "obviousness" that are very common in human-carried proofs).

2 A Brief of the Approach Results

In our approach we start with easy-to-verify Hoare total correctness assertions [1] for logical specification of imperative algorithms that implements the computer functions in "ideal" real arithmetic, and finish with computer-aided verification of the computer functions in computer fix-point arithmetic. Full details of our approach can be found in [6, 5].

In a journal (Russian) paper [6] an *adaptive* imperative algorithm implementing the Newton-Raphson method for a square root function $\sqrt{\ }$ has been specified by total correctness assertions and verified manually using Floyd-Hoare approach in both fix-point and floating-point arithmetics; the post-condition of the total correctness assertion states that the final overall truncation error is not greater that $2ulp$ where ulp is *Unit in the Last Place* — the unit of the last meaningful digit.

The paper [6] has reported also two steps towards computer-aided validation and verification of the used adaptive algorithm. In particular, an implementation of a fix-point data type according to the axiomatization can be found at `https://bitbucket.org/ainoneko/lib_verify/src/`; ACL2 computer-carried proofs of (i) the consistency of the computer fix-point arithmetic axiomatization, and (ii) the existence of a look-up table with initial approximations for $\sqrt{\ }$ are available at `https://github.com/apple2-66/c-light/tree/master/experiments/square-root`.

In a work-in-progress electronic preprint [5] platform-independent and incremental approach is applied for manual (pen-and-paper) verification (using Floyd-Hoare approach) of the computer functions `cos` and `sin` (that implement mathematical trigonometric functions cos and sin) for fix-point argument values in the rage $[-1, 1]$ (in radian measure); the post-condition of the total correctness assertion states that the final overall truncation error is not greater that $\frac{3n\times ulp}{2(1-ulp)}$ where $n = O(|\ln \varepsilon|)$ and $\varepsilon > 0$ is user-defined computational error (in ideal real arithmetic).

3 Fix-point Arithmetic

Below we present version axiomatization (modulo "ideal" arithmetic of real, rational and integer numbers) of a computer (platform-independent) fix-point arithmetic data type as in [6]. (Please remark that we explicitly admit that there may be several different fix-point data types simultaneously.)

A fix-point data-type (with Gaussian rounding) $\mathbb{D}$ satisfies the following axioms.

- The set of values $Val_\mathbb{D}$ is a finite set of rational numbers $\mathbb{Q}$ (and reals $\mathbb{R}$) such that
  - it contains the least $\inf_\mathbb{D} < 0$ and the largest $\sup_\mathbb{D} > 0$ elements,
  - altogether with
    - all rational numbers in $[\inf_\mathbb{D}, \sup_\mathbb{D}]$ with a step $\delta_\mathbb{D} > 0$,
    - all integers $Int_\mathbb{D}$ in the range $[-\inf_\mathbb{D}, \sup_\mathbb{D}]$.
- Admissible operations include machine addition $\oplus$, subtraction $\ominus$, multiplication $\otimes$, division $\oslash$, integer rounding up $\lceil\ \rceil$ and down $\lfloor\ \rfloor$.

  Machine addition and subtraction. If the exact result of the standard mathematical addition (subtraction) of two fix-point values falls within the interval $[\inf_\mathbb{D}, \sup_\mathbb{D}]$, then machine addition (subtraction respectively) of these arguments equals to the result of the mathematical operation (and notation $+$ and $-$ is used in this case).

  Machine multiplication and division. These operations return values that are nearest in $Val_\mathbb{D}$ to the exact result of the corresponding standard mathematical operation: for any $x, y \in Val_\mathbb{D}$

- if $x \times y \in Val_{\mathbb{D}}$ then $x \otimes y = x \times y$;
- if $x/y \in Val_{\mathbb{D}}$ then $x \oslash y = x/y$;
- if $x \times y \in [\inf_{\mathbb{D}}, \sup_{\mathbb{D}}]$ then $|x \otimes y - x \times y| \leq \delta_{\mathbb{D}}/2$;
- if $x/y \in [\inf_{\mathbb{D}}, \sup_{\mathbb{D}}]$ then $|x \oslash y - x/y| \leq \delta_{\mathbb{D}}/2$;

Integer rounding up and down are defined for all values in $Val_{\mathbb{D}}$.

