2nd INTERNATIONAL CONFERENCE

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ON MATHEMATICAL MODELLING
IN APPLIED SCIENCES

BSU Belgorod-Russia (August 20-24, 2019)

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BOOK of Abstracts

Editor

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Preface ━━━━━━━ ICMMAS 2019

The *2nd International Conference on Mathematical Modelling in Applied Sciences, ICMMAS'19*, is organized by Belgorod State University (BSU), Belgorod-Russia, during the period August 20-24, 20119 http://icmmas19.alpha-publishing.net/. Its first version, named as ICMMAS'17, was held at SPbPU, Saint Petersburg-Russia in July 24-28, 2017 http://icmmas.alpha-publishing.net/. The evaluation committee within a meeting headed by Prof. V. Antonov and including Prof. A. Debbouche, Prof. D. Baleanu, Prof. S. Capozziello, Prof. V.E. Fedorov, Prof. W. Sproessig, Prof. D.F.M. Torres and Prof. I. Area witnessed the success of the scientific event providing some positive facts, and heartily recommended to repeat it every two years.

The proposed Scientific Program of the conference is including plenary lectures, contributed oral talks, poster sessions and listeners. Five suggested special sessions/mini-symposium are also considered by the scientific committee. The areas of interest include but are not limited to:

TOPICS

Models Based on Analytical, Numerical and Experimental Solutions
Mathematical Modelling involving time fractional PDEs
Biological Systems and Cancer Dynamics
Ordinary and Partial Differential Equations: Theory and Applications
Integral Equations and Integral Transforms
Uncertainty Quantification in Mathematical Modelling
Control Theory, Optimization and their Applications
Probability, Statistics and Numerical Analysis
Inverse Problems: Modelling and Simulation
Modern Fractional Dynamic Systems and Applications
Computational Methods in Sciences and Engineering
Heat and Mass Transfer in Fractal Medium

ICMMAS'19 is also supported by the following high ranked well known international journals:

1. **Journal of Computational and Applied Mathematics**
   https://www.journals.elsevier.com/journal-of-computational-and-applied-mathematics
   Impact Factor: 1.883
   Guest Editors: Amar Debbouche, Michal Fečkan and Eduardo Hernández

2. **Mathematical Methods in the Applied Sciences**
   http://onlinelibrary.wiley.com/journal/10.1002/(ISSN)1099-1476
   Impact Factor: 1.533
   Guest Editors: Amar Debbouche and Vladimir Vasilyev

3. **Chaos, Solitons & Fractals**
   https://www.journals.elsevier.com/chaos-solitons-and-fractals
   Impact Factor: 3.064
   Guest Editors: Juan Carlos Cortés, Amar Debbouche and Sakthivel Rathinasamy

4. **The European Physical Journal Plus**
   https://www.springer.com/physics/applied+%26+technical+physics/journal/13360
   Impact Factor: 2.612
   Guest Editors: Amar Debbouche, Juan J. Nieto and Delfim F.M. Torres

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Which classification of fractional operators is the best?

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Abstract. During the last few years a lot of new fractional operators were suggested. One of the most successfully used and certified to many real world applications is the fractional operator with Mittag-Leffler kernel. At the moment there is a discussion regarding the so called classification of the fractional operators. In my talk I willl discuss the advantages and disadvantages of all existing classifications.

Keywords. fractional calculus; fractional derivatives;classification; non-singular fractional operators; Mittag-Leffler kernel.

MSC2010. 26A33; 33E12.

Introduction

The existence of many fractional operators it is not a surprise for researchers working in the field of fractional calculus [1, 2, 3]. It is clear that there is no chance to construct a unique fractional able to recover the existing ones as special cases. This open the gate of discussion to classify the fractional operators. So far, in the literature exist five attempts to classify the fractional operators [4, 5, 6, 7, 8] but the required conditions are very different

1 Main results

In this talk I would analyze the content of all suggested classifications of fractional operators. The advantages and disadvantages of each classification will be explained. In addition, the applications of fractional operators with Mittag-Leffler kernel [2, 3] will be given.

References

[1] D. Baleanu, K. Diethelm, E. Scalas and J. J Trujillo, *Fractional Calculus Models and Numerical Methods*, Series on Complexity, Nonlinearity and Chaos, World Scientific, 2012.

[2] D. Baleanu A. Atangana,New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, Therm. Sci. **20** (2016), 763–769.

[3] D. Baleanu A. Fernandez, On some new properties of fractional derivatives with Mittag-Leffler kernel, Comun. Nonlin. Numer.Simul.**59** (2018), 444–462.

[4] B. Ross, A brief history and exposition of the fundamental theory of fractional calculus, Fractional calculus and its applications, Lect. Notes Math. Springer-Verlag, Berlin, Heidelberg (1975), 1–36.

[5] M. Ortigueira J. A. Tenreiro Machado, What is a fractional derivative?, J. Comput. Phys. **293** (2015), 4–13.

[6] M. Caputo M. Fabrizio, On the notion of fractional derivative and applications to the hysteresis phenomena, Meccanica, **52** (2017), 3043–3052.

[7] D. Zhao M. Luo, Representations of acting processes and memory effects: general fractional derivative and its application to theory of heat conduction with finite wave speeds, Appl. Math. Comput. **346** (2019), 531–544.

[8] R. Hilfer Y. Luchko, Desiderata for fractional derivatives and integrals, Mathematics, **7** (2019), 149.

Hopf Bifurcations and Integrability in Some Biochemical Models

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Abstract. We propose an approach based on the elimination theory of computational algebra to find conditions for existence of Hopf bifurcations in polynomial systems of ordinary differential equations. We also describe an algorithmic approach which allows to find invariant surfaces and first integrals in such systems. Some applications to the investigation of biochemical models are discussed.

Keywords. limit cycle; bifurcation; first integral; invariant surface; conservation law.

MSC2010. 34C23; 37G15; 34C60

Introduction

The classical Lotka-Volterra system, also known as the predator-prey equations, describe an evolution of two competing species. A generalization to the case of three competing species was proposed by May and Leonard [3]. The model and its generalization, called the asymmetric May-Leonard model, is described by the system of differential equations

$$
\begin{aligned}
\dot{x} &= x(1 - x - \alpha_1 y - \beta_1 z), \\
\dot{y} &= y(1 - \beta_2 x - y - \alpha_2 z), \\
\dot{z} &= z(1 - \alpha_3 x - \beta_3 y - z),
\end{aligned}
\tag{1}
$$

where α_i, β_i ($1 \leq i \leq 3$) are non-negative parameters.

Tanabe and Namba [4] introduced a model of evolution of three species one of each is an omnivore, which can eat both a predator and a prey, given by the equations

$$
\begin{aligned}
\dot{x} &= x(1 - bx - y - z), \\
\dot{y} &= y(-c + x), \\
\dot{z} &= z(-e + fx + gy - \beta z).
\end{aligned}
\tag{2}
$$

Main results

A common feature of the models presented above (as well as many other biochemical models) is that both models are described by polynomial ODEs. However since the systems depend on many parameters already the detection of a Hopf bifurcation can be an extremely difficult

problem. Mostly such bifurcations in biochemical models are found numerically for heuristically chosen values of parameters or the methods based on the Routh-Hurwitz criterion are applied.

We describe a simple approach to find conditions for existence of Hopf bifurcations and center manifolds which is based on the elimination theory [2]. We describe it for system (2), but it can be applied to other polynomials systems of ODEs depending on parameters if the number of parameters is not too large.

We also propose a method to investigate the degenerate Hopf bifurcations (the Bautin bifurcations) without computation a normal form and transforming the matrix of the linearized system to a Jordan normal form and show that there are systems (2) with two limit cycles bifurcating from a stationary point with positive coordinates [5].

It is also demonstrated how algorithms from elimination theory can be used to find algebraic invariant surfaces in polynomial systems of ODEs. We look for first integrals of the Darboux type constructed using these invariant surfaces and find subfamilies of (1) admitting one or two independent first integrals [1].

Acknowledgments

References

[1] V. Antonov, W. Fernandes, V.G. Romanovski, N. L. Shcheglova, First integrals of the May-Leonard asymmetric system, Mathematics, **7** (2019), https://doi.org/10.3390/math7030292.

[2] Y. Li, V.G. Romanovski, Hopf bifurcations in a predator-prey model with an omnivore, Qualitative Theory of Dynamical Systems (2019), accepted.

[3] R. M. May, W. J. Leonard, Nonlinear aspects of competition between three species, SIAM J. Appl. Math., **29** (1975) 243–253.

[4] K. Tanabe and T. Namba, Omnivory creates chaos in simple food web models, Ecol. **86** (2005) 3411–3414.

[5] Y. Xia, M. Grašič, W. Huang and V. G. Romanovski, Limit cycles in a model of olfactory sensory neurons, Int. J. Bifurcat. Chaos, **29**(3) (2019) 1950038 (9 pages).

Transmutation operator method for practical solution of forward and inverse spectral problems

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Abstract. A new method for solving forward and inverse spectral problems on finite and infinite intervals for Sturm-Liouville equations is presented. It is based on classical objects of spectral theory: the transmutation operators and the Gel'fand-Levitan-Marchenko type equations. In the talk the case of the infinite intervals will be presented in detail.

Keywords. transmutations; Sturm-Liouville theory; Gel'fand-Levitan equation; one-dimensional scattering problem; short-range potentials.

MSC2010. 34L25, 34L40, 35Q53, 47A40, 65L09, 81U40.

The transmutation (transformation) operators represent one of the main theoretical tools of the spectral theory [2, 7-10]. In the talk a new approach is presented for solving the classical forward and inverse Sturm-Liouville problems on finite and infinite intervals. It is based on the Gel'fand-Levitan-Marchenko integral equations and recent results on the functional series representations for the transmutation integral kernels [1, 3-6]. In particular, a new representation for so-called Jost solutions is obtained reducing all the calculations related to spectral and scattering data to finite intervals instead of the half line or the whole line. This is the case of the spectral density function as well as of the reflection coefficient in the scattering problem. In a sense this reduction trivializes the classical spectral and scattering problems on infinite intervals previously considered as numerically challenging problems. Numerical methods based on the proposed approach for solving forward problems allow one to compute large sets of eigendata with a nondeteriorating accuracy. Solution of the inverse problems reduces directly to corresponding systems of linear algebraic equations. In the talk some numerical illustrations will be presented.

References.

1. B. B. Delgado, K. V. Khmelnytskaya, V. V. Kravchenko, Fourier-Laguerre series expansion of Levin's transmutation kernel and an efficient method for solving the spectral problem on the half line. Submitted.

2. V. V. Katrakhov, S. M. Sitnik The transmutation method and boundary–value problems for singular elliptic equations. Contemporary Mathematics. Fundamental Directions. Vol. 64, Iss. 2, pp. 211–426. (Russian).

3. V. V. Kravchenko, Construction of a transmutation for the one-dimensional Schrödinger operator and a representation for solutions. Applied Mathematics and Computation, v. 328 (2018) 75-81.

4. V. V. Kravchenko, On a method for solving the inverse scattering problem on the line. Mathematical Methods in the Applied Sciences 42 (2019) 1321-1327.

5. V. V. Kravchenko, On a method for solving the inverse Sturm-Liouville problem. Journal of Inverse and Ill-Posed Problems, Journal of Inverse and Ill-posed Problems, 27 (2019), 401-407.

6. V. V. Kravchenko, L. J. Navarro and S. M. Torba, Representation of solutions to the one-dimensional Schr odinger equation in terms of Neumann series of Bessel functions. Applied Mathematics and Computation, v. 314 (2017) 173-192.

7. B. M. Levitan, Inverse Sturm-Liouville problems, VSP, Zeist, 1987.

8. V. A. Marchenko, Sturm-Liouville Operators and Applications. Birkh auser, Basel, 1986.

9. S. M. Sitnik, E. L. Shishkina Method of Transmutations for Differential Equations with Bessel operators. 2019, Moscow, Fizmatlit, 224 pp. (Russian).

10. V. A. Yurko, Introduction to the theory of inverse spectral problems. Moscow, Fizmatlit, 2007, 384 pp. (Russian).

Computing the probability density function of non-stationary solutions for some significant random differential equations: Beyond means and variances

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Abstract. In this paper, we introduce some analytical methods able to compute reliable approximations of the first probability density function of the solution stochastic process for some relevant random differential equations. The methods are devised so that a wide range of differential equations with uncertainty in all inputs (initial/boundary conditions, source term and coefficients) can be treated. In the analysis, we will consider some significant random differential equations illustrating our main findings by means of simulations.

Keywords. Mean square random calculus; random differential equation; first probability density function; Karhunen-Loève expansion; Monte Carlo simulations.

MSC2010. 2010: 35R60; 60H15; 60H35; 35Q35; 65C30; 65C20.

Introduction

In dealing with deterministic differential equations, the main goal is to compute the solution either analytical or numerically. This is also a primer goal in solving random differential equations whose solution is a stochastic process. Unlike deterministic context, the determination of the main statistical functions associated to the solution stochastic process, such as the mean and variance functions, are also important goals to be achieved. In fact, the average or mean behaviour of the solution as well as its variability around the mean are obtained from these two statistical moments, respectively. Although this information is valuable, and most contributions focus on the computation of the solution stochastic process and its mean and variance/standard deviation functions, a more ambitious target includes the determination of the first probability density function (1-p.d.f.) of the solution. The 1-p.d.f. provides a full probabilistic description in each time instant of the solution stochastic process. Moreover, from the 1-p.d.f., both the mean and variance functions can be straightforwardly computed, but also asymmetry, kurtosis, and other higher unidimensional statistical moments provided they exist [1].

