CONTROLLABILITY AND VECTOR POTENTIAL
SIX LECTURES AT STEKLOV

Shiva Shankar [1]

Abstract. Kalman's fundamental notion of a controllable state space system, first described in Moscow [5], has been generalised to higher order systems by Willems [19], and further to distributed systems defined by partial differential equations [12]. It turns out, that for systems defined in several important spaces of distributions, controllability is now identical to the notion of vector potential in physics, or of vanishing homology in mathematics. These lectures will explain this relationship, and a few of its consequences. It will also pose an important question: does a controllable system, in any space of distributions, always admit a vector potential? In other words, is Kalman's notion of a controllable system, suitably generalised, nothing more - nor less - than the possibility of describing the dynamics of the system by means of a vector potential?

Contents

[1] shunyashankar@gmail.com

1. The Controllability Question

To bring into perspective the nature of the question that shall interest us here, we begin with a question that was of central importance in the 1950s and 60s.

The solvability question for systems of partial differential equations

Let $A = \mathbb{C}[\partial_1, \ldots, \partial_n]$ be the $\mathbb{C}$-algebra of constant coefficient partial differential operators, and $\mathcal{D}'$ the space of distributions, on $\mathbb{R}^n$. (We denote elements of A by lower case letters $a, p, q, \ldots$, or by $a(\partial), p(\partial), q(\partial)$, for emphasis.) The ring A acts on $\mathcal{D}'$ by differentiation, and gives it the structure of an A-module. Let $\mathcal{F}$ be an A-submodule of $\mathcal{D}'$. The attributes of dynamical systems that we study will assume values in $\mathcal{F}$, and we call it the 'space of signals'.

Let $P(\partial) = (p_{ij}(\partial))$ be an $\ell \times k$ matrix with entries in A; let its rows be denoted $r_1, \ldots, r_\ell$. It defines the differential operator

$$\text{(1)} \qquad \begin{array}{rccc} P(\partial): & \mathcal{F}^k & \longrightarrow & \mathcal{F}^\ell \\ & f = (f_1, \ldots, f_k) & \mapsto & (r_1 f, \ldots, r_\ell f), \end{array}$$

where $r_i f = \sum_{j=1}^k p_{ij}(\partial) f_j, \ i = 1, \ldots, \ell$. The solvability question for $P(\partial)$ in $\mathcal{F}$ asks: given $g \in \mathcal{F}^\ell$, is there an $f \in \mathcal{F}^k$ such that $P(\partial) f = g$?

Consider the set Q of all relations between the rows of $P(\partial)$, i.e. the set of all $(a_1, \ldots, a_\ell) \in A^\ell$ such that $a_1 r_1 + \cdots + a_\ell r_\ell = 0$. Q is an A-submodule of A^ℓ, and as A is Noetherian, is finitely generated, say by ℓ_1 elements. Let $Q(\partial)$ be the $\ell_1 \times \ell$ matrix whose rows are these ℓ_1 generators. Then, the sequence

$$\text{(2)} \qquad A^{\ell_1} \xrightarrow{Q^{\mathsf{T}}(\partial)} A^\ell \xrightarrow{P^{\mathsf{T}}(\partial)} A^k$$

is exact by construction (the superscript T denotes transpose). Applying the (left exact) functor[2] $\mathsf{Hom}_A(-,\ \mathcal{F})$ to the above sequence, gives the sequence

$$\text{(3)} \qquad \mathcal{F}^k \xrightarrow{P(\partial)} \mathcal{F}^\ell \xrightarrow{Q(\partial)} F^{\ell_1}.$$

This sequence is a complex, i.e. the image of the map $P(\partial)$ is contained in the kernel of the differential operator $Q(\partial)$. It provides a necessary condition for the solvability question: to solve $P(\partial) f = g$, it is necessary that $Q(\partial) g = 0$.

Further, suppose this sequence is also exact, i.e. suppose that the image of the map $P(\partial)$ is *equal* to the kernel of $Q(\partial)$. Then, a necessary *and* sufficient condition for the solvability of $P(\partial) f = g$ would be that $Q(\partial) g = 0$.

Thus, the solvability question for $P(\partial) : \mathcal{F}^k \to \mathcal{F}^\ell$ admits an answer precisely when its image equals a kernel (namely that of the map $Q(\partial) : \mathcal{F}^\ell \to \mathcal{F}^{\ell_1}$).

The problem of determining when images of partial differential operators are equal to kernels, was solved by Ehrenpreis, Malgrange, Hörmander, Palamodov and others. The solution depends on the A-module structure of $\mathcal{F}$, but before we summarise it below (for the classical spaces of distributions), we pose the dual problem.

---

[2]We refer to any standard book on Algebra for details, for instance [4].

Given $P(\partial)$ as above, now let R be the set of all relations between its columns. R is an A-submodule of A^k, and suppose that it is generated by k_1 elements, say. Let $R(\partial)$ be the $k \times k_1$ matrix whose columns are these generators. Then the sequence

$$A^\ell \stackrel{P^\mathsf{T}(\partial)}{\longrightarrow} A^k \stackrel{R^\mathsf{T}(\partial)}{\longrightarrow} A^{k_1} \tag{4}$$

is a complex (which is not necessarily exact). Again, applying the functor $\mathsf{Hom}_A(-,\ \mathcal{F})$ to this sequence gives the complex

$$\mathcal{F}^{k_1} \stackrel{R(\partial)}{\longrightarrow} \mathcal{F}^k \stackrel{P(\partial)}{\longrightarrow} \mathcal{F}^\ell. \tag{5}$$

Thus, the kernel of the map $P(\partial)$ contains the image of $R(\partial)$. We now ask:

($*$) *What is the question about $P(\partial) : \mathcal{F}^k \to \mathcal{F}^\ell$ for which the answer is 'precisely when its kernel equals an image (namely that of the map $R(\partial) : \mathcal{F}^{k_1} \to \mathcal{F}^k$)'?*

These lectures are precisely about this question.

Controllable Dynamical Systems: from Kalman to Willems

It was in Moscow that Kalman [5] first introduced the notion of a controllable state space system.

Definition 1.1. *A linear finite dimensional system whose evolution is described by*

$$\frac{\mathsf{d}}{\mathsf{dt}}x = Xx + Uu, \tag{6}$$

where the state (or phase) $x : \mathbb{R} \to \mathbb{C}^\ell$, and input $u : \mathbb{R} \to \mathbb{C}^m$, are smooth functions of time, and $X : \mathbb{C}^\ell \to \mathbb{C}^\ell$, $U : \mathbb{C}^m \to \mathbb{C}^\ell$ are linear maps, is said to be controllable if the following is satisfied: given x_1, x_2 in $\mathbb{C}^\ell$, and t_1, t_2 in $\mathbb{R}$ with $t_1 < t_2$, there is an input u such that the solution to the above equation with this input satisfies $x(t_1) = x_1,\ x(t_2) = x_2$.

This notion was quickly recognised to be fundamental, and on it was built the superstructure of post-war control theory. Nonetheless, there were foundational problems with the state space model itself. For one, the system's evolution had to be described by first order differential equations; another, more serious problem, was the need to specify, ab initio, a causal structure that declared some signals to be inputs. These, and other problems with the Kalman paradigm, were overcome in a spectacular fashion by J.C.Willems [19], who proposed a far larger class of models for linear dynamical systems, and a definition of controllability which faithfully generalised Kalman's definition. To motivate this development, we first make a few comments about the nature of models.

A model is a picture of reality, and the closer it is to reality, the better the picture it will be, and more effective the theory will be that describes this model. A model seeks to represent a certain phenomenon whose attributes are certain qualities that are varying with space, time etc. The closest we can get to this reality, this phenomenon, is to take all possible variations of the attributes of the phenomenon, itself, as the model. This collection, considered all together in our minds, is in engineering parlance, the *behaviour* of the phenomenon.

Indeed, we could have said:

The behaviour is all that is the case.

A priori, any variation of these attributes could perhaps have occured, but the laws governing the phenomenon limit the variations to those that can, and do, occur. We shall consider phenomena described by local laws, i.e. laws expressed by differential equations. Here by local we mean that the variation of the attributes of the phenomenon at a point depends on the values of the attributes in arbitrarily small neighbourhoods of the point, and not on points far away.

This is a familiar situation in physics, mathematics and engineering. For instance:

1. Planetary motion. A priori, the earth could perhaps have traversed any trajectory around the sun, but Kepler's laws restrict it to travel an elliptic orbit, sweeping equal areas in equal times, and such that the square of the period of revolution is proportional to the cube of the major semi-axis. Kepler's laws are of course local, namely Newton's equations of motion.

2. Magnetic fields. A priori, any function B could perhaps have been a magnetic field, but in fact must satisfy the law that there do not exist magnetic monopoles in the universe. By Gauss, this law is also local: $\mathsf{div}\, B = 0$.

3. Complex analysis. Complex analysis studies those functions which admit a convergent power series about each point. This is the law that defines the subject.

This law is again local, namely the Cauchy-Riemann equations.

We now further confine ourselves to phenomena which are described not only by local laws, but laws which are also linear and shift invariant. The behaviour of such phenomena can be modelled as kernels of linear maps defined by constant coefficient partial differential operators.

Thus, for example, suppose that the attributes of a phenomenon are described by some k-tuple of complex numbers, which depend smoothly on some n independent real valued parameters. Let these attributes be denoted by $f = (f_1, \ldots, f_k)$, where each $f_i : \mathbb{R}^n \to \mathbb{C}$ is smooth. Then, a law the phenomenon obeys is a partial differential operator $p(\partial) = (p_1(\partial), \ldots, p_k(\partial))$ such that $p(\partial)f = \sum p_i(\partial)f_i = 0$. In other words a law is an operator $p(\partial) : (\mathcal{C}^\infty)^k \to \mathcal{C}^\infty$, and to say that f obeys this law is to say that $p(\partial)f = 0$, or that f is a zero of $p(\partial)$. The behaviour B of the phenomenon is thus

$$B \;=\; \bigcap_{\text{all laws } p(\partial)} \{f \mid p(\partial)f = 0\},$$

the common zeros of all the laws governing the phenomenon. Shift invariance implies that the entries $p_1(\partial), \ldots, p_k(\partial)$ of $p(\partial)$ are all in $A = \mathbb{C}[\partial_1, \ldots, \partial_n]$. As the sum of two laws, and the scaling of a law by an element of A, are also laws, it follows that the set P of all laws governing the phenomenon is an A-submodule of A^k. If P is generated by $r_1, \ldots, r_\ell$, then writing these ℓ elements of A^k as rows of a matrix, say $P(\partial)$, implies that B is the kernel of

$$P(\partial) : (\mathcal{C}^\infty)^k \to (\mathcal{C}^\infty)^\ell,$$

which is Equation (1) for $\mathcal{F} = \mathcal{C}^\infty$.

Thus, the phenomena we study are exactly those whose behaviours can be modelled as kernels of constant coefficient partial differential operators (as in Equation (1)).

These kernels define the class of *distributed systems* (or *distributed behaviours).* When $n = 1$, i.e. when $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, they define the class of *lumped systems.*

Remark: The kernel $\mathsf{Ker}_{\mathcal{F}}(P(\partial))$ of the operator of Equation (1) depends upon the submodule $P \subset A^k$ generated by the rows of the matrix $P(\partial)$, and not on a specific choice of generators of P such as the rows of $P(\partial)$. Indeed,

$$\begin{array}{ccc} \mathsf{Hom}_A(A^k/P,\ \mathcal{F}) & \longrightarrow & \mathsf{Ker}_{\mathcal{F}}(P(\partial)) \\ \phi & \mapsto & (\phi(\bar{e_1}), \ldots, \phi(\bar{e_k})) \end{array}$$

is an A-module isomorphism; here, $\bar{e_1}, \ldots, \bar{e_k}$ denote the images of the standard basis $e_1, \ldots, e_k$ of A^k in A^k/P. Hence, we denote this kernel by $\mathsf{Ker}_{\mathcal{F}}(P)$, and call it the kernel of P in $\mathcal{F}$.

The following statements are immediate.

Lemma 1.1. *(i) If $P_1 \subset P_2$ are submodules of A^k, then $\mathsf{Ker}_{\mathcal{F}}(P_2) \subset \mathsf{Ker}_{\mathcal{F}}(P_1)$ in $\mathcal{F}^k$. ($\mathsf{Ker}_{\mathcal{F}}(P_2)$ is said to be a sub-system (or sub-behaviour) of $\mathsf{Ker}_{\mathcal{F}}(P_1)$.)*
(ii) If $\{P_i\}$ is a collection of submodules of A^k, then $\mathsf{Ker}_{\mathcal{F}}(\sum_i P_i) = \bigcap_i \mathsf{Ker}_{\mathcal{F}}(P_i)$, and $\sum_i \mathsf{Ker}_{\mathcal{F}}(P_i) \subset \mathsf{Ker}_{\mathcal{F}}(\bigcap P_i)$.

Example 1.1: Let $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$. The collection of all signals x and u in Kalman's state space system of Definition 1.1, is the kernel of the operator

$$(7) \qquad \begin{array}{rccc} P(\frac{\mathsf{d}}{\mathsf{dt}}) = \left(\frac{\mathsf{d}}{\mathsf{dt}}I_\ell - X,\ -U\right): & (\mathcal{C}^\infty)^{\ell+m} & \longrightarrow & (\mathcal{C}^\infty)^\ell \\ & (x,u) & \mapsto & P(\frac{\mathsf{d}}{\mathsf{dt}}) \left[\begin{array}{c} x \\ u \end{array}\right], \end{array}$$

but now, the difference between state and input has been obliterated. In other words, we do not impose any causal structure on the system. □

In our more general setting, controllability cannot any longer be the ability to move from one state to another in finite time (Definition 1.1), as there is now no notion of state. We need another definition altogether, which specialises to the Kalman definition in the case of Example 1.1 above. We make the definition first for the case when $\mathcal{F}$ is either $\mathcal{D}'$ or $\mathcal{C}^\infty$, and relegate a more general definition to an appendix at the end of the lecture.

Definition 1.2. *(J.C.Willems [19]) A system $\mathsf{Ker}_{\mathcal{F}}(P)$, where P is a submodule of A^k and $\mathcal{F}$ either $\mathcal{D}'$ or $\mathcal{C}^\infty$, is controllable if given any two subsets U_1 and U_2 of $\mathbb{R}^n$ whose closures do not intersect, and any two elements f_1 and f_2 of the system, then there is an element f in $\mathsf{Ker}_{\mathcal{F}}(P)$ such that $f = f_1$ on some neighbourhood of U_1 and $f = f_2$ on some neighbourhood of U_2.*

Such an f is said to patch f_1 on U_1 with f_2 on U_2.

Remark: The above definition is equivalent to the following: $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable if given any f in it, and any $U \subset V \subset \mathbb{R}^n$ such that the closure of U is contained in the interior of V, then there is an f_c in $\mathsf{Ker}_{\mathcal{F}}(P)$ such that $f_c = f$ on some neighbourhood of U, and equal to 0 on the complement of V.

Such an f_c is said to be a cutoff of f with respect to $U \subset V$ (in $\mathsf{Ker}_{\mathcal{F}}(P)$).

For emphasis, we sometimes refer to this notion as *behavioural controllability.*

The following propositions are elementary.

Proposition 1.1. *Let $\mathsf{Ker}_{\mathcal{F}}(P)$ be a controllable system as in the above definition. Then, its compactly supported elements are dense in it (here $\mathcal{D}'$ and $\mathcal{C}^\infty$ are equipped with their standard weak–$*$ and Fréchet topologies respectively).*

Proof: Let $V_1 \subset \cdots \subset V_i \subset V_{i+1} \subset \cdots$ be an exhaustion of $\mathbb{R}^n$ by compact sets such that V_i is contained in the interior of V_{i+1}. Let f be any element in $\mathsf{Ker}_{\mathcal{F}}(P)$, and let $f_i \in \mathsf{Ker}_{\mathcal{F}}(P)$ be a cutoff of f with respect to $V_i \subset V_{i+1}$. Then the sequence $\{f_i\}$ converges to f. □

Proposition 1.2. *Let $\mathcal{F}$ be either $\mathcal{D}'$ or $\mathcal{C}^\infty$. Suppose that a system $\mathsf{Ker}_{\mathcal{F}}(P)$ is given as the kernel of Equation (1) (thus the rows of $P(\partial)$ in (1) generate the submodule $P \subset A^k$). If it is equal to the image of an operator $R(\partial) : \mathcal{F}^{k_1} \to \mathcal{F}^k$ (i.e. if (5) is an exact sequence), then it is controllable.*

Proof: Given data f and $U \subset V \subset \mathbb{R}^n$ as in the remark above, let $g \in \mathcal{F}^{k_1}$ be any element such that $R(\partial)g = f$. Let g_c be any cutoff (in $\mathcal{F}^{k_1}$) with respect to $U \subset V$. Then $R(\partial)g_c$ is a cutoff of f with respect to $U \subset V$ in $\mathsf{Ker}_{\mathcal{F}}(P)$. □

Example 1.2: The collection of all static magnetic fields is the system defined by Gauss' Law, and is controllable because it admits the vector potential curl. In other words, the following sequence

$$(\mathcal{C}^\infty)^3 \xrightarrow{\mathsf{curl}} (\mathcal{C}^\infty)^3 \xrightarrow{\mathsf{div}} \mathcal{C}^\infty$$

is exact. Here, $\mathsf{div} = (\partial_1, \partial_2, \partial_3)$, and $\mathsf{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}$.

This example prompts the following definition [13].

Definition 1.3. *A system given by the kernel of a differential operator is said to admit a vector potential if it equals the image of some differential operator.*

The primary goal of these lectures is to prove that the converse of the above proposition is also true, that a distributed system in $\mathcal{D}'$ or $\mathcal{C}^\infty$ is controllable if and only if it admits a vector potential. This result is also true for some other spaces of distributions, with a more careful definition of controllability described in the appendix below.

Thus, the notion of a controllable system provides the answer to the question posed in $(*)$.

The argument depends upon the answers to the solvability question we started our discussion with. We therefore first summarise these classical results due to Malgrange [8, 9] and Palamodov [11]; they describe the A-module structure of some important spaces of distributions.

The Algebraic Structure of Some Spaces of Distributions

Definition 1.4. *An A-module $\mathcal{F}$ is injective if* $\mathsf{Hom}_A(-,\ \mathcal{F})$ *is an exact functor. It is flat if* $-\otimes_A \mathcal{F}$ *is an exact functor.*

Thus, if $\mathcal{F}$ is an injective A-submodule of $\mathcal{D}'$, then the sequence (3) above is always exact, and we can answer the solvability question for every $P(\partial)$.

Theorem 1.1. *The spaces $\mathcal{D}'$, $\mathcal{C}^\infty$, and $\mathcal{S}'$ (tempered distributions), are injective A-modules.*

This is the celebrated Fundamental Principle of Malgrange and Palamodov.

Remark: The special case of the above theorem when $\ell = k = 1$, are the classical existence theorems of Ehrenpreis-Malgrange for $\mathcal{D}'$ and $\mathcal{C}^\infty$, and of Hörmander for $\mathcal{S}'$. They assert that $p(\partial) : \mathcal{F} \to \mathcal{F}$ is surjective for any nonzero $p(\partial)$, and $\mathcal{F}$ one of these three spaces. In particular, any nonzero $p(\partial)$ has a fundamental solution which is a tempered distribution.

Their duals, the spaces $\mathcal{D}$ of test functions, $\mathcal{E}'$ of compactly supported distributions, and the Schwartz space $\mathcal{S}$ (of smooth functions, which together with all derivatives, are rapidly decreasing), are not injective, not even divisible A-modules. For instance, $\frac{\mathsf{d}}{\mathsf{dt}} : \mathcal{D} \to \mathcal{D}$ is not surjective, as $f \in \mathcal{D}$ is in the image if and only if its integral over $\mathbb{R}$ equals 0. It is also not surjective when $\mathcal{D}$ is replaced by $\mathcal{E}'$ or $\mathcal{S}$.

Theorem 1.2. [8] *The spaces $\mathcal{D}$, $\mathcal{E}'$, and $\mathcal{S}$ are flat A-modules.*

Our focus in these lectures will be mainly on the signal spaces $\mathcal{D}'$, $\mathcal{C}^\infty$, $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{E}'$ and $\mathcal{D}$, which we collectively refer to as the *classical spaces*.

Remark: Sometimes we will think of the ring A as the polynomial ring in n indeterminates, i.e. as $A = \mathbb{C}[x_1, \ldots, x_n]$. We will then identify the differential operator $p(\partial)$ with the corresponding polynomial $p(x)$, and consider its affine variety $\mathcal{V}(p(x))$ in $\mathbb{C}^n$. We call it the affine variety of $p(\partial)$, or simply, the affine variety of p. A point ξ is in $\mathcal{V}(p(x))$ if and only if the corresponding exponential is a solution of $p(\partial)$, i.e. $p(\partial)e^{<\xi,x>} = 0$. More generally, the exponential solutions of $p(\partial)$ are solutions of the form $q(x)e^{<\xi,x>}$, where $q(x)$ is a polynomial, and ξ a point on $\mathcal{V}(p(x))$. A theorem of Malgrange states that the set of solutions of $p(\partial)$ in $\mathcal{D}'$ is the closed convex hull of its exponential solutions.

We also identify a matrix $P(\partial)$ with the corresponding matrix $P(x)$ by identifying each entry $p_{ij}(\partial)$ with the polynomial $p_{ij}(x)$.

Definition 1.5. *An injective A-module $\mathcal{F}$ is a cogenerator if for every nonzero A-module M,* $\mathsf{Hom}_A(M,\ \mathcal{F})$ *is nonzero. A flat A-module $\mathcal{F}$ is faithfully flat if for every nonzero A-module M, $M \otimes_A \mathcal{F}$ is nonzero.*

Proposition 1.3. *The A-modules $\mathcal{D}'$ and $\mathcal{C}^\infty$ are cogenerators.*

Proof: Let M be a nonzero A-module. Let $m(\partial)$ be any nonzero element of M and let m be the cyclic submodule generated by it. Thus m is isomorphic to A/i for some proper ideal i of A. As $\mathcal{C}^\infty$ is an injective A-module, $\mathsf{Hom}_A(M,\ \mathcal{C}^\infty)$ surjects onto $\mathsf{Hom}_A(A/i,\ \mathcal{C}^\infty) \simeq \mathsf{Ker}_{\mathcal{C}^\infty}(i)$. Therefore to prove the proposition, i.e. to show that

$\mathsf{Hom}_A(M,\ \mathcal{C}^\infty)$ is nonzero, it suffices to show that $\mathsf{Ker}_{\mathcal{C}^\infty}(i)$ is nonzero. But this is elementary, for as i is a proper ideal of A, its variety $\mathcal{V}(i)$ (in $\mathbb{C}^n$) is nonempty. Let ξ be any point in it. Then $m(\partial)e^{<\xi,x>} = 0$, and so $\mathsf{Ker}_{\mathcal{C}^\infty}(i) \neq 0$. □

Remark: The injective module $\mathcal{S}'$ is not a cogenerator. For example, let $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, and p the ideal generated by $\frac{\mathsf{d}}{\mathsf{dt}} - 1$; then $\mathsf{Hom}_A(A/p,\ \mathcal{S}') = 0$, as e^t is not tempered.

Proposition 1.4. *$\mathcal{D}$ and $\mathcal{E}'$ are faithfully flat A-modules.*

Proof: A standard result states that a flat A-module M is faithfully flat if and only if $mM \neq M$, for every maximal ideal m of A.

The Fourier transforms of the distributions in $m\mathcal{E}'$, which extend to functions holomorphic on $\mathbb{C}^n$ by the Paley-Weiner Theorem, all vanish at the zero of m. Thus $m\mathcal{E}' \neq \mathcal{E}'$ for every maximal ideal m, hence $\mathcal{D}$ is faithfully flat. This argument also proves that $\mathcal{D}$ is faithfully flat. □

Remark: The flat module $\mathcal{S}$ is not however faithfully flat. For let $p(x)$ be any polynomial such that $p(x)^{-1}f$ is in $\mathcal{S}$, for every $f \in \mathcal{S}$, for instance, the polynomial $1+x_1^2+\cdots+x_n^2$. Then Fourier transformation shows that the morphism $p(\partial) : \mathcal{S} \to \mathcal{S}$ is surjective, so that $m\mathcal{S} = \mathcal{S}$, for any maximal ideal m of A that contains $p(\partial)$. □

Proposition 1.5. *Suppose that the A-submodule $\mathcal{F} \subset \mathcal{D}'$ is an injective cogenerator. Then there is an inclusion reversing bijection between A-submodules $P \subset A^k$ and distributed systems $\mathsf{Ker}_{\mathcal{F}}(P) \subset \mathcal{F}^k$.*

Proof: We have already observed in Lemma 1.1 that the corespondence between submodules and systems is inclusion reversing. Now suppose that $M \subsetneq N$, then $0 \to N/M \to A^k/M \to A^k/N \to 0$ is exact. Applying the functor $\mathsf{Hom}_A(-,\ \mathcal{F})$ to this sequence yields the exact sequence $0 \to \mathsf{Ker}_{\mathcal{F}}(N) \to \mathsf{Ker}_{\mathcal{F}}(M) \to \mathsf{Hom}_A(N/M,\ \mathcal{F}) \to 0$. As $\mathcal{F}$ is a cogenerator, $\mathsf{Hom}_A(N/M,\ \mathcal{F}) \neq 0$, hence $\mathsf{Ker}_{\mathcal{F}}(N) \subsetneq \mathsf{Ker}_{\mathcal{F}}(M)$.