- Admissible binary relations include all standard equalities and inequalities (within $[\inf_{\mathbb{D}}, \sup_{\mathbb{D}}]$) denoted in the standard way $=$, $\neq$, $\leq$, $\geq$, $<$, $>$.

References

[1] K.R. Apt, F.S. de Boer, and E.-R. Olderog. *Verification of Sequential and Concurrent Programs.* Springer-Verlag, 2009.

[2] G. Grohoski. Verifying oracle's sparc processors with acl2. slides of the invited talk for 14th international workshop on the acl2 theorem prover and its applications. http://www.cs.utexas.edu/users/moore/acl2/workshop-2017/slides-accepted/grohoski-ACL2_talk.pdf, 2017.

[3] J. Harrison. Formal verification of floating point trigonometric functions. *Lecture Notes in Computer Science*, 1954:217–233, 2000.

[4] G.J. Holzmann. Mars code. *Commun. ACM*, 57(2):64–73, 2014.

[5] N. V. Shilov, B. L. Faifel, S. O. Shilova, and A. V. Promsky. Towards platform-independent specification and verification of the standard trigonometry functions. arXiv:1901.03414, https://arxiv.org/abs/1901.03414, 2019.

[6] N. V. Shilov, D. A. Kondratyev, I. S. Anureev, E. V. Bodin, and A. V. Promsky. Platform-independent specification and verification of the standard mathematical square root function. *Modeling and Analysis of Information Systems*, 25(6):637–666, 2018. (In Russian. English translation: Automatic Control and Computer Sciences, 2019, Vol. 53, No. 7, pp. 595–616.).

On a Possibility of Finite Characterizations for Kripke Complete Non-Recursively Axiomatizable Superintuitionistic Predicate Logics

Dmitrij Skvortsov

It is well known that there exist many natural classes $\mathcal{F}$ of predicate Kripke frames for which the corresponding superintuitionistic predicate logics $\mathbf{L}[\mathcal{F}]$ are non-recursively axiomatizable (see e.g. [1, 2, 3]). Here we consider a possibility of finite semantical characterizations for some logics of this kind.

Denote the superintuitionistic predicate logic of a (predicate) Kripke frame F (i.e., the set of formulas valid in F) by $\mathbf{L}F$. Recall that a predicate logic $\mathbf{L}$ is called *Kripke complete* if $\mathbf{L} = \mathbf{L}[\mathcal{F}]$ for some class $\mathcal{F}$ of Kripke frames, where $\mathbf{L}[\mathcal{F}] = \bigcap(\mathbf{L}F : F \in \mathcal{F})$. The *Kripke completion* of a logic $\mathbf{L}$ is the smallest (w.r.t. the inclusion) Kripke complete extension of $\mathbf{L}$.

Let $\mathbf{L}$ and $\mathbf{L}_0$ be two predicate logics, $\mathbf{L}$ being Kripke complete. We say that $\mathbf{L}_0$ *semantically generates* $\mathbf{L}$ if $\mathbf{L}$ is the Kripke completion of $\mathbf{L}_0$.

A logic $\mathbf{L} = \mathbf{L}[\mathcal{F}]$ of a class $\mathcal{F}$ of Kripke frames is called *finitely* (or *recursively*) *semantically generated* if there exists a finitely (or, resp., recursively) axiomatizable logic $\mathbf{L}_0$ semantically generating $\mathbf{L}$. In this case we also can say that the class $\mathcal{F}$ itself is finitely (resp., recursively) semantically generated. This notion can be regarded as a semantical analogue to finite / recursive axiomatizability; namely, semantical comletion (i.e., 'semantical closure') is used instead of deductive closure (i.e., deductive consequence). Clearly, this notion can be interesting only for natural classes of Kripke frames.

Obviously, every finitely / recursively axiomatizable Kripke complete logic is trivially finitely / recursively semantically generated: just take $\mathbf{L}_0 = \mathbf{L}$. Here we show that natural Kripke complete non-recursively axiomatizable logics can be still finitely (hence, recursively) semantically generated.