The aim of this talk is to present a some new result advances about the computation of the 1-p.d.f. for some relevant random differential equations appearing in Mathematical Modelling whose inputs (initial/boundary conditions, source term and coefficients) can be random variable or stochastic processes.

1 Main results

Some of the new results that will be presented in this talk will focus on the random linear advection equation that describes the concentration of a chemical substance transported by a one-dimensional fluid

$$\begin{cases} \frac{\partial}{\partial t}Q(x,t) + V(t)\frac{\partial}{\partial x}Q(x,t) = 0, & t > 0, \quad x \in \mathbb{R}, \\ Q(x,0) = Q_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $V(t)$ is the velocity and $Q_0(x)$ is the initial concentration of the substance at the spatial point. This partial differential equation has also been used to model the flux of a two-phase equal viscosity miscible fluid in a porous media. In [2], a randomized version of this problem has been successfully studied in the case that both $V(t)$ and $Q_0(x)$ are stochastic processes being $V(t)$ Gaussian. In our contribution, we will apply novel results on the stochastic chain rule and we will solve the random linear advection equation in the mean square sense [3]. Furthermore, we will provide a new representation for the p.d.f. of the solution stochastic process, which can be accurately computed by means of Monte Carlo simulations, and which does not require the specific probability distribution of the integral of the velocity. This will allow us to solve the non-Gaussian velocity case, that to best of our knowledge has not been yet dealt in the extant literature.

One of our main results, that will be shown in this contribution, will consist of deriving a theoretical partial differential equation for $f_Q(q;x,t)$ that extend [2, Prop. 3.1] in the case that $V(t)$ is stochastic process. This result is stated in the following

Theorem 1 *Suppose that $Q_0(x)$ is a stochastic process with absolutely continuous law for each $x \in \mathbb{R}$. Let $V(t)$ be any stochastic process, $t \in [0,T]$. Assume that Q_0 and V are independent processes. Then the probability density function of $Q(x,t)$, $f_Q(q;x,t)$, satisfies the following partial differential equation for $x \in \mathbb{R}$ and $t \in [0,T]$:*

$$\frac{\partial}{\partial t}f_Q(q;x,t) + \frac{\partial}{\partial x}\left\{\mathbb{E}[V(t)|Q(x,t) = q]f_Q(q;x,t)\right\} = 0.$$

Acknowledgments

References

[1] T.T. Soong, *Random Differential Equations in Sciences and Engineering*, Academic Press, New York, 1973.

[2] F. A. Dorini and M. C. C. Cunha, On the linear advection equation subject to random velocity fields, Math. Comput. Simul. **82**(4) (2011) 679–690.

[3] L. Villafuerte and C.A. Braumann and J.-C. Cortés and L. Jódar, Random differential operational calculus: Theory and applications, Comput. Math. Appl. **59**(1) (2010) 115–125.

Integral Transforms Composition Method (ITCM) in Transmutation Theory

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Abstract. The report paper is devoted to the study of Integral Transforms Composition Method (ITCM) in transmutation theory. This method is based on applications of classical integral transforms compositions and leads to unified theory of transmutations. It is applied to connection formulas for solutions of singular differential equations, including Bessel operators.

Keywords. transmutations; integral transforms; Fourier transform; Mellin transform; Hankel transform; Bessel operator; singular differential equations, connection formulas for PDE solutions.

MSC2010. 42C15, 46E30, 46E35.

In the report paper we study applications of integral transforms composition method (ITCM) [1–4] for obtaining transmutations via integral transforms. It is possible to derive wide range of transmutation operators by this method. Classical integral transforms are involved in the integral transforms composition method (ITCM) as basic blocks, among them are Fourier, sine and cosine–Fourier, Hankel, Mellin, Laplace and some generalized transforms. The ITCM and transmutations obtaining by it are applied to deriving connection formulas for solutions of singular differential equations and more simple non–singular ones. We consider well–known classes of singular differential equations with Bessel operators, such as classical and generalized Euler–Poisson–Darboux equation and the generalized radiation problem of A.Weinstein. Methods of this paper are applied to more general linear partial differential equations with Bessel operators, such as multivariate Bessel–type equations, GASPT (Generalized Axially Symmetric Potential Theory) equations of A.Weinstein, Bessel–type generalized wave equations with variable coefficients,ultra B–hyperbolic equations and others. So with many results and examples the main conclusion of this paper is illustrated: *the integral transforms composition method (ITCM) of constructing transmutations is very important and effective tool also for obtaining connection formulas and explicit representations of solutions to a wide class of singular differential equations, including ones with Bessel operators.*

In transmutation theory explicit operators were derived based on different ideas and methods, often not connecting altogether. So there is an urgent need in transmutation theory to develop a general method for obtaining known and new classes of transmutations. The *integral transform composition method* (ITCM) gives the algorithm not only for constructing new transmutation operators, but also for all now explicitly known classes of transmutations, including Poisson, Sonine, Vekua–Erdelyi–Lowndes, Buschman-Erdelyi, Sonin–Katrakhov and

Poisson–Katrakhov ones, as well as the classes of elliptic, hyperbolic and parabolic transmutation operators introduced by R. Carroll, cf. [1–4].

The formal algorithm of ITCM is the next. Let us take as input a pair of arbitrary operators A, B, and also connecting with them generalized Fourier transforms F_A, F_B, which are invertible and act by the formulas

$$F_A A = g(t) F_A, \quad F_B B = g(t) F_B, \tag{1}$$

t is a dual variable, g is an arbitrary function with suitable properties. It is often convenient to choose $g(t) = -t^2$ or $g(t) = -t^\alpha$, $\alpha \in \mathbb{R}$.

Then the essence of ITCM is to obtain formally a pair of transmutation operators P and S as the method output by the next formulas:

$$S = F_B^{-1} \frac{1}{w(t)} F_A, \qquad P = F_A^{-1} w(t) F_B \tag{2}$$

with arbitrary function $w(t)$. When P and S are transmutation operators intertwining A and B:

$$SA = BS, \qquad PB = AP. \tag{3}$$

A formal checking of (3) can be obtained by direct substitution. The main difficulty is the calculation of compositions (2) in an explicit integral form, as well as the choice of domains of operators P and S.

Let us list main advantages of Integral Transform Composition Method (ITCM).

- Simplicity — many classes of transmutations are obtained by explicit formulas from elementary basic blocks, which are classical integral transforms.

- ITCM gives by a unified approach all previously explicitly known classes of transmutations.

- ITCM gives by a unified approach many new classes of transmutations for different operators.

- ITCM gives a unified approach to obtain both direct and inverse transmutations in the same composition form.

- ITCM directly leads to estimates of norms of direct and inverse transmutations using known norm estimates for classical integral transforms on different functional spaces.

- ITCM directly leads to connection formulas for solutions to perturbed and unperturbed differential equations.

References.

1. Katrakhov V.V., Sitnik S.M. The transmutation method and boundary–value problems for singular elliptic equations. Contemporary Mathematics. Fundamental Directions. Vol. 64, Iss. 2, pp. 211–426. (in Russian).

2. Sitnik S.M., Shishkina E.L. Method of Transmutations for Differential Equations with Bessel operators. 2019, Moscow, Fizmatlit, 224 P.

3. Fitouhi A., Jebabli I., Shishkina E.L., Sitnik S.M. Applications of integral transforms composition method to wave-type singular differential equations and index shift transmutations. Electron. J. Differential Equations, Vol. 2018 (2018), No. 130, pp. 1-27.

4. Katrakhov V.V., Sitnik S.M. Composition method for constructing B–elliptic, B–hyperbolic, and B–parabolic transformation operators. Russ. Acad. Sci., Dokl. Math. 1995, Vol. 50, No. 1, pp. 70–77.

Fuzzy Transforms Through the Prism of Galerkin Methods

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Abstract. The aim of the contribution is to propose and justify a new methodology in the construction of spaces of test functions used in a weak formulation of the Boundary Value Problem. The construction is based on a fuzzy partition of the problem domain and selected test functions are connected with the construction of higher degree F-(fuzzy) transform components. The main advantage of this so called "two dimensional" approach consists in the independent selection of the key parameters, aiming at achieving a requested quality of approximation with a reasonable complexity. We give theoretical justification and illustration on examples that confirm our methodology.

Keywords. fuzzy partition; standard BVP; F-transform; Galerkin method.

MSC2010. 35D30, 65M60

1   Introduction

Fuzzy modeling is known for its ability to create logically justified discrete models of continuous dynamic processes. However, there is one particular theory originated from fuzzy modeling but expressed in the language of functional analysis and approximation theory. It is named as *fuzzy transforms* (F-transforms in short).

This theory has many applications in data analysis, image processing, and numerical analysis. The latter is confirmed by many smart and efficient algorithms for numerical solutions of ODEs and PDEs [3, 4].

In short, the *F-transform* (FzT) is a particular integral transform, whose peculiarity is in using a fuzzy partition of a universe of discourse as a free parameter. The F-transform has two phases: direct and inverse. The direct FzT is applied to functions from $L^2(\mathbb{R})$ and maps them linearly onto sequences (originally finite) of numeric/functional components. Each component is a weighted orthogonal projection of a given function on a certain linear subspace of $L^2(\mathbb{R})$. The *inverse F-transform* (iFzT) is applied to a sequence of components and transforms it linearly into a function from $L^2(\mathbb{R})$. In general, it is different from the original one. In [2], it has been shown that the iFzT can approximate a continuous function with an arbitrary precision.

In classical numerical analysis, the similar abstract idea – converting a continuous operator problem (e.g., differential equation) to a discrete one, is known under the name *Galerkin method* [1]. This method was also discovered by Walther Ritz, to whom Galerkin refers.

2 Main Results

In the talk, we first recall the notion of a **weak solution** and the operator based technique in connection with the standard BVP problem. Then, we concentrate on finding appropriate approximating spaces - sources of test functions and feasible solutions. We introduce spaces that are based on a fuzzy partition of a problem domain and recall the notion of a higher degree F-transform.

We show that any L^2 function can be efficiently decomposed into a sequence of F^m-transform components. Then, we show that the corresponding to the direct F-transform linear operator is a bounded operator that maps $L^2(a, b)$ in a direct sum of Hilbert spaces determined by the fuzzy partition. Finally, we show that the sequence of F^m-transform components of any L^2 function converges to the decomposition of this function determined by the fuzzy partition. All these facts guarantee that the weak solution to the standard BVP can be efficiently computed using approximation spaces stemmed from a fuzzy partition. We support our theoretical claims by examples.

Acknowledgments

References

[1] B.G. Galerkin, On electrical circuits for the approximate solution of the Laplace equation. Vestnik Inzh. **19** (1915) 897–908 (In Russian).

[2] I. Perfilieva and M. Daňková and B. Bede, Towards a higher degree f-transform, Fuzzy Sets and Systems **180** (2011) 3–19.

[3] A. Khastan and I. Perfilieva and Z. Alijani, A new fuzzy approximation method to Cauchy problems by fuzzy transform. Fuzzy Sets and Systems **288** (2016) 75–95.

[4] Irina Perfilieva and Petra Števuliáková and Radek Valášek, F-transform-based shooting method for nonlinear boundary value problems. Soft Computing, **21** (2017) 3493–3502.

Fractional differentiation and integration above power law

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Abstract. Very recently new trends of fractional differentiation and integration were introduced and applied in many fields of science, technology and engineering with great success. We present in this talk their properties, methods and applications.

Keywords. Exponential decay law, power law, generalized Mittag-Leffler function, non-singular kernels.

MSC2010. 26A33.

References

[1] Abdon Atangana, Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties. Physica A: Statistical Mechanics and its Applications 505, 688-706(2018)

[2] Abdon Atangana. Blind in a commutative world: Simple illustrations with functions and chaotic attractors. Chaos, Solitons & Fractals 114, 347- 363(2018)

[3] A Atangana, JF Gómez-Aguilar. Fractional derivatives with no-index law property: application to chaos and statistics. Chaos, Solitons & Fractals 114, 516-535(2018)

[4] A Atangana, JF Gómez-Aguilar. Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena. The European Physical Journal Plus 133, 1-22(2018)

A numerical method for solving a type of variable order partial differential equations

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Abstract. In this work, we consider a type variable orders partial differential equations (VOPDEs) where the fractional derivative is in Caputo scene.
We apply the operational matrix method based on orthonormal Bernstein polynomials to obtain numerical solution of the above equations. With the help of operational matrices and collocation method, we can reduce VOPDEs to an algebraic system, then we obtain the approximate solution.

Keywords. Variable orders partial differential equations; Orthonormal polynomials; The operational matrix; Collocation method.

MSC2010. 35R11;65N35.