Next suppose that $M \not\subset N$; then $N \subsetneq M + N$, hence $\mathsf{Ker}_{\mathcal{F}}(M) \cap \mathsf{Ker}_{\mathcal{F}}(N) = \mathsf{Ker}_{\mathcal{F}}(M+N) \subsetneq \mathsf{Ker}_{\mathcal{F}}(N)$ by the above paragraph, hence $\mathsf{Ker}_{\mathcal{F}}(M) \neq \mathsf{Ker}_{\mathcal{F}}(N)$. This establishes the proposition. □

Remark: For $\mathcal{F}$ any A-submodule of $\mathcal{D}'$, and any distributed system $\mathsf{Ker}_{\mathcal{F}}(P) \subset \mathcal{F}^k$, let $\mathcal{M}(\mathsf{Ker}_{\mathcal{F}}(P)) \subset A^k$ be the submodule of all the elements in A^k that map every element in $\mathsf{Ker}_{\mathcal{F}}(P)$ to 0. Clearly $P \subset \mathcal{M}(\mathsf{Ker}_{\mathcal{F}}(P))$, indeed it is the largest submodule of A^k whose kernel equals $\mathsf{Ker}_{\mathcal{F}}(P)$. The determination of $\mathcal{M}(\mathsf{Ker}_{\mathcal{F}}(P))$ is the Nullstellensatz problem for systems of PDE in $\mathcal{F}$ [14]. The above proposition avers that if $\mathcal{F}$ is an injective cogenerator, then $P = \mathcal{M}(\mathsf{Ker}_{\mathcal{F}}(P))$, for every submodule $P \subset A^k$.

The assignment $\mathcal{M}$ is also inclusion reversing, i.e. if $B_1 \subset B_2$ are two distributed systems in $\mathcal{F}^k$, then $\mathcal{M}(B_2) \subset \mathcal{M}(B_1)$. Thus we have two inclusion reversing assignments, $\mathsf{Ker}_{\mathcal{F}}$ and $\mathcal{M}$, which define a Galois connection between the partially ordered sets of submodules of A^k and systems in $\mathcal{F}^k$.

Example 1.3: Both the ideals (1) and $(\frac{\mathsf{d}}{\mathsf{dt}} - 1)$ of $\mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$ define the 0 system in $\mathcal{S}'$, hence $\mathcal{M}(\mathsf{Ker}_{\mathcal{S}'}(\frac{\mathsf{d}}{\mathsf{dt}} - 1)) = (1)$.

An argument, similar to the one in the above proposition, proves the following:

Proposition 1.6. *Suppose that $\mathcal{F}$ is an injective cogenerator. Then the complex $M \xrightarrow{f} N \xrightarrow{g} P$ is exact if and only if $\mathsf{Hom}_A(P,\ \mathcal{F}) \xrightarrow{-\circ g} \mathsf{Hom}_A(N,\ \mathcal{F}) \xrightarrow{-\circ f} \mathsf{Hom}_A(M,\ \mathcal{F})$ is exact. Hence, $0 \to M \to N \to P \to 0$ is split exact if and only if $0 \to \mathsf{Hom}_A(P,\ \mathcal{F}) \to \mathsf{Hom}_A(N,\ \mathcal{F}) \to \mathsf{Hom}_A(M,\ \mathcal{F}) \to 0$ is split exact.*

Proof: Suppose $\mathsf{Im}(f) \subsetneq \mathsf{Ker}(g) \subset N$. Then, by injectivity of $\mathcal{F}$, $\mathsf{Hom}_A(N/\,\mathsf{Im}(f),\ \mathcal{F})$ maps onto $\mathsf{Hom}_A(\mathsf{Ker}(g)/\,\mathsf{Im}(f),\ \mathcal{F})$, where the second A-module is nonzero because $\mathcal{F}$ is also a cogenerator. Thus, there exists an A-module map $\phi : N \to \mathcal{F}$ which restricts to 0 on $\mathsf{Im}(f)$, but which is nonzero on $\mathsf{Ker}(g)$. This ϕ maps to 0 in $\mathsf{Hom}_A(M,\ \mathcal{F})$, as $\phi \circ f = 0$. However, it is not in the image of $- \circ g$, as ϕ is nonzero on $\mathsf{Ker}(g)$.

The second assertion now follows immediately. □

Example 1.4: The above proposition is not true for an injective A-module which is not a cogenerator. For instance $0 \to (\frac{\mathsf{d}}{\mathsf{dt}} - 1) \to \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}] \to \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]/(\frac{\mathsf{d}}{\mathsf{dt}} - 1) \to \mathsf{0}$ is not split exact, yet $0 \to \mathsf{Ker}_{\mathcal{S}'}(\frac{\mathsf{d}}{\mathsf{dt}} - 1) \to \mathcal{S}' \to \mathsf{Hom}_{\mathcal{S}'}(\frac{\mathsf{d}}{\mathsf{dt}} - 1,\ \mathcal{S}') \to \mathsf{0}$ splits because $\mathsf{Ker}_{\mathcal{S}'}(\frac{\mathsf{d}}{\mathsf{dt}} - 1) = \mathsf{0}$.

Remark: We have seen that $\mathcal{D}$ and $\mathcal{E}'$ and $\mathcal{S}$ are flat A-modules. By the equational criterion of flatness [4], the kernel of every $P(\partial) : \mathcal{F}^k \to \mathcal{F}^\ell$ is an image, i.e. sequence (5) is always exact if $\mathcal{F}$ is flat. Thus, it turns out, that with an appropriate definition of controllability (described in the appendix below), every distributed system in these spaces is controllable.

We shall therefore not study systems in these spaces, except to use them to construct counter examples to statements that are true for systems in injective modules, but not in general. (We refer to [16] for a discussion on the relationship between the structure of a topological A-module and its dual.)

Appendix

Willems' definition of a controllable system cannot be carried over *in toto* to an arbitrary A-submodule $\mathcal{F}$ of $\mathcal{D}'$, because it might be that with respect to some $U \subset V$, with the closure of U contained in the interior of V (as in the remark following Definition 1.2), there is no cutoff f_c of some f in $\mathsf{Ker}_{\mathcal{F}}(P)$, even in $\mathcal{F}^k$.

Example 1.4: Let $\mathcal{F}$ be the Schwartz space $\mathcal{S}$ of rapidly decreasing functions on $\mathbb{R}^2$. Let $U = \{(x, y) \in \mathbb{R}^2 |\ y < 0\}$ and $V = \{(x, y) \in \mathbb{R}^2 |\ y < e^{-x^2}\}$. Then, the closure of U is contained in the interior of V, but no cutoff 1_c of the constant function 1, with respect to this $U \subset V$ can be in $\mathcal{S}$ (the derivatives of any such cutoff would not be rapidly decreasing).

We therefore generalize the cutoff formulation of controllability to an arbitrary $\mathcal{F}$ in the following way:

Definition 1.6. *Let $\mathcal{F}$ be an A-submodule of $\mathcal{D}'$. A pair of subsets $U \subset V$ of $\mathbb{R}^n$ is said to be admissible with respect to $\mathcal{F}$, if every $f \in \mathcal{F}$ admits a cutoff $f_c \in \mathcal{F}$ with respect to it.*

A system $\mathsf{Ker}_{\mathcal{F}}(P) \subset \mathcal{F}^k$ *is said to be controllable if for every* $U \subset V$ *admissible with respect to* $\mathcal{F}$*, every* f *in* $\mathsf{Ker}_{\mathcal{F}}(P)$ *admits a cutoff* f_c *in* $\mathsf{Ker}_{\mathcal{F}}(P)$ *with respect to it.*

Thus $U \subset V$ of $\mathbb{R}^2$ in the above example is not admissible with respect to $\mathcal{S}$. Some examples of admissible $U \subset V$ are given in the following proposition.

Proposition 1.7. *(i) Any* $U \subset V$*, with the closure of* U *contained in the interior of* V*, is admissible with respect to* $\mathcal{D}'$, $\mathcal{C}^\infty$, $\mathcal{E}'$ *or* $\mathcal{D}$.

(ii) Suppose that the distance between U *and the complement* V^c *of* V *is bounded away from 0, i.e.* $\|x - y\| \geqslant \epsilon > 0$ *for all* $x \in U, y \in V^c$ *(for instance, suppose the boundary of* U *is compact). Then such a pair is admissible with respect to* $\mathcal{S}$ *or* $\mathcal{S}'$.

Proof: (i) is elementary, for let ρ be any smooth function which equals 1 on some neighborhood U' of U contained in the interior of V, and 0 on the complement of V. Then for an element f in $\mathcal{C}^\infty$ or in $\mathcal{D}$, ρf is a cutoff of f with respect to $U \subset V$.

Now let g be in $\mathcal{D}'$ or in $\mathcal{E}'$. As $\rho g(f) = g(\rho f)$ for every $f \in \mathcal{D}$ or $\mathcal{C}^\infty$ respectively, it follows that ρg is the required cutoff.

(ii) Let U' be an open subset containing U and contained in the interior of V, such that the distance between U' and the complement of V is $\frac{\epsilon}{2}$. Let χ be the characteristic function of U', and let κ be a smooth 'bump' function supported in the ball of radius $\frac{\epsilon}{4}$ centered at the origin, i.e. κ is identically 1 in some smaller ball about the origin. Then the convolution $\rho = \kappa \star \chi$ is a smooth function that is identically 1 on U', 0 on the complement of V, and such that all its derivatives are bounded on $\mathbb{R}^n$. Thus for any f in $\mathcal{S}$ or $\mathcal{S}'$, ρf is the required cutoff. □

With this definition of controllability, Proposition 1.2, and its proof, carry over to any A-submodule $\mathcal{F}$ of $\mathcal{D}'$.

(∗∗) *Thus, the existence of a vector potential always implies controllability.*

2. Necessary and Sufficient Conditions for Controllability

Our immediate goal is to answer question $(*)$ of Lecture One.

We make the following standing assumption about the A-module $\mathcal{F}$:
Let V be a bounded subset of $\mathbb{R}^n$. Then there exists a subset U whose closure is contained in the interior of V such that the pair $U \subset V$ is admissible with respect to $\mathcal{F}$ (as in Definition 1.6).

Thus, every $f \in \mathcal{F}$ admits a cutoff $f_c \in \mathcal{F}$ with respect to such a pair.

By Proposition 1.7, the A-modules $\mathcal{D}'$, $\mathcal{C}^\infty$, $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{E}'$, and $\mathcal{D}$ satisfy this assumption, whereas the space $\mathcal{O}$ of real analytic functions does not.

Elementary consequences of this assumption are the following corollaries.

Corollary 2.1. *An $\mathcal{F}$ satisfying the above assumption is a faithful A-module. In other words, if $p(\partial)$ is a nonzero element of A, then the kernel of $p(\partial) : \mathcal{F} \to \mathcal{F}$ is strictly contained in $\mathcal{F}$.*

Proof: The above assumption implies that there are compactly supported elements in $\mathcal{F}$. No such element can be a homogenous solution of $p(\partial)$ by the Paley-Wiener Theorem. □

Corollary 2.2. *The only sub-systems of $\mathcal{F}$ that are controllable are* 0 *and $\mathcal{F}$ itself.*

Proof: The sub-system 0, which is the kernel of the map $1 : \mathcal{F} \to \mathcal{F}$, is trivially controllable. So is $\mathcal{F}$, the kernel of $0 : \mathcal{F} \to \mathcal{F}$, by Definition 1.6.

Suppose $p(\partial) \neq 0$ is such that the kernel of $p(\partial) : \mathcal{F} \to \mathcal{F}$ is nonzero. By the standing assumption above, there is a relatively compact pair $U \subset V$ admissible with respect to $\mathcal{F}$. No nonzero $f \in \mathsf{Ker}_{\mathcal{F}}(p(\partial))$ can admit a cutoff with respect to this pair (in $\mathsf{Ker}_{\mathcal{F}}(p(\partial))$). □

Proposition 2.1. *Sequence (4) is exact if and only if A^k/P is torsion free.*

Proof: To recollect notation: P is the submodule of A^k generated by the rows of the $\ell \times k$ matrix $P(\partial)$ of Equation (1), $R \subset A^k$ is the submodule of relations between the columns of $P(\partial)$ and it is generated by the k_1 columns of $R(\partial)$. Let the kernel of $R^{\mathsf{T}}(\partial) : A^k \to A^{k_1}$ be $P_1 \subset A^k$; clearly $\mathsf{Im}(P^{\mathsf{T}}(\partial)) = P \subset P_1$. Thus (4) is a complex, and is exact if and only if $P = P_1$.

Let K be the field of fractions of the doman A. Tensoring complex (4) by K gives the complex $K^\ell \xrightarrow{P^{\mathsf{T}}(\partial)} K^k \xrightarrow{R^{\mathsf{T}}(\partial)} K^{k_1}$ of finite dimensional K-vector spaces, as the A-module maps $P^{\mathsf{T}}(\partial)$ and $R^{\mathsf{T}}(\partial)$ are also K-linear. As localisation is a flat functor, the image of $P^{\mathsf{T}}(\partial)$ now is $K \otimes_A P$, and the kernel of $R^{\mathsf{T}}(\partial)$ is now $K \otimes_A P_1$. If the rank of $P^{\mathsf{T}}(\partial)$ equals r, then the rank of $R^{\mathsf{T}}(\partial)$ equals $k - r$ and its kernel has dimension r. Thus $K \otimes_A P = K \otimes_A P_1$, hence $K \otimes_A P_1/P = 0$ and so P_1/P is a torsion module.

We next claim that the torsion submodule of A^k/P equals P_1/P. For suppose $x \in A^k$ is such that $ax \in P$ for some nonzero $a \in A$. Then $1 \otimes ax \in K \otimes_A P = K \otimes_A P_1$, hence $1 \otimes ax = \alpha_1 \otimes p_1 + \cdots + \alpha_r \otimes p_r$, for some $\alpha_i \in K$ and $p_i \in P_1$, for all i. Clearing the denominators of the α_i, we have $1 \otimes bax = a_1 \otimes p_1 + \cdots + a_r \otimes p_r$, where the $b, a_1, \ldots, a_r$ are all in A. As P_1 has no torsion, this equality implies that $bax = a_1p_1 + \cdots + a_rp_r \in P_1$.

Thus, $R^{\mathsf{T}}(\partial)(bax) = 0$, and as $ba \neq 0$, $R^{\mathsf{T}}(\partial)x = 0$, which is to say that x is itself in P_1.

Thus it follows that (4) is exact, i.e. $P = P_1$, if and only if A^k/P is torsion free. □

Example 2.1: Let $A = \mathbb{C}[\partial_1, \partial_2, \partial_3]$, and m its maximal ideal $(\partial_1, \partial_2, \partial_3)$. Consider the following sequence

$$0 \to A \xrightarrow{\mathsf{div}^{\mathsf{T}}} A^3 \xrightarrow{\mathsf{curl}^{\mathsf{T}}} A^3 \xrightarrow{\mathsf{grad}^{\mathsf{T}}} A \xrightarrow{\pi} A/m \to 0 \tag{8}$$

where π is the canonical projection, and $\mathsf{grad} = \mathsf{div}^{\mathsf{T}}$ (Example 1.2). As A is an integral domain, the map $\mathsf{div}^{\mathsf{T}}$ is an injection. The module of relations between the columns of the matrix curl is a cyclic module generated by the one column of grad, and the module of relations between the columns of div is generated by the columns of curl. Let C and D be the submodules of A^3 generated by the rows of curl and div respectively. As A^3/C and A^3/D are both torsion free, it follows from the above proposition that the above sequence is exact, and hence a resolution of A/m.

We can now answer question $(*)$.

Theorem 2.1. *([12]) Let $\mathcal{F} \subset \mathcal{D}'$ be an A-submodule, and P a submodule of A^k.*
(i) If $\mathcal{F}$ is an injective A-module, then $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable if A^k/P is torsion free.
(ii) If $\mathcal{F}$ is also a cogenerator, then $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable if and only if A^k/P is torsion free.

Proof: (i) If A^k/P is torsion free, then sequence (4) is exact by the above proposition, and this in turn implies sequence (5) is exact because $\mathcal{F}$ is an injective A-module. Thus $\mathsf{Ker}_{\mathcal{F}}(P)$ is an image, and hence controllable (by Proposition 1.2).

(ii) Now suppose that $\mathcal{F}$ is an injective cogenerator, and suppose that A^k/P is not torsion free. Let $q(\partial) \in A^k \setminus P$ be such that $a(\partial)q(\partial) \in P$, for some nonzero $a(\partial) \in A$. Denote by q and aq the cyclic submodules of A^k generated by $q(\partial)$ and $a(\partial)q(\partial)$ respectively. Then $P \subsetneq P + q$, hence $\mathsf{Ker}_{\mathcal{F}}(P) \not\subset \mathsf{Ker}_{\mathcal{F}}(q)$, by Lemma 1.1 and Proposition 1.5. However, $\mathsf{Ker}_{\mathcal{F}}(P) \subset \mathsf{Ker}_{\mathcal{F}}(aq)$, as $aq \subset P$.

Let $f \in \mathsf{Ker}_{\mathcal{F}}(P) \setminus \mathsf{Ker}_{\mathcal{F}}(q)$. Let U be a bounded open subset of $\mathbb{R}^n$ such that $q(\partial)f \neq 0$ on U. Let V be any compact set containing the closure of U in its interior; thus $U \subset V$ is admissible with respect to $\mathcal{F}$ by our standing assumption. Suppose $\mathsf{Ker}_{\mathcal{F}}(P)$ were controllable, then by Definition 1.6, some cutoff f_c of f with respect to $U \subset V$ would be in $\mathsf{Ker}_{\mathcal{F}}(P)$. Clearly $q(\partial)f_c \neq 0$, and it has compact support.

As $a(\partial)q(\partial) \in P$, it follows that $a(\partial)(q(\partial)f_c) = 0$. This is a contradiction, as $a(\partial)$ cannot have a compactly supported homogeneous solution by the Paley-Wiener Theorem. □

Remark: If an injective A-submodule $\mathcal{F}$ of $\mathcal{D}'$ is not a cogenerator, then $\mathsf{Ker}_{\mathcal{F}}(P)$ could be an image, and hence controllable, even though A^k/P is not torsion free. For instance $A/(\frac{\mathsf{d}}{\mathsf{dt}} - 1)$ is a torsion module, yet $\mathcal{S}' \xrightarrow{0} \mathcal{S}' \xrightarrow{(\frac{\mathsf{d}}{\mathsf{dt}}-1)} \mathcal{S}'$ is exact.

Example 2.2 (deRham complex on $\mathbb{R}^3$): (i) Suppose that $\mathcal{F}$ is $\mathcal{D}'$, $\mathcal{C}^\infty$ or $\mathcal{S}'$. As these are injective A-modules, applying the functor $\mathsf{Hom}(-,\ \mathcal{F})$ to the resolution (8) of Example

2.1 yields the exact sequence

$$0 \to \mathbb{C} \xrightarrow{\pi^{\mathsf{T}}} \mathcal{F} \xrightarrow{\mathsf{grad}} \mathcal{F}^3 \xrightarrow{\mathsf{curl}} \mathcal{F}^3 \xrightarrow{\mathsf{div}} \mathcal{F} \to 0,$$

where π^{T} is the injection of $\mathsf{Hom}_A(A/m,\ \mathcal{F}) \simeq \mathbb{C}$, the space of constant functions, into $\mathcal{F}$. Manifestly $\mathsf{Ker}_{\mathcal{F}}(\mathsf{div})$ and $\mathsf{Ker}_{\mathcal{F}}(\mathsf{curl})$ are controllable.

The ideal of A generated by the rows of grad is the maximal ideal $m = (\partial_1, \partial_2, \partial_3)$. As A/m is a torsion module, $\mathsf{Ker}_{\mathcal{F}}(\mathsf{grad})$ is not controllable when $\mathcal{F}$ is $\mathcal{D}'$ or $\mathcal{C}^\infty$, by Theorem 2.1 (ii). (Clearly we cannot patch together distinct constant functions within the space of constant functions.) This is also true when $\mathcal{F} = \mathcal{S}'$.

(ii) Suppose now that $\mathcal{F}$ is $\mathcal{D}$, $\mathcal{E}'$ or $\mathcal{S}$. These are flat A-modules, hence tensoring (8) with $\mathcal{F}$ yields the exact sequence

$$0 \to \mathcal{F} \xrightarrow{\mathsf{grad}} \mathcal{F}^3 \xrightarrow{\mathsf{curl}} \mathcal{F}^3 \xrightarrow{\mathsf{div}} \mathcal{F} \xrightarrow{\pi} \mathcal{F}/m\mathcal{F} \to 0$$

Now $\mathsf{Ker}_{\mathcal{F}}(\mathsf{grad})$ is also controllable because it equals 0. This is in agreement with the remark at the end of Lecture 1.

We now have two definitions of a controllable state space system, namely Kalman's Definition 1.1, and Willems' Definition 1.2 when the system (6) is rewritten as equation (7) of Example 1.1. We show below that the two definitions coincide; thus Willems' definition is a perfect generalisation of Kalman's.

We use the Popov-Belevitch-Hautus (PBH) test for Kalman controllability of state space systems.

PBH Test: Let $P(x) = (xI_\ell - X, -U)$, where $P(\frac{\mathsf{d}}{\mathsf{dt}}) \in \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$ is the operator of equation (7) in Example 1.1. Then the state space system (6) of Definition 1.1 is controllable if and only if $P(x)$ is of full rank for every $x \in \mathbb{C}$.

Thus, to determine whether (6) is controllable, it suffices to calculate all the maximal minors of $P(x)$, i.e. the determinants of its $\ell \times \ell$ submatrices – there are $\binom{\ell+m}{\ell}$ of them – and then to check that these minors do not have a common zero in $\mathbb{C}$. In other words, is the ℓ-th determinantal ideal $\mathfrak{i}_\ell(P(x))$, generated by the determinants of the $\ell \times \ell$ submatrices of $P(x)$, equal to (1)?

We will generalise this test to the more general situation of $A = \mathbb{C}[\partial_1, \ldots, \partial_n]$. Henceforth, we adopt the following convention: given a submodule $P \subset A^k$, let $\ell \geqslant 0$ be such that P can be generated by ℓ elements, but not by any set of $\ell - 1$ elements. Then, any set of ℓ generators for P is said to be minimum. We will always represent P by an $\ell \times k$ matrix $P(\partial)$, where the ℓ rows is a minimum set of generators. We call this integer ℓ the minimum number of generators for P.

Given such a $P(\partial)$, consider the ideal of $\mathbb{C}[x_1, \ldots, x_n]$ generated by all the $\ell \times \ell$ minors of $P(x)$. If $\ell > k$, this ideal is equal to 0, otherwise it is the $(k-\ell)$-th Fitting ideal of A^k/P, and is therefore independent of the choice of the matrix $P(\partial)$ whose ℓ rows is a minimum set of generators for P. It is the *cancellation* ideal of P, or of $P(\partial)$ or $\mathsf{Ker}_{\mathcal{F}}(P)$, and we denote it by $\mathfrak{i}_\ell(P)$.

Similarly, the ideal generated by all the $k \times k$ minors of $P(x)$, where $P(\partial)$ is any matrix whose rows generate P, is the 0-th Fitting ideal of A^k/P. It is the characteristic ideal of P, or of $\mathsf{Ker}_{\mathcal{F}}(P)$, and is denoted $\mathfrak{i}_k(P)$.

The affine varieties $\mathcal{V}(\mathfrak{i}_\ell(P))$ and $\mathcal{V}(\mathfrak{i}_k(P))$ in $\mathbb{C}^n$ are the cancellation and characteristic varieties of $\mathsf{Ker}_{\mathcal{F}}(P)$, respectively.

Definition 2.1. *Suppose that the submodule $P \subset A^k$ can be generated by not more than k elements, i.e. $\ell \leqslant k$ above, then $\mathsf{Ker}_{\mathcal{F}}(P)$ is underdetermined. It is strictly underdetermined if $\ell < k$. If $\ell > k$, $\mathsf{Ker}_{\mathcal{F}}(P)$ is an overdetermined system.*

Thus, the characteristic ideal $\mathfrak{i}_k(P)$ equals 0 if $\mathsf{Ker}_{\mathcal{F}}(P)$ is strictly underdetermined, and the cancellation ideal $\mathfrak{i}_\ell(P)$ equals 0 if it is overdetermined.