Proposition. *Let* $\mathbf{L} = \mathbf{L}[\mathcal{F}]$ *be the logic of a class of Kripke frames* $\mathcal{F}$*, let* $\mathbf{L}_0$ *be a predicate logic, and let* $\mathcal{F}_{\mathbf{L}_0}$ *be the class of Kripke frames validating* $\mathbf{L}_0$*. Then:* $\mathbf{L}_0$ *semantically generates* $\mathbf{L}$ *iff the following two conditions hold:*

(i) $\mathcal{F} \subseteq \mathcal{F}_{\mathbf{L}_0}$ *or, equivalently,* $\mathbf{L}_0 \subseteq \mathbf{L}(= \mathbf{L}[\mathcal{F}])$*;*

(ii) $\mathbf{L} \subseteq \mathbf{L}F$ *for every frame* F *from* $\mathcal{F}_{\mathbf{L}_0} \backslash \mathcal{F}$*.*

Proof. Let $\mathbf{L}^* = \mathbf{L}[\mathcal{F}_{\mathbf{L}_0}]$ be the Kripke completion of $\mathbf{L}_0$; hence

$\mathbf{L}_0$ semantically generates $\mathbf{L}$ iff (0) $\mathbf{L}^* = \mathbf{L}$.

The condition (ii) means that $\mathbf{L} \subseteq \mathbf{L}[\mathcal{F}_{\mathbf{L}_0} \backslash \mathcal{F}]$.

Therefore, the condition (0) obviously implies (ii), as well as (i).

On the other hand, (i) implies that $\mathbf{L}^* = \mathbf{L} \cap \mathbf{L}[\mathcal{F}_{\mathbf{L}_0} \backslash \mathcal{F}]$, so (ii) gives (0). $\square$

Let $\mathbf{Fin}^c$ be the class of frames with constant domains over finite posets; let $\mathbf{P}^c_\infty$ be the class of frames of finite height with constant domains; finally, let $\mathbf{WF}^c_d$ be the class of frames with constant domains over dually well-founded (i.e., Nötherian) posets.

Theorem. *(The logics of) the classes of frames* $\mathbf{Fin}^c$*,* $\mathbf{P}^c_\infty$*,* $\mathbf{WF}^c_d$ *are finitely semantically generated.*

Cf. [4], Proposition 1 and footnote 9 from Sect. 3.2, and Lemma 3 from Sect. 4.3. Note that here the condition (*ii*) from our Proposition follows from the subsequent claim (cf. [4], Corollary 2 from Sect. 3.1):

Claim. $\mathbf{L}[\mathbf{Fin}^c] \subseteq \mathbf{L}F$ *for every frame* F *with a finite constant domain.*

We do not know if the logics of the corresponding classes of frames with expanding domains are finitely (or at least recursively) semantically generated.

References

[1] Skvortsov D. On axiomatizability of some intermediate predicate logics (summary). Reports on Math. Logic 22 (1988), pp. 115–116.

[2] Skvortsov D. The predicate logic of finite Kripke frames is not recursively axiomatizable. Journal of Symbolic Logic 70 (2005), pp. 451–459.

[3] Skvortsov D. On non-axiomatizability of superintuitionistic predicate logics of some classes of well-founded and dually well-founded Kripke frames. Journ. of Logic and Computation 16, No.5 [Special Issue: Computer Science Applications of Modal Logic (ed.V.B.Shehtman)] (2006), pp.685-695.

[4] Skvortsov D. A remark on the superintuitionistic predicate logic of Kripke frames of finite height with constant domains: A simpler Kripke complete logic that is not strongly complete. Advances in Modal Logic, Vol.12, College Publications, 2018, pp.577-590 [Intern. Conf. AiML 2018, 27 - 31 Aug. 2018, Bern, Switzerland].

Definability of graph properties in modal languages

Vladislav Slyusarev[1]

Moscow Institute of Physics and Technology, Moscow, Russia
vladislav.slyusarev@phystech.edu

Graphs provide a descriptively effective theoretical framework for a lot of branches of mathematics and computer science. Despite the high level of generality from the descriptive point of view, graph theory lacks common methods that could be used to check arbitrary graph properties in a similar way. Usually each graph problem has to be solved and each graph property has to be tested with a specific method that usually does not generalize to other different problems or properties.