1   Introduction

Variable derivative is a new definition that Samko and Ross [1] introduced in 1993, that the order of differential (or integral) operator is not a constant but it is a function of space, time or other variables. Since derivative operator has a kernel of the variable order, it is not simply task to obtain the solution of such equations. Recently, different methods are presented for solving VOPDEs and such for solving variable order of the differential equations [2]. The aim of this paper is to investigate numerically VOPDEs in the following form: in the following form:

$$
\begin{aligned}
&\frac{\partial^{\kappa_i(x,t)}\chi(x,t)}{\partial t^{\kappa_i(x,t)}} + \frac{\partial^{\upsilon_i(x,t)}\chi(x,t)}{\partial x^{\upsilon_i(x,t)}} = F\left(x,t,\chi(x,t),\frac{\partial^m\chi(x,t)}{dt^m},\frac{\partial^n\chi(x,t)}{dx^n}\right), \\
&\chi(x,0) = f_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 < x < 1 \\
&\chi(0,t) = f_1(t), \ \chi(1,t) = f_2(t), \qquad\qquad\qquad\qquad\quad\ 0 < t < 1,
\end{aligned} \tag{1}
$$

where $q_i - 1 < \kappa_i(t) < q_i, q_i' - 1 < \upsilon_i(x) < q_i'$ and $q_i, q_i', m, \nu \in \mathbb{N}$. $\dfrac{\partial^{\kappa_i(x,t)}\chi(x,t)}{\partial t^{\kappa_i(x,t)}}$ and $\dfrac{\partial^{\upsilon_i(x,t)}\chi(x,t)}{\partial x^{\upsilon_i(x,t)}}$ are the time-fractional derivatives and the space-fractional derivatives. by Bernstein polynomials.

Definition 1 *The time-fractional derivative and the space-fractional derivative are defined as the Caputo fractional derivatives as following:*

$$
\frac{\partial^{\kappa_i(x,t)}\chi(x,t)}{\partial t^{\kappa_i(x,t)}} = \frac{1}{\Gamma(q_i - \kappa_i(x,t))}\int_0^t (t-s)^{q_i - \kappa_i(x,t) - 1}\ \frac{\partial^{q_i}\chi(x,s)}{\partial s^{q_i}}\ ds, \tag{2}
$$

$$
\frac{\partial^{\upsilon_i(x,t)}\chi(x,t)}{\partial x^{\upsilon_i(x,t)}} = \frac{1}{\Gamma(q_i' - \upsilon_i(x,t))}\int_0^x (x-s)^{q_i' - \upsilon_i(x,t) - 1}\ \frac{\partial^{q_i'}\chi(s,t)}{\partial s^{q_i'}}\ ds. \tag{3}
$$

14

2 Operational matrix of Bernstein polynomials

We obtain the operational matrix for the fractional order derivative operator as:

$$D^{\kappa(x,t)}\varphi(t) = D^{\kappa(x,t)}[AT_n(t)] = A\ D^{\kappa(x,t)}[1\ t\ t^2\ \cdots\ t^n]^T \tag{4}$$

We take $p = \lceil \kappa(x,t) \rceil$, $p < n$, then

$$D^{\kappa(x,t)}\varphi(t) = A[0\ 0\ \cdots\ 0\ \frac{\Gamma(p+1)}{\Gamma(p+1-\kappa(x,t))}t^{p-\kappa(x,t)}\ \frac{\Gamma(p+2)}{\Gamma(p+2-\kappa(x,t))}t^{p+1-\kappa(x,t)}\ \cdots\ \frac{\Gamma(n+1)}{\Gamma(n+1-\kappa(x,t))}t^{n-\kappa(x,t)}]^T$$

$$= A\Psi A^{-1}\varphi(t),$$

$A\Psi A^{-1}$ is called the operational matrix of $D^{\kappa(x,t)}\varphi(t)$. Similarly

$$\frac{\partial^{\kappa_i(x,t)}X(x,t)}{\partial t^{\kappa_i(x,t)}} = \varphi(x)^T\ K\ D_t^{\kappa_i(x,t)}\varphi(t) = \varphi(x)^T\ K\ (A\Psi_i A^{-1})\varphi(t) = \varphi(x)^T\ K\ \Omega_i\varphi(t), \tag{5}$$

and

$$\frac{\partial^{v_i(x,t)}X(x,t)}{\partial x^{v_i(x,t)}} = D^{v_i(x,t)}\varphi(x)^T\ K\ \varphi(t) = (A\Phi_i A^{-1})\ \varphi(x)^T\ K\ \varphi(t) = \Upsilon_i\ \varphi(x)^T\ K\ \varphi(t). \tag{6}$$

where $\Omega_i = A\Psi_i A^{-1}$ and $\Upsilon_i = A\Phi_i A^{-1}$.

3 Proposed method for solving Eq. (1)

To solving the Eq. (1), we use the operational matrix method. Substituting (5) and (6) in Eq. (1), we have

$$\varphi(x)^T\ K\ \Omega_i\ \varphi(t) + \Upsilon_i\ \varphi(x)^T\ K\ \varphi(t) = F\left(x, t, \varphi(x)^T\ K\ \varphi(t), \varphi(x)^T\ K\ D^m\varphi(t), D^n\varphi(x)^T\ K\ \varphi(t)\right),$$

$$\varphi(x)^T\ K\ \varphi(0) = f_0(x),\ 0 < x < 1, \tag{7}$$

$$\varphi(0)^T\ K\ \varphi(t) = f_1(t),\ \varphi(1)^T\ K\ \varphi(t) = f_2(t),\ 0 < t < 1, \tag{8}$$

We define the residual function:

$$R(x,t) = \varphi(x)^T\ K\ \Omega_i\ \varphi(t) + \Upsilon_i\ \varphi(x)^T\ K\ \varphi(t) - F(.) = 0 \tag{9}$$

Substituting points x_i and t_j $(i = j = 0, 1, \ldots, n)$ in (7), (8) and (9), we obtain

$$R(x_i, t_j) = \varphi(x_i)^T\ K\ \Omega_i\ \varphi(t_j) + \Upsilon_i\ \varphi(x_i)^T\ K\ \varphi(t_j)$$

$$- F\left(x_i, t_j, \varphi(x_i)^T\ K\ \varphi(t_j), \varphi(x_i)^T\ K\ D^m\varphi(t_j), D^n\varphi(x_i)^T\ K\ \varphi(t_j)\right) = 0, \tag{10}$$

$$\varphi(x_i)^T\ K\ \varphi(0) = f_0(x_i),\ i = 0, 1, \ldots, n, \tag{11}$$

$$\varphi(0)^T\ K\ \varphi(t_j) = f_1(t_j),\ \varphi(1)^T\ K\ \varphi(t_j) = f_2(t_j),\ j = 1, 2, \ldots, n, \tag{12}$$

By solving system Eq. (10), (11) and Eq. (12), we obtain K, then the approximation solution obtain for Eq. (1).

References

[1] S.G. Samko and B. Ross, Integration and differentiation to a variable fractional order, Integral Transforms and Special Functions, **1**(4), (1993), 277–300.

[2] X. Li and B. Wu, A numerical technique for variable fractional functional boundary value problems, Appl. Math. Lett., **43**, (2015) 108–113.

Family of Bessel operator functions

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Abstract. A family of Bessel operator functions is introduced into consideration. As well as in theory semigroups and operator cosine functions, this family is investigated independently of the differential equations with which it will eventually be connected.

1   Introduction

Study of differential equations with unbounded operator coefficients acting in the Banach space E, stimulates the development of the theory of resolving operators of the corresponding initial problems. As a result of evolutionary studies first-order equations $u'(t) = Au(t)$ have semigroups of linear operators $T(t)$, and when studying an equation of second order (abstract wave equation) $u''(t) = Au(t)$ are operator cosine functions $C(t)$. Weakening requirements on resolving Cauchy problem operators for abstract differential equations of first and second orders led to the notion of an integrated semigroup and an integrated cosine operator function. The operator functions of Bessel and Struve were introduced into consideration in [1, 2, 3] as resolving operators for study, respectively, of the Euler-Poisson-Darboux and Bessel-Struve equations.

This paper presents a different approach to constructing a family of Bessel operator functions. As well as in theory semigroups and operator cosine functions, the Bessel family of operator functions is first investigated independently of the differential equations with which it will eventually be connected. An important role in the construction of a family is played by the operator depending on the parameter $k > 0$ generalized shift T_s^t defined by equality (see [4])

$$T_s^t Y(s) = \frac{\Gamma(k/2 + 1/2)}{\Gamma(1/2)\Gamma(k/2)} \int\limits_0^\pi Y\left(\sqrt{s^2 + t^2 - 2st\cos\varphi}\right) \sin^{k-1}\varphi \, d\varphi.$$

where $\Gamma(\cdot)$ is the Euler gamma function $s, t \geq 0$.

2   Main results

Let $k > 0$ and $Y_k(\cdot) : [0, \infty) \to B(E)$ be an operator function acting in the space of linear bounded operators $B(E)$.

Definition 1. Dependent on the parameter $k > 0$, a strongly continuous family of bounded linear operators $Y_k(t) : [0, \infty) \to B(E)$ is called the Bessel operator function (OFB) if:

i) $Y_k(0) = I$; ii) $Y_k(t)Y_k(s) = T_s^t Y_k(s)$, $s, t \geq 0$; iii) $\exists M \geq 1$, $\omega \geq 0 : \|Y_k(t)\| \leq Me^{\omega t}$, $t \geq 0$.

The Bessel differential operator is closely related to the OFB family $B_k = \dfrac{d^2}{dt^2} + \dfrac{k}{t}\dfrac{d}{dt}$.

Definition 2. The OFB $Y_k(t)$ generator is the operator A with the definition domain $D(A)$ consisting of those $x \in E$, for which the function $Y_k(t)x$ is twice differentiable at $t = 0$ and defined by $Ax = \lim\limits_{t \to 0+} B_k Y_k(t)x$.

Theorem 1. If the operator A is a OFB generator $Y_k(t)$, then the domain of $D(A)$ is dense in E. Moreover, in E there is a dense set of elements on which all powers of the operator A are defined.

Theorem 2. Let $Y_k(t)$ be an OFB and A be its generator. Then for any $t, s \geq 0$ and $x \in D(A)$ fair equality $Y_k(t)Y_k(s) = Y_k(s)Y_k(t)$, $AY_k(t)x = Y_k(t)Ax$.

Theorem 3. Let $Y_k(t)$ be an OFB and A be its generator. Then if $x \in D(A)$ and $t > 0$, then $Y_k(t)x \in D(A)$ and $AY_k(t)x = B_k Y_k(t)x$.

We now turn to the Cauchy problem, with which the OFB $Y_k(t)$ is associated.

Definition 3. By solving a second-order linear differential equation with an operator coefficient A is called the function $u(t)$, which for $t \geq 0$ is twice continuously differentiable, for $t > 0$ takes the values belonging to $D(A)$, i.e., $u(t) \in C^2(\bar{R}_+, E) \cap C(R_+, D(A))$, and satisfies this equation.

Theorem 4. Let $Y_k(t)$ be OFB, A be its generator, and $u_0 \in D(A)$. Then the function $Y_k(t)u_0$ is solving the EPD equation

$$B_k u(t) \equiv u''(t) + \frac{k}{t}u'(t) = Au(t), \quad t > 0$$

and satisfies the initial conditions

$$u(0) = u_0, \ u'(0) = 0.$$

Theorem 5. If $Y_k(t)$ is OFB, then its generator A is closed.

Acknowledgments

References

[1] Glushak, A.V., The Bessel operator function, Dokl. Math., 1997, vol. 55, no. 1, pp. 103–105.

[2] Glushak, A.V. and Pokruchin, O.A., Criterion for the solvability of the Cauchy problem for an abstract Euler-Poisson-Darboux equation, Differ. Equations, 2016, vol. 52, no. 1, pp. 39–57.

[3] Glushak, A.V., Abstract Cauchy Problem for the Bessel-Struve Equation, Differential Equations, 2017, Vol. 53, No. 7, pp. 864–878.

[4] Levitan, B.M., Expansion in Fourier Series and Integrals with Bessel Functions, Uspekhi Mat. Nauk, 1951, vol. 1, no. 2 (42), pp. 102–143 (in Russian).

A novel method for boundary layer flow of a Powell-Eyring non-Newtonian fluid

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Abstract. In this paper, the boundary layer flow of a Powell-Eyring non-Newtonian fluid over a stretching sheet has been researched. Reproducing kernel functions are used to get the better solutions. The approximate solutions are shown and the proposed method is compared with some well known techniques. Convergence analysis of the technique is given.

Keywords. reproducing kernel Hilbert space method; non-Newtonian fluid; reproducing kernel functions.

MSC2010. 46E22; 76A05; 12H20

1 Introduction

The main goal of this work is to apply reproducing kernel Hilbert space method using reproducing kernel functions for investigating the nonlinear differential equation of Powell-Eyring problem, in an unbounded domain. This method has been applied to many problems successfully [1, 2].