When $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, a principal ideal domain, every submodule of A^k is free (being finitely generated and torsion free) and has a basis not larger in number than k. Then, every matrix $P(\frac{\mathsf{d}}{\mathsf{dt}})$ representing P as above, i.e. whose rows is now a basis for P, will be of full (row) rank. In this case, the PBH test generalizes perfectly to arbitrary lumped systems:

Proposition 2.2. *([19]) Let $\mathcal{F}$ be an injective cogenerator, and let P be a submodule of $(\mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}])^k$ of rank ℓ. Then $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable (in the sense of Definition 1.2) if and only if the PBH condition, $\mathfrak{i}_\ell(P) = (1)$, is satisfied.*

Proof: The structure theory for modules over a PID implies that there is a basis $e_1, \ldots, e_k$ for A^k, $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, and elements $a_1(\frac{\mathsf{d}}{\mathsf{dt}}), \ldots, a_\ell(\frac{\mathsf{d}}{\mathsf{dt}})$ of A, such that $a_1(\frac{\mathsf{d}}{\mathsf{dt}})e_1, \ldots,$ $a_\ell(\frac{\mathsf{d}}{\mathsf{dt}})e_\ell$ is a basis for P, namely the Smith form of any matrix $P(\frac{\mathsf{d}}{\mathsf{dt}})$ whose ℓ rows generate P. Thus, for $\xi \in \mathbb{C}$, the rank of $P(x)$ at $x = \xi$ equals ℓ if and only $a_1(\xi), \ldots, a_\ell(\xi)$ are all nonzero. This latter condition holds for every ξ in $\mathbb{C}$ if and only if they are all nonzero constants. In other words, $P(x)$ has full row rank for every $\xi \in \mathbb{C}$ if and only if a basis for P extends to a basis for A^k.

This last condition is in turn equivalent to saying that the exact sequence

$$0 \to P \longrightarrow A^k \longrightarrow A^k/P \to 0$$

splits (the condition asserts that $0 \to P \to A^k$ splits). This implies that A^k/P is a submodule of A^k, hence torsion free (and therefore free). By Theorem 2.1 (ii), this is equivalent to the controllability of $\mathsf{Ker}_F(P)$. □

Specializing to the case of Example 1.1 shows that Willems' definition of behavioural controllability is a faithful generalization of Kalman's state controllability.

Corollary 2.3. *([19]) A state space behaviour is controllable in the sense of Definition 1.2 if and only if it is controllable in the sense of Definition 1.1.* □

We return to the case of $A = \mathbb{C}[\partial_1, \ldots, \partial_n]$, and ask the following questions.

Let the rows of the matrix $P(\partial)$ be a minimum set of generators for P. What does it mean if $P(x)$ has full row rank for each $x \in \mathbb{C}^n$? Is this still a necessary and sufficient condition for controllability of $\mathsf{Ker}_{\mathcal{F}}(P)$? If not, what is the analogue of the PBH test?

Proposition 2.3. *([10]) Let P be a submodule of A^k, and let $\ell \geqslant 0$ be the cardinality of a minimum set of generators for P. Then A^k/P is free (of rank $k - \ell$) if and only* *if the PBH condition,* $\mathsf{i}_\ell(P) = (1)$*, is satisfied. Thus, the PBH condition is sufficient for* $\mathsf{Ker}_{\mathcal{F}}(P)$ *to be controllable when* $\mathcal{F}$ *is* $\mathcal{D}'$, $\mathcal{C}^\infty$ *or* $\mathcal{S}'$*.*

Proof: Clearly, either of the above statements implies that $\ell \leqslant k$; for suppose that A^k/P is free, then $A^k \to A^k/P \to 0$ splits, hence P is a direct summand of A^k, hence a projective A-module. Therefore, by Quillen-Suslin,[3] P is free, of rank $\ell \leqslant k$. The other statement, $\mathsf{i}_\ell(P) = 1$, manifestly implies the inequality. We may also assume that $\ell > 0$, because otherwise P is the 0 submodule of A^k, and then $\mathsf{i}_0(P)$ is the k-th Fitting ideal of A^k, which by definition equals (1).

Let $P(\partial) = (p_{ij})$ be any $\ell \times k$ matrix whose ℓ rows is a minimum set of generators for P, and $P(x)$ the corresponding matrix with polynomial entries. Now, the value $p_{ij}(\xi)$ of $p_{ij}(x)$ at $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ is the image of $p_{ij}(\partial)$ under the morphism $A \to A/m_\xi$, where m_ξ is the maximal ideal $(\partial_1 - \xi_1, \dots, \partial_n - \xi_n)$. Thus $\mathsf{rank}(P(\xi)) = \ell$ if and only if there is a minor, say the determinant $\mathsf{det}(M(\xi))$ of an $\ell \times \ell$ submatrix $M(\xi)$ of $P(\xi)$, which is not 0. This implies that $\mathsf{det}(M(\partial))$ does not belong to m_ξ, and is therefore a unit in the localization A_{m_ξ}. Hence there is a basis such that the matrix of the localization $P_{m_\xi}(\partial) : A^k_{m_\xi} \to A^\ell_{m_\xi}$ (of the morphism $P(\partial) : A^k \to A^\ell$) has an $\ell \times \ell$ submatrix equal to the identity I_ℓ. By row and column operations, all other entries of $P_{m_\xi}(\partial)$ can be made zero. This implies that

$$0 \to P_{m_\xi} \longrightarrow A^k_{m_\xi} \longrightarrow A^k_{m_\xi}/P_{m_\xi} \to 0$$

splits. Thus $A^k_{m_\xi}/P_{m_\xi} \simeq ((A^k/P)_{m_\xi}$ is projective, and as this is true for every maximal ideal m_ξ, it follows that A^k/P is projective. Again by Quillen-Suslin, A^k/P is free (of rank $k - \ell$). □

Remark: In Example 2.1, $\mathsf{i}_\ell(D) = (\partial_1, \partial_2, \partial_3)$, hence D does not satisfy the PBH condition (as $\mathsf{div}(x)$ drops rank at $0 \in \mathbb{C}^3$), and A^3/D is not free.

Remark: The condition in Proposition 2.3, that A^k/P be free, is sufficient but not necessary for $\mathsf{Ker}_{\mathcal{F}}(P)$ to be controllable when $\mathcal{F}$ is an injective A-module. Besides, A^k/P is free if and only if it, and therefore also P, are direct summands of A^k. Then $A^k \simeq P \oplus A^k/P$, hence $\mathsf{Ker}_{\mathcal{F}}(P) \simeq \mathsf{Hom}_A(A^k/P,\ \mathcal{F})$ is a direct summand of $\mathcal{F}^k \simeq \mathsf{Hom}_A(A^k,\ \mathcal{F})$, $\mathcal{F}$ any A-submodule of $\mathcal{D}'$. Such a system is said to be strongly controllable. By Proposition 2.2, every controllable lumped system is strongly controllable.

Thus PBH is a sufficient condition for the strong controllability of $\mathsf{Ker}_{\mathcal{F}}(P)$, for $\mathcal{F}$ any A-submodule of $\mathcal{D}'$.

Now suppose that $\mathcal{F}$ is an injective cogenerator. Then $\mathsf{Ker}_{\mathcal{F}}(P)$ is a direct summand of $\mathcal{F}^k$ if and only if P is a direct summand of A^k. Indeed, by Proposition 1.6, splittings of $A^k \to A^k/P \to 0$ are in bijective correspondence with splittings of $0 \to \mathsf{Ker}_{\mathcal{F}}(P) \to \mathcal{F}^k$. Thus the PBH condition is necessary and sufficient for strong controllability when $\mathcal{F} = \mathcal{D}'$, $\mathcal{C}^\infty$. In particular, when $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, a controllable system is also strongly controllable (namely Proposition 2.2).

[3]The theorem of Quillen-Suslin asserts that a finitely generated projective module over the polynomial ring, is free.

The above proposition is a statement about a projection of the system $\mathcal{F}^k$. We study projections of a general system in the appendix below.

The next result is the generalisation of the PBH Test to the case of $A = \mathbb{C}[\partial_1, \ldots, \partial_n]$, under the condition that the cancellation ideal be nonzero. Later, we show that this condition is generically satisfied in the set of underdetermined systems.

Theorem 2.2. *Let P be a submodule of A^k, and let $P(\partial)$ be any $\ell \times k$ matrix whose ℓ rows generate P. Suppose that the cancellation ideal $\mathfrak{i}_\ell(P) \neq 0$ (so that in particular $\ell \leqslant k$). Then A^k/P is torsion free if and only if $P(x)$ has full row rank for all x in the complement of an algebraic variety in $\mathbb{C}^n$ of dimension $\leqslant n-2$ (or in other words, that the Krull dimension of the ring $A/\mathfrak{i}_\ell(P)$ be less than or equal to $n-2$). Thus,*

(i) if $\mathcal{F}$ is an injective A-module, then the underdetermined system $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable if the dimension of the cancellation variety, $\mathcal{V}(\mathfrak{i}_\ell(P))$, is less than or equal to $n-2$.

(ii) If $\mathcal{F}$ is an injective cogenerator, then $\mathsf{dim}(\mathcal{V}(\mathfrak{i}_\ell(P))) \leqslant n-2$ is also necessary for $\mathsf{Ker}_{\mathcal{F}}(P)$ to be controllable.

Proof: Suppose to the contrary that $\mathsf{rank}(P(x))$ is less than ℓ for x on an algebraic variety of dimension $n-1$ in $\mathbb{C}^n$ - it cannot be less than ℓ on all of $\mathbb{C}^n$ because $\mathfrak{i}_\ell(P)$ is nonzero, by assumption. Let V be an irreducible component, and let it be the zero locus of the irreducible polynomial $p(x)$ (a codimension 1 prime ideal in a unique factorization domain is a principal ideal, generated by a prime element). This means that each of the $\binom{k}{\ell}$ generators of $\mathfrak{i}_\ell(P)$ is divisible by $p(\partial)$. Let $\mathfrak{p}$ be the prime ideal $(p(\partial))$, and let $P_\mathfrak{p}(\partial)$ be the image of the matrix $P(\partial)$ in the localization $A_\mathfrak{p}$ of A at $\mathfrak{p}$. Suppose that every entry of some row of $P_\mathfrak{p}(\partial)$, say the first row, is divisible by $p(\partial)$. As divisibility by $p(\partial)$ in A and $A_\mathfrak{p}$ are equivalent, the corresponding row of $P(\partial)$ is also divisible by $p(\partial)$, and then clearly A^k/P has a torsion element. Otherwise, at least one element of this first row of $P_\mathfrak{p}(\partial)$, say the first element, is not divisible by $p(\partial)$, and is therefore a unit in $A_\mathfrak{p}$. By row and column operations, every other element of the first column, and of the first row, can be made zero. Let the matrix so obtained after these row and column operations, with a unit in the (1,1) entry and the other entries of the first row and column equal to 0, be denoted $P^1_\mathfrak{p}(\partial)$, and let $P^1_\mathfrak{p}$ be the submodule of $A^k_\mathfrak{p}$ generated by its rows.

The above argument for $P_\mathfrak{p}(\partial)$ can be repeated now for $P^1_\mathfrak{p}(\partial)$, and it follows that either $p(\partial)$ divides every element of some row, say the second, or that some element in that row, say the (2,2) entry (the (2,1) entry is zero) is a unit in $A_\mathfrak{p}$. In the first case, $A^k_\mathfrak{p}/P^1_\mathfrak{p}$ has a torsion element, hence so does A^k/P; otherwise all other elements in the second row and second column can be made zero by column and row operations. Eventually, after at most $\ell-1$ such steps, the resultant matrix has units in the (j,j) entries, $1 \leqslant j \leqslant \ell-1$, and the other entries zero except in positions $(\ell, j), \ell \leqslant j \leqslant k$. If now $p(\partial)$ does not divide every entry of the ℓ-th row, then it also does not divide the generators of the cancellation ideal $\mathfrak{i}_\ell(P)$, which is a contradiction.

Conversely, suppose that A^k/P has a torsion element. This implies that there is an element $x(\partial) = (a_1(\partial), \ldots, a_k(\partial))$ in $A^k \setminus P$ such that $r(\partial) = p(\partial)x(\partial) = (p(\partial)a_1(\partial), \ldots, p(\partial)a_k(\partial))$ is in P, for some nonzero $p(\partial)$ which can be assumed to be irreducible (we comment on this further in the next lecture). As $r(\partial)$ is in P, it is an A-linear combination of the ℓ rows $r_1(\partial), r_2(\partial), \ldots, r_\ell(\partial)$ of $P(\partial)$, say $r(\partial) =$

$b_1(\partial)r_1(\partial) + \cdots + b_\ell(\partial)r_\ell(\partial)$. Clearly the $b_j(\partial)$ are not all zero, and are also not all divisible by $p(\partial)$, because otherwise it would imply that $x(\partial)$ belongs to P, contrary to its choice. Thus, without loss of generality, let $b_1(\partial)$ be nonzero and not divisible by $p(\partial)$. Now let $B(\partial)$ be the followin $\ell \times \ell$ matrix: its first row is $(b_1(\partial), \ldots, b_\ell(\partial))$, it has 1 in the (j, j) entries, $2 \leqslant j \leqslant \ell$, and all other entries are 0. Then the product $B(\partial)P(\partial)$ is an $\ell \times k$ matrix whose first row is $r(\partial)$ and whose other rows are the rows $r_2(\partial), \ldots, r_\ell(\partial)$ of $P(\partial)$. By construction of $B(\partial)$, its determinant $b_1(\partial)$ is not divisible by $p(\partial)$, and as every generator of the ℓ-th determinantal ideal of $B(\partial)P(\partial)$ is divisible by $p(\partial)$, it follows that every generator of the cancellation ideal $\mathfrak{i}_\ell(P)$ is also divisible by $p(\partial)$. This implies that $\mathsf{rank}(P(x)) < \ell$ at points x in $\mathbb{C}^n$ where $p(x) = 0$, an algebraic variety of dimension $n - 1$.

Statements (i) and (ii) now follow from Theorem 2.1. □

We have seen in Proposition 2.3 that the cancellation ideal $\mathfrak{i}_\ell(P)$ of $P \subset A^k$ equals (1) if and only if P is a direct summand of A^k. The condition that $\mathfrak{i}_\ell(P)$ be nonzero in the above theorem also admits an elementary interpretation.

Proposition 2.4. *Let P be a submodule of A^k. Then its cancellation ideal $\mathfrak{i}_\ell(P)$ is nonzero if and only if P is free (of rank ℓ).*

Proof: Let $P(\partial)$ be an $\ell \times k$ matrix whose ℓ rows is a minimum set of generators for P, then $\mathfrak{i}_\ell(P)$ is generated by the $\ell \times \ell$ minors of $P(x)$. If $\ell > k$, then there is nothing to be done, and so we assume $\ell \leqslant k$.

Now if P is not free, then there is a nontrivial relation between the rows of $P(x)$, hence all the maximal minors equal 0.

Conversely, suppose that $\mathfrak{i}_\ell(P) = 0$. This means that after localizing at the 0 ideal, i.e over the function field $K = \mathbb{C}(x_1, \ldots, x_n)$, the ℓ rows of $P(x)$ span a subspace $\bar{P}$ of K^k, whose projections to K^ℓ, given by choosing ℓ of the k coordinates, are all subspaces of dimension strictly less than ℓ. If $k = \ell$, then the rows of $P(x)$ are K-linearly dependent, hence there is a relation between the rows of $P(x)$, and P is not free (as ℓ is the minimum number of elements needed to generate it). Otherwise, there are $\binom{k}{\ell} > \ell$ such projections, hence $\bar{P}$ must be of dimension strictly less than ℓ, so that again, by definition of ℓ, P is not free. □

Theorem 2.2 implies that if the system determined by a free submodule is controllable, then its cancellation ideal cannot be principal. In other words, we have the following corollary.

Corollary 2.4. *Suppose that $\mathcal{F}$ is an injective cogenerator, and suppose that the minimum number of elements required to generate the submodule $P \subset A^k$ is equal to k (i.e. suppose $\ell = k$ in our notation). If $\mathsf{Ker}_\mathcal{F}(P)$ is controllable, then P is not free.*

Proof: Let $P(\partial)$ be a $k \times k$ matrix whose rows is a minimum set of generators for P. Then the cancellation ideal $\mathfrak{i}_k(P)$ is a principal ideal of A, generated by the determinant of $P(x)$. If P is free, then $\mathfrak{i}_k(P) \neq 0$ by the above proposition. Hence, the cancellation variety has dimension $n-1$, and $\mathsf{Ker}_\mathcal{F}(P)$ is not controllable by Theorem 2.2 (ii). □

Example 2.3: Neither the above corollary, nor Theorem 2.2, provide answers when P is not free, i.e. when $\mathfrak{i}_\ell(P) = 0$. For instance, let P_1 and P_2 be the submodules of A^3

generated by the rows of the matrices

$$P_1(\partial) = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \quad \text{and} \quad P_2(\partial) = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_1\partial_2 & \partial_1^2 & 0 \end{pmatrix} .$$

The three rows of each matrix is a minimal set of generators for P_1 and P_2, and the cancellation ideals $\mathfrak{i}_3(P_i)$, $i = 1, 2$, are both equal to 0. While A^3/P_1 is torsion free, A^3/P_2 has torsion; hence, $\mathsf{Ker}_{\mathcal{F}}(P_1)$ is controllable, whereas $\mathsf{Ker}_{\mathcal{F}}(P_2)$ is not. □

We shall continue to study the ideals $\mathfrak{i}_\ell(P)$ and $\mathfrak{i}_k(P)$ in the next lecture.

Appendix

We study the projection of a kernel $\mathsf{Ker}_{\mathcal{F}}(P)$ to a subset of its coordinates, and ask if this projection is also a differential kernel. This is a PDE analogue of the 'elimination problem' of algebraic geometry.

Consider the split exact sequence

$$0 \to A^p \underset{\xleftarrow{\pi_1}}{\xrightarrow{i_1}} A^{p+q} \underset{\xleftarrow{i_2}}{\xrightarrow{\pi_2}} A^q \to 0 ,$$

and the corresponding split exact sequence

$$0 \to \mathcal{F}^p \underset{\xrightarrow{i_1}}{\xleftarrow{\pi_1}} \mathcal{F}^{p+q} \underset{\xrightarrow{\pi_2}}{\xleftarrow{i_2}} \mathcal{F}^q \to 0 .$$

Let $P \subset A^{p+q}$ be any submodule, and $\mathsf{Ker}_{\mathcal{F}}(P) \subset \mathcal{F}^{p+q}$ the corresponding differential kernel.

Proposition 2.5. *Let $\mathcal{F}$ be an A-submodule of $\mathcal{D}'$. Then*

$$i_2^{-1}(\mathsf{Ker}_{\mathcal{F}}(P)) = \mathsf{Ker}_{\mathcal{F}}(\pi_2(P)) \subset \pi_2(\mathsf{Ker}_{\mathcal{F}}(P)) \subset \mathsf{Ker}_{\mathcal{F}}(i_2^{-1}(P)) .$$

If $\mathcal{F}$ is an injective A-submodule of $\mathcal{D}'$, then $\pi_2(\mathsf{Ker}_{\mathcal{F}}(P)) = \mathsf{Ker}_{\mathcal{F}}(i_2^{-1}(P))$, so that a projection of a distributed system is also a distributed system.

Proof: The chain of equality and inclusions in the statement is elementary. For instance, suppose that $p_2(\partial) \in i_2^{-1}(P)$. Then $(0, p_2(\partial)) \in P$, so that $(0, p_2(\partial))(f) = 0$ for all $f \in \mathsf{Ker}_{\mathcal{F}}(P)$. It follows that $p_2(\partial)(\pi_2(f)) = 0$, hence $\pi_2(\mathsf{Ker}_{\mathcal{F}}(P)) \subset \mathsf{Ker}_{\mathcal{F}}(i_2^{-1}(P))$.

The first split sequence above implies that the sequence

$$0 \to A^q/i_2^{-1}(P) \xrightarrow{i_2} A^{p+q}/P \xrightarrow{\pi_1} A^p/\pi_1(P) \to 0$$

is exact. Thus, if $\mathcal{F}$ is an injective A-submodule of $\mathcal{D}'$, then

$$0 \to \mathsf{Ker}_{\mathcal{F}}(\pi_1(P)) \xrightarrow{i_1} \mathsf{Ker}_{\mathcal{F}}(P) \xrightarrow{\pi_2} \mathsf{Ker}_{\mathcal{F}}(i_2^{-1}(P)) \to 0$$

is also exact. □

Example 2.4: Let $A = \mathbb{C}[\partial_x, \partial_y]$. Consider the Cauchy-Riemann equation

$$\begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix} : \begin{array}{ccc} (\mathcal{C}^\infty)^2 & \longrightarrow & (\mathcal{C}^\infty)^2 \\ (u,v) & \mapsto & (\partial_x u - \partial_y v,\ \partial_y u + \partial_x v) \end{array}$$

An elment (u, v) in its kernel defines the holomorphic function $u + iv$.

If $P \subset A^2$ is the submodule generated by the rows of the C-R matrix above, then $i_2^{-1}(P)$ is the principal ideal $(\partial_x^2 + \partial_y^2)$ generated by the Laplacian. As $\mathcal{C}^\infty$ is an injective A-module, the above proposition states that the projection $\pi_2 : (u, v) \mapsto v$ maps the kernel of the C-R equation onto the kernel of the Laplacian, namely the set of harmonic functions.

Remark: In modelling the attributes $f \in \mathcal{F}^q$ of a phenomenon, it is sometimes convenient to include additional variables $g \in \mathcal{F}^p$. For instance, if we wish to model the voltages that appear across various branches of a circuit, it might be convenient to also include the currents through these branches. These additional variables are the latent variables, and the attributes of the phenomenon are the manifest variables, of the model [19]. The modelling process then leads to equations of the form

$$P(\partial)f = Q(\partial)g\ .$$

If $\mathcal{F}$ is an injective A-module, then those f that appear in a solution (f, g) of the above equation, the manifest variables, is also a differential kernel by the above proposition. In other words, we can 'eliminate' the latent variables g from the model.

3. The Controllable-Uncontrollable Decomposition

We continue our study of the influence of the cancellation and characteristic ideals on the behaviour of a distributed system.

We first obseve that the condition $\mathfrak{i}_\ell \neq 0$ of Theorem 2.2 is superfluous when $n = 1, 2$:

$n = 1$: Now every submodule P of $\mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]^{\mathsf{k}}$ is free (being finitely generated and torsion free). Moreover, an algebraic subset of $\mathbb{C}$ of dimension $n-2$ is empty. Hence, Theorem 2.2 specializes without qualification to the classical Hautus test, as well as to Proposition 2.2.

$n = 2$: Now $A = \mathbb{C}[\partial_1, \partial_2]$. Let $P \subset A^k$ be a submodule such that $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable, where $\mathcal{F}$ is an injective cogenerator. Then, A^k/P is torsion free, hence it injects into a free module A^{k_1}, for some k_1. Let $P(\partial)$ be an $\ell \times k$ matrix whose rows is a minimum set of generators for P. The following sequence is then a minimal resolution

$$\to A^{\ell_1} \longrightarrow A^{\ell} \stackrel{P^t(\partial)}{\longrightarrow} A^k \longrightarrow A^{k_1} \longrightarrow A^{k_1}/(A^k/P) \to 0 \ ,$$

and hence $A^{\ell_1} = 0$ as the global dimension of the ring A equals 2. The morphism $P^t(\partial)$ is injective, and this implies that its image P is free. Thus, a controllable $2-D$ system is defined by a free submodule, so that the cancellation ideal $\mathfrak{i}_\ell(P)$ is always nonzero. Theorem 2.2(ii) then implies the following result of Wood et al. ([20]) on $2-D$ systems:

Corollary 3.1. *Let $n = 2$, and $\mathcal{F}$ an injective cogenerator. Then $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable if and only if $P(x)$ drops rank at most at a finite number of points in $\mathbb{C}^2$ (where the ℓ rows of $P(\partial)$ is a minimum set of generators for P).*

Proof: By the above discussion it is unnecessary to assume that $\mathfrak{i}_\ell(P) \neq 0$, so that by Theorem 2.2, $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable if and only if the dimension of the cancellation variety of $\mathfrak{i}_\ell(P)$ is equal to 0. The finite set of points of this variety is precisely where $P(x)$ drops rank. □

The above discussion also implies that when $n = 2$ and $\mathcal{F}$ an injective cogenerator, there cannot be a phenomenon such as the controllable system $\mathsf{Ker}_{\mathcal{F}}(\mathsf{curl})$.

Corollary 3.2. *Let $n = 2$, and $\mathcal{F}$ an injective cogenerator. Then any nonzero controllable system is strictly underdetermined.*

We return to the case of general n. The following example explains the nomenclature 'cancellation ideal' for $\mathfrak{i}_\ell$:

Example 3.1: Consider the special case of a scalar system given by $p_1(\partial)f_1 = p_2(\partial)f_2$, i.e. the system defined by the kernel of the map

$$\begin{array}{rccc} P(\partial) = (p_1(\partial), -p_2(\partial)) : & \mathcal{F}^2 & \longrightarrow & \mathcal{F} \\ & (f_1, f_2) & \mapsto & (p_1(\partial), -p_2(\partial)) \left[\begin{array}{c} f_1 \\ f_2 \end{array}\right] , \end{array}$$

where $\mathcal{F}$ is an injective cogenerator. The cancellation ideal, here $\ell = 1$, is the ideal $\mathfrak{i}_1(P) = (p_1, p_2)$ generated by p_1 and p_2. Theorem 2.2 asserts that this kernel is controllable if and only if there is no non-constant p which divides both p_1 and p_2. This is the classical pole-zero cancellation criterion.