In the last few decades, logical languages, and particularly *modal languages*, have attracted the attention as the means for generalized solution of graph problems. Modal logics are preferred due to Kripke semantics that allow to evaluate modal formulas in structures that are essentially directed graphs. Graph properties can be defined in modal languages through the concept of *validity in a frame*. Given a graph G and a formula ϕ, we use the notation $G \models \phi$ for "ϕ is valid in G", considering a directed graph as a Kripke frame with a single relation defined by the edge set of G. The problem of determining if $G \models \phi$ for a given modal formula ϕ and a graph G, known as the *frame-checking problem*, is decidable. Its computational complexity is estimated in [3]: it is shown that the frame-checking problem is $PTIME$ (linear) in the length of formula and $EXPTIME$ in the size of frame.

There are several approaches to define a graph property with logical formulas.

1. A property is called finitely definable if there exists a single formula ϕ such that a graph G has the desired property if and only if $G \models \phi$.

2. A property is called definable if there exists a (possibly infinite) set of formulas Φ such that a graph G has the desired property if and only if $\forall \phi \in \Phi\ (G \models \phi)$.

3. A property is called co-definable if there exists a (possibly infinite) set of formulas Φ such that a graph G has the desired property if and only if $\exists \phi \in \Phi\ (G \models \phi)$.

In particular, a graph property is co-definable if there exists a countable set of formulas $\{\phi_n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ any graph G_n of cardinality n has the property is and only if $G_n \models \phi_n$. This important special case is a common means of defining complex graph properties such as being Hamiltonian.

A study of graph properties from the point of view of modal definability is presented in [1], [2], and [3]. In [1], where the problem of modal definability for graph properties was first stated, the author proves that the basic modal language fails to express several important properties, such as being connected, planar or Eulerian. Extensions of the modal language, including hybrid logics, are discussed in [3], and examples of formulas defining graph properties are given. These results are generalized in [2]: it is shown that every NP graph property is co-definable (with a countable set of formulas dependent on graph size) in the language $\mathcal{FHL}$ (Full Hybrid Logics).

The latter result shows the effectiveness of the chosen method but still can be improved and extended. First, the considered language is complex and therefore computationally non-optimal compared to the known algorithms for several graph properties. Second, it does not cover the important question of finite definability for graph properties.

In this talk we will discuss different approaches to defining graph properties, such as acyclicity, connectivity, planarity, etc., with the language of modal logics and its extensions. We consider the basic modal language $\mathcal{ML}$ with propositional variables, Boolean connectives and the modal operator $\Diamond$:

$$\mathcal{ML} ::= \top \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi.$$

Then we define more expressive languages by enriching it with additional modal operators and symbols, including:

1. the inverse modality $\Diamond^{-1}$;
2. the transitive modality $\Diamond^{+}$;
3. the inverse transitive modality $\Diamond^{-}$;
4. the global modality E [4];
5. the nominals i and the satisfaction operators $@_i$ [4];
6. the state variables x and the bind operators $\downarrow x.\cdot$ [4].

We combine these extensions to define the extended modal languages as follows:

$$\begin{aligned}
\mathcal{ML}^{\pm} &::= \top \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid \Diamond^{+}\phi \mid \Diamond^{-1}\phi \mid \Diamond^{-}\phi;\\
\mathcal{PH} &::= \top \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi;\\
\mathcal{PH}(@) &::= \top \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid @_i\phi;\\
\mathcal{PH}(\mathsf{E}) &::= \top \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid E\phi;\\
\mathcal{PH}(@,\downarrow) &::= \top \mid i \mid x \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid @_i\phi \mid @_x\phi \mid \downarrow x.\phi;\\
\mathcal{H} &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi;\\
\mathcal{H}(@) &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid @_i\phi;\\
\mathcal{H}(\mathsf{E}) &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid E\phi;\\
\mathcal{H}(@,\downarrow) &::= \top \mid p \mid i \mid x \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid @_i\phi \mid @_x\phi \mid \downarrow x.\phi;\\
\mathcal{HGL} &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid \Diamond^{+}\phi \mid @_i\phi;\\
\mathcal{HGL}(\downarrow) &::= \top \mid p \mid i \mid x \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid \Diamond^{+}\phi \mid @_i\phi \mid @_x\phi \mid \downarrow x.\phi;\\
\mathcal{FHL} &::= \top \mid p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi \mid \Diamond^{-1}\phi \mid E\phi \mid @_i\phi \mid @_x\phi \mid \downarrow x.\phi.
\end{aligned}$$

The study of various modal languages in the context of graph property definability has proved to yield some valuable notions and examples, illustrating their comparative expressiveness. We will discuss the known results in this field, including the following.