We consider a stretching sheet with a linear velocity $V_\tau t$. $V_\tau = \alpha$ is the linear stretching velocity and t is the distance from the slit. The shear tensor in a Powell-Eyring model is given as [3]:

$$\gamma_{ij} = \eta \frac{\partial v_i}{\partial t_j} + \frac{1}{\omega} \sin h^{-1} \left(\frac{1}{P} \frac{\partial v_i}{\partial t_j} \right) \tag{1}$$

The second order approximation of function is presented as [3]:

$$\sin h^{-1} \left(\frac{1}{P} \frac{\partial v_i}{\partial t_j} \right) \simeq \frac{1}{P} \frac{\partial v_i}{\partial t_j} - \frac{1}{6} \left(\frac{1}{P} \frac{\partial v_i}{\partial t_j} \right)^3, \quad \left| \frac{1}{P} \frac{\partial v_i}{\partial t_j} \right| << 1. \tag{2}$$

The boundary layer problems for an incompressible fluid based on Powell-Eyring model is given as [3]:

$$\frac{\partial v}{\partial t} + \frac{\partial g}{\partial s} = 0, \tag{3}$$

$$v \frac{\partial v}{\partial t} + g \frac{\partial v}{\partial s} = \left(\mu + \frac{1}{\rho \omega P} \right) \frac{\partial^2 v}{\partial s^2} - \frac{1}{2 \rho \omega P^3} \left(\frac{\partial v}{\partial s} \right)^2 \frac{\partial^2 v}{\partial s^2}. \tag{4}$$

The kinematic viscosity is given as $\mu = \frac{\eta}{\rho}$. For Eqs.(3) and (4), the boundary conditions are given as [3]:

$$\begin{cases} v = V_\tau t = \alpha t, & g = 0 \quad s = 0, \\ \\ v \to 0 \quad \text{as} \quad s \to \infty. \end{cases} \tag{5}$$

The following equations are acquired by dimensionless stream function $z(\zeta)$, where ζ is the similarity variable:

$$v = \frac{\partial \varphi}{\partial s}, \quad g = -\frac{\partial \varphi}{\partial t}, \tag{6}$$

where

$$\varphi = (\alpha\mu)^{\frac{1}{2}} t z(\zeta), \quad \zeta = \left(\frac{\alpha}{\mu}\right)^{\frac{1}{2}} s. \tag{7}$$

Then, we obtain [3]:

$$(1+\varepsilon)z'''(\zeta) - \varepsilon\delta z''^2(\zeta)z'''(\zeta) - z'^2(\zeta) + z(\zeta)z''(\zeta) = 0, \tag{8}$$

in which ε and δ are the material fluid parameters. These quantities have the following definitions:

$$\varepsilon = \frac{1}{\eta\omega P}, \quad \delta = \frac{\alpha^3 t^2}{2P^2\mu}. \tag{9}$$

The boundary conditions for Eq.(8) are acquired by Eq.(5) as [3]:

$$\begin{cases} z(0) = 0, \\ z'(0) = 1, \\ z'(\zeta) = 0 \quad \zeta \to \infty. \end{cases} \tag{10}$$

We investigate Eq.(8) with its boundary conditions Eq.(10) in the reproducing kernel Hilbert space in this paper.

References

[1] Azarnavid B, Parand K, Abbasband S (2018). An iterative kernel based method for fourth order nonlinear equation with nonlinear boundary condition. Communications In Nonlinear Science And Numerical Simulation Volume: 59 544-552.

[2] Li X, Wu B (2018). A new reproducing kernel collocation method for nonlocal fractional boundary value problems with non-smooth solutions. Applied Mathematics Letters Volume: 86 194-199.

[3] Parand K, Bahramnezhad A, Farahani H (2018). A numerical method based on rational Gegenbauer functions for solving boundary layer flow of a Powell-Eyring non-Newtonian fluid. Computational and Applied Mathematics.

Boundary Layer Solution for an Elliptic Problem with a Singular Neumann Boundary Condition

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Abstract. We consider an elliptic reaction-diffusion equation in a two-dimensional domain with a singular Neumann boundary condition. The existence of a solution with boundary layer is proved and the asymptotic approximation of the solution is obtained.

1   Introduction

We consider a singularly perturbed problem for an elliptic equation in a closed simply connected domain D with a sufficiently smooth boundary ∂D

$$\varepsilon^2 \Delta u = f(u, x, \varepsilon), \quad x = (x_1, x_2) \in D, \quad \varepsilon \left. \frac{\partial u}{\partial n} \right|_{\partial D} = h(x). \tag{1}$$

Here $\varepsilon > 0$ is a small parameter, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$ is the Laplacian operator and $\dfrac{\partial}{\partial n}$ denoting the outward normal derivative on ∂D.

We point out the singular boundary condition. Problems with similar conditions on a segment were considered in the papers [1] and [2].

We assume to be satisfied the following conditions:

(A_0) The functions $f(u, x, \varepsilon)$ and $h(x)$ are sufficiently smooth.

(A_1) The degenerate equation $f(u, x, 0) = 0$ has in the domain D a single root $u = \varphi(x)$ such that $f_u(\varphi(x), x, 0) > 0$.

Our goals were to prove the existence of a solution with a boundary layer in the problem (1) and obtain an asymptotic approximation of the solution. To prove the existence theorem, we used the asymptotic method of differential inequalities [4], [3]. The method assumes the justification of the solution with the boundary layer through the construction of the upper and lower solutions based on the asymptotics.

The solution is represented as a sum of two types of functions:

$$u(x, \varepsilon) = \bar{u}(x, \varepsilon) + Q(\xi, \theta, \varepsilon),$$

where $\bar{u}(x,\varepsilon) = \bar{u}_0(x) + \varepsilon\bar{u}_1(x) + \ldots$ is regular part; the functions $Q(\xi,\theta,\varepsilon) = Q_0(\xi,\theta) + \varepsilon Q_1(\xi,\theta) + \ldots$ describe the solution in the neighborhood of the boundary ∂D. Local variables (l,θ) are introduced near the boundary ∂D by the Lyusternik-Vishik method and $\xi = \varepsilon^{-1}l$ is stretched variable.

The equations for the regular part functions and boundary layer functions are compiled by the Vasilieva method (see [5]).

Acknowledgments

References

[1] A. V. Nesterov, The asymptotic behavior of the solution to a parabolic equation with singularly perturbed boundary conditions, Comput. Math. Math. Phys., **37** (1997), 1051-1057.

[2] V. F. Butuzov, N. N. Nefedov, L. Recke and K. R. Schneider, Periodic solutions with a boundary layer of reaction-diffusion equations with singularly perturbed Neumann boundary conditions, International Journal of Bifurcation and Chaos. **24**(2014), 1440019.

[3] N. N. Nefedov, An asymptotic method of differential inequalities for the investigation of periodic contrast structures: Existence, asymptotics, and stability, Diff. Eq. **36** (2000), 298-305.

[4] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York and London, 1992.

[5] A. B. Vasileva, V. F. Butuzov and L. V. Kalachev, *The boundary function method for singular perturbation problems*, SIAM, Phyladelphia, 1995.

Linear dynamical equations with uncertainty on time Scales

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Abstract. In this paper, we study the first order fuzzy dynamic equations on time scales. We investigate the existence of two solutions to first order linear fuzzy dynamic equations using the concept of strongly generalized differentiability.

Keywords. Generalized Hukuhara differentiability; Fuzzy dynamic equations, time scales.

MSC2010. 34A07.

References

[1] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic,* Springer, London, 2013.

[2] M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Springer Science & Business Media, 2012.

[3] V. Lupulescu, *Hukuhara differentiability of interval-valued functions and interval differential equations on time scales*, Information Sciences, 248 (2013) 50–67.

Mathematical Problems of Atmospheric Electricity

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Abstract. Various formulations of mathematical problems arising in the modeling of the global electric circuit are discussed. Initial- boundary value problems for the system of Maxwell equations in the electric non-relativistic approximation are considered. The new formulation of problems of determination of quasi-stationary electromagnetic fields, covering non-relativistic electric and magnetic approximations, is presented. The relation between different problems depending on physical parameters is investigated.

Keywords. atmospheric electricity; global electric circuit; quasi-stationary approximation for Maxwell equations; initial-boundary value problem.

MSC2010. 35Q60.

In recent years, there has been a significant increase in interest in mathematical, physical and numerical modeling of the global electric circuit (GEC), on the concept of which the modern theory of atmospheric electricity is based [1]-[15].

Most of the existing models of the GEC are aimed at finding the distribution of the density of the so-called quasi-stationary current, which, according to the hypothesis proposed by Wilson [16, 17], is supported by a constant charge separation in electrified clouds. In this case, to describe the electromagnetic fields in the lower layers of the atmosphere, a non-relativistic electric approximation for the Maxwell equations is used [18], which neglects the time variation of the magnetic field, that is, the electric field is assumed to be potential.

The papers [6, 15, 19, 20] considered problems for the equation of the global atmospheric electrical circuit, formulated in a quasi-stationary electrical approximation using the electric field and the scalar potential for various types of boundary conditions motivated by applications.

In this paper, we discuss some new problem statements that arise in the modeling of the global electrical circuit. In particular, a quasi-stationary approximation is formulated for the system of Maxwell's equations covering classical non-relativistic electric and magnetic approximations and suitable for the description of essentially inhomogeneous media including regions with high and low conductivity.

References

[1] R. G. Roble and P. B. Hays, A quasi-static model of global atmospheric electricity: 2. Electrical coupling between the upper and lower atmosphere, J. Geophys. Res. **84** N A12 (1979), 7247–7256.

[2] T. Ogawa, Fair-weather electricity, J. Geophys. Res. **90** N D4 (1985), 5951–5960.

[3] M. J. Rycroft, R. G. Harrison, K. A. Nicoll and E. A. Mareev, An overview of Earth's global electric circuit and atmospheric conductivity Space Sci. Rev., **137** 1-4 (2008), 83–105.

[4] E. Williams and E. Mareev, Recent progress on the global electrical circuit Atmos. Res., **135-136** (2014), 208–227.

[5] E. A. Mareev, Achievements and prospects of global electric circuit research, UFN, **180** N 5 (2010), 527–534.

[6] A. V. Kalinin, N. N. Slyunyaev, E. A. Mareev and A. A. Zhidkov, Stationary and nonstationary models of the global electric circuit: well-posedness, analytical relations, and numerical implementation, Izv. Atmos. Ocean. Phys. **50** 3 (2014), 314–322.

[7] R. Markson, The global circuit intensity: its measurement and variation over the last 50 years, Bull. Am.Met. Soc. **88** N2 (2007), 223–241.

[8] E. A. Mareev, S. A. Yashunin, S. S. Davydenko et al. On the role of transient currents in the global electric circuit, Geophys. Res. Lett. **35** N 15 (2008), L15810.

[9] B. A. Tinsley, The global atmospheric electric circuit and its effects on cloud microphysics, Rep. Progr. Phys. **71** N 6 (2008), 066801.

[10] E. R. Williams, The global electrical circuit: a review, Atmos. Res. **91** N 2-4 (2009), 140–152.

[11] C. Liu, E. R. Williams, E. J. Zipser and G. Burns, Diurnal variation of global thunderstorms and electrified shower clouds and their contribution to the global electrical circuit, J. Atmos. Sci. **67** N 2 (2010), 309–323.

[12] L. Zhou and B. A. Tinsley, Global circuit model with clouds, J. Atmos. Sci. **67** N 4 (2010), 1143–1156.

[13] M. J. Rycroft and R. G. Harrison, Electromagnetic atmosphere plasma coupling: the global atmospheric electric circuit, Space Sci. Rev. **168** N 1-4 (2011), 363–384.

[14] S. A. Mallios and V. P. Pasko, Charge transfer to the ionosphere and to the ground during thunderstorms, J. Geophys. Res. **117** N A8 (2012), A08303.

[15] A. V. Kalinin and N. N. Slyunyaev, Initial-boundary value problems for the equations of the global atmospheric electric circuit, J. Math. Anal. Appl. **450** N 1 (2017), 112–136.

[16] T. R. Wilson, Investigations on lightning discharges and on the electric field of thunderstorms, Phil. Trans. Roy. Soc. Lond. A., **221** (1921), 73–115.

[17] T. R. Wilson, The electric field of a thundercloud and some of its effects, Proc. Phys. Soc. London. **37** (1924), 32D–37D.

[18] V. V. Tolmachev, A. M. Golovin and V. S. Potapov, *Thermodynamics and electrodynamics of continuous media*, MSU, Moscow, 1988.

[19] A. A. Zhidkov and A. V. Kalinin, The correctness of one mathematical problem of atmospheric electricity, Vestn. Nizhegorod. Univ. im. N.I. Lobachevskogo, **4** (2009), 122–129.

[20] A. A. Zhidkov and A. V. Kalinin, Some problems of mathematical and numerical modelling of the global electric circuit in the atmosphere, Vestn. Nizhegorod. Univ. im. N.I. Lobachevskogo, **6** 1 (2009), 150–158.

Uniqueness of the Solution in the Inverse Problem for an Evolution Equation of Arbitrary Order

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Abstract. A linear inverse problem for an abstract evolution equation of arbitrary order is studied. The inhomogeneous term of the equation is unknown. An additional condition at the final point in time is given. For the stated problem, a uniqueness criterion for the solution is obtained. It is expressed through zeros of a special entire function of the Mittag-Leffler type. The result is fair for the most general case with minimal restrictions on equation type.

Keywords. Abstract differential equation; inverse problem; Mittag-Leffler entire functions; generalized λ-hyperbolic functions; uniqueness criterion of the solution.

MSC2010. 30D20; 35R30; 49K20.

Let E be a Banach space and let A be a linear closed operator in E with a domain $D(A) \subset E$ (not necessarily dense in E). Given numbers $T > 0$ and $n \in \mathbb{N}$, we consider the differential equation

$$u^{(n)}(t) = Au(t) + g, \qquad 0 < t < T, \tag{1}$$

with an unknown element g from E. To find a function $u \colon [0, T] \to E$ and an element $g \in E$, we add to (1) standard Cauchy conditions

$$u^{(j)}(0) = u_j, \qquad \text{for all} \quad j \in \{0, \ldots, n-1\}, \tag{2}$$

and a special final overdetermination

$$u^{(q)}(T) = u_n \tag{3}$$

with a fixed value $q \in \{0, \ldots, n-1\}$. Elements $u_0, \ldots, u_n$ are given in E. The problem (1)–(3) belongs to the class of inverse problems. A pair $(u(t), g)$ is a solution of (1)–(3) if

$$u \in C^n((0, T), E) \cap C^{n-1}([0, T], E), \qquad u(t) \in D(A) \quad \text{for} \quad 0 < t < T, \qquad g \in E,$$

and all the relations (1)–(3) are satisfied.