We now study the structure of a distributed system that is not necessarily controllable. We first observe that it contains a maximal controllable sub-system, and that the quotient is isomorphic to an 'uncontrollable' system in the following sense.

Definition 3.1. *A system is uncontrollable if none of its nonzero sub-systems is controllable.*

Thus, a sub-system of an uncontrollable system is also uncontrollable. As the zero system is (trivially) uncontrollable, and also controllable, we use these adjectives only for nonzero systems.

We have the following characterisation.

Proposition 3.1. *Let $\mathcal{F}$ be an injective cogenerator. Then $\mathsf{Ker}_{\mathcal{F}}(P)$ is uncontrollable if and only if A^k/P is a torsion module.*

Proof: Sub-systems of $\mathsf{Ker}_{\mathcal{F}}(P)$ are in bijective correspondence with submodules $P' \subset A^k$ that contain P. A nonzero sub-system, say corresponding to $P' \subsetneq A^k$, is controllable if and only if A^k/P' is torsion free. However, $A^k/P' \subset A^k/P$, and hence is a torsion module as A^k/P is. □

By Proposition 1.1, a controllable system is the closure of its compactly supported elements. In contrast, there is no compactly supported element in an uncontrollable system.

Proposition 3.2. *Let $\mathsf{Ker}_{\mathcal{F}}(P)$ be an uncontrollable system as in the above proposition. Then there is no nonzero compactly supported element in it.*

Proof: Suppose f is a nonzero compactly supported element in $\mathsf{Ker}_{\mathcal{F}}(P)$. As $\mathcal{F}$ is an injective cogenerator, there is an element $q(\partial) \in A^k \setminus P$ such that $q(\partial)f \neq 0$. Let $a(\partial)$ be a nonzero element in A such that $a(\partial)q(\partial) \in P$ (such an $a(\partial)$ exists because A^k/P is a torsion module). Then $a(\partial)(q(\partial)f) = 0$, but this contradicts the Paley-Wiener Theorem as $q(\partial)f$ has compact support. □

Remark: An uncontrollable system does not also contain rapidly decreasing elements in it. The proof is identical to the proof above, and it only requires us to observe that because the Fourier transform maps $\mathcal{S}$ to itself, no nonzero element of A can map an element of $\mathcal{S}$ to zero.

Recollect that a prime ideal p of a commutative ring R is an associated prime of an R-module M if it is equal to the annihilator $\mathsf{ann}(x)$ of some $x \in M$; the set of its associated primes is denoted $\mathsf{Ass}(M)$. An element r of R is a zero divisor for M if there is a nonzero x in M such that $rx = 0$. The maximal elements of the family $\{\mathsf{ann}(x) \mid 0 \neq x \in \mathcal{M}\}$ are associated primes of M, hence the union of all the associated primes is the set of all the zero divisors for M (for instance [4]). It then follows that M is torsion free if and only if 0 is its only associated prime, and is torsion if and only if 0 is not an associated prime.

Now let P be an A-submodule of A^k. Let P_0 be the submodule $\{x \in A^k \mid \exists\ a \neq 0 \textit{ with } ax \in P\}$. Then P_0 contains P, and the quotient P_0/P is the submodule of

A^k/P consisting of its torsion elements. The following sequence

$$0 \to P_0/P \longrightarrow A^k/P \longrightarrow A^k/P_0 \to 0 \tag{9}$$

is exact, where A^k/P_0 is torsion free. In general, given a short exact sequence of A-modules, the associated primes of the middle term is contained in the union of the associated primes of the other two modules. However, here it is clear that we have equality.

Lemma 3.1. $\mathsf{Ass}(A^k/P) = \mathsf{Ass}(P_0/P) \sqcup \mathsf{Ass}(A^k/P_0)$ *(disjoint union). Hence, if* $P \subsetneq P_0 \subsetneq A^k$*, then* $\mathsf{Ass}(A^k/P_0) = \{0\}$ *and* $\mathsf{Ass}(P_0/P)$ *is the set of all the nonzero associated primes of* A^k/P. □

Applying the functor $\mathsf{Hom}_A(-,\ \mathcal{F})$, where $\mathcal{F}$ is an injective cogenerator, to the above sequence gives the exact sequence

$$0 \to \mathsf{Ker}_{\mathcal{F}}(P_0) \longrightarrow \mathsf{Ker}_{\mathcal{F}}(P) \longrightarrow \mathsf{Hom}_A(P_0/P,\ \mathcal{F}) \to 0$$

As A^k/P_0 is torsion free, $\mathsf{Ker}_{\mathcal{F}}(P_0)$ is a controllable sub-system of $\mathsf{Ker}_{\mathcal{F}}(P)$. If P_1 is any A-submodule of A^k such that $P \subset P_1 \not\supseteq P_0$, then A^k/P_1 has torsion elements and $\mathsf{Ker}_{\mathcal{F}}(P_1)$ is not controllable. Hence $\mathsf{Ker}_{\mathcal{F}}(P_0)$ is the largest controllable sub-system of $\mathsf{Ker}_{\mathcal{F}}(P)$ in the sense that any other controllable sub-system is contained in it. We call it the controllable part of the system.

Suppose P_0/P can be generated by r elements; then $P_0/P \simeq A^r/R$, for some submodule $R \subset A^r$, hence $\mathsf{Hom}_A(P_0/P,\ \mathcal{F}) \simeq \mathsf{Ker}_{\mathcal{F}}(R)$. As A^r/R is a torsion module, $\mathsf{Ker}_{\mathcal{F}}(R)$ is uncontrollable. This system is a quotient, and not a sub-system of $\mathsf{Ker}_{\mathcal{F}}(P)$ (unless the above short exact sequence splits). We now show that there is a sub-system of $\mathsf{Ker}_{\mathcal{F}}(P)$, which is not however canonically determined, that is isomorphic to this quotient.

Recollect that a submodule N of an R-module M is a primary submodule of M if every zero divisor $r \in R$ for M/N satisfies $r \in \sqrt{\mathsf{ann}(M/N)}$. It then follows that $\mathsf{Ass}(M/N)$ contains only one elment, and if $\mathsf{Ass}(M/N) = \{p\}$, then N is a p-primary submodule of M. We also note that as A is a noetherian domain, N is 0-primary if and only if M/N is torsion free.

The result we need is the following:

Theorem: Let R be a noetherian ring, and M a finitely generated R-module. Then every proper submodule N of M admits an irredundant primary decompostion, which is to say that $N = N_1 \cap \cdots \cap N_r$, where N_i is a p_i-primary submodule of M, $p_i \neq p_j$ for $i \neq j$, and where none of the N_i can be omitted (i.e. $N \neq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_r$, for any i). $\mathsf{Ass}(M/N)$ then equals $\{p_1, \ldots, p_r\}$. If p_i is a minimal element in $\mathsf{Ass}(M/N)$, then the p_i-primary component N_i is uniquely determined.

We improve the second part of Lemma 1.1(ii) when $\mathcal{F}$ is injective, and when the collection of submodules of A^k is finite.

Lemma 3.2. *Let* $\mathcal{F}$ *be an injective* A*-module, then* $\sum_{i=1}^r \mathsf{Ker}_{\mathcal{F}}(P_i) = \mathsf{Ker}_{\mathcal{F}}(\bigcap_{i=1}^r P_i)$.

Proof: Consider the following exact 'Mayer-Vietoris' sequence

$$0 \to A^k/(P_1 \cap P_2) \xrightarrow{\ i\ } A^k/P_1 \oplus A^k/P_2 \xrightarrow{\ \pi\ } A^k/(P_1 + P_2) \to 0\ ,$$

where $i([x]) = ([x], -[x])$, $\pi([x],[y]) = [x+y]$ and where $[x]$ indicates the class of x in various quotients. As $\mathcal{F}$ is an injective A-module, it follows that

$$0 \to \mathsf{Hom}_A(A^k/(P_1+P_2),\ \mathcal{F}) \longrightarrow \mathsf{Hom}_A(A^k/P_1,\ \mathcal{F}) \oplus \mathsf{Hom}_A(A^k/P_2,\ \mathcal{F}) \longrightarrow$$
$$\mathsf{Hom}_A(A^k/(P_1 \cap P_2),\ \mathcal{F}) \to 0$$

is also exact. But $\mathsf{Hom}_A(A^k/(P_1+P_2),\ \mathcal{F}) \simeq \mathsf{Ker}_{\mathcal{F}}(P_1+P_2) = \mathsf{Ker}_{\mathcal{F}}(P_1) \cap \mathsf{Ker}_{\mathcal{F}}(P_2)$ by the first part of Lemma 1.1(ii), hence the above exact sequence implies that $\mathsf{Ker}_{\mathcal{F}}(P_1) + \mathsf{Ker}_{\mathcal{F}}(P_2) = \mathsf{Ker}_{\mathcal{F}}(P_1 \cap P_2)$.

The lemma now follows by induction. □

Theorem 3.1. *Let $\mathcal{F}$ be an injective cogenerator, and P a proper submodule of A^k. Then $\mathsf{Ker}_{\mathcal{F}}(P)$ can be written as the sum of its controllable part and an uncontrollable sub-system.*

Proof: If A^k/P is either torsion free or a torsion module, then there is nothing to be done. Otherwise, let $P = P_0 \cap P_1 \cap \cdots \cap P_r$, $r \geqslant 1$, be an irredundant primary decomposition of P in A^k, where P_0 is a 0-primary submodule of A^k, and the P_i, $i \geqslant 1$, are p_i-primary submodules for nonzero primes p_i. Then A^k/P_0 is torsion free, and $A^k/(P_1 \cap \cdots \cap P_r)$ is torsion; hence $\mathsf{Ker}_{\mathcal{F}}(P) = \mathsf{Ker}_{\mathcal{F}}(P_0 \cap P_1 \cap \cdots \cap P_r) = \mathsf{Ker}_{\mathcal{F}}(P_0) + \mathsf{Ker}_{\mathcal{F}}(P_1 \cap \cdots \cap P_r)$, where $\mathsf{Ker}_{\mathcal{F}}(P_0)$ is controllable, and $\mathsf{Ker}_{\mathcal{F}}(P_1 \cap \cdots \cap P_r) = \mathsf{Ker}_{\mathcal{F}}(P_1) + \cdots + \mathsf{Ker}_{\mathcal{F}}(P_r)$ is uncontrollable. As 0 is a minimal associated prime of A^k/P, P_0 is uniquely determined, and in fact equals the P_0 of Lemma 3.1. Hence, $\mathsf{Ker}_{\mathcal{F}}(P_0)$ is the controllable part of $\mathsf{Ker}_{\mathcal{F}}(P)$. □

Propositions 1.1 and 3.2 now imply the following corollary.

Corollary 3.3. *Let $\mathcal{F}$ be either $\mathcal{D}'$ or $\mathcal{C}^\infty$ with their standard topologies, and P an A-submodule of A^k. Then $\mathsf{Ker}_{\mathcal{F}}(P)$ can be written as the sum of two sub-systems: one, which is the closure of the set of compactly suported elements, and another, which contains no nonzero compactly supported, nor rapidly decreasing, element.*

We now describe the nonzero associated primes of A^k/P that determine the uncontrollable part of $\mathsf{Ker}_{\mathcal{F}}(P)$ in the above theorem, in the case when P is a free proper submodule of A^k.

Proposition 3.3. *Let P be a free submodule of A^k. Then the nonzero associated primes of A^k/P are the principal ideals generated by the irreducible $p(\partial)$ that divide every element of $\mathfrak{i}_\ell(P)$.*

Proof: By Proposition 2.3, the cancellation ideal $\mathfrak{i}_\ell(P)$ of P is nonzero. Now, the first half of Theorem 2.2 demonstrates that an irreducible $p(\partial)$ which divides every generator of $\mathfrak{i}_\ell(P)$, and hence every one of its elements, is a zero divisor for A^k/P. Conversely, suppose that $a(\partial)$ is a zero divisor for A^k/P, and that x is an element of $A^k \backslash P$ such that $a(\partial)x \in P$. Because A is a UFD, $a(\partial)$ is a unique product of irreducible factors, say $p_1(\partial) \cdots p_s(\partial)$ (where the p_i are not necessarily distinct). Then, there is an i, $1 \leqslant i \leqslant r$, such that $p_1(\partial) \cdots p_i(\partial)x \in P$, but $p_1(\partial) \cdots p_{i-1}(\partial)x \notin P$. Thus it follows that the maximal elements of the family of principal ideals generated by zero divisors of A^k/P are the principal ideals generated by irreducible zero divisors. Furthermore, by the second half of the proof of Theorem 2.2, every irreducible zero divisor divides every generator of $\mathfrak{i}_\ell(P)$. Together with the above observation, this shows that the set of irreducible zero divisors for A^k/P is precisely the set of irreducible common factors of the $\binom{k}{\ell}$ generators of the cancellation ideal $\mathfrak{i}_\ell(P)$. As the latter set is finite, it follows that the set of irreducible zero divisors for A^k/P is a finite set.

This in turn implies that the nonzero associated primes of A^k/P are precisely the principal ideals generated by the irreducible zero divisors described above. For suppose that some associated prime p is not principal. Let $p_1(\partial)$ be an irreducible element of p such that its degree is minimum amongst all the (nonzero) elements in p. Let $p_2(\partial)$ be another irreducible element in p, not a multiple of $p_1(\partial)$, again of minimum degree amongst all elements in $p \setminus p_1(\partial)$. Then $p_1(\partial) + \alpha p_2(\partial)$ is irreducible, for all $\alpha \in \mathbb{C}$. These infinite number of ireducible elements are all in p, and hence are all zero divisors for A^k/P, contradicting the assertion above. □

By Theorem 2.2(ii), the system defined by a free submodule of A^k is not controllable if and only if its cancellation variety is of codimension 1. In this case, the above proposition, together with the proof of Theorem 3.1, implies the following corollary.

Corollary 3.4. *Let P be a free submodule of A^k, and $\mathcal{F}$ an injective cogenerator. Then, the codimension 1 irreducible components of the cancellation variety $\mathcal{V}(\mathfrak{i}_\ell(P))$ determine the uncontrollable part of $\mathsf{Ker}_\mathcal{F}(P)$ in Theorem 3.1.*

These results, which follow from Theorem 2.2, are valid only when the cancellation ideal is nonzero, which is a specific (though generic) case of underdetermined systems. They are not valid for underdetermined systems whose cancellation ideal equals 0.

We now study distributed systems from the point of view of their characteristic ideals.

For any submodule $P \subset A^k$, its characteristic ideal $\mathfrak{i}_k(P)$ and characteristic variety $\mathcal{V}(\mathfrak{i}_k(P)) \subset \mathbb{C}^n$ are fundamental objects that determine $\mathsf{Ker}_\mathcal{F}(P)$. Indeed, the Integral Representation Theorem of Palamadov [11], a vast generalization of Malgrange's Theorem quoted earlier, states that an element in $\mathsf{Ker}_{\mathcal{D}'}(P)$ is an absolutely convergent integral of its exponential solutions, and these exponential solutions are determined by the points on its characteristic variety.

Our study is based on the following elementary lemma.

Lemma 3.3. $\mathfrak{i}_k(P) \subset \mathsf{ann}(A^k/P) \subset \sqrt{\mathfrak{i}_k(P)}$, *and thus* $\mathcal{V}(\mathfrak{i}_k(P)) = \mathcal{V}(\mathsf{ann}(A^k/P))$.

Proof: Let d be the determinant of a $k \times k$ matrix D all whose rows belong to the submodule $P \subset A^k$. Let D' be the matrix adjoint to D, so that $D'D$ is the $k \times k$ diagonal matrix whose diagonal elements are all equal to d. But the rows of $D'D$ are also in P, and as these rows are $de_1, \ldots, de_k$ ($\{e_i,\ 1 \leqslant i \leqslant k\}$ is the standard basis for A^k), it follows that d is in $\mathsf{ann}(A^k/P)$. This implies that $\mathfrak{i}_k(P) \subset \mathsf{ann}(A^k/P)$.

Conversely, let a be in $\mathsf{ann}(A^k/P)$ so that $ae_1, \ldots, ae_k$ are all in P. The determinant of the $k \times k$ matrix whose rows are these ae_i equals a^k. Thus a^k is in $\mathfrak{i}_k(P)$ which implies that a is in $\sqrt{\mathfrak{i}_k(P)}$. □

Thus, $\mathsf{Ker}_\mathcal{F}(P)$ is overdetermined if $\mathsf{ann}(A^k/P) \neq 0$.

Corollary 3.5. *Let $\mathcal{F}$ be an injective cogenerator, and P a proper submodule of A^k. Then $\mathsf{Ker}_\mathcal{F}(P)$ is uncontrollable if and only if $\mathfrak{i}_k(P) \neq 0$.*

Proof: By the above lemma, $\mathsf{i}_k(P) \neq 0$ if and only if $\mathsf{ann}(A^k/P) \neq 0$. As A^k/P is finitely generated, $\mathsf{ann}(A^k/P) \neq 0$ is equivalent to saying that A^k/P is a torsion module. The corollary now follows from Proposition 3.1. □

Corollary 3.6. *Suppose P is a p-primary submodule of A^k, $\mathcal{F}$ an injective cogenerator. Then $\mathsf{Ker}_{\mathcal{F}}(P)$ is uncontrollable if p is a nonzero prime, and controllable otherwise.*

Proof: Suppose p is nonzero. As p equals the set of zero divisors for A^k/P, it follows that $\sqrt{\mathsf{ann}(A^k/P)} = p$. Thus $\mathsf{ann}(A^k/P)$ is nonzero, and by the above lemma, so is $\mathsf{i}_k(P)$.

Conversely, P is 0-primary if and only if A^k/P is torsion free. Then, and only then, is $\mathsf{Ker}_{\mathcal{F}}(P)$ controllable. □

Corollary 3.7. *The controllable part of a strictly underdetermined system is nonzero ($\mathcal{F}$ as above).*

Proof: The characteristic ideal of a strictly underdetermined system (i.e. $\ell < k$) is equal to 0. □

The condition $\mathsf{ann}(A^k/P) = 0$ is weaker than the condition guaranteeing controllability of $\mathsf{Ker}_{\mathcal{F}}(P)$, namely that A^k/P be torsion free. We now explain one implication of this condition on the dynamics of the system.

Proposition 3.4. *Let P be a submodule of A^k, and $\mathcal{F}$ an injective A-module. Let $\pi_j : \mathcal{F}^k \to \mathcal{F}$, $(f_1, \ldots, f_k) \mapsto f_j$ be the projection onto the j-th coordinate. Then $\mathsf{ann}(A^k/P) = 0$ if and only if there is a j, $1 \leqslant j \leqslant k$, such that the restriction $\pi_j : \mathsf{Ker}_{\mathcal{F}}(P) \to \mathcal{F}$ is surjective.*

Proof: Let $i_j : A \to A^k$, $a \mapsto (0, \ldots, a, \ldots, 0)$ be the inclusion into the j-th coordinate. Now, $\mathsf{ann}(A^k/P) = 0$ is equivalent to saying that there is a j, $1 \leqslant j \leqslant k$, such that $i_j^{-1}(P) = 0$. By Proposition 2.5, $\pi_j(\mathsf{Ker}_{\mathcal{F}}(P)) = \mathsf{Ker}_{\mathcal{F}}(i_j^{-1}(P))$, as $\mathcal{F}$ is injective. This is in turn equivalent to $\pi_j(\mathsf{Ker}_{\mathcal{F}}(P)) = \mathcal{F}$ by Corollary 2.1. □

As the only controllable sub-systems of $\mathcal{F}$ are 0 and $\mathcal{F}$ itself, the surjectivity statement above is termed coordinate controllability. Thus a system $\mathsf{Ker}_{\mathcal{F}}(P)$ is coordinate controllable (where $\mathcal{F}$ is an injective A-module) if and only if $\mathsf{ann}(A^k/P) = 0$. Lemma 3.3 then implies the following supplement to the previous corollary.

Corollary 3.8. *A strictly underdetermined system is coordinate controllable.* □

To formulate Willems' behavioural controllability in these terms, consider a general homothety: given $r(\partial) = (a_1(\partial), \ldots, a_k(\partial))$ in A^k, let $i_r : A \to A^k$ be the morphism $a(\partial) \mapsto a(\partial)r(\partial)$ (which is an injection when $r(\partial) \neq 0$). Applying $\mathsf{Hom}_A(-, \mathcal{F})$ to this morphism gives $r(\partial) : \mathcal{F}^k \to \mathcal{F}$, mapping $f = (f_1, \ldots, f_k)$ to $a_1(\partial)f_1 + \cdots + a_k(\partial)f_k$ (and which is a surjection when $r(\partial) \neq 0$); thus, the injections i_j and the projections π_j in the above proposition correspond to $r(\partial) = (0, \ldots, 1, \ldots, 0)$. Behavioural controllability is then an assertion about the restrictions of the maps $r(\partial)$ above to $r(\partial) : \mathsf{Ker}_{\mathcal{F}}(P) \to \mathcal{F}$, for all $r(\partial)$ in A^k.

Proposition 3.5. *Let P be a submodule of A^k, and $\mathcal{F}$ an injective cogenerator. The system $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable if and only if every image $r(\partial)(\mathsf{Ker}_{\mathcal{F}}(P))$ is controllable (in other words if and only if every $r(\partial)(\mathsf{Ker}_{\mathcal{F}}(P))$ is either 0 or all of $\mathcal{F}$).*

Proof: Consider the map

$$\begin{array}{rccc} m_r : & A & \longrightarrow & A^k/P \\ & 1 & \mapsto & [r(\partial)] \end{array}$$

defined by $r(\partial) \in A^k$. Applying $\mathsf{Hom}_A(-, \mathcal{F})$ to it gives the restriction $r(\partial) : \mathsf{Ker}_{\mathcal{F}}(P) \to \mathcal{F}$ described above. The map m_r is the 0 map if $r(\partial) \in P$, hence the image $r(\partial)(\mathsf{Ker}_{\mathcal{F}}(P))$ equals 0, and is controllable.

Suppose now that $\mathsf{Ker}_{\mathcal{F}}(P)$ is controllable so that A^k/P is torsion free. Then m_r is an injective map for every $r(\partial) \notin P$, hence $r(\partial)(\mathsf{Ker}_{\mathcal{F}}(P))$ equals $\mathcal{F}$ (as $\mathcal{F}$ is an injective module). Conversely, suppose $\mathsf{Ker}_{\mathcal{F}}(P)$ is not controllable. Let $r(\partial) \notin P$ be a torsion element in A^k/P; then m_r is not injective, neither is it the 0 map. Hence $r(\partial)(\mathsf{Ker}_{\mathcal{F}}(P))$ is not all of $\mathcal{F}$, nor does it equal 0. Thus it is not controllable by Corollary 2.2. □

A coordinate onto which a system surjects can play the role of an input [19]. This raises the question: can the projection of a controllable system to several coordinates also be surjective? The example below shows that in general this need not be the case.

Example: Let $A = \mathbb{C}[\partial_1, \partial_2]$. The system given by the kernel of the map $(\partial_1, \partial_2) : \mathcal{F}^2 \to \mathcal{F}$ is controllable as its cancellation variety, the origin in $\mathbb{C}^2$, is of codimension 2). By Proposition 3.4, each $\pi_j : \mathsf{Ker}_{\mathcal{F}}((\partial_1, \partial_2)) \to \mathcal{F},\ j = 1, 2$, is surjective, but $\mathsf{Ker}_{\mathcal{F}}((\partial_1, \partial_2)) \to \mathcal{F}^2$ is clearly not.

A strongly controllable system, namely the second remark after Proposition 2.3, does however surject onto several of its coordinates. Indeed, if the rank of the free module A^k/P equals ℓ', then $\mathsf{Ker}_{\mathcal{F}}(P) \simeq \mathcal{F}^{\ell'}$.

Thus, we have identified three graded notions of controllability of $\mathsf{Ker}_{\mathcal{F}}(P)$, each corresponding to increasingly weaker requirements on A^k/P. Because these notions can be understood in terms of projections of the system to various coordinates (as the above propositions indicate), they amount to notions of inputs and outputs, and hence to notions of causality.

Table 1. Notions of controllability of $\mathsf{Ker}_{\mathcal{F}}(P)$

Strong controllability	$\Longleftrightarrow$	A^k/P is free
Behavioural controllability	$\Longleftrightarrow$	A^k/P is torsion free
Coordinate controllability	$\Longleftrightarrow$	$\mathsf{ann}(A^k/P) = 0$

(In the above table, the first equivalence is valid for an arbitrary A-module $\mathcal{F}$, the second for $\mathcal{F}$ that are injective cogenerators, and the last for injective $\mathcal{F}$.)

We conclude this lecture with a few additional comments on systems which do not admit inputs, and whose evolution is therefore autonomous.

As suggested above, our intuitive understanding of an input is a signal which is not restricted in any way, and which can therefore be any element in $\mathcal{F}$. By Proposition 3.4, uncontrollable systems are precisely those which do not admit inputs. These are also called autonomous systems [12, 19].

We now study a special class of autonomous systems, namely those distributed systems whose behaviours resemble overdetermined systems of lumped behaviours.