1. No useful graph properties are definable in $\mathcal{ML}$ and $\mathcal{ML}^{\pm}$.

2. Full graphs and graphs without loops are finitely-definable in $\mathcal{PH}$ (and, consequently, in richer languages) [4].

3. Strongly connected graphs are finitely-definable in $\mathcal{H}(\mathsf{E})$.

4. Weak connectivity, k-regularity, clique number, independence number, diameter, radius, girth, minimum and maximum vertex degree are not definable in $\mathcal{H}(@)$

5. Strongly connected and acyclic graphs are finitely-definable in $\mathcal{HGL}$ [3].

6. Hamiltonian graphs are co-definable in $\mathcal{HGL}$ [3].

7. Every graph property in NP is co-definable in $\mathcal{FHL}$.

We will compare different extensions of the modal language from the point of view of graph problems, and provide proof sketches for the main results.

References

[1] M. R. F. Benevides. Modal logics for finite graphs. *Logic for concurrency and synchronisation*, 2003.

[2] F. M. Ferreira, C. M. Freire, M. R. F. Benevides, L. M. Schechter, and A. T. Martins. Hybrid logics and NP graph properties. 2011.

[3] L. M. Schechter M. R. F. Benevides. Using modal logics to express and check global graph properties. 2009.

[4] Balder ten Cate. Model theory for extended modal languages. 2005.

TOWARDS CAPTURING PTIME WITH NO COUNTING CONSTRUCT (BUT WITH A VERSION OF HILBERT'S CHOICE OPERATOR EPSILON)

EUGENIA TERNOVSKA

The central open question in Descriptive Complexity is whether there is a logic that characterizes deterministic polynomial time (PTIME) on relational structures. In this talk, I introduce my work on this question. I define a logic that is obtained from first-order logic with fixed points, FO (FP), by a series of transformations that include restricting logical connectives and adding a dynamic version of Hilbert's Choice operator Epsilon. The formalism can be viewed either as an algebra of binary relations or a linear-time modal dynamic logic, where algebraic expressions describing "proofs" or "programs" appear inside the modalities.

Many typical polynomial time properties such as cardinality, reachability and those requiring "mixed" propagations (that include linear equations modulo two) are axiomatizable in the logic, and an arbitrary PTIME Turing machine can be encoded. For each fixed Choice function, the data complexity of model checking is in PTIME. However, there can be exponentially many such functions. "Naive evaluations" refer to a version of this model checking procedure where the Choice function variable Epsilon is simply treated as a constant. A crucial question is under what syntactic conditions on the algebraic terms such a naive evaluation works, that is, provides a certain answer to the original model checking problem. The two views of the formalism support application of both automata-theoretic and algebraic techniques to the study of this question.

Simon Fraser University, Canada
Email address: `ter@sfu.ca`

Unrefutability by clause set cycles

Joint work with Stefan Hetzl

Jannik Vierling

Vienna University of Technology
Institute of Discrete Mathematics and Geometry

jannik.vierling@tuwien.ac.at

The subject of automated inductive theorem proving (AITP) aims at automating the process of finding proofs by mathematical induction. The automation of proof by mathematical induction has applications in formal methods for software engineering and in the formalization of mathematics. A great variety of methods has been developed for the automation of proof by induction. Typically each method operates in a more or less different setting. Furthermore, the design of these methods is driven by efficiency and ease of automation and therefore many AITP methods exist mainly at lower levels of abstraction. Because of their technical nature, AITP methods are traditionally analyzed empirically, and formal results backing the empirical observations are still rare. In particular, there are currently only very few negative results and it is difficult to classify AITP systems by their strength.