Taking into account the results [1]–[4] and the methods of [5], we obtain a universal criterion for the uniqueness of the solution in the problem (1)–(3). A criterion is suitable for all n, q. An important role in the study is played by generalized λ-hyperbolic functions (see [6]) of the form

$$y_{n,k}(t, \lambda) = \frac{t^k}{k!} + \lambda \frac{t^{n+k}}{(n+k)!} + \lambda^2 \frac{t^{2n+k}}{(2n+k)!} + \ldots = \sum_{m=0}^{\infty} \lambda^m \frac{t^{mn+k}}{(mn+k)!}, \tag{4}$$

where $t \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

For $t = 1$ and $\lambda = z$, we obtain from (4) a special family of entire functions

$$Y_{n,k}(z) = \frac{1}{k!} + \frac{z}{(n+k)!} + \frac{z^2}{(2n+k)!} + \cdots = \sum_{m=0}^{\infty} \frac{z^m}{(mn+k)!}, \qquad z \in \mathbb{C}. \tag{5}$$

The functions (5) belong to the class of entire functions of Mittag-Leffler type (see [7]). They are named functions of a variable $z \in \mathbb{C}$ with parameters $\rho > 0$ and $\mu \in \mathbb{C}$ of the form

$$E_\rho(z; \mu) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\rho^{-1} + \mu)}, \tag{6}$$

where Γ is the gamma function. For the family (5) we have $Y_{n,k}(z) = E_{1/n}(z; k+1)$, i.e. $Y_{n,k}(z)$ is an entire function of Mittag-Leffler type with $\rho = 1/n$ and $\mu = k + 1$. Given $n \in \mathbb{N}$, $k \in \mathbb{Z}$, we define a set of zeros

$$\Lambda_{n,k} \equiv \left\{ z \in \mathbb{C}: \ Y_{n,k}(z) = 0 \right\} = \left\{ z_j(n,k), \ j \in J(n,k) \subset \mathbb{Z} \right\} \tag{7}$$

with an indexation $j \in J(n,k)$ without taking multiplicity into account. By the notions (4)–(7) we can establish the next result for the inverse problem (1)–(3).

Theorem 1 *Let A be a linear closed operator in a Banach space E. The inverse problem (1)–(3) with fixed n, q has at most one solution, if and only if none of the numbers*

$$\lambda_j = z_j(n, n-q)/T^n, \qquad z_j(n, n-q) \in \Lambda_{n, n-q},$$

is an eigenvalue of the operator A. Here $\Lambda_{n, n-q}$ is the set of zeros of the functions $Y_{n, n-q}(z)$ from the family (5).

Using the results on the distribution of zeros of Mittag-Leffler functions (see [7]), we can obtain sufficient conditions of the uniqueness of the solution in the inverse problem (1)–(3).

References

[1] I. V. Tikhonov and Yu. S. Eidelâman, The unique solvability of a two-point inverse problem for an abstract differential equation with an unknown parameter. Differ. Equ. **36:8** (2000), 1256–1258. [Russian: Differ. Uravn. Vol. 36. No. 8. 2000. P. 1132–1133].

[2] I. V. Tikhonov and Yu. S. Eidelâman, An inverse problem for a differential equation in Banach space and the distribution of zeros of an entire Mittag-Leffler function. Differ. Equ. **38:5** (2002), 669–677. [Russian: Differ. Uravn. Vol. 38. No. 5. 2002. P. 637–644].

[3] M. Almohamed, A uniqueness criterion of the solution in the inverse problem for an abstract differential equation of second order. In: Sovremennye metodi teorii funktcii i smezhnye problemi. Voronezh. Puplisher: VSU, 2019. P. 19–20. [in Russian].

[4] I. V. Tikhonov and M. Almohamed, Inverse problem for a high order differential equation in Banach space. In: Nekotorye aktualnye problemi sovremennoi matematiki i matematicheskogo obrazovaniya. SPb. Puplisher: Herzen RGPU, 2019. P. 91–95. [in Russian].

[5] I. V. Tikhonov, A generalized Ward problem for abstract differential equations. Differ. Equ. **41:3** (2005), 340–351. [Russian: Differ. Uravn. Vol. 41. No. 3. 2005. P. 325–336].

[6] M. E. Muldoon and A. A. Ungar, Beyond Sin and Cos. Math. Mag. **69:1** (1996), 3–14.

[7] A. Yu. Popov and A. M. Sedletskii, Distribution of roots of Mittag-Leffler functions. J. Math. Sciences, **190:2** (2013), 209–409. [Russian: Sovremennaya Matematika. Fundamentalânye Napravleniya. Vol. 40. 2011. P. 3–171].

Mathematical model involving time-fractional partial differential equations

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Abstract. In this work, we shall investigate a class of time-fractional Keller–Segel models with boundary Dirichlet conditions. We use Faedo–Galerkin method with some compactness arguments to show the existence results of weak solutions. Further, we establish the Mittag–Leffler stability of these weak solutions for the indicated models.

Keywords. Fractional PDE; Keller–Segel model; Faedo–Galerkin method; Weak solution; Mittag–Leffler stability.

MSC2010. 35R11; 34A08; 35B35; 35D30.

References

[1] Y. Zhou, J. Manimaran, L. Shangerganesh, A. Debbouche, Weakness and Mittag-Leffler Stability of Solutions for Time-Fractional Keller-Segel Models, International Journal of Nonlinear Sciences and Numerical Simulation, Vol. 19, no.7-8 (2018) 753–761.

[2] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of the Fractional Differential Equations, Elsevier, Amsterda, 2006.

[3] Y. Li, Y. Chen and I. Podlubny, Mittag–Leffler stability of fractional order nonlinear dynamics systems, Automatica, **45** (2009) 1965–1969.

[4] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, 1999.

Exact Solutions for Generlized Benjamin-Bona-Mahony Equation Using Direct Methods

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Abstract. In this article we construct exact solutions for the generalized Benjamin-Bona-Mahony equation using some direct methods.

Keywords. Nonlinear PDEs, generalized Benjamin-Bona-Mahony equation, Exact solutions

MSC2010. 47D60, 47B10, 47B65.

1 Introduction

The Benjamin–Bona–Mahony (BBM) equation has been studied by Benjamin, Bona, and Mahony in 1972 as the improved KdV equation for the description of long surface gravity waves having a small amplitude. They have also investigated the stability and uniqueness of solutions to the BBM equation [1]. The BBM equation describe many phenomena such as the drift of waves in plasma physics, the propagation of wave in semi-conductors and optical devices, and the behavior of Rossby waves in rotating fluids. We will consider the following form of the generlized Benjamin-Bona-Mahony (GBBM) equation

$$u_t + \alpha u_x + \left(\beta u^\theta + \gamma u^{2\theta} \right) u_x - \delta u_{xxt} = 0.$$

where $\alpha, \beta, \gamma, \delta,$ and θ are constant.

2 Main results

Recently several efficient techniques for solving nonlinear PDEs have been developed, such as the Hirotas bilinear method, the simplified Hirotas method, the Darboux transformation method, the variational iteration method, the exp-function method, the generalized exponential rational function method, and extended tanh-function methods [1, 2,3,4].

Some multi-soliton solutions for the GBBM equation were found when $\gamma = 0$ and $\theta = 1$.

Ghanbari *et al.* found exact solutions for the GBBM equation using the generalized exponential rational function method [5].

In this work we will apply some other direct methods to find new exact solutions for the GBBM equation

$$u_t + \alpha u_x + \left(\beta u^\theta + \gamma u^{2\theta}\right) u_x - \delta u_{xxt} = 0.$$

for several values of $\alpha, \beta, \gamma, \delta,$ and θ.

Acknowledgments

References

[1] Benjamin, T.B.; Bona, J.L.; Mahony, J.J. Model equations for long waves in nonlinear dispersive systems.Philos. Trans. R. Soc. Lond. Ser. A 1972, 272, 47.

[2] Wazwaz, A.M. Partial Differential Equations and Solitary Waves Theory; Higher Education Press, Springer: Berlin, Germany, 2009.

[3] W. Hereman and A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, Math. Comput. Simulation, 4313–27, (1997).

[4] Fan, E. Extended tanh-function method and its applications to nonlinear equations. Phys. Lett. A 2000, 277, 212–218.

[5] Behzad Ghanbari, Dumitru Baleanu, and Maysaa Al Qurashi. New Exact Solutions of the Generalized Benjamin–Bona–Mahony Equation, Symmetry 2019, 11, 20.

Local dynamics of the dispersion version of the Cahn-Hilliard equation

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Abstract. The dispersion version of the Cahn-Hilliard equation with the periodic boundary conditions is considered. The existence of invariant manifold formed by t periodic solutions is proved. Conditions, where it is local attractor are found. At the same time it is proved that all solutions belonging to this attractor are unstable.

Keywords. boundary value problem;periodic solutions; local attractor; instability; asymptotic formulas.

MSC2010. 35B32,35B41

1   Introduction

The equation

$$u_t + u_{xxxx} + bu_{xx} + au_{xxx} = c(u^2)_{xx} + (u^3)_{xx}, \tag{1}$$

where $a, b, c \in R$ is called the Cahn–Hilliard equation or the generalized Cahn–Hilliard equation [1-4]. Generally, equation (1) was considered for $a = 0$. If $a \neq 0$, then this equation includes a term that takes into account dispersion effects.

Let $a \neq 0$. In this case, we study equation (1) together with periodic boundary conditions

$$u(t, x + 2\pi) = u(t, x). \tag{2}$$

The boundary value problem (1)–(2) has a family of homogeneous equilibrium states $u(t, x) = \beta \in R$. Equilibrium state $u(t, x) = \alpha \, (\alpha \in R)$ is stable, if $b_\alpha = b - 2c\alpha - 3\alpha^2 < 1$ and unstable, if $b_\alpha > 1$.

2   Main results

Let $b_\alpha = 1 + \gamma\varepsilon, \varepsilon \in (0, \varepsilon_0), \gamma = 1$ or -1 and is chosen in the process of proof. Suppose that $D = c^2 + 3(b - 1) > 0$.

Theorem 1 *There exists such $\varepsilon_0 > 0$, that for any $\varepsilon \in (0, \varepsilon_0)$ the boundary value problem (1)–(2) has t periodic solutions for which the asymptotic representations are valid*

$$u(t, x, \varepsilon) = \alpha(\varepsilon) + 2\varepsilon^{1/2}\rho\cos\psi + \varepsilon\rho^2\nu(2\cos 2\psi - a\sin 2\psi) + o(\psi), \tag{3}$$

where $\alpha(\varepsilon)$ is equal to one of two values

$$\alpha(\varepsilon) = -\frac{c}{3} \pm \frac{\sqrt{D}}{3}(1 + \frac{\gamma}{2D}\varepsilon) + o(\varepsilon),\ \psi = \sigma(\varepsilon)t + x + \varphi_0,\ \sigma(\varepsilon) = a + l_2\rho^2\varepsilon,$$

$$\nu = -\frac{4}{3(4+a^2)}(c + 3\alpha),\ \rho = \sqrt{-\frac{\gamma}{l_1}},\ l_1 = -3 + \frac{8D}{3(4+a^2)},\ l_2 = \frac{4Da}{3(4+a^2)},\ \varphi_0 \in R.$$

Let $C(\varepsilon)$ is a cycle generated by periodic solutions and $V(\alpha(\varepsilon)) = \underset{\varepsilon\in(0,\varepsilon_0)}{\cup} C(\varepsilon)$. This invariant manifold $V(\alpha(\varepsilon))$ for solutions of the boundary value problem (1)–(2) $V(\alpha(\varepsilon))$ is a local attractor, if $l_1 < 0(\gamma = 1)$ and it has a saddle type if $l_1 > 0(\gamma = -1)$.

Theorem 2 *Periodic solutions* (1) *are unstable in the sense of A.M.Lyapunov definition.*

They are also unstable when $V(\alpha(\varepsilon))$ is an attractor, i.e. for $l_1 < 0$. If $l_1 < 0$ we obtain a local attractor with the following properties:

1) all solutions on it are periodic functions of variable t;

2) all these solutions are unstable in the sense of A.M.Lyapunov definition.

The proof of Theorems 1,2 uses the method of invariant manifolds in combination with the apparatus of normal Poincare forms and asymptotic methods of analysis [4,5,6].

Acknowledgments

References

[1] J.W. Cahn and J.E. Hilliard, Free Energy of a Nonuniform System I. Interfacial Free Energy, J. Chem. Phys. **28** (1968), 258–267.

[2] A. Podolny, M.A. Zaks, B.Y. Rubinstein, A.A. Golovin and ï¿½.ï¿½. Nepomnyahchy, Dynamics of domain walls governed by the convective Cahn–Hilliard equation, Physica D: Nonlinear Phenomena **201** (2005), 291–305.

[3] A. Novick-Cohen and L.ï¿½. Segal, Nonlinear aspects of the Cahn–Hilliard, Physica D: Nonlinear Phenomena **10** (1984), 277–298.

[4] A. Kulikov and D. Kulikov, Spatially ingomogeneous solitions in two boundary value problems for the Cahn–Hilliard equation, Nauchniye vedomosti Belgorodskogo universiteta. Seria: Mathematica. Fizika **51** (2019), 21–32.

[5] A. Kulikov and D. Kulikov, Local bifurcations in the Cahn – Hilliard and Kuramoto – Sivashinsky equations and in their generalizations, Computational Mathematics and Mathematical Physics **59** (2019), 630-643.

[6] A. Kulikov and D. Kulikov, Local bifurcations in the periodic boundary value problem for the generalized Kuramoto-Sivashinsky equation, Automation and Remote Control **78** (2017), 1955-1966.