Definition 3.2. [12] *A system* $\mathsf{Ker}_{\mathcal{F}}(P)$ *is strongly autonomous if its characteristic variety* $\mathcal{V}(\mathfrak{i}_k(P))$ *has dimension 0.*

We first observe that if m is a maximal ideal of $A = \mathbb{C}[\partial_1, \dots, \partial_n]$, say $m = (\partial_1 - \xi_1, \dots, \partial_n - \xi_n)$, then $\mathsf{Ker}_{\mathcal{D}'}(m)$ is the 1-dimensional $\mathbb{C}$-subspace of $\mathcal{D}'$ spanned by $e^{<\xi,x>}$, where $\xi = (\xi_1, \dots, \xi_n)$. As $\mathsf{Ker}_{\mathcal{F}}(m) = \mathsf{Ker}_{\mathcal{D}'}(m) \cap \mathcal{F}$, it's dimension equals 1 if $e^{<\xi,x>}$ is in $\mathcal{F}$, and 0 otherwise.

Lemma 3.4. *Let* $i = (p_1(\partial), \dots, p_i(\partial), \dots, p_r(\partial))$ *be an ideal of* $\mathbb{C}[\partial_1, \dots, \partial_n]$ *such that* $\mathsf{Ker}_{\mathcal{F}}(i)$ *is a finite dimensional* $\mathbb{C}$*-vector space. Let* $j = (p_1(\partial), \dots, p_i^2(\partial), \dots, p_r(\partial))$*. Then* $\mathsf{Ker}_{\mathcal{F}}(j)$ *is also finite dimensional.*

Proof: Let i' be the ideal $(p_1(\partial), \dots, \widehat{p_i(\partial)}, \dots, p_r(\partial))$, where $\widehat{p_i(\partial)}$ means that $p_i(\partial)$ has been omitted. Consider the following $\mathbb{C}$-linear map:

$$P_i : \begin{array}{ccc} \mathsf{Ker}_{\mathcal{F}}(i') & \longrightarrow & \mathcal{F} \\ f & \mapsto & p_i(\partial) f \end{array}$$

Then $\mathsf{Ker}_{\mathcal{F}}(i)$ equals the kernel of P_i, which is finite dimensional by assumption. Hence $\mathsf{Ker}_{\mathcal{F}}(j) = P_i^{-1}(\mathsf{Ker}_{\mathcal{F}}(i))$ is also finite dimensional. □

Corollary 3.9. *Let* i *be an ideal of* $\mathbb{C}[\partial_1, \dots, \partial_n]$*. Then* $\mathsf{Ker}_{\mathcal{F}}(i)$ *is a finite dimensional* $\mathbb{C}$*-vector space if and only if* $\mathsf{Ker}_{\mathcal{F}}(\sqrt{i})$ *is.*

Proof: If $\mathsf{Ker}_{\mathcal{F}}(\sqrt{i})$ is finite dimensional, then so is the system defined by every power of $\sqrt{i}$ by the above lemma. As i is finitely generated, some power of $\sqrt{i}$ is contained in i. Hence $\mathsf{Ker}_{\mathcal{F}}(i)$ is also finite dimensional. The other implication is obvious. □

Proposition 3.6. *Let* $\mathcal{F}$ *be either* $\mathcal{D}'$ *or* $\mathcal{C}^\infty$*, and* P *a submodule of* A^k*. Then* $\mathsf{Ker}_{\mathcal{F}}(P)$ *is strongly autonomous if and only if it is a finite dimensional* $\mathbb{C}$*-vector subspace of* $\mathcal{F}^k$*.*

Proof: Suppose $\mathsf{Ker}_{\mathcal{F}}(P)$ is strongly autonomous, so that its characteristic variety $\mathcal{V}(\mathfrak{i}_k(P))$ is a finite set of points. Hence the ideal of this variety, which is the radical of of $\mathfrak{i}_k(P)$, is a finite intersection of maximal ideals. The exponential solutions of $\sqrt{\mathfrak{i}_k(P)}$ are then the union of the exponential solutions corresponding to each of these maximal ideals, and this spans a finite dimensional $\mathbb{C}$-vector space. By the theorem of Malgrange quoted in the remark after Theorem 1.2, this space is equal to $\mathsf{Ker}_{\mathcal{F}}(\sqrt{\mathfrak{i}_k(P)})$. By the above corollary, $\mathsf{Ker}_{\mathcal{F}}(\mathfrak{i}_k(P))$ is also finite dimensional.

Suppose that $\mathfrak{i}_k(P) = (a_1(x), \dots, a_r(x))$. Then every component of an element $f = (f_1, \dots, f_k)$ in $\mathsf{Ker}_{\mathcal{F}}(P)$ is also a homogenous solution of $a_i(\partial), i = 1, \dots, r$, and hence belongs to $\mathsf{Ker}_{\mathcal{F}}(\mathfrak{i}_k(P))$. It follows that $\mathsf{Ker}_{\mathcal{F}}(P)$ is finite dimensional as well.

Conversely, suppose that $\mathsf{Ker}_{\mathcal{F}}(P)$ is not strongly autonomous. Let $P(\partial)$ be any matrix whose rows generate the submodule P. At each point ξ in the characterisic variety of P, the columns $c_1, \ldots, c_k$ of $P(x)$ is $\mathbb{C}$-linearly dependent, say $\sum \alpha_i c_i = 0$. Then $f = (\alpha_1 e^{<\xi,x>}, \ldots, \alpha_k e^{<\xi,x>})$ is in $\mathsf{Ker}_{\mathcal{F}}(P)$. The set of these exponential elements, corresponding to the infinite points in the characteristic variety, is linearly independent. □

It follows from the above proof that every solution of a strongly autonomous system is entire. Thus no nonzero element in it can vanish on any open subset of $\mathbb{R}^n$.

We compare this with the behaviour of a lumped system which either contains elements that are not entire, or is finite dimensional. In the case of distributed systems, a behaviour could consist entirely of entire functions, yet be infinite dimensional (as in the case of an elliptic operator).

4. Genericity

We now ask how large is the class of controllable systems in the set of all systems? More precisely, we topologise the set of systems, and ask whether the subset of controllable systems is open dense. The results below are with respect to a very coarse topology, the Zariski topology.

This question is motivated by the following considerations. A system's behaviour is modelled by differential equations which involve parameters that are only approximately known, for they are determined by measurements only upto a certain accuracy. Hence, the conclusions and predictions of a theory must be *stable* with respect to perturbations of the differential equations appearing in the model, in order for it to be an effective theory. These ideas go back to the notion of structural stability of autonomous systems due to Andronov and Pontryagin [1].

In contrast, control systems usually admit inputs, and are therefore non-autonomous. However, our needs are modest compared to the foundational work in [1], where a topological congugacy is sought to be established between the flow of a vector field and the flow of the perturbed vector field. Instead, all we ask for here is the persistence of controllability under perturbations. In other words, if a system is controllable, is a perturbation of it also controllable? As we work with linear phenomena, we are able to answer this question with respect to perturbations given by a very coarse topology, even though the dynamics of the system are described by partial differential equations.

In this section, we restrict our attention to the case when the A-module $\mathcal{F}$ is an injective cogenerator. By Proposition 1.5, the set of distributed systems in $\mathcal{F}^k$ is in (inclusion reversing) bijection with the set of all A-submodules of A^k. As we wish to study the variation of a system's properties with respect to variations in the differential equations that describe it, we need to topologise the set of all submodules of A^k, and towards this we first topologise the set of all matrices with k columns and entries in A.

Let $A(d),\ d \geqslant 0$, be the subset of $A = \mathbb{C}[\partial_1, \ldots, \partial_n]$ consisting of differential operators of degree at most d. There are $N(d) := \binom{n+d}{n}$ monomials of degree at most d in the partial derivatives $\partial_1, \ldots, \partial_n$, hence we identify $A(d)$ with the affine space $\mathbb{C}^{N(d)}$ equipped with the Zariski topology. There is a natural inclusion of $A(d)$ into $A(d+1)$ as a Zariski closed subset. Thus $\{A(d), d \geqslant 0\}$ is a directed system, and its (strict) direct limit, $\varinjlim A(d)$, is the topological space of all differential operators. We denote it by $\mathcal{A}$.

An element of the ring A is a sum of monomial terms $\partial^d := \partial_1^{d_1} \cdots \partial_n^{d_n}$ with complex coefficients; in the topological space $\mathcal{A}$ it corresponds to the point whose 'coordinates' are these coefficients (if a monomial does not appear in the sum, then its coefficient is 0). Thus, there is a 'coordinate axis' in the space $\mathcal{A}$ corresponding to each monomial in the ring A. We denote the indeterminate, and the coordinate axis, corresponding to the monomial ∂^d by X_d or $X_{d_1 \cdots d_n}$. The points of the space $\mathcal{A}$ corresponding to the units in the ring A are the points on the axis $X_{0\cdots 0}$ except for the origin. Its closure is this axis, and is a proper Zariski closed subset of $\mathcal{A}$.

The space $\mathcal{A}$ is not Noetherian, for instance the descending sequence of Zariski closed subspaces $A \supsetneq \{X_{1\cdots 1} = 0\} \supsetneq \{X_{1\cdots 1} = 0, X_{2\cdots 2} = 0\} \supsetneq \ldots$ does not stabilise. If $\{X_d\}$

denotes the union of the indeterminates $X_{d_1 \cdots d_n}$, then the coordinate ring of the space $\mathcal{A}$ is $\mathbb{C}[\{X_d\}]$.

Now let k be a fixed positive integer. Let $M_{\ell,k}$ be the set of differential operators defined by matrices with k columns and ℓ rows, and with entries from the ring A. Let $M_{\ell,k}(d)$ be the set of those matrices in $M_{\ell,k}$ whose entries are all bounded in degree by d. We identify it with the affine space $\mathbb{C}^{\ell k N(d)}$ with the Zariski topology, and just as above, we have the directed system $\{M_{\ell,k}(d),\ d \geqslant 0\}$, where $M_{\ell,k}(d) \hookrightarrow M_{\ell,k}(d+1)$ continuously as a Zariski closed subset. The direct limit, $\varinjlim M_{\ell,k}(d)$, is $M_{\ell,k}$ with the Zariski topology, and we denote it by $\mathcal{M}_{\ell,k}$ (the space $\mathcal{A}$, in this notation, is $\mathcal{M}_{1,1}$).

Let $S_{\ell,k}$ be the set of submodules of A^k that can be generated by ℓ elements, and let the set of distributed systems they define in $\mathcal{F}^k$ be denoted by $B_{\ell,k}$. We define $\Pi_{\ell,k} : \mathcal{M}_{\ell,k} \to S_{\ell,k}$ by mapping a matrix to the submodule generated by its rows. We equip $S_{\ell,k}$ with the quotient topology, and denote this topological space by $\mathcal{S}_{\ell,k}$. The map $\Pi_{\ell,k}$ is then continuous. We carry over this topology to $B_{\ell,k}$ via the bijection $P \mapsto \mathsf{Ker}_{\mathcal{F}}(P)$, and denote this space by $\mathcal{B}_{\ell,k}$.

For $k = 1$, we denote $\mathcal{M}_{\ell,1}$ by $\mathcal{A}^\ell$, so that a point is a column with ℓ entries. The space $\mathcal{S}_{\ell,1}$ of ideals of the ring A that can be generated by ℓ elements, we denote by $\mathcal{I}_\ell$. In this notation, $\Pi_{\ell,1} : \mathcal{A}^\ell \to \mathcal{I}_\ell$ maps an element of $\mathcal{A}^\ell$ to the ideal generated by its ℓ entries.

A subset of differential operators in $\mathcal{M}_{\ell,k}$, or of submodules in $\mathcal{S}_{\ell,k}$ or systems in $\mathcal{B}_{\ell,k}$, is said to be *generic* if it contains an open dense subset in this topology.

For each $d \geqslant 0$, $M_{\ell,k}(d)$ is irreducible (being isomorphic to affine space), hence so is the direct limit $\mathcal{M}_{\ell,k}$ irreducible. It follows that $\mathcal{S}_{\ell,k}$ and $\mathcal{B}_{\ell,k}$ are also irreducible. Thus every nonempty open subset of these spaces is also dense. Nonetheless, we continue to use the phrase 'open dense' for emphasis.

We collect a few elementary properties of these spaces.

Lemma 4.1. *The element $0 \in \mathcal{I}_r$, namely the 0 ideal of A, is closed in $\mathcal{I}_r$, for every $r > 0$.*

Proof: In the above notation, $\Pi_{r,1}^{-1}(0)$ equals $0 \in \mathcal{A}^r$. As $\{0\}$ is closed in $A^r(d)$ for every d, it is closed in $\mathcal{A}^r$. It follows that $\{0\}$ is a closed point of $\mathcal{I}_r$. □

Lemma 4.2. *Let $I = \{i_1, \ldots, i_{\ell'}\}$ be a set of ℓ' indices between 1 and ℓ, and $J = \{j_1, \ldots, j_{k'}\}$ a set of k' indices between 1 and k. Let $s : \mathcal{M}_{\ell,k} \to \mathcal{M}_{\ell',k'}$ map an $\ell \times k$ matrix with entries in A to its $\ell' \times k'$ submatrix determined by the indices I and J. Then s is continuous and open.*

Proof: The map $s(d) : M_{\ell,k}(d) \to M_{\ell',k'}(d)$ induced by s, is continuous and open for every d as it is a projection $\mathbb{C}^{\ell k N(d)} \to \mathbb{C}^{\ell' k' N(d)}$ of an affine space onto the affine space given by a subset of its coordinates. Hence it follows that s is continuous and open. □

Lemma 4.3. *The map $\mathsf{det} : \mathcal{M}_{r,r} \to \mathcal{A}$, mapping a square matrix of size r with entries in A to its determinant, is continuous. Hence, the subset of matrices in $\mathcal{M}_{r,r}$ whose determinant is nonzero, is open dense.*

Proof: For each $d \geqslant 0$, the map det restricts to a map $M_{r,r}(d) \to A(rd)$ of affine spaces, which is continuous with respect to the Zariski topology as it is given by algebraic operations. Thus the set of matrices with nonzero determinant is open dense in $M_{r,r}(d)$ for each d, and hence in $\mathcal{M}_{r,r}$. □

Let $\ell \leqslant k$. Define $\mathfrak{m}_{\ell,k} : \mathcal{M}_{\ell,k} \to \mathcal{A}^{\binom{k}{\ell}}$ by mapping a matrix to its $\ell \times \ell$ minors (written as a column in some fixed order). This map is continuous by the above lemmas. If the rows of two matrices in $\mathcal{M}_{\ell,k}$ generate the same submodule of A^k, then their images in $\mathcal{I}_{\binom{k}{\ell}}$, under the composition $\Pi_{\binom{k}{\ell},1} \circ \mathfrak{m}_{\ell,k}$, are equal. Thus the map $\mathfrak{m}_{\ell,k}$ descends to a map $\mathfrak{i}_{\ell,k} : \mathcal{S}_{\ell,k} \to \mathcal{I}_{\binom{k}{\ell}}$, mapping a submodule P to the ideal generated by the maximal minors of any matrix in $\mathcal{M}_{\ell,k}$ whose rows generate P . In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{\ell,k} & \xrightarrow{\Pi_{\ell,k}} & \mathcal{S}_{\ell,k} \\ {\scriptstyle \mathfrak{m}_{\ell,k}}\downarrow & & \downarrow{\scriptstyle \mathfrak{i}_{\ell,k}} \\ \mathcal{A}^{\binom{k}{\ell}} & \xrightarrow{\Pi_{\binom{k}{\ell},1}} & \mathcal{I}_{\binom{k}{\ell}} \end{array} \tag{10}$$

Remark: If $P \in \mathcal{S}_{\ell,k}$ can be generated by fewer than ℓ elements, then clearly $\mathfrak{i}_{\ell,k}(P) = 0$. On the other hand, if the minimum number of generators for P is ℓ, then $\mathfrak{i}_{\ell,k}(P)$ is the cancellation ideal $\mathfrak{i}_\ell(P)$ of P.

Lemma 4.4. *The map $\mathfrak{i}_{\ell,k} : \mathcal{S}_{\ell,k} \to \mathcal{I}_{\binom{k}{\ell}}$ is continuous.*

Proof: The maps $\mathfrak{m}_{\ell,k}$, $\Pi_{\ell,k}$ and $\Pi_{\binom{k}{\ell},1}$ are continuous, and as the space $\mathcal{S}_{\ell,k}$ is a quotient of $\mathcal{M}_{\ell,k}$, so is $\mathfrak{i}_{\ell,k}$ also continuous. □

Proposition 4.1. *Let $\ell \leqslant k$. Then the set of submodules in $\mathcal{S}_{\ell,k}$ that are free of rank ℓ is open dense in $\mathcal{S}_{\ell,k}$.*

Proof: By Proposition 2.4, a submodule in $\mathcal{S}_{\ell,k}$ is free of rank ℓ, if and only if its cancellation ideal is nonzero. By Lemmas 4.1 and 4.4, the complement of this set of submodules, $\mathfrak{i}_{\ell,k}^{-1}(0)$, is closed in $\mathcal{S}_{\ell,k}$. □

Corollary 4.1. *The hypothesis in Theorem 2.2, that the cancellation ideal be nonzero, holds for a Zariski open dense set of submodules of $\mathcal{S}_{\ell,k}$.*

Similar results hold, when $\ell \geqslant k$, for characteristic ideals of submodules of A^k in $\mathcal{S}_{\ell,k}$. Now let $\mathfrak{m}^{\mathsf{T}}_{\ell,k} : \mathcal{M}_{\ell,k} \to \mathcal{A}^{\binom{\ell}{k}}$ map a matrix to its $k \times k$ minors (written as a column in some fixed order). This map is continuous. It again descends to a continuous map $\mathfrak{i}^{\mathsf{T}}_{\ell,k} : \mathcal{S}_{\ell,k} \to \mathcal{I}_{\binom{\ell}{k}}$, mapping a submodule P to the ideal generated by the $k \times k$ minors of any matrix in $\mathcal{M}_{\ell,k}$ whose rows generate P. Its image $\mathfrak{i}^{\mathsf{T}}_{\ell,k}(P)$ is the characteristic ideal $\mathfrak{i}_k(P)$ of the submodule P.

Analogous to Proposition 4.1, we have the following result.

Proposition 4.2. *Let $\ell \geqslant k$. Then the set of submodules in $\mathcal{S}_{\ell,k}$ whose characteristic ideals are nonzero, is open dense.*

We can immediately conclude a genericity result.

Theorem 4.1. *Let $\ell \geqslant k$. Then the set of uncontrollable systems in $\mathcal{B}_{\ell,k}$ is open dense. In other words, uncontrollability is generic in $\mathcal{B}_{\ell,k}$ when $\ell \geqslant k$.*

Proof: By Corollary 3.5, a system $\mathsf{Ker}_{\mathcal{F}}(P)$ is uncontrollable if and only if its characteristic ideal $\mathfrak{i}_k(P)$ is nonzero. □

The above construction of the topological space $\mathcal{M}_{\ell,k}$, and of the spaces $\mathcal{S}_{\ell,k}$ and $\mathcal{B}_{\ell,k}$, assumes that a perturbation of a system governed by ℓ laws results again in a system governed by the same number ℓ of laws. We could replace this with the more general assumption that perturbations might result in a change in the number of laws that govern the system. We incorporate this assumption as follows. We consider a differential operator in $\mathcal{M}_{\ell,k}$ as an element in $\mathcal{M}_{\ell',k}$, where $\ell' > \ell$, by appending to its ℓ rows the $(\ell' - \ell) \times k$ matrix $0_{\ell'-\ell,k}$, whose every entry is the 0 operator in A. Then $\mathcal{M}_{\ell,k} \hookrightarrow \mathcal{M}_{\ell',k}$ continuously, with image a proper Zariski closed subspace. This construction descends to the level of submodules of A^k, via the projections $\{\Pi_{\ell,k}, \ell \geqslant 1\}$, and hence to distributed systems in $\mathcal{F}^k$. Thus we have the chain of inclusions

$$\mathcal{M}_{1,k} \hookrightarrow \mathcal{M}_{2,k} \hookrightarrow \cdots \hookrightarrow \mathcal{M}_{\ell-1,k} \hookrightarrow \mathcal{M}_{\ell,k} \hookrightarrow \cdots$$

and the corresponding chains

$$\mathcal{S}_{1,k} \hookrightarrow \mathcal{S}_{2,k} \hookrightarrow \cdots \hookrightarrow \mathcal{S}_{\ell-1,k} \hookrightarrow \mathcal{S}_{\ell,k} \hookrightarrow \cdots$$

$$\mathcal{B}_{1,k} \hookrightarrow \mathcal{B}_{2,k} \hookrightarrow \cdots \hookrightarrow \mathcal{B}_{\ell-1,k} \hookrightarrow \mathcal{B}_{\ell,k} \hookrightarrow \cdots$$

of submodules of A^k and systems in $\mathcal{F}^k$. We can now consider the directed systems given by $\{\mathcal{M}_{\ell,k}\}, \{\mathcal{S}_{\ell,k}\}, \{\mathcal{B}_{\ell,k}\},\ \ell = 1, 2, \ldots$. Their direct limits, denoted $\mathcal{M}(k)$, $\mathcal{S}(k)$ and $\mathcal{B}(k)$, are the spaces of matrices with k columns and entries in A, of submodules of A^k, and of distributed systems in $\mathcal{F}^k$, respectively, all with the Zariski topolgy.

Just as above, a subset of differential operators in $\mathcal{M}(k)$ (or of submodules in $\mathcal{S}(k)$ or systems in $\mathcal{B}(k)$) is said to be generic if it contains an open dense subset in this topology.

We can therefore either study perturbations of a system defined by ℓ laws within the space of systems all defined by ℓ laws, i.e. within $\mathcal{M}_{\ell,k}$, or instead study perturbations which result in arbitrary numbers of laws by studying the inclusion $\mathcal{M}_{\ell,k} \hookrightarrow \mathcal{M}(k)$.

However, as $\mathcal{M}_{\ell-1,k}$ embeds in $\mathcal{M}_{\ell,k}$ as a proper Zariski closed subset, the topology we have defined implies that, generically, the number of laws governing a system cannot decrease, but can only increase. (This is akin to the statement that the number 0 can be easily perturbed to become nonzero, but it is unlikely that a nonzero number, when perturbed, will become 0.) In other words, in the space of systems that are described by ℓ laws, those systems that could be described by a fewer number of laws is a Zariski closed subset, and those that need to be described by ℓ laws is open dense.

Here we are concerned only with genericity questions, or in other words, concerned about properties of systems belonging to an open dense set. Furthermore, the answer to the controllability question depends upon whether the system is underdetermined or overdetermined, for instance Theorem 4.1 is applicable to overdetermined systems. Hence we confine ourselves to the case where the number of laws defining a system remains constant under perturbations. In other words, we study genericity questions within $\mathcal{M}_{\ell,k}$.

For the rest of this lecture, we consider strictly underdetermined systems, i.e. the case when $\ell < k$, and show that controllability is generic in $\mathcal{B}_{\ell,k}$.

The proof rests on Theorem 2.2(ii). The cancellation ideal $\mathfrak{i}_\ell(P)$ of a submodule P in $\mathcal{S}_{\ell,k}$ is generically nonzero by Proposition 4.1, and is generated by the $\binom{k}{\ell}$ many maximal minors of any $P(\partial) \in \mathcal{M}_{\ell,k}$, whose ℓ rows is a minimum set of generators for P. A theorem of Macaulay (for instance [4]) states that the codimension of such a determinantal ideal, if proper, is bounded above by $k - \ell + 1$. We show that when $(k-\ell+1) \leqslant n$, there is a Zariski open subset of $\mathcal{M}_{\ell,k}$ where either the cancellation ideal attains this codimension, or is equal to A. On the other hand, when $(k - \ell + 1) > n$, we show that there is a Zariski open subset of $\mathcal{M}_{\ell,k}$ where the cancellation ideal equals A. Together, this will imply that controllability is generic in $\mathcal{B}_{\ell,k}$, for all $\ell < k$, because $k - \ell + 1$ is at least equal to 2.

We begin the proof with a few elementary observations.

Lemma 4.5. *Let $\mu : \mathcal{A}^r \times \mathcal{A}^r \to \mathcal{A}$ be the map $((a_i), (b_i)) \mapsto \sum_1^r a_i b_i$. Then $\mu^{-1}(0)$ is a proper Zariski closed subset of the space $\mathcal{A}^r \times \mathcal{A}^r$.*

Proof: The map μ restricts to $\mu(d) : A(d)^r \times A(d)^r \to A(2d)$. It is given by adding and multiplying the coefficients of a_i and b_i, and hence it is continuous in the Zariski topology. The point 0 is closed in $A(2d)$, hence $\mu(d)^{-1}(0)$ is closed in $A(d)^r \times A(d)^r$. The direct limit of $\{\mu(d)^{-1}(0)\}_{d=0,1,\ldots}$ equals $\mu^{-1}(0)$, hence it is closed in $\mathcal{A}^r \times \mathcal{A}^r$. □

We need to study the image of the map μ.