We address this situation by analyzing AITP methods formally. The first step of such an analysis consists in abstracting an AITP method, or a family of methods, by a logical theory. Such an abstraction can then be analyzed by applying results and techniques from mathematical logic. In this way we can measure the strength of AITP systems and compare them with each other. Furthermore, abstracting AITP systems by logical theories allows us to obtain negative results which are particularly valuable in revealing the logical features that a given method lacks.

The n-clause calculus [KP13] is an AITP method that extends the superposition calculus by a cycle detection mechanism. A cycle detected by the n-clause calculus represents an argument by infinite descent that establishes the inconsistency of the given clause set and thus terminates the refutation. In [HV20] we have analyzed the n-clause calculus by abstracting its comparatively technical cycle detection mechanism by the notion of clause set cycles. In the following we will recall the notion of refutation by a clause set cycle and some important results. By $0/0$ and $s/1$ we denote the function symbols representing the natural number 0 and the successor function for natural numbers, respectively. Furthermore, we fix a special, fresh constant symbol η on which arguments by infinite descent take place.

Definition 1. *Let L be a first-order language. An $L \cup \{\eta\}$ clause set $\mathcal{C}(\eta)$ is called an L clause set cycle if it satisfies the following conditions*

$$\mathcal{C}(s(\eta)) \models \mathcal{C}(\eta), \tag{C1}$$

$$\mathcal{C}(0) \models \bot. \tag{C2}$$

An $L \cup \{\eta\}$ clause set $\mathcal{D}(\eta)$ is refuted by a clause set cycle $\mathcal{C}(\eta)$ if

$$\mathcal{D}(\eta) \models C(\eta). \tag{C3}$$

By dualizing the definition of clause set cycles and observing that clause set cycles are essentially parameter-free we can we can show that refutations by a clause set cycle can be simulated by the parameter-free induction rule for $\exists_1$ formulas.

Theorem 2 ([HV])**.** *Let $\mathcal{D}(\eta)$ be an $L \cup \{\eta\}$ clause set. If $\mathcal{D}(\eta)$ is refuted by an L clause set cycle, then $[\varnothing, \exists_1(L)^{-}\text{-IND}^R] + \mathcal{D}(\eta)$ is inconsistent.*

As mentioned above this upper bound is optimal in terms of the quantifier complexity of the induction formulas. In other words clause set cycles cannot be simulated by quantifier-free induction.

Theorem 3 ([HV20])**.** *There exists a language L and an $L \cup \{\eta\}$ clause set $\mathcal{D}(\eta)$ such that $\mathcal{D}(\eta)$ is refuted by an L clause set cycle, but* $\text{Open}(L)\text{-IND} + \mathcal{D}(\eta)$ *is consistent.*

These results give rise to the question whether clause set cycles are at least as strong as induction for quantifier-free formulas. Empirical evidence has led us to conjecture that refutation by a clause set cycle is incomparable with induction for quantifier-free formulas.

In this talk we will show that this conjecture has a positive answer. We define a candidate clause set in the setting of linear arithmetic. The language of linear arithmetic consists of the symbols $0/0$, $s/1$, $p/1$, and $+/2$, where the latter two represent the predecessor function and the addition of natural numbers, respectively. Let $\mathcal{T}$ be the theory axiomatized by the universal closure of $0 \neq s(x)$ and the defining equations of $p/1$ and $+/2$, then the clause set $\mathcal{I}(\eta)$ is given by

$$\mathcal{I}(\eta) \coloneqq \mathit{cnf}(\mathcal{T}) \cup \{\{\eta + \eta = \eta\}, \{\eta \neq 0\}\}.$$

Intuitively, the clause set $\mathcal{I}(\eta)$ asserts the existence of a non-zero additive idempotent. By making use of the upper bound of Theorem 2 we can show the unrefutability of $\mathcal{I}(\eta)$ by a clause set cycle by proving the following independence result.

Theorem 4. $[\mathcal{T}, \exists_1(L(\mathcal{T}))^{-}\text{-IND}^R] \nvdash x + x = x \rightarrow x = 0$.