Solvability of fractional α–Voigt model with memory along the motion trajectories

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Abstract. We establish the existence of weak solutions to one fractional α–Voigt type model of viscoelastic fluid. This model takes into account a memory along the motion trajectories. Also we show the convergence of sequence of solutions of the α–models family to the weak solution of the original initial-boundary value problem. The investigation is based on the theory of regular Lagrangean flows, approximation of the problem under consideration and the following passage to the limit.

Keywords. fluid dynamics; non-Newtonian fluid; α-model; Voigt model; fractional derivative; existens theorem.

MSC2010. 35Q35; 76A05.

1 Introduction

Let Ω be bounded domain in the space $\mathbb{R}^n$, $n = 2, 3$, with a smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial v}{\partial x_i} - \mu_0 \Delta v - \mu_1 \frac{1}{\Gamma(1-\lambda)} \mathrm{Div} \int_0^t (t-s)^{-\lambda} \mathcal{E}(v)(s, z(s;t,x)) \, ds + \nabla p = f; \quad (1)$$

$$z(\tau; t, x) = x + \int_t^\tau v(s, z(s;t,x)) \, ds, \quad t, \tau \in [0, T], \quad x \in \overline{\Omega}; \quad (2)$$

$$u = (I - \alpha^2 \Delta)^{-1} v, \quad t \in [0, T], \quad x \in \Omega; \quad (3)$$

$$\mathrm{div}\, v(t, x) = 0, \quad t \in [0, T], \quad x \in \Omega; \qquad v|_{\partial\Omega} = 0, \qquad v|_{t=0} = v_0. \quad (4)$$

Here, $v = (v_1(t,x), \ldots, v_n(t,x))$ is un unknown vector-valued velocity function of particles in the fluid, $u = (u_1(t,x), \ldots, u_n(t,x))$ is a modified velocity vector function defined by the equality (3), $p = p(t,x)$ is un unknown pressure, $f = f(t,x)$ is the external force. The divergence $\mathrm{Div}\, C$ of the tensor $C = (c_{ij}(t,x))$ is the vector with with coordinates $(\mathrm{Div}\, C)_j = \sum_{i=1}^{n}(\partial c_{ij}/\partial x_i)$;

$$\mathcal{E}(v) = (\mathcal{E}_{ij}(v)), \quad \mathcal{E}_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad i, j = \overline{1, n},$$

is the strain-rate tensor; $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} \, dt$ is the Euler's Gamma function; $\mu_0 > 0$ is the fluid viscosity, $\mu_1 \geq 0$, $\lambda > 0$ and $\alpha > 0$ are positive constance, v_0 is given function.

The system (1)–(4) describes the motion of viscoelastic fractional α–Voigt model with memory along the motion trajectories. To get acquainted with the model can be at [1]. Note the papers [2]–[3] with similar results on α-models of non-Newtonian fluid dynamics.

We need the following space in order to define a weak solution and give the main theorems for the problem (1)–(4):

$$W = \{v \in L_2(0,T;V^1) \cap L_\infty(0,T;V^0), \quad v' \in L_{4/3}(0,T;V^{-1})\}.$$

By $\Delta_\alpha : V^\beta \to V^{\beta-2}$ we denote the operator $\Delta_\alpha = J + \alpha^2 A$, where $J = PI$. The operator Δ_α is invertible. Apply the Leray projection $P : L_2(\Omega) \to V^0$ to equality $v = (I - \alpha^2\Delta)u$ for $\beta = 3$. Since $v(t) \in V^1$ for almost all $t \in [0,T]$, then $v = \Delta_\alpha u$. Therefore, $u = (J + \alpha^2 A)^{-1}v = \Delta_\alpha^{-1}v$ and $u(t) \in V^3$ for almost all $t \in [0,T]$.

Definition 1 *Let $f \in L_2(0,T;V^{-1})$ and $v_0 \in V^0$. The function $v \in W$ is call a weak solution to the initial-boundary value problem (1)-(4), if it satisfies for all $\varphi \in V^1$ and almost all $t \in (0,T)$ the equality*

$$\langle v', \varphi \rangle - \int_\Omega \sum_{i,j=1}^n (\Delta_\alpha^{-1}v)_i v_j \frac{\partial \varphi_j}{\partial x_i}\, dx + \mu_0 \int_\Omega \nabla v : \nabla \varphi\, dx +$$

$$\frac{\mu_1}{\Gamma(1-\lambda)}\Big(\int_0^t (t-s)^{-\lambda}\mathcal{E}(v)(s, z(v)(s;t,x))\, ds, \mathcal{E}(\varphi)\Big) = \langle f, \varphi \rangle,$$

and the inital condition $v|_{t=0} = v_0$. Here $z(v)$ is the regular Lagrangean flows associated to the function v.

Theorem 1 *Let $f \in L_2(0,T;V^{-1})$ and $v_0 \in V^0$. Then viscoelastic fractional α–Voigt model with memory along the motion trajectories (1)–(4) has at least one weak solution $v \in W$.*

Theorem 2 *Let the conditions of Theorem 1 are hold. Moreover, if we consider a family of α–models that depends on the parameter α_m, then when $\alpha_m \to 0$ there exists a sequence of solutions v_m of the α–models family, which converges to the weak solution $v \in W$ of the original initial-boundary value problem.*

Acknowledgments

References

[1] V. Zvyagin and V. Orlov, Weak solvability of fractional Voigt model of viscoelasticity, Discrete and Continuous Dnamical Systems–A. **38**:12 (2018), 6327–6350.

[2] A.V. Zvyagin, Solvability of Thermoviscoelastic Problem for Leray Alpha-Model, Russian Mathematics. **60**:10 (2016), 59–63.

[3] A.V. Zvyagin, Optimal Feedback Control for Leray and Navier-Stokes Alpha Models, Doklady Mathematics. **99**:3 (2019), 299-302.

Degenerate Fractional Order Differential Equations in Banach Spaces and Applications

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Abstract. We consider a class of semilinear equations in Banach spaces, which are not solved with respect to the highest order fractional Riemann — Liouville derivative, with a pair of operators, generating analytic in a sector resolving family of operators. The unique solvability theorems for the Showalter — Sidorov problem to such equation are proved. They are applied to the unique solvability study of initial boundary value problems for partial differential equations and systems of equations, not solved with respect to the fractional Riemann — Liouville derivative.

Keywords. fractional derivative; degenerate evolution equation; Showalter — Sidorov type problem; sectorial pair of operators.

MSC2010. 34G20; 35R11.

Let $L, M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$ (linear closed and densely defined operators) have domains D_L, D_M, $\ker L \neq \{0\}$. Let us consider the equation

$$D_t^\alpha L x(t) = M x(t) + N(t, D_t^{\alpha-m} x(t), D_t^{\alpha-m+1} x(t), \ldots, D_t^{\alpha-1} x(t)) + f(t), \qquad (1)$$

where D_t^α is the Riemann — Liouville derivative, X is an open set in $\mathbf{R} \times \mathcal{X}^m$, $N : X \to \mathcal{Y}$, $f[t_0, T] \to \mathcal{Y}$. Equation (1) is called degenerate, because it is supposed that $\ker L \neq \{0\}$.

A function $x : [t_0, t_1] \to D_M \cap D_L$ is called a solution of equation (1) on a segment $[t_0.t_1]$, if $g_{m-\alpha} * Lx \in C^m((t_0, t_1]; \mathcal{Y})$, $Mx \in C((t_0, t_1]; \mathcal{Y})$ and equality (1) is valid. A solution x of (1) is called a solution to the Showalter — Sidorov type problem

$$D_t^{\alpha-m+k} L x(t_0) = y_k, \quad k = 0, 1, \ldots, m-1, \qquad (2)$$

for equation (1), if $g_{m-\alpha} * Lx \in C^{m-1}([t_0, t_1]; \mathcal{X})$ and Lx satisfies conditions (2).

By $\rho^L(M)$ the set of $\mu \in \mathbf{C}$ is denoted, for which the mapping $\mu L - M : D_L \cap D_M \to \mathcal{Y}$ is injective, and $R_\mu^L(M) := (\mu L - M)^{-1} L \in \mathcal{L}(\mathcal{X})$ (linear bounded operator), $L_\mu^L(M) := L(\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y})$.

Let $L, M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$. A pair of operators (L, M) belongs to the class $\mathcal{H}_\alpha(\theta_0, a_0)$, if the following two conditions are valid:

(i) there exist $\theta_0 \in (\pi/2, \pi)$ and $a_0 \geq 0$, such that for all $\lambda \in S_{\theta_0, a_0}$ we have $\lambda^\alpha \in \rho^L(M)$;

(ii) for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists a constant $K = K(\theta, a) > 0$, such that for all $\mu \in S_{\theta, a}$ $\max\{\|R_{\mu^\alpha}^L(M)\|_{\mathcal{L}(\mathcal{X})}, \|L_{\mu^\alpha}^L(M)\|_{\mathcal{L}(\mathcal{Y})}\} \leq \frac{K(\theta, a)}{|\mu^{\alpha-1}(\mu-a)|}$.

The unique solvability of problem (2) to the linear equation (1) ($N \equiv 0$) with $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$ is researched in [1].

Denote $\ker R_\mu^L(M) = \mathcal{X}^0$, $\ker L_\mu^L(M) = \mathcal{Y}^0$. By $\mathcal{X}^1$ ($\mathcal{Y}^1$) the closure of the subspace $\mathrm{im} R_\mu^L(M)$ ($\mathrm{im} L_\mu^L(M)$) in the norm of $\mathcal{X}$ ($\mathcal{Y}$) is denoted. The projection on the subspace $\mathcal{X}^1$ ($\mathcal{Y}^1$) along $\mathcal{X}^0$ ($\mathcal{Y}^0$) is denoted by P (Q). By L_k (M_k) we denote the restriction L (M) on $D_{L_k} := D_L \cap \mathcal{X}^k$ ($D_{M_k} := D_M \cap \mathcal{X}^k$), $k = 0, 1$. Introduce also the denotations $S = L_1^{-1} M_1 : D_S \to \mathcal{X}^1$, $D_S = \{x \in D_{M_1} : M_1 x \in \mathrm{im} L_1\}$; $T = M_1 L_1^{-1} : D_T \to \mathcal{Y}^1$, $D_T = \{y \in \mathrm{im} L_1 : L_1^{-1} y \in D_{M_1}\}$.

Theorem 1 [2]. *Let Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ be reflexive, $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$. Then*

(i) $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, $\mathcal{Y} = \mathcal{Y}^0 \oplus \mathcal{Y}^1$;

(ii) *the projection P (Q) on the subspace $\mathcal{X}^1$ ($\mathcal{Y}^1$) along $\mathcal{X}^0$ ($\mathcal{Y}^0$) has the form $P = s\text{-} \lim\limits_{n \to \infty} n R_n^L(M)$ ($Q = s\text{-} \lim\limits_{n \to \infty} n L_n^L(M)$);*

(iii) $L_0 = 0$, $M_0 \in Cl(\mathcal{X}^0; \mathcal{Y}^0)$, $L_1, M_1 \in Cl(\mathcal{X}^1; \mathcal{Y}^1)$;

(iv) *there exist inverse operators $L_1^{-1} \in Cl(\mathcal{Y}^1; \mathcal{X}^1)$, $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$;*

(v) *if $L_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, or $M_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, then $S \in Cl(\mathcal{X}^1)$;*

(vi) *if $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, or $M_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, then $T \in Cl(\mathcal{Y}^1)$.*

Theorem 2 *Let Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ be reflexive, $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$, $L_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, or $M_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, X be an open set in $\mathbf{R} \times \mathcal{X}^m$, $N : X \to \mathcal{Y}^1$, $L_1^{-1} N \in C(X; D_{L_1^{-1} M_1})$ be locally Lipschitzian in $\overline{x}$, $f \in C([t_0, T]; \mathcal{Y})$ for some $T > t_0$, $L_1^{-1} Q f \in C([t_0, T]; D_{L_1^{-1} M_1})$, $g_{m-\alpha} * M_0^{-1}(I - Q) f \in C^{m-1}([t_0, T]; \mathcal{X})$, $y_k \in L[D_{L_1^{-1} M_1}]$ for all $k = 0, 1, \ldots, m - 1$. Then for some $t_1 \in (t_0, T]$ there exists a unique solution of problem (1), (2) on the segment $[t_0, t_1]$.*

Theorem 3 *Let Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ be reflexive, $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, X be an open set in $\mathbf{R} \times \mathcal{X}^m$, $N \in C(X; D_{M_1 L_1^{-1}})$ be locally Lipschitz continuous in $\overline{x}$, $f \in C([t_0, T]; \mathcal{Y})$ for some $T > t_0$, $Q f \in C([t_0, T]; D_{M_1 L_1^{-1}})$, $g_{m-\alpha} * M_0^{-1}(I - Q) f \in C^{m-1}([t_0, T]; \mathcal{X})$, $y_k \in L[D_M]$, $k = 0, 1, \ldots, m - 1$. Then for some $t_1 \in (t_0, T]$ there exists a unique solution of problem (1), (2) on the segment $[t_0, t_1]$.*

The theorems are applied to the unique solvability study of initial boundary value problems for partial differential equations and systems of equations, not solved with respect to the fractional Riemann — Liouville derivative.

Acknowledgments

References

[1] Fedorov V.E., Avilovich A.S., The Cauchy type problem for degenerate equation with the Riemann — Liouville derivative in the sectorial case, Siberian Mathematical Journal, **60** (2019), No 2.