Lemma 4.6. *Let I be a proper ideal of A. Then I is a proper Zariski closed subset of the space $\mathcal{A}$. Hence the set of elements of $\mathcal{A}$ which are not in I is an open dense subset of $\mathcal{A}$.*

Proof: Let I be generated by $\{a_1, \ldots, a_r\}$, and let a denote the point $(a_1, \ldots, a_r) \in A^r$. Define the map $\mu_a : \mathcal{A}^r \to \mathcal{A}$ by $\mu_a(b_1, \ldots, b_r) = \sum a_i b_i$. This is a $\mathbb{C}$-linear map as the coefficients of the operator $\sum a_i b_i$ are $\mathbb{C}$-linear combinations of the coefficients of the b_i. Its image is precisely the ideal I.

Let the maximum of the degrees of the a_i be s, then $s \geqslant 1$ as I is proper. For each d, the map μ_a restricts to a map $\mu_a(d) : A(d)^r \to A(d+s)$. Its image, say $I_a(d+s)$, is contained in $I \cap A(d+s)$. As I is proper, $I_a(d+s)$ is a proper linear subspace of $A(d+s)(\simeq \mathbb{C}^{N(d+s)})$, for all $d \gg 0$. It is then a proper Zariski closed subset of $A(d+s)$, and its vanishing ideal is generated by linear forms (in the indeterminates $\{X_{d_1 \cdots d_n}\}$, and whose coefficients are polynomial functions of the coefficients of the operators $a_1, \ldots, a_r$). Its complement is then open dense. For $d_1 < d_2$, the map $\mu_a(d_2)$ restricts to $\mu_a(d_1)$, hence $I_a(d_1 + s) \subset I_a(d_2 + s)$. The direct limit of the closed subspaces $\{I_a(d+s)\}_{d=0,1,\ldots}$ is I, hence I is a proper Zariski closed subset of $\mathcal{A}$, and its complement is open dense. □

Remark: More generally, every $\mathbb{C}$-linear subspace of $\mathcal{A}$ is Zariski closed, given by the common zeros of linear forms in the indeterminates $\{X_d\}$. So is therefore every affine linear subset of $\mathcal{A}$.

Proposition 4.3. *For $a = (a_1, \dots, a_r) \in \mathcal{A}^r$, let I_a be the Zariski closed subset of $\mathcal{A}$ consisting of the elements in the ideal $(a_1, \dots, a_r)$ generated by the coordinates $a_1, \dots, a_r$ of a (namely, the above lemma). Then the set $G = \{(a, a_{r+1}) \mid a \in \mathcal{A}^r,\ a_{r+1} \notin I_a\}$ contains a Zariski open subset of $\mathcal{A}^r \times \mathcal{A}$.*

Proof: The proof is an explanation of the parenthetical sentence in the proof of the above lemma.

Fix $s \geqslant 0$. Each $x \in \mathcal{A}(s)^r$ defines, for every $d \geqslant 0$, a $\mathbb{C}$-linear map $\mu_x(d) : A(d)^r \to A(d+s)$. If we represent this map by a matrix, say $M_x(d)$, the image of $\mu_x(d)$ is the column span of $M_x(d)$, and we denote it by $I_x(d+s)$ (as in the above lemma). There is a Zariski open dense subset of $A(s)^r$, say $U(s)$, such that the matrix $M_x(d)$, corresponding to every $x \in U(s)$, is of full column rank. The column span of $M_x(d)$ is a linear subspace of $A(d+s)$, and is a variety defined by linear forms. These linear forms belong to the kernel of the transpose $M_x^{\mathsf{T}}(d)$ (as the annihilator of an image of a linear map equals the kernel of its transpose). By 'Cramer's rule', this kernel is spanned by multi-linear forms in the coefficients of $M_x(d)$. Hence, as x varies in $U(s)$, the subset $\{(x, x_{r+1}) \mid x \in U(s), x_{r+1} \in I_x(d+s)\}$ of $A(s)^r \times A(d+s)$ is locally closed (i.e. the intersection of a closed set with an open set). The subset $\{(x, x_{r+1}) \mid x \in U(s), x_{r+1} \notin I_x(d+s)\}$ is then open in $U(s) \times A(d+s)$, for all d, and thus its direct limit (with respect to d), $G(s) = \{(x, x_{r+1}) \mid x \in U(s), x_{r+1} \notin I_x\}$, is open in $U(s) \times \mathcal{A}$.

For $s < s'$, $G(s) \subset G(s')$, and the direct limit of $\{G(s)\}_{s=0,1,\dots}$ equals G. This proves the proposition. □

Lemma 4.7. *Let I be a proper ideal of A. Then the set of zero divisors on A/I is closed in the space $\mathcal{A}$, and hence the set of nonzero divisors on A/I is open dense in $\mathcal{A}$.*

Proof: The set of zero divisors on A/I is the union of its finite number of associated primes, and this finite union is closed in A (by Lemma 4.6). □

Suppose I is a maximal ideal of $A = \mathbb{C}[\partial_1, \dots, \partial_n]$, it can then be generated by n elements (as $\mathbb{C}$ is algebraically closed), say $I = (a_1, \dots, a_n)$. The ideal $I + (a)$ is equal to A if and only if the element a does not belong to I. Thus $I + (a) = A$, for a in an open dense subset of $\mathcal{A}$ (by Lemma 4.6). This is analogous to saying that the set of units in $\mathbb{C}$ is Zariski open, because $A/I \simeq \mathbb{C}$.

Furthermore, let M be the set of points $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ such that the ideal $(a_1, \dots, a_n)$ generated by its coordinates is maximal (we show below that M contains an open dense subset of $\mathcal{A}^n$). Then by the above proposition, the set of points $(a_1, \dots, a_n, a_{n+1}) \in A^{n+1}$ such that $(a_1, \dots, a_n) \in M$ and the ideal generated by these $n+1$ coordinates is proper, is closed in the subspace $M \times \mathcal{A}$ of $\mathcal{A}^n \times \mathcal{A}$.

The opposite is however the case for proper ideals generated by fewer than n elements, namely the proposition below, and is suggested by the following heuristic:

Let I be a proper ideal generated by $r < n$ elements, then the codimension of I is at most r (by Krull's Theorem), and the dimension of A/I is at least $n - r$. By Noether normalization (for instance [4]), A/I is isomorphic to an integral extension of a polynomial ring with number of indeterminates at least $n - r$. Therefore, the set of units in A/I is contained in a proper Zariski closed set, and the set of elements $a \in A$

such that the sum $I + (a)$ is not equal to A contains an open dense subset of the space $\mathcal{A}$. Indeed, we have the following proposition.

Proposition 4.4. *Let $r \leq n$. Then the set P_r of elements $(a_1, \ldots, a_r) \in \mathcal{A}^r$ such that the ideal $(a_1, \ldots, a_r)$ generated by its coordinates is a proper ideal of A, contains an open dense subset of $\mathcal{A}^r$.*

We use the following result of Brownawell [2]:
Theorem (Brownawell): *Suppose the ideal generated by the polynomials $p_1, \ldots, p_m$ equals $\mathbb{C}[x_1, \ldots, x_n]$, where the degree of the p_i is less than or equal to d. Then there are polynomials $q_1, \ldots, q_m$ such that $\sum p_i q_i = 1$, where the degree of the q_i is less than or equal to $D = n^2 d^n + nd$.*

Proof of proposition: It suffices to prove the statement for $r = n$, for by Lemma 4.2, if P_n contains an open dense subset of $\mathcal{A}^n$, then its projection to $\mathcal{A}^r$ also contains an open dense subset. The coordinates of these points generate proper ideals of A. Thus this projection of P_n is contained in P_r.

So let I_a be the ideal generated by the coordinates of a point $a = (a_1, \ldots, a_n) \in A^n$. Let the degrees of the a_i be bounded by d. To say that $I \subsetneq A$ is to say that the map $\sigma_a : \mathbb{C}^n \to \mathbb{C}^n$ defined by $x \mapsto (a_1(x), \ldots, a_n(x))$ has nonempty inverse image $\sigma_a^{-1}(0)$. Clearly there is some point a_o in $A(d)^n$ such that the corresponding map σ_{a_o} has a regular point x in $\sigma_{a_o}^{-1}(0)$. This means that the rank of σ_{a_o} equals n at x, hence all smooth maps sufficiently close to σ_{a_o}, in the compact-open topology, also include 0 in their images (Inverse Function Theorem). Restricting to maps given by elements in $A(d)^n$ as above, this means that there is an open neighbourhood of a in the *euclidean topology* on $A(d)^n$ such that the variety of the ideal generated by the coordinates of every point in it, is nonempty. These ideals are thus proper ideals of A.

Thus $P_n \cap A(d)^n$ contains a euclidean open subset of $A(d)^n$. We now show that in fact it contains a nonempty Zariski open subset.

Suppose to the contrary that I_a were not a proper ideal of A. Then by the theorem of Brownawell, there are elements $b_1, \ldots, b_n$ of $\mathcal{A}$, of degree bounded by $D = n^2 d^n + nd$, such that $\sum a_i b_i = 1$. This is a Zariski closed condition on the coefficients of the a_i and b_i (as in Lemma 4.5), and it defines a proper affine variety in the affine space $A(d)^n \times A(D)^n$. Its projection C to the first n coordinates in $A(d)^n$ is a constructible set (by Chevalley's theorem (EGA IV, 1.8.4), the image of a variety is a constructible set, i.e., a finite union of locally closed sets). The coordinates of a point in C generates the unit ideal A, and the coordinates of points in the complement $C' = A(d)^n \setminus C$, which is also constructible, generate proper ideals of A.

We have shown at the outset that C' contains a euclidean open subset; as it is constructible, it must therefore contain a nonempty Zariski open subset of $A(d)^n$. This is true for every d, hence the set of points in A^n which generate a proper ideal of A, contains an open dense subset of $\mathcal{A}^n$. □

Proposition 4.5. *Let $r < n$. The set U_{r+1} of points $(a_1, ..., a_{r+1}) \in P_{r+1}$ such that $a_1 \neq 0$ in A, $a_2 \neq 0$ in $A/(a_1)$, ..., $a_{r+1} \neq 0$ in $A/(a_1, \ldots, a_r)$, is open dense in P_{r+1}, and hence contains an open dense subset of $\mathcal{A}^{r+1}$.*

Proof: The point 0 is closed in $\mathcal{A}$, hence the statement is true for $r = 0$. Assume by induction that U_r is open dense in P_r. The subset $\{(a, b_{r+1}) \mid a \in U_r,\ b_{r+1} \in I_a\}$ is contained in the complement of the set G of Proposition 4.3, it is thus contained in a closed subset of $U_r \times A$. Its complement is then open dense in $U_r \times A$; hence U_{r+1}, which is the intersection of this complement with P_{r+1}, is open dense in P_{r+1}, and hence contains a dense open subset of $\mathcal{A}^{r+1}$. □

Proposition 4.6. *Let $r < n$. The set N_{r+1} of points $(a_1, \ldots, a_{r+1}) \in U_{r+1}$ such that a_1 is a nonzero divisor (nzd) on A, a_2 is a nzd on $A/(a_1)$, ..., a_{r+1} is a nzd on $A/(a_1, \ldots, a_r)$ is open dense in U_{r+1}, and hence contains an open dense subset of $\mathcal{A}^{r+1}$.*

Proof: The statement is true for $r = 0$ as now N_1 equals U_1 (A is an integral domain). Assume by induction that N_r is open dense in U_r. For $a = (a_1, \ldots, a_r)$ in N_r, let Z_a be the set of zero divisors on $A/(a_1, \ldots, a_r)$. Z_a is a closed subset of the space $\mathcal{A}$ by Lemma 4.7. We need to show that the set $Z = \{(a, b_{r+1}\} \mid a \in N_r, b_{r+1} \in Z_a\}$ is closed in $N_r \times A$, for its complement in U_{r+1} is precisely N_{r+1}. To show this, it suffices by Proposition 4.3 to show that every element in the vanishing ideal of Z_a is a polynomial function of elements in the vanishing ideal of the closed set I_a of the space $\mathcal{A}$.

Let $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be the multiplication map, mapping (a, b) to ab (the map of Lemma 4.5 for $r = 1$). Denote also by μ its restriction $\mu : A \times (A \setminus I_a) \to A$. Then $V = \mu^{-1}(I_a)$ is locally closed in $\mathcal{A} \times \mathcal{A}$, as it is is a Zariski closed subset of the space $A \times (A \setminus I_a)$. Let its vanishing ideal be J; its elements are polynomial functions of the elements of the vanishing ideal of I_a, and therefore polynomial functions of the coefficients of the components $a_1, \ldots, a_r$ of a.

The set Z_a is the projection of V to the first factor $\mathcal{A}$. In general a projection is not Zariski closed as the space $\mathcal{A}$ is not complete, but here Z_a is indeed closed. Hence its vanishing ideal equals $\iota^{-1}(J)$, where ι is the inclusion of $\mathcal{A}$ in $\mathcal{A} \times (\mathcal{A} \setminus I_a)$.

As ι is an algebraic map, it follows that elements of $\iota^{-1}(J)$ are polynomial functions of the coefficients of $a_1, \ldots, a_r$. This completes the proof of the proposition. □

Recall the definition of a Cohen-Macaulay ring: a sequence $a_1, \ldots, a_t$ in a Noetherian ring R is regular if (i) the ideal $(a_1, \ldots, a_t) \subsetneq R$, and (ii) a_1 is a nonzero divisor in R, and for each i, $2 \leq i \leq t$, a_i is a nonzero divisor on $R/(a_1, \ldots, a_{i-1})$ (thus, the above proposition asserts that for $r < n$, the set of points in $\mathcal{A}^{r+1}$ corresponding to regular sequences contains an open dense subset). The depth of an ideal I is the length of any maximal regular sequence in I. The ring R is Cohen-Macaulay if for every ideal I of R, $\mathsf{depth}(I) = \mathsf{codim}(I)$.

Proposition 4.7. *For $r \leqslant n$, the set of points $(a_1, \ldots, a_r) \in \mathcal{A}^r$ such that the ideal generated by its coordinates has codimension r, contains an open dense subset. In particular, the set M of points $(a_1, \ldots, a_n)$ in $\mathcal{A}^n$ such that the ideal $(a_1, \ldots, a_n)$ is maximal, contains an open dense subset of $\mathcal{A}^n$.*

Proof: The set of points in $\mathcal{A}^r$ corresponding to regular sequences contains an open dense subset of $\mathcal{A}^r$. As the ring A is Cohen-Macaulay, the codimension of an ideal generated by such a sequence equals r. The second statement of the proposition now follows because the dimension of A equals n. □

Corollary 4.2. *Let $r > n$. Then the set of points $(a_1, \ldots, a_r)$ in $\mathcal{A}^r$ such that the ideal $(a_1, \ldots, a_r)$ equals A, contains a Zariski open subset of $\mathcal{A}^r$.*

Proof: It suffices to prove the statement for $r = n+1$. By the remarks preceding Proposition 4.4, the set of points $a \in M \times \mathcal{A}$ such that its coordinates generate a proper ideal is a proper closed subset, hence its complement is open in $M \times A$, and so contains an open dense subset of $\mathcal{A}^{n+1}$. □

We can now prove the main theorem of the lecture.

Theorem 4.2. *(i) Let $\ell < k$ be such that $k - \ell + 1 \leqslant n$. Then the set of differential operators in $\mathcal{M}_{\ell,k}$ which define controllable systems contains an open dense subset of $\mathcal{M}_{\ell,k}$.*

(ii) Let $k - \ell + 1 > n$. Then an open dense set of matrices in $\mathcal{M}_{\ell,k}$ define strongly controllable systems.

Proof: (i) Set $r = \binom{k}{\ell}$ and $s = k - \ell + 1$. Let $\mathcal{M}^*_{\ell,k}$ be the subset of those operators in $\mathcal{M}_{\ell,k}$ whose cancellation ideals are nonzero. By Lemmas 4.2 and 4.3, $\mathcal{M}^*_{\ell,k}$ is an open dense subset of $\mathcal{M}_{\ell,k}$.

Suppose $P(\partial) \in \mathcal{M}^*_{\ell,k}$ is such that its cancellation ideal $\mathfrak{i}_\ell(P(\partial))$ is equal to (1). Then by Proposition 2.3, $\mathsf{Ker}_{\mathcal{F}}(P)$ is strongly controllable.

Assume now that the cancellation ideal of $P(\partial)$ is a nonzero proper ideal of A. By a theorem of Macaulay quoted earlier, its codimension is bounded by s. We show that this bound is attained by an open subset of $\mathcal{M}^*_{\ell,k}$.

Consider the following matrix in $\mathcal{M}_{\ell,k}$

$$P(\partial) = \begin{pmatrix} a_1 & a_2 & \cdots & a_s & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & & & \\ \vdots & \vdots & & \vdots & & E_{\ell-1} & \\ 0 & 0 & \cdots & 0 & & & \end{pmatrix}$$

where $E_{\ell-1}$ is the $(\ell-1)\times(\ell-1)$ identity matrix, and the a_i are arbitrary nonzero elements of A. Its nonzero maximal minors are $\{a_1, a_2, \dots, a_s\}$, hence its cancellation ideal equals $(a_1, a_2, \dots, a_s)$.

As $s \leq n$, the image of the composition $\mathcal{M}^*_{\ell,k} \xrightarrow{\mathfrak{m}_{\ell,k}} \mathcal{A}^r \xrightarrow{\pi} \mathcal{A}^s$ contains the open set of points in $\mathcal{A}^s$ whose coordinates are regular sequences of length s (Proposition 4.7); here the second map is the projection of $\mathcal{A}^r$ to $\mathcal{A}^s$, where the indices of s are determined by the nonzero maximal minors described above, in the ordered set of all the r maximal minors. Hence there is an open subset of operators in $\mathcal{M}^*_{\ell,k}$ with the property that the ideal generated by the s minors described above is of codimension s. The cancellation ideals of all the operators in this open subset of $\mathcal{M}^*_{\ell,k}$ then have codimension at least s, and hence codimension equal to s. As $s \geqslant 2$, these operators define controllable systems by Theorem 2.2(ii).

(ii) If $s = k - \ell + 1 > n$, then by Corollary 4.2 there is an open dense subset of points in A^s whose coordinates generate the unit ideal in A. Let $(a_1, \dots, a_s)$ be such a point; there is a matrix just as in (i) above, whose maximal minors are these a_i. Hence the set U of matrices in $\mathcal{M}^*_{\ell,k}$ such that for every $P(\partial) \in U$, the ideal generated by the corresponding s minors equals A, is nonempty open, and so open dense. Thus $\mathfrak{i}_{\ell,k}(P(\partial)) = A$, and $\mathsf{Ker}_{\mathcal{F}}(P)$ is strongly controllable. □

This genericity result in the space of operators descends to the space of distributed systems.

Theorem 4.3. *(i) (i) Let $\ell < k$ be such that $k-\ell+1 \leqslant n$. Then the set of distributed systems in $\mathcal{B}_{\ell,k}$ which define controllable systems contains an open dense subset of $\mathcal{B}_{\ell,k}$.*

(ii) Let $k-\ell+1 > n$. Then an open dense set of systems in $\mathcal{B}_{\ell,k}$ define strongly controllable systems.

Proof: (i) We show that the image of the set of operators defining controllable systems in $\mathcal{M}_{\ell,k}$ under the map $\Pi_{\ell,k}$ contains an open subset of $\mathcal{S}_{\ell,k}$. We refer to the commutative diagram (10) in the argument below.

Let $s = k-\ell+1$ as before, and let $X \subset \mathcal{I}_s$ be the subset of those ideals whose codimensions equal s. By the above theorem, it suffices to show that X contains an open subset of $\mathcal{I}_s$. As $\mathcal{I}_s$ is equipped with the quotient topology given by the surjection $\Pi_{s,1} : \mathcal{A}_s \to \mathcal{I}_s$, we need to show that $U := \Pi_{s,1}^{-1}(X)$ contains an open subset of $\mathcal{A}_s$.

Let $I \in X$, and let $(a_1, \ldots, a_s) \in \Pi_{s,1}^{-1}(I)$. If $a_1, \ldots, a_s$ is a regular sequence, then there is nothing to be done by Proposition 4.6. Suppose that it is not. As A is Cohen-Macaulay, $\mathsf{depth}(I) = s$, hence there is a regular sequence $a'_1, \ldots, a'_s$ in I. Each a'_i is an A-linear combination of $a_1, \ldots, a_s$, hence it follows that there is an A-linear map $L : A^s \to A^s$ such that $L(a_1, \ldots, a_s) = (a'_1, \ldots, a'_s)$. Again by Proposition 4.6, there is a neighbourhood W of $(a'_1, \ldots, a'_s)$ such that the coordinates of each point in it is a regular sequence. The map L is continuous, hence $L^{-1}(W)$ contains an open neighbourhood of $(a_1, \ldots, a_s)$; let $(b_1, \ldots, b_s)$ be a point in it. Then $L(b_1, \ldots, b_s) = (b'_1, \ldots, b'_s)$ is in W, and so the sequence $b'_1, \ldots, b'_s$ is regular. Each b'_i belongs to the ideal $J = (b_1, \ldots, b_s)$, hence $\mathsf{depth}(J) = s$, and so also the codimension of J equals s. Thus $L^{-1}(W) \subset U$, and U is open.

(ii) The proof follows from Theorem 4.2(ii); it suffices to observe that by Corollary 4.2, the ideal (1) is open in $\mathcal{I}_s$, as $s > n$. □

We summarise the principal results of this lecture (Theorems 4.1 and 4.3) in the following statement.

Theorem 4.4. *Controllability is generic in $\mathcal{B}_{\ell,k}$ when $\ell < k$. On the other hand, uncontrollability is generic in $\mathcal{B}_{\ell,k}$ when $\ell \geqslant k$.*

Appendix

We can improve the results of Proposition 4.4, 4.5 and 4.6 to subsets of irreducible elements of the ring A.

Proposition 4.8. *The set of irreducible elements in $\mathcal{A}$ contains an open dense subset when $n > 1$.*

Proof: It suffices to show that the complement of the set of irreducible elements in $A(d)$ is contained in a proper Zariski closed subset, for all d.

As the degree of a product of two elements in $\mathcal{A}$ is the sum of the two degrees, every element in $\mathcal{A}$ is irreducible. Now let $d > 1$, and let d_1, d_2, be integers greater than or equal to 1, such that $d_1 + d_2 = d$. Let $\mu : A(d_1) \times A(d_2) \to A(d)$ be the restriction of the multiplication map in Lemma 4.5. As it is an algebraic map, it follows that $\mathsf{dim}(\overline{\mu(A(d_1) \times A(d_2))}) \leq \mathsf{dim}(A(d_1) \times A(d_2))$ (for instance [4]). But $\mathsf{dim}(A(d_1) \times A(d_2)) = \binom{n+d_1}{n} + \binom{n+d_2}{n}$, whereas $\mathsf{dim}(A_+(d_1 + d_2)) = \binom{n+d_1+d_2}{n}$. As

$n > 1$, it follows that for d sufficiently large, $\binom{n+d_1+d_2}{n} > \binom{n+d_1}{n} + \binom{n+d_2}{n}$ (whereas the reverse inequality is true for $n = 1$); therefore the image of the map μ is contained in a proper Zariski closed subset of $A(d)$. There are $d - 1$ instances of integers d_1, d_2 as above which sum to d, hence the union of the images of the corresponding multiplication maps is also contained in a proper Zariski closed subset of $A(d)$. This image contains all the elements of degree d that are not irreducible. □

Corollary 4.3. *In Proposition 4.4, the subset of P_r consisting of elements $(a_1, \ldots, a_r)$ where each a_i is irreducible, also contains an open dense subset of $\mathcal{A}^r$ (and similar statements for U_{r+1} and N_{r+1} in Propositions 4.5 and 4.6 respectively).* □

5. Pathologies

We have already pointed out several instances where some fact which is true for a signal space $\mathcal{F}$ that is an injective cogenerator, is not true in general (for instance Examples 1.3 and 1.4). We now point out in more detail, in the context of the Sobolev spaces, the problems that can arise.

Let k be a field and R a commutative k-algebra. Let F be an R-module and $\mathsf{D}(F) = \mathsf{Hom}_k(F,\ k)$ its algebraic dual. The functor D, from the category of R-modules to itself, is contravariant and exact, . Let

$$0 \to M_1 \longrightarrow M_2 \longrightarrow M_3 \to 0$$

be an exact sequence of R-modules. Suppose that F is a flat R-module. Then

$$0 \to F \otimes_R M_1 \longrightarrow F \otimes_R M_2 \longrightarrow F \otimes_R M_3 \to 0$$

is exact, so that

$$0 \to \mathsf{D}(F \otimes_R M_3) \longrightarrow \mathsf{D}(F \otimes_R M_2) \longrightarrow \mathsf{D}(F \otimes_R M_1) \to 0$$

is also exact. But $\mathsf{D}(F \otimes_R M_i) = \mathsf{Hom}_k(F \otimes_R M_i,\ k) \simeq \mathsf{Hom}_R(M_i,\ \mathsf{Hom}_k(F,\ k))$ by the adjointness of the pair $(\mathsf{Hom},\ \otimes)$. Thus

$$0 \to \mathsf{Hom}_R(M_3,\ \mathsf{D}(F)) \longrightarrow \mathsf{Hom}_R(M_2,\ \mathsf{D}(F)) \longrightarrow \mathsf{Hom}_R(M_1,\ \mathsf{D}(F)) \to 0$$

is exact, which is to say that $\mathsf{D}(F)$ is an injective R-module. Reversing the above argument shows that F is flat if and only if its algebraic dual $\mathsf{D}(F)$ is injective.