We will proceed by constructing a model M with non-zero idempotents whose domain consists of one copy of $\mathbb{N}$ and $|\mathbb{N}|$ copies of $\mathbb{Z}$. In particular, we will show that for every true, p-free, $\exists_1$ formula $\varphi(x)$, there exists on every non-standard chain an infinite, strictly descending sequence of elements $(z_i)_{i\in\mathbb{N}}$ such that

$$M \models \varphi(z_i), \text{ for all } i \in \mathbb{N}.$$

The unrefutability of $\mathcal{I}(\eta)$ by clause set cycles shows that clause set cycles are very weak and can not even deal with formulas such as $x+x = x \rightarrow x = 0$ that have a straightforward proof by quantifier-free induction. However, the situation may even be much worse. A clause set cycle $\mathcal{C}(\eta)$ corresponds roughly speaking to an inductive $\exists_1$ lemma $\varphi_{\mathcal{C}}(x)$. However, the notion of refutation by a clause set cycle only uses this lemma to infer the instance $\varphi_{\mathcal{C}}(\eta)$. Therefore, we conjecture that refutation by a clause set cycle is even incomparable with parameter-free induction for quantifier-free formulas. The intuition for this is, that proving a sentence like $0 + (\eta + \eta) = \eta + \eta$ requires the lemma $0 + x = x$ and the instance $x \mapsto \eta + \eta$. The conjectured relations between the refutational strength of clause set cycles and some related theories with induction are shown in Figure 1.

References

[HV] Stefan Hetzl and Jannik Vierling. An unprovability result for clause set cycles. In preparation.

[HV20] Stefan Hetzl and Jannik Vierling. Clause Set Cycles and Induction. *Logical Methods in Computer Science*, 16(4):11:1–11:17, November 2020.

[KP13] Abdelkader Kersani and Nicolas Peltier. Combining superposition and induction: A practical realization. In Pascal Fontaine, Christophe Ringeissen, and Renate A. Schmidt, editors, *Frontiers of Combining Systems*, volume 8152 of *Lecture Notes in Computer Science*, pages 7–22. Springer, 2013.

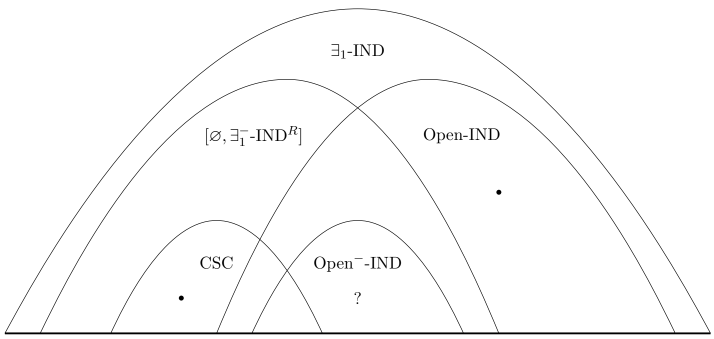

Figure 1: Conjectured relation between the refutational strength of various induction systems. The dots indicate that the surrounding area is not empty, the question mark indicates that we conjecture the are to be non-empty.

Presburger Arithmetic and Visser's Conjecture

Alexander Zapryagaev

National Research University Higher School of Economics, 6, Usacheva Str., Moscow, 119048, Russian Federation

Abstract

Presburger Arithmetic the true theory of natural numbers with addition. We show that the interpretations of Presburger Arithmetic in itself are definably isomorphic to the trivial one, confirming the conjecture of A. Visser. To prove that, we develop a characterization of linear orderings interpretable in $(\mathbb{N}, +)$. We show that all interpretable linear orderings can be expressed as a restriction of the lexicographical ordering on $\mathbb{Z}^k$ for some k to some Presburger-definable set. This generalizes the results of [10] where the one-dimensional result was proven.

This talk is based on a joint work with Fedor Pakhomov.

Presburger Arithmetic **PrA** [7] is the true theory of natural numbers with addition. Unlike Peano Arithmetic **PA**, it is complete, decidable and admits quantifier elimination in an extension of its language.

A *reflexive* arithmetical theory ([9, p. 13]) is a theory that can prove the consistency of all its finitely axiomatizable subtheories. Peano Arithmetic **PA** and Zermelo-Fraenkel set theory **ZF** are among well-known reflexive theories. For sequential theories reflexivity implies that the theory cannot be interpreted in any of its finite subtheories. A. Visser has conjectured that this purely interpretational-theoretic property holds for **PrA** as well. Note that Presburger Arithmetic, unlike sequential theories, cannot encode tuples of natural numbers by single natural numbers, and thus, for interpretations in Presburger Arithmetic it is important whether individual objects are interpreted by individual objects (one-dimensional interpretations) or by tuples of objects of some fixed length m (m-dimensional interpretations).