[2] V. E. Fedorov, E. A. Romanova and A. Debbouche, Analytic in a sector resolving families of operators for degenerate evolution fractional equations, J. Math. Sc. **228** (2018), 380–394.

Linear Inverse Problem for a Class of Degenerate Fractional Differential Equations in Banach Spaces

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Abstract. We consider linear inverse problems for differential equations in Banach spaces with a degenerate operator at the fractional order Gerasimov — Caputo derivative. The criteria of the unique solvability is obtained for the inverse problems to a class of fractional order degenerate evolution equations with a relatively bounded pair of operators.

Keywords. inverse problem; fractional order Gerasimov — Caputo derivative; degenerate evolution equation.

MSC2010. 34R30; 35R11; 34K29.

Let $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, $\mathcal{L}(\mathcal{X};\mathcal{Y})$ be the Banach space of linear continuous operators, $\mathcal{Cl}(\mathcal{X};\mathcal{Y})$ be the set of all linear closed densely defined operators. Let $L \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $\ker L \neq \{0\}$, $M \in \mathcal{Cl}(\mathcal{X};\mathcal{Y})$ has a domain of the definition D_M.

Consider degenerate fractional order differential equation

$$D_t^\alpha L x(t) = M x(t) + B(t)u + y(t), \quad t \in [0, T]. \tag{1}$$

Here D_t^α is the fractional Gerasimov —Caputo derivative, $B : [0, T] \to \mathcal{L}(\mathcal{U};\mathcal{Y})$, $y : [0, T] \to \mathcal{Y}$, $u \in \mathcal{U}$, $T > 0$. Endow equation (1) by the Cauchy conditions

$$x^{(k)}(0) = x_k, \quad k = 0, 1, \ldots, m - 1, \tag{2}$$

and by the overdetermination condition

$$\int_0^T x(t)d\mu(t) = x_T. \tag{3}$$

The scalar-valued function μ has a bounded variation on the segment $[0, T]$. The integral in condition (3) is understood as vector integral of Riemann — Stieltjes. The elements x_k, $k = 0, 1, \ldots, m - 1$, x_T are known.

For $\alpha, \beta > 0$ we shall use the Mittag-Leffler function $E_{\alpha,\beta}(v) = \sum_{n=0}^{\infty} \frac{v^n}{\Gamma(\alpha n + \beta)}$. We also introduce the notations $\rho^L(M) = \{\lambda \in \mathbf{C} : (\lambda L - M)^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})\}$, $\sigma^L(M) = \mathbf{C} \setminus \rho^L(M)$, $R_\lambda^L(M) = (\lambda L - M)^{-1}L$, $L_\lambda^L(M) = L(\lambda L - M)^{-1}$.

At $p \in \mathbf{N}_0$ an operator M is called (L, σ)-*bounded*, if $\sigma^L(M) \subset \{\lambda \in \mathbf{C} : |\lambda| \leq a\}$ for some $a > 0$. Under the condition of (L, p)-boundedness of the operator M there exist projections

$P = \frac{1}{2\pi i} \int_\gamma R_\lambda^L(M) d\lambda \in \mathcal{L}(\mathcal{X})$, $Q = \frac{1}{2\pi i} \int_\gamma L_\lambda^L(M) d\lambda \in \mathcal{L}(\mathcal{Y})$, where $\gamma = \{\lambda \in \mathbf{C} : |\lambda| = a + 1\}$.

Denote by $\mathcal{X}^0$ ($\mathcal{Y}^0$) the kernel $\ker P$ ($\ker Q$), and by $\mathcal{X}^1$ ($\mathcal{Y}^1$) the image $\mathrm{im} P$ ($\mathrm{im} Q$) of the projection P (Q). Let M_k (L_k) be the restriction of the operator M (L) on $D_{M_k} = \mathcal{X}^k \cap D_M$ ($\mathcal{X}^k$), $k = 0, 1$. Under the condition of (L, σ)-boundedness of the operator M we have that $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, $\mathcal{Y} = \mathcal{Y}^0 \oplus \mathcal{Y}^1$; $L_k \in \mathcal{L}(\mathcal{X}^k; \mathcal{Y}^k)$, $k = 0, 1$, $M_0 \in \mathcal{Cl}(\mathcal{X}^0; \mathcal{Y}^0)$, $M_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$; there exist operators $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$. An operator M is called (L, p)-bounded [1], if it is (L, σ)-bounded, $H^p \neq 0$, $H^{p+1} = 0$, where $H = M_0^{-1} L_0$.

Denote

$$\chi(L_1^{-1} M_1) := \int_0^T d\mu(t) \int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(L_1^{-1} M_1 (t - s)^\alpha) L_1^{-1} Q B(s) ds \in \mathcal{L}(\mathcal{U}; \mathcal{X}^1),$$

$$\psi(L_1^{-1} M_1) := P x_T - \int_0^T \sum_{k=0}^{m-1} t^k E_{\alpha, k+1}(L_1^{-1} M_1 t^\alpha) P x_k d\mu(t) -$$

$$- \int_0^T d\mu(t) \int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(L_1^{-1} M_1 (t - s)^\alpha) L_1^{-1} Q y(s) ds \in \mathcal{X}^1.$$

Theorem 1 *Let M (L, p)-bounded, $B \in C([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $y \in C([0, T]; \mathcal{Y})$, $\mu : [0, T] \to \mathbb{R}$ has a bounded variation, $(D_t^\alpha H)^n M_0^{-1} Q_0 B \in C([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{X}))$, $(D_t^\alpha H)^n M_0^{-1} Q_0 y \in C([0, T]; \mathcal{X})$ for all $n = 0, 1, \ldots, p$, $\chi(L_1^{-1} M_1)^{-1} \in \mathcal{L}(\mathcal{X}^1; \mathcal{U})$, $x_k \in \mathcal{X}$, $k = 0, 1, \ldots, m - 1$, $x_T \in \mathcal{X}$ such that*

$$P_0 x_k = -D_t^k|_{t=0} \sum_{n=0}^{p} (D_t^\alpha H)^n M_0^{-1} Q_0 [B(t) \chi(L_1^{-1} M_1)^{-1} \psi(L_1^{-1} M_1) + y(t)], \ k = 0, 1, \ldots, m - 1,$$

and $P_0 x_T = -\int_0^T \sum_{n=0}^{p} (D_t^\alpha H)^n M_0^{-1} Q_0 [B(t) \chi(L_1^{-1} M_1)^{-1} \psi(L_1^{-1} M_1) + y(t)] d\mu(t)$. Then there exists a unique solution of problem (1)–(3), moreover

$$\|u\|_\mathcal{U} \leq C \left(\sum_{k=0}^{m-1} \|P x_k\|_\mathcal{X} + \|P x_T\|_\mathcal{X} + \|Q y\|_{C([0, T]; \mathcal{Y})} \right).$$

Theorem 2 *Let M (L, p)-bounded, $B \in C([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $\mathrm{im} B \subset \mathcal{Y}^1$, $y \in C([0, T]; \mathcal{Y}^1)$, function $\mu : [0, T] \to \mathbb{R}$ has a bounded variation. Then problem (1)–(3) is uniquely solvable for any $x_k \in \mathcal{X}^1$, $k = 0, 1, \ldots, m - 1$, $x_T \in \mathcal{X}^1$ if and only if $\chi(L_1^{-1} M_1)^{-1} \in \mathcal{L}(\mathcal{X}^1; \mathcal{U})$. Moreover*

$$\|u\|_\mathcal{U} \leq C \left(\sum_{k=0}^{m-1} \|x_k\|_\mathcal{X} + \|x_T\|_\mathcal{X} + \|y\|_{C([0, T]; \mathcal{Y})} \right).$$

Acknowledgments

References

[1] Fedorov, V.E., Gordievskikh, D.M., Plekhanova, M.V.: Equations in Banach spaces with a degenerate operator under a fractional derivative. Differential Equations **51** (2015), 1360–1368.

Free boundary problems and homogenization on non-periodic structures

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Abstract. The proposed talk is focused at the study of mathematical modelling of free boundary problems for non-periodic structures, like the growth of biological tissues cultivated on artificial nutrient media or uranium mining and cleaning a bottom hole zone of oil (gas) wells by in-situ leaching. Corresponding mathematical models consist of free boundary problems with non-periodic structure. We suggest a new method for homogenization of the problem at the microscopic level based on the special approximation and fixed point theorem.

Keywords. Free boundary problems, two-scale convergence, homogenization of non-periodic structures.

MSC2010. 76R10

1 Introduction

We consider free boundary problems for non-periodic structures, like the growth of biological tissues cultivated on artificial nutrient media or uranium mining. Both physical processes have almost coincident mathematical models and by this reason we concentrate ourself on the uranium mining. Uranium mining by leaching are very important economic problems. Real uranium deposits or hydrocarbon reservoirs are complicated geological heterogeneous bodies. An understanding of the movement of fluids media and dissolution mechanism of rocks by acids is therefore fundamental. Currently the leaching of rocks describes by the large range of mathematical models describing the physical processes at the macroscopic level (see [1] and references there in). Namely, in these models at each point of a continuous medium there are the solid skeleton and the liquid in pores. Such models are also called macroscopic (phenomenological) models.

2 Main results

All existing phenomenological models describing these processes are based on the same principle. Fluid dynamics is usually described by Darcy system of filtration or some its modification. The migration of acid and products of chemical reactions are described by somehow postulated convection-diffusion equations for the appropriate concentrations. The main point of these postulates is the form of the coefficients of differential equations. Right here we see a lot of variety depending on the tastes and preferences of the authors. It is quite explainable because

the basic mechanism of the physical process is concentrated on the unknown (free) boundary between the pore space and the solid skeleton and exactly this basic mechanism is not prescribed in the proposed macroscopic models. Dissolution of rocks takes place exactly there. It changes the concentration of the injected acid and the geometry of the pore space and free boundary and creates the flow of products of chemical reactions inward the pore space. All these principally important changes occur at the microscopic scale, corresponding to a mean size of the pores or cracks in rocks. At the same time, any of the proposed macroscopic mathematical models operate with other scales (much larger) and just do not "see" neither free boundary nor peculiarities of the interaction of the acid with rocks. This explains such a variety of macroscopic mathematical models. The authors of these models simply have no the exact method, describing the physical processes at the microscopic level on the basis of the fundamental laws of continuum mechanics and chemistry. They also have no the opportunity to take into account the microstructure in macroscopic models. So they have to be limited by some plausible contemplative considerations. R. Burridge and J. B. Keller [2] and E. Sanchez-Palencia [3] were the first to state explicitly that mathematical models for filtrations and acoustics must be derived rigorously from the microstructure. To this end, one should: (a) describe the physical process under consideration most precisely at microscopic level (exact model), (b) distinguish a set of small parameters, (c) derive the macroscopic model as the asymptotic limit of the exact model.

Various particular cases of the exact models for filtration and acoustics have been intensively investigated by many authors: Buchanan, J.L., Gilbert, R.P. [4] and others. Most systematically, using Nguenseng's two-scale convergent method [5], the exact models has been studied by Meirmanov [6]. But all these problems at the microscopic level had a periodic structure that did not depend on time. For in-situ leaching the situation is completely different. Acid in the pores dissolves the solid skeleton and changes its structure. In this way appear free boundaries and non-periodic structures. For the special form of the pore space, given by the function $r(x,t)$ we consider special approximations $r^\varepsilon(x,t)$ depending on the fast variable $\dfrac{x}{\varepsilon}$ and defining the pore space. Then we construct approximate models consisting of Stokes equations for liquid velocity $v^\varepsilon(x,t)$ and pressure $p^\varepsilon(x,t)$ in pores and diffusion-convection equation for concentration $c^\varepsilon(x,t)$ of the acid in the given domain. Here we did not use additional free boundary condition. Separately we consider formal homogenization of the exact model and show that final result is the same for both exact and approximate models. The homogenization of the additional free boundary condition gives us the operator $\overline{r} = F(r)$ and fixed points of this operator defines solution for the desired mathematical models at the macroscopic level.

References

[1] N. Kalia and V. Balakotaiah, Effect of medium heterogeneities on reactive dissolution of carbonates. Chemical Engineering Science. **64**, (2009), 376–390.

[2] R. Burridge and J. B. Keller, Poroelasticity equations derived from microstructure. Journal of Acoustic Society of America. **70**, (1981), 1140–1146.

[3] E. Sanchez-Palencia, Non-Homogeneous Media and Vibration Theory. Lecture Notes in Physics. **129**, (1980).

[4] J.L. Buchanan and R. P. Gilbert, Transition loss in the farfield for an ocean with a Biot sediment over an elastic substrate. ZAMM. **77**, (1997), 121–135.

[5] G. Nguenseng, A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. **20**, (1989), 608–623. 2014.

[6] A. Meirmanov, *Mathematical models for poroelastic flows*, Atlantic Press, Paris, 2014.

Chirped Bright Solitons of Nonlinear Schrödinger Models in Optical Fibers using Sech Scheme

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Abstract. This work presents new types of bright soliton solutions with nonlinear chirp for a derivative nonlinear Schrödinger models. These models are Chen-Lee-Liu equation, FokasâLenells equation, and Lakshmanan-Porsezian-Daniel equation. Sech scheme retrieves plane wave solutions to the models. The existence criteria of these waves are also presented

Keywords. Solitary wave solutions, Sech method, FokasâLenells equation, Chen-Lee-Liu equation, Lakshmanan-Porsezian-Daniel equation.

MSC2010. 35B10.