Suppose further that F is faithfully flat. Then if M is any nonzero module, $F \otimes_R M$ is nonzero, so that $\mathsf{D}(F \otimes_R M) \simeq \mathsf{Hom}_R(M,\ \mathsf{D}(F))$ is also nonzero, which is to say that $\mathsf{D}(F)$ is an injective cogenerator. Reversing this shows that F is faithfully flat if and only if $\mathsf{D}(F)$ is an injective cogenerator.

Thus one could say that in the category of R-modules, flatness and injectivity are adjoint properties, as are also the properties of faithful flatness and injective cogeneration.

Suppose now that $k = \mathbb{C}$ and $R = A = \mathbb{C}[\partial_1, \ldots, \partial_n]$. We recollect the A-module structure of the classical spaces of distributions from Lecture 1, on which this entire theory of control rests:

1. The locally convex topological vector space $\mathcal{D}'$ of distributions, as well as the space $\mathcal{C}^\infty$ of smooth functions, are injective cogenerators as A-modules;

2. Their *topological* duals $\mathcal{D}$ and $\mathcal{E}'$, respectively, of compactly supported smooth functions and distributions are faithfully flat A-modules;

3. The space $\mathcal{S}'$ of tempered distributions is an injective A-module that is not a cogenerator. Its topological dual $\mathcal{S}$, the Schwartz space of rapidly decreasing smooth functions, is a flat module that is not faithfully flat.

As all the topological A-modules listed above are reflexive topological k-vector spaces, this raises the question whether flatness and injectivity are also adjoint properties in the category of topological A-modules.

We show that the answer to this question is negative by exhibiting an elementary counter-example of a flat topological module whose topological dual is not injective, but which is again flat.

For every s in $\mathbb{R}$, the Sobolev space $\mathcal{H}^s$ on $\mathbb{R}^n$ of order s is the space of temperate distributions f whose Fourier transform $\hat{f}$ is a measurable function such that

$$\|f\|_s = \big(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi\big)^{\frac{1}{2}} < \infty$$

$\mathcal{H}^s$ is a Hilbert space with norm $\|\ \|_s$. When $s > t$, $\mathcal{H}^s \hookrightarrow \mathcal{H}^t$ is a continuous inclusion. If $p(\partial)$ is an element of A of order r, then it maps $\mathcal{H}^s$ into $\mathcal{H}^{s-r}$. If the family $\{\mathcal{H}^s, s \in \mathbb{R}\}$ is considered an increasing family of vector spaces indexed by the directed set $\mathbb{R}$, then its inductive limit $\overrightarrow{\mathcal{H}} := \varinjlim \mathcal{H}^s$ is the union $\cup_{s\in\mathbb{R}}\mathcal{H}^s$ of all the Sobolev spaces and is an A-module that is strictly contained in $\mathcal{S}'$. Instead, if the family $\{\mathcal{H}^s, s \in \mathbb{R}\}$ is considered a decreasing family of vector spaces, then its projective limit $\overleftarrow{\mathcal{H}} := \varprojlim \mathcal{H}^s$ is the intersection $\bigcap_{s\in\mathbb{R}}\mathcal{H}^s$ of the Sobolev spaces, and is again an A-module. This intersection contains the Schwartz space $\mathcal{S}$ but is strictly larger than it. As $\mathbb{Z}$ is cofinal in $\mathbb{R}$ in either of the above two situations, these limits are also the inductive and projective limits of the countable family $\{\mathcal{H}^s, s \in \mathbb{Z}\}$.

Further, if each $\mathcal{H}^s$ is given its Hilbert space topology, then the union $\overrightarrow{\mathcal{H}}$ will be equipped with the inductive limit topology, i.e. the strongest topology so that each $\mathcal{H}^s \hookrightarrow \overrightarrow{\mathcal{H}}$ is continuous. (As the topology that $\mathcal{H}^s$ inherits from $\mathcal{H}^t$ for $s > t$ is strictly weaker, this inductive limit is *not* a strict inductive limit.) With this topology $\overrightarrow{\mathcal{H}}$ is a locally convex space which is bornological and barrelled. On the other hand, the intersection $\overleftarrow{\mathcal{H}}$ with the projective limit topology, which is the weakest topology such that each inclusion $\overleftarrow{\mathcal{H}} \hookrightarrow \mathcal{H}^s$ is continuous, is a Fréchet space. As the dual of $\mathcal{H}^s$ is $\mathcal{H}^{-s}$, $\overrightarrow{\mathcal{H}}$ and $\overleftarrow{\mathcal{H}}$ are duals of each other. Thus $(\overrightarrow{\mathcal{H}})' = \overleftarrow{\mathcal{H}}$ and $(\overleftarrow{\mathcal{H}})' = \overrightarrow{\mathcal{H}}$, so that they are both reflexive.

Lemma 5.1. *$\overrightarrow{\mathcal{H}}$ and $\overleftarrow{\mathcal{H}}$ are both torsion free A-modules.*

Proof: Suppose f is any element in $\overrightarrow{\mathcal{H}}$ or $\overleftarrow{\mathcal{H}}$ such that $p(\partial)f = 0$ for some nonzero $p(\partial)$. Fourier transform implies that $p(\xi)\hat{f}(\xi) = 0$, which implies that the support of the measurable function $\hat{f}$ is contained in the real variety of the polynomial $p(\xi)$, a set of measure zero. Hence $\hat{f} = 0$, and so is therefore $f = 0$. □

Corollary 5.1. *Let $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, the $\mathbb{C}$-algebra of ordinary differential operators. Then the Sobolev limits $\overrightarrow{\mathcal{H}}(\mathbb{R})$ and $\overleftarrow{\mathcal{H}}(\mathbb{R})$ are both flat A-modules, but not faithfully flat.*

Proof: As $\mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$ is a principal ideal domain, torsion free implies flat [4]. Thus it remains to show that the Sobolev limits are not faithfully flat.

Let $p(\frac{\mathsf{d}}{\mathsf{dt}}) = 1 + \frac{\mathsf{d}^2}{\mathsf{dt}^2}$. Then for any element f in either of the two Sobolev limits, the Fourier inverse of $(1+\xi^2)^{-1}\hat{f}(\xi)$ is also in the corresponding Sobolev limit. This implies that $p(\frac{\mathsf{d}}{\mathsf{dt}})$ defines a surjective morphism on the Sobolev limits, so that $\mathfrak{m}\overrightarrow{\mathcal{H}}(\mathbb{R}) = \overrightarrow{\mathcal{H}}(\mathbb{R})$ and $\mathfrak{m}\overleftarrow{\mathcal{H}}(\mathbb{R}) = \overleftarrow{\mathcal{H}}(\mathbb{R})$ for the maximal ideals $\mathfrak{m}$ of $\mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$ that contain $p(\frac{\mathsf{d}}{\mathsf{dt}})$. This shows that the two limits are not faithfully flat. □

Lemma 5.2. *$\overrightarrow{\mathcal{H}}$ and $\overleftarrow{\mathcal{H}}$ are not divisible (hence not injective) A-modules.*

Proof: Let f in $\mathcal{S}'$ be such that $\hat{f}(\xi)$ equals $\sqrt{\xi}$ in a neighbourhood of 0 and is rapidly decreasing at infinity. Such an f is in every Sobolev space $\mathcal{H}^s$, and so is in both $\overrightarrow{\mathcal{H}}$ as well as $\overleftarrow{\mathcal{H}}$. However, the Fourier inverse of $\xi^{-1}\hat{f}(\xi)$ is not in any Sobolev space and is therefore not in either of the Sobolev limits. This implies that $\frac{\mathsf{d}}{\mathsf{dt}}$ does not define a surjective morphism on the Sobolev limits, so that they are not divisible (hence not injective) modules. □

Thus, we conclude the following result.

Theorem 5.1. *Flatness and injectivity are not adjoint properties in the category of topological A-modules.* □

While the Sobolev limits on $\mathbb{R}$ are flat modules over the ring of differential operators, the corresponding fact is false in $\mathbb{R}^n, n \geqslant 2$.

Example 5.1 (The Sobolev-deRham complex on $\mathbb{R}^3$): Recollect the deRham complex from Lecture 2, and the exact sequences of Example 2.2. We now show that the Sobolev-deRham complex

$$\overrightarrow{\mathcal{H}} \xrightarrow{\mathsf{grad}} (\overrightarrow{\mathcal{H}})^3 \xrightarrow{\mathsf{curl}} (\overrightarrow{\mathcal{H}})^3 \xrightarrow{\mathsf{div}} \overrightarrow{\mathcal{H}}$$

is *not* exact, and similarly for $\overleftarrow{\mathcal{H}}$. Thus we exhibit an element in the kernel of $\mathsf{curl} : (\overrightarrow{\mathcal{H}})^3 \to (\overrightarrow{\mathcal{H}})^3$ which is not in the image of $\mathsf{grad} : \overrightarrow{\mathcal{H}} \to (\overrightarrow{\mathcal{H}})^3$, and similarly for $\overleftarrow{\mathcal{H}}$. By the equational criterion for flatness [4], this will show that the Sobolev limits are not flat A-modules.

Let h be an element in the kernel of $\mathsf{curl} : (\overrightarrow{\mathcal{H}})^3 \to (\overrightarrow{\mathcal{H}})^3$. As the deRham complex is exact in the space $\mathcal{S}'$ of temperate distributions, there is certainly an f in $\mathcal{S}'$ (and unique upto an additive constant) such that $\mathsf{grad}(f) = h$. Thus $h = (\partial_x f, \partial_y f, \partial_z f)$. We claim that there is an f in $\mathcal{S}'$ which is not in any Sobolev space but such that $\partial_x f$, $\partial_y f$ and $\partial_z f$ are all three in every one of them.

The Sobolev spaces are subspaces of $\mathcal{S}'$ on which is defined the Fourier transform. Hence by Fourier transformation, the operators grad and curl become matrix operators with polynomial entries -

$$\widehat{\mathsf{grad}} = \imath \begin{pmatrix} x \\ y \\ z \end{pmatrix}; \widehat{\mathsf{curl}} = \imath \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

It now suffices to find an $\hat{f}$ (here $\hat{}$ denotes the Fourier transform) which is not in $\mathcal{L}^2(\mathbb{R}^3)$ but such that $x\hat{f}$, $y\hat{f}$ and $z\hat{f}$ are all in $\mathcal{L}^2(\mathbb{R}^3)$, with respect to the measure $(1+r^2)^s d\mathsf{vol}$, for every s (where $r = \sqrt{x^2+y^2+z^2}$). But $x\hat{f}$, $y\hat{f}$ and $z\hat{f}$ are all in $\mathcal{L}^2$ if and only if $r\hat{f}$ is in $\mathcal{L}^2$. Thus it suffices to find an $\hat{f}$ which is not in $\mathcal{L}^2$ but such that $r\hat{f}$ is in $\mathcal{L}^2$.

This is elementary, for let $\hat{f}$ be any function which is rapidly decreasing at infinity, and which at 0 is $O(r^\alpha)$, $-\frac{5}{2} < \alpha \leqslant -\frac{3}{2}$. Then

$$\int_{\mathbb{R}^3} |\hat{f}|^2 (1+r^2)^s d\mathsf{vol} = \int_{\mathbb{R}^3} |\hat{f}|^2 (1+r^2)^s r^2 dr d\theta d\phi$$

This integral, in some neighbourhood Ω of 0, is therefore of the order of

$$\int_\Omega r^{2\alpha+2} dr d\theta d\phi$$

which is not finite.

On the other hand

$$\int_{\mathbb{R}^3} r^2|\hat{f}|^2(1+r^2)^s d\mathsf{vol} = \int_{\mathbb{R}^3} |\hat{f}|^2 r^4(1+r^2)^s dr d\theta d\phi$$

is finite. Thus this f is not in any $\mathcal{H}^s$ whereas $\mathsf{grad}(f)$ is in every $\mathcal{H}^s$. The image of grad is therefore strictly contained in the kernel of curl. Similarly the image of curl is also strictly contained in the kernel of div. The Sobolev-deRham complex is therefore not exact at either place.

Thus, in the case of the Sobolev limits, there is an obstruction to a kernel being an image, other than the existence of torsion elements (A^3/C is torsion free, where C is the submodule of A^3 generated by the rows of curl, as in Examples 2.1 and 2.2). □

Some of our results require only that $\mathcal{F}$ be an injective A-module, and are thus true for $\mathcal{S}'$, as well as for $\mathcal{D}'$ and $\mathcal{C}^\infty$. Proposition 2.5 on the elimination problem for PDE is one such result. However, elimination is not always possible as the example below demonstrates.

Example 5.2: If $\mathcal{F}$ is not injective, then the projection of a system may not be a system. For instance, let $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, let $\mathcal{F} = \mathcal{D}$ (the space of smooth compactly supported functions), and let $\pi_2 : \mathcal{D}^2 \to \mathcal{D}$ be the projection onto the second factor. Let $P \subset A^2$ be the cyclic submodule generated by $(\frac{\mathsf{d}}{\mathsf{dt}}, -1)$. Then $\mathsf{Ker}_{\mathcal{D}}(P) = \{(f, \frac{\mathsf{d}}{\mathsf{dt}} f) \mid f \in \mathcal{D}\}$, but $\pi_2(\mathsf{Ker}_{\mathcal{D}}(P)) = \{\frac{\mathsf{d}}{\mathsf{dt}} f \mid f \in \mathcal{D}\}$ is not a differential kernel in $\mathcal{D}$, as $\frac{\mathsf{d}}{\mathsf{dt}} : \mathcal{D} \to \mathcal{D}$ is not surjective.

In general, the obstruction to the projection $\pi_2(\mathsf{Ker}_{\mathcal{F}}(P))$ in Proposition 2.5 being a differential kernel, lies in $\mathsf{Ext}^1_A(A^p/\pi_1(P),\ \mathcal{F})$.

We point out a few other pathologies. For instance, Lemma 3.2 improves containment in Lemma 1.1(ii) to equality, when $\mathcal{F}$ is injective. In general, containment is strict, as in the example below.

Example 5.3: Let $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, and let $\mathcal{F}$ be the space $\mathcal{D}$ of compactly supported smooth functions. Let P_1 and P_2 be cyclic submodules of A^2 generated by $(1, 0)$ and $(1,\ -\frac{\mathsf{d}}{\mathsf{dt}})$, respectively. Then $P_1 \cap P_2$ is the 0 submodule, so that $\mathsf{Ker}_{\mathcal{D}}(P_1 \cap P_2)$ is all of $\mathcal{D}^2$. On the other hand, $\mathsf{Ker}_{\mathcal{D}}(P_1) = \{(0, f) \mid f \in \mathcal{D}\}$ and $\mathsf{Ker}_{\mathcal{D}}(P_2) = \{(\frac{\mathsf{d}}{\mathsf{dt}} g,\ g) \mid g \in \mathcal{D}\}$. Thus an element (u, v) in $\mathcal{D}^2$ is in $\mathsf{Ker}_{\mathcal{D}}(P_1) +\ \mathsf{Ker}_{\mathcal{D}}(P_2)$ only if $u = \frac{\mathsf{d}}{\mathsf{dt}}(g),\ v = f + g$, where f and g are arbitrary elements in $\mathcal{D}$. Let now u be any (nonzero) non-negative compactly supported smooth function. Then $(u, 0)$, which is in $\mathsf{Ker}_{\mathcal{D}}(P_1 \cap P_2)$, is, however, not in $\mathsf{Ker}_{\mathcal{D}}(P_1) + \mathsf{Ker}_{\mathcal{D}}(P_2)$, as $g(t) = \int_{-\infty}^t dg = \int_{-\infty}^t u\ dt$ is not compactly supported.

Thus $\mathsf{Ker}_{\mathcal{D}}(P_1) + \mathsf{Ker}_{\mathcal{D}}(P_2) \subsetneq \mathsf{Ker}_{\mathcal{D}}(P_2)$. In fact, it turns out that that $\mathsf{Ker}_{\mathcal{D}}(P_1) + \mathsf{Ker}_{\mathcal{D}}(P_2)$ is not the kernel of any differential operator (viz., Corollary 6.3 below).

In general there is an obstruction to a sum of two systems in $\mathcal{F}$ being a system, and this obstruction can be located in $\mathsf{Ext}^1_A(A^k/(P_1 + P_2), \mathcal{F})$, which, for an injective $\mathcal{F}$, equals zero. Indeed, this is Lemma 3.2.

The sum of two systems, if a system, is the one obtained by connecting the two in 'parallel'. Their intersection is the system corresponding to a 'series' connection.

By Lemma 1.1(ii), the series connection of infinitely many systems is again a system (defined by the sum of the submodules defining these systems), and Lemma 3.2 states that the parallel connection of finitely many systems is a system, when the signal space is an injective A-module. However, the example below shows that the parallel connection of infinitely many systems need not be one, even when $\mathcal{F}$ is an injective cogenerator.

Example 5.4: Let $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, and let $\mathcal{F} = \mathcal{C}^\infty(\mathbb{R})$. Let I be the principal ideal $(\frac{\mathsf{d}}{\mathsf{dt}})$, and let $P_i = I^i$, $i \geqslant 0$. It is easy to see that $\bigcap_{i\geqslant 0} P_i = 0$ (for instance, A is an integral domain, hence its I-adic completion is Hausdorff by a theorem of Krull). It follows that $\mathsf{Ker}_{\mathcal{C}^\infty}(\bigcap_{i\geqslant 0} P_i) = \mathcal{C}^\infty$. However, $\mathsf{Ker}_{\mathcal{C}^\infty}(P_i)$ equals the $\mathbb{C}$-vector space of polynomials of degree less than i, and hence the sum $\sum_{i\geqslant 0} \mathsf{Ker}_{\mathcal{C}^\infty}(P_i)$ equals the space of all polynomials, which is strictly contained in $\mathsf{Ker}_{\mathcal{C}^\infty}(\bigcap_{i\geqslant 0} P_i)$. It is clear that this infinite sum of systems is not a system at all.

6. The Nullstellensatz for Systems of PDE

We have studied controllable systems defined in a signal space $\mathcal{F}$ which is an injective cogenerator ($\mathcal{D}'$ and $\mathcal{C}^\infty$, for instance), and we now turn our attention to other signal spaces of importance. If $\mathcal{F}$ is a flat A-module (for instance $\mathcal{D}$, $\mathcal{E}'$, or $\mathcal{S}$), then every differential kernel is an image by the equational crterion of flatness [4]; every system in such a signal space is thus controllable by $(**)$ of Appendix 1. Here we focus on the space $\mathcal{S}'$ of tempered distributions. It is an injective A-module, but not a cogenerator (namely the remark after Proposition 1.3). The system $\mathsf{Ker}_{\mathcal{S}'}(P)$, defined by $P \subset A^k$, is controllable if A^k/P is torsion free, but this is no longer a necessary condition (Theorem 2.1(i) and the remark following it).

The first problem we encounter when studying systems in a space $\mathcal{F}$ which is not an injective cogenerator is that Proposition 1.5 is not valid, and there is no longer a bijective correspondence between systems in $\mathcal{F}^k$ and submodules of A^k. This prompted the definition of $\mathcal{M}(\mathsf{Ker}_\mathcal{F}(P))$, for P a submodule of A^k, in the remark following it, and we recall it now.

Definition 6.1. *Let $P \subset A^k$. Then $\mathcal{M}(\mathsf{Ker}_\mathcal{F}(P))$ is the submodule of all $p(\partial) \in A^k$ such that the kernel of the map $p(\partial) : \mathcal{F}^k \to \mathcal{F}$ contains $\mathsf{Ker}_\mathcal{F}(P)$. It is the Willems closure (or simply, the closure) of P with respect to $\mathcal{F}$. We sometimes denote it by $\bar{P}_\mathcal{F}$ (or simply by $\bar{P}$). P is said to be closed with respect to $\mathcal{F}$ if $P = \bar{P}$.*

Thus for instance, Example 1.3 states that $\overline{(\frac{\mathsf{d}}{\mathsf{dt}} - 1)} = (1)$ with respect to $\mathcal{S}'$. We have also observed that every submodule P is closed when $\mathcal{F}$ is an injective cogenerator.

We give another useful description of $\bar{P}_\mathcal{F}$. As $\mathsf{Hom}_A(A^k/P,\ \mathcal{F}) \simeq \mathsf{Ker}_\mathcal{F}(P) = \mathsf{Ker}_\mathcal{F}(\bar{P}_\mathcal{F}) \simeq \mathsf{Hom}_A(A^k/\bar{P}_\mathcal{F},\ \mathcal{F})$, it follows that every $\phi : A^k \to \mathcal{F}$ which vanishes on P also vanishes on $\bar{P}_\mathcal{F}$. Conversely, if for $p \in A^k$, there is some $\phi : A^k \to \mathcal{F}$ which vanishes on P, but $\phi(p) \neq 0$, then this $p \notin \bar{P}_\mathcal{F}$.

More generally, we also let $\mathcal{M}(X)$, for any subset X of $\mathcal{F}^k$, to equal the submodule of all elements $p(\partial) \in A^k$ such that the kernel of $p(\partial) : \mathcal{F}^k \to \mathcal{F}$ contains X.

If P_1 and P_2 are submodules of A^k, then clearly $\mathsf{Ker}_\mathcal{F}(P_1) = \mathsf{Ker}_\mathcal{F}(P_2)$ if and only if $\bar{P}_1 = \bar{P}_2$. We collect a few other elementary observations below.

Lemma 6.1. *Let $P \subsetneq A^k$, and $\mathcal{F}$, an A-submodule of $\mathcal{D}'$. Then $\bar{P}_\mathcal{F}$ is contained in the 0-primary component P_0 of P (in the notation of exact sequence (9)).*

Proof: If A^k/P is a torsion module, then $P_0 = A^k$, and there is nothing to be done. Otherwise, let $p \in A^k \setminus P_0$, and let $[p]$ denote its class in A^k/P_0 or in A^k/P. We show that there is a map $\phi : A^k/P \to \mathcal{F}$ which does not vanish at $[p]$.

As A^k/P_0 is finitely generated and torsion free, it embeds into a free A-module, say A^r. Let the image of $[p]$ in A^r be $q(\partial) = (q_1(\partial), \ldots, q_r(\partial))$, where we assume, without loss of generality, that $q_1(\partial)$ is nonzero. As $\mathcal{F}$ is a faithful A-module, namely Corollary 2.1, there is an $f \in \mathcal{F}$ such that $q_1(\partial)f \neq 0$. Then the composition $A^k/P_0 \hookrightarrow A^r \xrightarrow{\pi_1} A$, where π_1 is the projection to the first factor, yields a map $A^k/P_0 \to \mathcal{F}$ which does not vanish at $[p]$. As $P \subset P_0$, this map induces a map $\phi : A^k/P \to \mathcal{F}$ which does not vanish at $[p]$. □

Lemma 6.2. *Let $\{B_i\}$ be a collection of systems in $\mathcal{F}^k$, then $\sum_i \mathcal{M}(B_i) \subset \mathcal{M}(\bigcap_i B_i)$, and $\mathcal{M}(\sum_i B_i) = \bigcap_i \mathcal{M}(B_i)$.*

Proof: This lemma is analogous to lemma 1.1(ii), and is elementary. □

Lemma 6.3. *For any signal space $\mathcal{F}$, $\mathsf{Ker}_{\mathcal{F}} \circ \mathcal{M}$ is the identity map on systems.*

Proof: Let $B = \mathsf{Ker}_{\mathcal{F}}(P)$; then $\mathcal{M}(B) = \bar{P}$, and $\mathsf{Ker}_{\mathcal{F}}(\bar{P}) = B$. □

Corollary 6.1. *For any $\mathcal{F}$, the closure of P with respect to $\mathcal{F}$ is closed with respect to $\mathcal{F}$.*

Proof: $\bar{\bar{P}} = \mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(\mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(P)) = \mathcal{M}(\mathsf{Ker}_{\mathcal{F}} \circ \mathcal{M}(\mathsf{Ker}_{\mathcal{F}}(P)) = \mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(P) = \bar{P}$. □

While the inclusion in Lemma 6.2 might be strict, there is equality of closures.

Lemma 6.4. *Suppose $\{B_i\}$ is a collection of systems in $\mathcal{F}^k$, then the closure of $\sum_i \mathcal{M}(B_i)$ with respect to $\mathcal{F}$ equals $\mathcal{M}(\bigcap_i B_i)$.*

Proof: By Lemmas 1.1(ii) and 6.3, $\mathsf{Ker}_{\mathcal{F}}(\sum_i \mathcal{M}(B_i)) = \bigcap_i \mathsf{Ker}_{\mathcal{F}}(\mathcal{M}(B_i)) = \bigcap_i B_i = \mathsf{Ker}_{\mathcal{F}}\,\mathcal{M}(\bigcap_i B_i)$. Hence the closure of $\sum_i \mathcal{M}(B_i)$ equals the closure of $\mathcal{M}(\bigcap_i B_i)$, and this equals $\mathcal{M}(\bigcap_i B_i)$ by the above corollary. □

By Lemma 6.3, there is now a bijective correspondence between systems in $\mathcal{F}^k$ and submodules of A^k closed with respect to $\mathcal{F}$. This is completely analogous to the familiar Galois correspondence between affine varieties in $\mathbb{C}^n$ and radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$. The notion of closure here is the analogue of the radical of an ideal, and its calculation is the analogue of the Hilbert Nullstellensatz.