As shown in [11], Visser's conjecture follows from the following statement:

Theorem 1. *Let $\mathfrak{A}$ be a model of* **PrA** *interpreted in $(\mathbb{N}, +)$. Then $\mathfrak{A}$ is isomorphic to $(\mathbb{N}, +)$ and, moreover, the isomorphism is definable in $(\mathbb{N}, +)$.*

In the work [10], we have established that Visser's conjecture holds for one-dimensional interpretations. We establish that by studying the interpretation of the ordering on $\mathfrak{A}$ induced by the interpretation.

We showed that each linear order that is interpretable in $(\mathbb{N}, +)$ is *scattered*, i.e. it doesn't contain a dense suborder. Moreover, we are able to give an estimation for Cantor-Bendixson ranks of the orders [3] (for a more precise estimation we use a slightly different notion of VD_*-rank from [5]):

Theorem 2 ([10], Theorem 4.3)**.** *All linear orderings m-dimensionally interpretable in $(\mathbb{N}, +)$ have the VD_*-rank at most m.*

Already for $n \geq 2$ the rank condition is far from sufficient. In order to produce a multidimensional version of Theorem 1, we establish a necessary and sufficient condition on the linear ordering interpretability. Turns out that the following holds:

Theorem 3. *A linear ordering $(L, \prec)$ is m-dimensionally interpretable in $(\mathbb{N}, +)$ for some $m \geq 1$ if and only if there exists some $k \in \mathbb{N}$ and a* **PrA***-definable set $\mathcal{D} \in \mathbb{Z}^k$ such that L is isomorphic to the restriction of the lexicographic ordering on $\mathbb{Z}^k$ to $\mathcal{D}$.*

From this description we derive the multi-dimensional Visser's conjecture.

Theorem 4 (Visser's Conjecture, Multi-Dimensional Version)**.** *For any model* $\mathfrak{A}$ *of* **PrA** *that is* m*-dimensionally interpreted in the model* $(\mathbb{N},+)$ *(*$m \geq 2$*),* $\mathfrak{A}$ *is* definably *isomorphic to* $(\mathbb{N},+)$.

The results are provided in the article [6].

References

[1] Cluckers, R.: Presburger sets and p-minimal fields. *The Journal of Symbolic Logic*, 68(1), 153-162 (2003)

[2] Ginsburg, S., Spanier, E.: Semigroups, Presburger formulas, and languages. Pacific J. Math. 16.2, 285-296 (1966)

[3] Hausdorff, F.: Grundzüge einer Theorie der geordneten Mengen. Math. Ann. 65.4, 435-505 (1908)

[4] Ito, R.: Every semilinear set is a finite union of disjoint linear sets. J. Comput. System Sci. 3.2, 221-231 (1969)

[5] Khoussainov, B., Rubin, S., Stephan, F.: Automatic linear orders and trees. ACM Trans. Comput. Log. 6.4, 675-700 (2005)

[6] Pakhomov F., Zapryagaev, A.: Multi-Dimensional Interpretations of Presburger Arithmetic in Itself. Under review in *Journal of Logic and Computation*

[7] Presburger M.: Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. Comptes Rendus du I congrès de Mathématiciens des Pays Slaves 92–101 (1929) – English translation in [8]

[8] Stansifer, R.: Presburger's Article on Integer Arithmetic: Remarks and Translation (Technical Report). Cornell University (1984)

[9] Visser, A.: An overview of interpretability logic. In Kracht M., de Rijke M., Wansing H., Zakharyaschev M., ed., Advances in Modal Logic. CSLI Lecture Notes 1.87, 307-359 (1998)

[10] Zapryagaev, A., Pakhomov F.: Interpretations of Presburger arithmetic in itself. International Symposium on Logical Foundations of Computer Science. Springer, Cham (2018)

[11] Zoethout, J.: Interpretations in Presburger Arithmetic. BS thesis. Utrecht University (2015)