Lemma 6.5. *Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be signal spaces, and P a submodule of A^k. If P is closed with respect to $\mathcal{F}_1$, then it is also closed with respect to $\mathcal{F}_2$.*

Proof: If P is not closed with respect to $\mathcal{F}_2$, then it is strictly contained in its closure $\bar{P}_{\mathcal{F}_2}$. It follows that $\mathsf{Ker}_{\mathcal{F}_1}(P) = \mathsf{Ker}_{\mathcal{F}_2}(P) \cap \mathcal{F}_1^k = \mathsf{Ker}_{\mathcal{F}_2}(\bar{P}_{\mathcal{F}_2}) \cap \mathcal{F}_1^k = \mathsf{Ker}_{\mathcal{F}_1}(\bar{P}_{\mathcal{F}_2})$, so that P is also not closed with respect to $\mathcal{F}_1$. □

Lemma 6.6. *Let $\mathcal{F}$ be a signal space contained in $\mathcal{D}'$. If $P \subset A^k$ is not closed with respect to $\mathcal{F}$, then $\mathsf{Ker}_{\mathcal{F}}(P)$ is not dense in $\mathsf{Ker}_{\mathcal{D}'}(P)$.*

Proof: By assumption, P is strictly contained in its closure $\bar{P}_{\mathcal{F}}$. Thus the closure of $\mathsf{Ker}_{\mathcal{F}}(P)$ in $(\mathcal{D}')^k$ is contained in $\mathsf{Ker}_{\mathcal{D}'}(\bar{P}_{\mathcal{F}})$, a closed subspace of $\mathcal{D}'^k$ that is strictly contained in $\mathsf{Ker}_{\mathcal{D}'}(P)$ (as every submodule of A^k is closed with respect to $\mathcal{D}'$). □

Lemma 6.7. *Let $\{P_i\}$ be a collection of submodules of A^k, each closed with respect to $\mathcal{F}$. Then $\bigcap_i P_i$ is also closed with respect to $\mathcal{F}$.*

Proof: As each P_i is closed, it follows that $\bigcap_i P_i \subset \mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(\bigcap_i P_i) \subset \mathcal{M}(\sum_i \mathsf{Ker}_{\mathcal{F}}(P_i)) = \bigcap_i \mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(P_i) = \bigcap_i P_i$, where the second inclusion follows by Lemma 1.1(ii) and the inclusion reversing nature of the assignment $\mathcal{M}$. This implies equality everywhere, and thus that $\bigcap_i P_i = \mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(\bigcap_i P_i)$. $\square$

A question more general than the above lemma is whether $\mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(\bigcap_i P_i)$ equals $\bigcap_i \mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(P_i)$ for an arbitrary collection of submodules $\{P_i\}$ of A^k. In other words, is the closure of an intersection equal to the intersection of the closures? (In our suggestive notation, is $\overline{\bigcap_i P_i} = \bigcap_i \bar{P}_i$?) It is trivially true when $\mathcal{F}$ is an injective cogenerator, but in general it is only the containment $\mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(\bigcap_i P_i) \subset \bigcap_i \mathcal{M}\,\mathsf{Ker}_{\mathcal{F}}(P_i)$ that holds. The other inclusion is not always true, as the following example demonstrates.

Example 6.1: Let $A = \mathbb{C}[\frac{\mathsf{d}}{\mathsf{dt}}]$, and let I be any nonzero proper ideal of A. Let $P_i = I^i, i \geqslant 0$, and let $\mathcal{F} = \mathcal{D}(\mathbb{R})$, the space of smooth compactly supported functions. As each P_i is a nonzero ideal, $\mathsf{Ker}_{\mathcal{D}}(P_i) = 0$ by Paley-Wiener. Thus each $\bar{P}_i = A$, and hence $\bigcap_i \bar{P}_i = A$. On the other hand, $\bigcap_i P_i = 0$ (as in Example 5.4), and hence $\overline{\bigcap_i P_i} = 0$ (as observed in Corollary 2.1).

The question therefore is whether the we have equality when the intersection is finite, i.e. does $\overline{\bigcap_{i=1}^m P_i} = \bigcap_{i=1}^m \bar{P}_i$ hold? This would be analogous to the fact that the radical of a finite intersection of ideals equals the intersection of the individual radicals. The only way to answer such a question seems to be (to this author) to first characterise the closures of submodules with respect to the space $\mathcal{F}$, and we turn to this problem next.

Proposition 6.1. *Let P be a proper submodule of A^k, and let $\mathcal{F}$ be $\mathcal{D}$, $\mathcal{E}'$, or $\mathcal{S}$. Let $P = P_0 \cap P_1 \cap \cdots \cap P_r$ be an irredundant primary decomposition of P in A^k, where P_0 is a 0-primary submodule of A^k, and P_i, $i \geqslant 1$, is a p_i-primary submodule, p_i a nonzero prime. Then the closure $\bar{P}_{\mathcal{F}}$ of P with respect to $\mathcal{F}$ equals P_0. Thus P is closed with respect to $\mathcal{F}$ if and only if A^k/P is torsion free.*

Proof: Suppose P does not have a 0-primary component. Then A^k/P is a torsion module, $\mathsf{Ker}_{\mathcal{D}'}(P)$ is uncontrollable, and there is no nonzero compactly supported or rapidly decaying decaying element in it (Proposition 3.2 and the remark following it). Thus $\mathsf{Ker}_{\mathcal{F}}(P) = 0$, and $\bar{P}_{\mathcal{F}} = A^k$.

Now assume that P has a 0-primary component P_0. We have already noted that it is uniquely determined by P, independent of the primary decomposition of P. We first show that $\mathsf{Ker}_{\mathcal{F}}(P) = \mathsf{Ker}_{\mathcal{F}}(P_0)$, and hence that $\bar{P}_{\mathcal{F}} = \bar{P}_{0\mathcal{F}}$.

As $\mathcal{F}$ is a flat A-module, $\mathsf{Ker}_{\mathcal{F}}(P)$ is an image, say of the map $R(\partial) : \mathcal{F}^{k_1} \to \mathcal{F}^k$. Consider the extension of this map to $\mathcal{D}'$, namely $R(\partial) : (\mathcal{D}')^{k_1} \to (\mathcal{D}')^k$. Its image is equal to the controllable part of $\mathsf{Ker}_{\mathcal{D}'}(P)$, which equals $\mathsf{Ker}_{\mathcal{D}'}(P_0)$ by Theorem 3.1. Thus, the image $R(\partial)(\mathcal{F}^{k_1})$ is contained in $\mathsf{Ker}_{\mathcal{F}}(P_0)$, and is hence equal to it.

We now claim that P_0 is closed with respect to $\mathcal{F}$, so that $\bar{P}_{0\mathcal{F}} = P_0$. For if not, then by Lemma 6.5, $\mathsf{Ker}_{\mathcal{F}}(P_0)$ is not dense in $\mathsf{Ker}_{\mathcal{D}'}(P_0)$. But this contradicts Proposition 1.1 because $\mathsf{Ker}_{\mathcal{D}'}(P_0)$ is controllable as A^k/P_0 is torsion free. $\square$

Remark: We can also describe the above closure as follows. Let $\pi : A^k \to A^k/P$ be the canonical surjection. Then the closure $\bar{P}_{\mathcal{F}}$ equals $\pi^{-1}(T(A^k/P)$, where $T(A^k/P)$ is the submodule of torsion elements of A^k/P.

We turn now to the space $\mathcal{S}'$ of tempered distributions. We use the following result on primary decompositions.

Suppose $R = \bigcap_{i=1}^{r} P_i$ is a primary decomposition of R in A^k, where P_i is p_i-primary. Suppose I is an ideal such that $I \subset p_i$ for $i = q+1, \ldots, r$, and that I is not contained in the other p_i. Then the ascending chain of submodules $(R : I) \subset (R : I^2) \subset \cdots$ stabilizes to the submodule $\bigcap_{i=1}^{q} P_i$, and hence this submodule is independent of the primary decomposition of R.

Proposition 6.2. *Let P be a submodule of A^k, and let $\mathcal{F} = \mathcal{S}'$. Let $P = P_1 \cap \cdots \cap P_r$ be an irredundant primary decomposition of P in A^k, where P_i is p_i-primary. Let $\mathcal{V}(p_i)$, the variety of the ideal p_i in $\mathbb{C}^n$, contain purely imaginary points for $i = 1, \ldots, q$, and not for $i = q+1, \ldots, r$, (by a purely imaginary point in $\mathbb{C}^n$, we mean a point in $\imath\mathbb{R}^n$). Then the closure $\bar{P}_{\mathcal{S}'}$ of P with respect to $\mathcal{S}'$ equals $P_1 \cap \cdots \cap P_q$. Thus P is closed with respect to $\mathcal{S}'$ if and only if every $\mathcal{V}(p_i)$ contains purely imaginary points.*

Proof: If we set $I = p_{q+1} \cap \cdots \cap p_r$ in the result quoted just above, then it follows that $P' := P_1 \cap \cdots \cap P_q$ is independent of the primary decomposition of P.

We first claim that $\mathsf{Ker}_{\mathcal{S}'}(P')$ equals $\mathsf{Ker}_{\mathcal{S}'}(P)$. As $P \subset P'$, it suffices to show that $\mathsf{Ker}_{\mathcal{S}'}(P) \subset \mathsf{Ker}_{\mathcal{S}'}(P')$. If this is not true, then there is some f in $\mathsf{Ker}_{\mathcal{S}'}(P)$, and a $p(\partial) \in P' \setminus P$, such that $p(\partial)f \neq 0$. However, for every $a(\partial)$ in the ideal $(P : p(\partial))$, $a(\partial)(p(\partial)f) = 0$. Taking Fourier transforms gives $a(\imath x)(\widehat{p(\partial)f})(x) = 0$, hence $\imath(\mathsf{supp}(\widehat{p(\partial)f}))$ is contained in $\mathcal{V}(a) \cap \imath\mathbb{R}^n$ (where supp denotes support). As $P = \bigcap_{i=1}^{r} P_i$, $(P : p(\partial)) = \bigcap_{i=1}^{r}(P_i : p(\partial))$. The submodule P_i is p_i-primary in A^k, and so it follows that $(P_i : p(\partial))$ either equals A (if $p(\partial) \in P_i$), or it equals p_i (if $p(\partial) \notin P_i$). Hence $(P : p(\partial))$ is equal to the intersection of a subset of $\{p_{q+1}, \ldots, p_r\}$. It now follows that $\imath(\mathsf{supp}(\widehat{p(\partial)f}))$ is contained in $(\bigcup_{i=q+1}^{r} \mathcal{V}(p_i)) \cap \imath\mathbb{R}^n$. By assumption, none of the $\mathcal{V}(p_i)$, $i = q+1, \ldots, r$, intersects $\imath\mathbb{R}^n$, hence the support of $\widehat{p(\partial)f}$ is empty. Thus $p(\partial)f = 0$, contrary to assumption.

We now show that $P' = \bar{P}_{\mathcal{S}'}$. So let $p(\partial)$ be any element of $A^k \setminus P'$, and consider the exact sequence

$$0 \to A/(P' : p(\partial)) \xrightarrow{p} A^k/P' \xrightarrow{\pi} A^k/P' + (p(\partial)) \to 0$$

where the morphism p above maps $[a(\partial)]$ to $[a(\partial)p(\partial)]$, and π is the canonical surjection. Applying the functor $\mathsf{Hom}_A(\cdot\, , \mathcal{S}')$ gives the sequence

$$0 \to \mathsf{Hom}_A(A^k/P' + (p(\partial)), \mathcal{S}') \longrightarrow \mathsf{Hom}_A(A^k/P', \mathcal{S}') \xrightarrow{p(\partial)} \mathsf{Hom}_A(A/(P' : p(\partial)), \mathcal{S}') \to 0$$

which is exact because $\mathcal{S}'$ is an injective A-module. Observe now that $V((P' : p(\partial)))$ is the union of a some of the varieties $\mathcal{V}(p_1), \ldots, \mathcal{V}(p_q)$, hence by assumption there is a purely imaginary point, say ξ on it. Then the function $e^{<\xi,x>}$ is tempered, and so belongs to the last term $\mathsf{Hom}_A(A/(P' : p(\partial)),\ \mathcal{S}')$ above. It is therefore nonzero, and this implies that $\mathsf{Ker}_{\mathcal{S}'}(P' + p(\partial))$ is strictly contained in $\mathsf{Ker}_{\mathcal{S}'}(P')$. The proposition follows. □

As we have already explained, these results are analogues of Hilbert's Nullstellensatz, and the closure we have defined is analogous to the radical of an ideal. We put together these results below.

Theorem 6.1. *(Nullstellensatz for PDE) (i) Every submodule of A^k is closed with respect to $\mathcal{D}'$ and $\mathcal{C}^\infty$.*
(ii) Let P be a submodule of A^k, and let $\mathcal{F} = \mathcal{S}'$. Let $P = P_1 \cap \cdots \cap P_r$ be an irredundant primary decomposition of P in A^k, where P_i is p_i-primary. Let $\mathcal{V}(p_i)$, the variety of the ideal p_i in $\mathbb{C}^n$, contain purely imaginary points for $i = 1, \ldots, q$, and not for $i = q+1, \ldots, r$. Then the closure $\bar{P}_{\mathcal{S}'}$ of P with respect to $\mathcal{S}'$ equals $P_1 \cap \cdots \cap P_q$. Thus P is closed with respect to $\mathcal{S}'$ if and only if every $\mathcal{V}(p_i)$ contains purely imaginary points.
(iii) Let P be a submodule of A^k, and let $\mathcal{F}$ be $\mathcal{D}$, $\mathcal{E}'$, or $\mathcal{S}$. Let $P = P_0 \cap P_1 \cap \cdots \cap P_r$ be an irredundant primary decomposition of P in A^k, where P_0 is a 0-primary submodule of A^k, and P_i, $i \geqslant 1$, is a p_i-primary submodule, p_i a nonzero prime. Then the closure $\bar{P}_{\mathcal{F}}$ of P with respect to $\mathcal{F}$ equals P_0. Thus P is closed with respect to $\mathcal{F}$ if and only if A^k/P is torsion free.

We can now answer the question following Example 6.1, when $\mathcal{F}$ is one of the classical spaces.

Corollary 6.2. *Let P_i, $i = 1, \ldots, m$, be submodules of A^k, and let $\mathcal{F}$ be a classical space. Then $(\overline{\bigcap_{i=1}^m P_i})_{\mathcal{F}} = \bigcap_{i=1}^m \bar{P}_{i\mathcal{F}}$.*

Proof: By induction, it suffices to prove the statement for $m = 2$.

(i) When $\mathcal{F} = \mathcal{D}'$ or $\mathcal{C}^\infty$, the statement is trivial, because now every submodule of A^k is closed.

As the inclusion $(\overline{P_1 \cap P_2})_{\mathcal{F}} \subset \bar{P}_{1\mathcal{F}} \cap \bar{P}_{2\mathcal{F}}$ is true for every $\mathcal{F}$, we need to show the reverse inclusion for $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{E}'$ and $\mathcal{D}$.

(ii) Let $P_1 = \bigcap_{i=1}^{r_1} Q_i$ and $P_2 = \bigcap_{j=1}^{r_2} Q'_j$ be irredundant primary decompositions of P_1 and P_2 in A^k respectively, where Q_i and Q'_j are p_i-primary and p'_j-primary, respectively, $i = 1, \ldots, r_1,\ j = 1, \ldots, r_2$. Suppose that the varieties of $p_1, \ldots, p_{q_1}, p'_1, \ldots, p'_{q_2}$ contain purely imaginary points, whereas those of $p_{q_1+1}, \ldots, p_{r_1}, p'_{q_2+1} \ldots, p'_{r_2}$ do not. By Theorem 6.1(ii), $\bar{P}_{1\mathcal{S}'} \cap \bar{P}_{2\mathcal{S}'} = (\bigcap_{i=1}^{q_1} Q_i) \cap (\bigcap_{j=1}^{q_2} Q'_j)$.

On the other hand, $(\bigcap_{i=1}^{r_1} Q_i) \cap (\bigcap_{j=1}^{r_2} Q'_j)$ is a primary decomposition of $P_1 \cap P_2$, though perhaps not irredundant. An irredundant primary decomposition can however be obtained from it by omitting, if necessary, some of the Q_i or Q'_j. Thus the set of associated primes of $P_1 \cap P_2$ is a subset of $\{p_1, \ldots, p_{r_1}, p'_1, \ldots, p'_{r_2}\}$. This implies that those associated primes of $P_1 \cap P_2$ whose varieties contain purely imaginary points is a subset of $\{p_1, \ldots, p_{q_1}, p'_1, \ldots, p'_{q_2}\}$. Clearly then, by Theorem 6.1(ii), $\bar{P}_{1\mathcal{S}'} \cap \bar{P}_{2\mathcal{S}'} \subset (\overline{P_1 \cap P_2})_{\mathcal{S}'}$, which proves the corollary for $\mathcal{S}'$.

(iii) Now let $\mathcal{F} = \mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$. Let $p \in A^k$ be in $\bar{P}_{1\mathcal{F}} \cap \bar{P}_{2\mathcal{F}}$. Then there exist nonzero a_1 and a_2 in A such that $a_1 p \in P_1$ and $a_2 p \in P_2$ (by the remark following Proposition 6.1). As A is a domain, the product $a_1 a_2 \neq 0$, and $a_1 a_2 p \in P_1 \cap P_2$, which is to say that $p \in (\overline{P_1 \cap P_2})_{\mathcal{F}}$. □

Corollary 6.3. *Let $\mathcal{F}$ be a classical space, and P_i, $i = 1, \ldots, r$, submodules of A^k. Then $\mathsf{Ker}_{\mathcal{F}}(\bigcap_{i=1}^r P_i)$ is the smallest kernel containing $\sum_{i=1}^r \mathsf{Ker}_{\mathcal{F}}(P_i)$.*

Proof: By Lemma 3.2, the two expressions are equal if $\mathcal{F}$ is an injective A-module. Now let $\mathcal{F} = \mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$. Suppose $\mathsf{Ker}_{\mathcal{F}}(P)$ contains both $\mathsf{Ker}_{\mathcal{F}}(P_1)$ and $\mathsf{Ker}_{\mathcal{F}}(P_2)$, for some $P \subset A^k$. Then $\bar{P}_{\mathcal{F}}$ is contained in both $\bar{P}_{1\mathcal{F}}$ and $\bar{P}_{2\mathcal{F}}$, and hence in $\bar{P}_{1\mathcal{F}} \cap \bar{P}_{2\mathcal{F}}$, which equals $(\overline{P_1 \cap P_2})_{\mathcal{F}}$ by the above corollary. It now follows that

$$\mathsf{Ker}_{\mathcal{F}}(P_1 \cap P_2) = \mathsf{Ker}_{\mathcal{F}}((\overline{P_1 \cap P_2})_{\mathcal{F}}) \subset \mathsf{Ker}_{\mathcal{F}}(\bar{P}_{\mathcal{F}}) = \mathsf{Ker}_{\mathcal{F}}(P).$$

Thus, any kernel which contains both $\mathsf{Ker}_{\mathcal{F}}(P_1)$ and $\mathsf{Ker}_{\mathcal{F}}(P_2)$ also contains $\mathsf{Ker}_{\mathcal{F}}(P_1 \cap P_2)$. Thus $\mathsf{Ker}_{\mathcal{F}}(P_1 \cap P_2)$ is the smallest kernel that contains $\mathsf{Ker}_{\mathcal{F}}(P_1) + \mathsf{Ker}_{\mathcal{F}}(P_2)$. The lemma follows by induction. □

We can now provide a necessary condition for a system in $\mathcal{S}'$ to be controllable.

Theorem 6.2. *Let P be a submodule of A^k. The system $\mathsf{Ker}_{\mathcal{S}'}(P)$ is controllable if and only if $A^k/\bar{P}_{\mathcal{S}'}$ is torsion free, and hence if and only if the system admits a vector potential.*

Proof: Let $P = P_0 \cap P_1 \cap \cdots \cap P_r$ be an irredundant primary decomposition of P in A^k, where P_0 is 0-primary, and P_i is p_i-primary, p_i nonzero prime, for $i \geqslant 1$. Let $\mathcal{V}(p_i)$ contain purely imaginary points for $i = 0, \ldots, q$, and not for $i = q+1, \ldots, r$. Then the closure $\bar{P}_{\mathcal{S}'}$ of P with respect to $\mathcal{S}'$ equals $P_0 \cap P_1 \cap \cdots \cap P_q$. By definition $\mathsf{Ker}_{\mathcal{S}'}(P) = \mathsf{Ker}_{\mathcal{S}'}(\bar{P}_{\mathcal{S}'})$, and this in turn is equal to $\sum_{i=0}^{q} \mathsf{Ker}_{\mathcal{S}'}(P_i)$ by Lemma 3.2. In this sum, only the summand $\mathsf{Ker}_{\mathcal{S}'}(P_0)$ is controllable. Hence $\mathsf{Ker}_{\mathcal{S}'}(P)$ is controllable if and only if the index q is equal to 0, which is to say that $A^k/\bar{P}_{\mathcal{S}'}$ is torsion free. By the proof of Theorem 2.1(i), $\mathsf{Ker}_{\mathcal{S}'}(\bar{P}_{\mathcal{S}'})$ then admits a vector potential. □

We conclude with an important question. We have seen that in the case of the classical spaces $\mathcal{D}'$, $\mathcal{C}^\infty$, $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{E}'$ and $\mathcal{D}$, a controllable system always admits a vector potential. Thus, the converse of $(**)$, at the end of Appendix 1, is also true, and the notion of controllability is equivalent to the existence of a vector potential. We ask if this is so over any signal space $\mathcal{F}$? In other words, is Kalman's notion of a controllable system, suitably generalised, nothing more - nor less - than the possibility of describing the dynamics of the system by means of a vector potential?

References

[1] A.A. Andronov and L.S. Pontryagin (1937), Coarse systems (in Russian), Doklady Akademii Nauk SSSR. 14:247-250, 1937.

[2] W.D. Brownawell, Bounds for the degrees in the Nullstellensatz, Annals of Mathematics, 126:577-591, 1987.

[3] H. Bourlès, Algebraic analysis applied to the theory of linear dynamical systems, arXiv:1501.00656 [math.OC], 2015.

[4] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Mathematics 150, Springer-Verlag, 1995.

[5] R.E. Kalman, On the general theory of control systems, Proceedings, 1st World Congress of the International Federation of Automatic Control, Moscow, 481-493, 1960.

[6] V. Lomadze, Controllability as minimality, SIAM jl. Control and Optimization, 50:357-367, 2012.

[7] V. Lomadze and E. Zerz, Partial differential equations of Krull dimension zero, Proc. 14th Int. Symp. Mathematical Theory Networks Systems, 2002.

[8] B. Malgrange, Division des distributions, Séminaire L. Schwartz, Exposés, 2125, 1960.

[9] B. Malgrange, Systèmes différentiels à coefficients constants, Séminaire Bourbaki, vol. 1962/63:246.01-246.11, 1963.

[10] U. Oberst, Multidimensional constant linear systems, Acta Applicandae Mathematicae, 20:1-175, 1990.

[11] V.P. Palamodov, A remark on exponential representation of solutions of differential equations with constant coefficients (in Russian), Math. USSR Sbornik, 5: 401-416, 1968.

[12] H. Pillai and S. Shankar, A behavioural approach to control of distributed systems, SIAM jl. Control and Optimization, 37: 388-408, 1998.

[13] J-F. Pommaret, Partial differential control theory, volume II: control systems, Kluwer Academic Publishers, 2001.

[14] S. Shankar, The Nullstellensatz for systems of PDE, Advances in Applied Mathematics, 23:360-374, 1999.

[15] S. Shankar, The lattice structure of behaviours, SIAM jl. Control and Optimization, 39:1817-1832, 2001.

[16] S. Shankar, On the dual of a flat module in TOP, Linear Algebra and its Applications, 433:1077-1081, 2010.

[17] S. Shankar, The Hautus test and genericity results for controllable and uncontrollable behaviors, SIAM jl. Control and Optimization, 52:32-51, 2014.

[18] S. Shankar and P. Rocha, The generic degree of autonomy, SIAM jl. Applied Algebra Geometry, 2:410-427, 2018.

[19] J.C. Willems, The behavioral approach to open and interconnected systems, IEEE Control Systems Magazine, 27:46-99, 2007.

[20] J. Wood, E. Rogers and D.H. Owens, A formal theory of matrix primeness, Math. Control Signals Systems, 1998, 11:40-78.

[21] J. Wood, Key problems in the extension of module-behaviour duality, Linear Algebra and its Applications, 351-352: 761-798, 2002.

Index