1 Introduction

This paper studies pulse propagation engineering through optical fibers and PCF with a newly established equations that are Chen-Lee-Liu equation[14], FokasâLenells equation [15], and Lakshmanan-Porsezian-Daniel equation [16]. It has been studied to obtain soliton solutions of various kinds by using the complexâamplitude ansatz and other mechanisms. This paper will apply sech method to retrieve soliton solutions. It will be established that bright combo solitons.

1.1 Governing Equation of perturbed Chen-Lee-Liu equation

The Chen-Lee-Liu (CLL) equation of the form [14]:

$$iq_t + aq_{xx} + ib|q|^2 q_x = 0 \tag{1}$$

1.2 Governing Equation of perturbed FokasâLenells Equation FLE

The dimensionless form of perturbed FLE that has been proposed takes the form [15]:

$$iq_t + a_1 q_{xx} + a_2 q_{xt} + b|q|^2 q = i[\sigma|q|^2 q_x + \alpha q_x + \lambda(|q|^{2m} q)_x + \mu(|q|^{2m})_x q] \tag{2}$$

1.3 Governing Equation of perturbed Lakshmanan-Porsezian-Daniel equation

The Lakshmanan-Porsezian-Daniel LPD model in its dimensionless form is to be studied which is given by [16]:

$$iq_t + aq_{xx} + bq_{xt} + F(|q|^2)q = \sigma q_{xxxx} + \alpha(q_x)^2 q^* + \beta|q_x|^2 q + \gamma|q|^2 q_{xx} + \nu q^2 q_{xx}^* + \delta|q|^4 q. \tag{3}$$

2   Sech function method [Bright Solution]

The solutions of many nonlinear equations can be expressed in the form [17]:

$$u(\xi) = Asech^{\ddot{I}}(\sigma\xi) \tag{4}$$

References

[1] A. Biswas , D. Milovic. Traveling wave solutions of the non-linear Schrödinger's equation in non-Kerr law media. Communications in Nonlinear Science and Numerical Simulation. Volume 14, 1993â1998. (2009).

[2] A. J. M. Jawad, M. D. Petkovic , A. Biswas. Modified simple equation method for evolution equations. Applied Mathematics and Computation. Volume 217, Number 2, 869â877. (2010).

A generalized variational problem of point-to-point ICP

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Abstract. Two most applicable variants of the ICP error metrics are point-to-point and point-to-plane. It is known that the point-to-plane metric has been shown to perform better than the point-point metric in terms of accuracy and convergence rate. A closed-form solution to the point-to-plane case for orthogonal transformations is an open problem whereas the point-to-point variational subproblem has closed form solution. In the proposed paper we describe a new variant of the point-to-point variational subproblem. The proposed algorithm combines information on points and normal vectors. With the help of computer simulation, the proposed method is compared with known algorithms for the searching of optimal orthogonal transformation.

Keywords. Iiterative closest points (ICP); rigid ICP; point-to-point; point-to-plane; orthogonal transformations; surface reconstruction.

MSC2010. 49K30.

1 Introduction

The algorithm ICP (Iterative Closest Points) is the main one among the methods of alignment of three-dimensional models based on the use of exclusively geometric characteristics. The alignment is a geometric transformation which is the best way relative to the norm of L_2, connects two data sets (clouds) of points in R^3. The algorithm is widely used to record data obtained using 3D scanners. There are two main approaches to the choice of error metrics for pairs of points. The first approach (point-to-point) [1] uses the distance between the elements of the pair in R^3, the second approach (point-to-plane) [2] takes into account the distance between the point of the first cloud and the tangent plane to the corresponding point of the second cloud. For orthogonal transformations, the solution of this problem in closed form was obtained by Horn in [3]. The key element of the ICP algorithm is the search for an orthogonal or affine transformation, best in the sense of a quadratic metric combining two clouds of points with a given correspondence between the points (a variational subproblem of the ICP algorithm). In this paper is proposed the approach to the ICP variational subproblem that utilyzes information both points coordinates and normal vectors of point clouds. The proposed approach is intermediate between the standard point-to-point and the point-to-plane methods.

2 Main results

Let $P = \{p^1, \ldots, p^s\}$ be a template point cloud, and $Q = \{q^1, \ldots, q^s\}$ be a target point cloud in R^3. Suppose that the relationship between points in P and Q is given in such a manner that for each point p^i exists a corresponding point q^i. Denote by $S(P)$ and $S(Q)$ surfaces, constructed from the clouds P and Q respectively, by $T_P(p^i)$ and $T_Q(q^i)$ denote tangent planes of $S(P)$ at point p^i and $S(Q)$ at point q^i respectively.

The ICP algorithm is commonly considered as a geometrical transformation for rigid objects mapping P to Q

$$Rp_i + T, \tag{1}$$

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, \quad p^i = (p_1^i \ p_2^i \ p_3^i)^t, \quad q^i = (q_1^i \ q_2^i \ q_3^i)^t, \tag{2}$$

where R is a rotation (orthogonal) matrix, T is a translation vector, $i = 1, \ldots, s$. Denote by $J_h(R, T)$ the following functional: $J_h(R, T) = \sum_{i=1}^{s} \parallel Rp^i + T - q^i \parallel^2$. The standard point-to-point variation problem is formulated as follows:

$$arg \min_{R, T} J_h(R, T). \tag{3}$$

Denote by n_p^i and n_q^i the normal vectors to planes $T_P(p^i)$ and $T_Q(q^i)$ respectively, $i = 1, \ldots, s$. Consider the functional $J_g(R, T)$

$$J_g(R, t) = \sum_{i=1}^{s} \parallel Rp^i + T - q^i \parallel^2 + \lambda \sum_{i=1}^{s} \parallel Rn_p^i - n_q^i \parallel^2, \tag{4}$$

where λ is a parameter. Consider the generalized point-to-point variation problem

$$arg \min_{R, T} J_g(R, T). \tag{5}$$

Denote the matrix $p^i(q^i)^t$ by M^i and matrix D_p as $D_p = \sum_{i=1}^{s} M^i$.

Theorem 1 *The variational problem (5) takes the form*

$$arg \max_{R} < R, D > . \tag{6}$$

Remark 1 *The variation problem (6) is solved by the Horn method [3].*

Acknowledgments

References

[1] P. Besl and McKay N., A method for registration of 3-D shapes, IEEE Transactions of Pattern Analysis and Machine Intelligence **14**(2), (1992), 239-256.

[2] Y. Chen, and G. Medioni, Object modeling by registration of multiple range images, Image and Vision Computing **2**(10), (1992), 145-155.

[3] B. Horn, H. Hilden, and S. Negahdaripour, Closed-form solution of absolute orientation using orthonormal matrices, Journal of the Optical Society of America, Series A **5**(7) (1988), 1127-1135.

Approximate and Complete Controllability of Stochastic Systems with Impulsive Effects in Infinite Dimension

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Abstract

In this paper, we study the approximate and the complete controllability for nonlinear neutral impulsive stochastic integro-differential system in infinite dimensional spaces assuming controllability of associated linear systems. Sufficient conditions are established for each of these types of controllability. The results are obtained by using the Banach fixed point theorem. A numerical examples is provided to illustrate the technique.

Keywords. Approximate controllability, Complete controllability, Fixed point theorem, Stochastic neutral impulsive systems.

AMS subject classifications. 93B05, 93E03, 37C25, 60H10

1 Introduction

In this paper, we consider the following impulsive semilinear stochastic integrodifferential system

$$
\begin{cases}
dx(t) = Ax(t)dt + Bu(t)dt \\
\quad + F_1\left(t, x(t), f_{1,1}(\eta x(t)), f_{1,2}(\delta x(t)), f_{1,3}(\xi x(t))\right) dt \\
\quad + F_2\left(t, x(t), f_{2,1}(\eta x(t)), f_{2,2}(\delta x(t)), f_{2,3}(\xi x(t))\right) dw(t), \\
t \in J, \ t \neq t_k, \\
\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, ..., r, \\
x(0) = x_0 \in H,
\end{cases}
\tag{1}
$$

in a real separable Hilbert space H, where, for $i = 1, 2$

$$
\begin{aligned}
f_{i,1}(\eta x(t)) &= \int_0^t f_{i,1}(t, s, x(s))ds, \\
f_{i,2}(\delta x(t)) &= \int_0^T f_{i,2}(t, s, x(s))ds, \\
f_{i,3}(\xi x(t)) &= \int_0^t f_{i,3}(t, s, x(s))dw(s).
\end{aligned}
$$

1

Here, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $\{S(t),\ t \geq 0\}$ in H, B is bounded linear operator from U into H.

$$F_1 : J \times H \times H \times H \times H \to H, \qquad f_{i,1}, f_{i,2} : J \times J \times H \to H,\ I_k : H \to H,$$
$$F_2 : J \times H \times H \times H \times H \to \mathcal{L}_Q(E, H), \quad {}' \quad f_{i,3} : J \times J \times H \to \mathcal{L}_Q(E, H).$$

Furthermore, the fixed times t_k satisfies $0 = t_0 < t_1 < ... < t_r < T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$. And $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k, where I_k determines the size of the jump. The initial value x_0 is $\mathcal{F}_0$-measurable H-valued square-integrable random variable independent of w. The system (1) is in a very general form and it covers many possible models with various definitions of $f_{1,1}$, $f_{1,2}$, $f_{1,3}$, $f_{2,1}$, $f_{2,2}$, $f_{2,3}$.

References

[1] Nazim I Mahmudov. Controllability of linear stochastic systems. *Automatic Control, IEEE Transactions on*, 46(5):724–731, 2001.

[2] K Balachandran and S Karthikeyan. Controllability of stochastic integrodifferential systems. *International Journal of Control*, 80(3):486–491, 2007.01.

[3] R Subalakshmi, K Balachandran, and JY Park. Controllability of semilinear stochastic functional integrodifferential systems in hilbert spaces. *Nonlinear Analysis: Hybrid Systems*, 3(1):39–50, 2009.

[4] R Sakthivel, NI Mahmudov, and JH Kim. On controllability of second order nonlinear impulsive differential systems. *Nonlinear Analysis: Theory, Methods & Applications*, 71(1):45–52, 2009.

[5] R Subalakshmi and K Balachandran. Approximate controllability of nonlinear stochastic impulsive integrodifferential systems in hilbert spaces. *Chaos, Solitons & Fractals*, 42(4):2035–2046, 2009.

[6] Abbes Benchaabane. Controllability of neutral stochastic integro-differential systems with impulsive effects. *Electronic Journal of Differential Equations*, 2015(256):1–11, 2015.

[7] Abbes Benchaabane. Complete controllability of general stochastic integrodifferential systems. *MATHEMATICAL REPORTS*, 18(4):437–448, 2016.

Two Disjoint Convex Hulls

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Abstract. In this study, a geometric version of an NP-hard problem ("Almost $2 - SAT$" problem) is introduced which has potential applications in binary sensor networks, shape separation, image processing, etc. Furthermore, it has been illustrated that the new problem known as "Two Disjoint Convex Hulls" can be solved in polynomial time due to some combinatorial aspects and geometric properties. For this purpose, an $O(n^2)$ algorithm has also been presented which employs the Separating Axis Theorem (SAT) and the duality of points/lines.

Keywords. Convex Hull ; Algorithm ; Computational Geometry ; Binary Sensor Network ; SAT ; Separation Axis Theorem.

MSC2010. 52Cxx; 19L64; 05C85; 68Uxx.

1   Introduction

Detecting and testing the intersection between geometric objects are among the most important applications of computational geometry. It is one of the main questions addressed in Shamos' article that lays the groundwork for computational geometry [**?**], the first application of the plane sweep technique [**?**]. It is hard to overstate the importance of finding efficient algorithms for intersection testing or collision detection as this class of problems has many applications in Robotics, Computer Graphics, Computer-Aided Design and VLSI design [**?**, **?**, **?**]. In the plane, Shamos [**?**] presents an optimal linear algorithm to construct the intersection of a pair of convex polygons. Another linear time algorithm is later presented by O Rourke et al. [**?**].

For the testing version of the problem, Lemma **??** has been proved [**?**].

Lemma 1 *Let P and Q be two convex polygons with n and m vertices, respectively. The 2D-algorithm determines if P and Q intersect in $O(\log n + \log m)$ time[**?**].*

As discussed earlier, the problems are applicable to binary sensor networks and target tracking. Assume that some of the sensors do not work properly in a binary sensor network; these sensors send incorrect signs. Clearly, if two convex hulls overlap, it means that some of the sensors are making mistakes, but the reverse is not Necessarily right.
In some applications, the minimum sensors are ignored because they are prone to make a wrong sign at the moment. This problem is summarized as Problem **??**.

Problem 1 *Minimum Sign to Remove: There are a plus set P_1 and a minus set P_2 of points on the plane. For an integer $K \leq min\{|P_1|, |P_2|\}$, are there K points in $P_1 \cup P_2$ whose removing yields two disjoint convex hulls for P_1 and P_2?*

2 Main results

For some given point sets which constructs corners of the shapes reconstructing binary images with disjoint components is an important problem, for which there are several applications in digital/graphical games. Similar problems arise in the face of binary sensor networks. For all these applications, it is worth detecting the minimum number of noises which could be removed and caused construction of some separate components.

In this paper, a new problem entitled "Two Disjoint Convex Hulls" is introduced. A useful application of binary sensor networks along with some observations are also discussed in detail. Furthermore, a naive algorithm ($O(n^3)$) and a faster one ($O(n^2)$) are presented.
We can conclude the main results in Theorem **??**.

Theorem 2 *Let A and B be two sets of points in the plane, $|A| = m$ and $|B| = n$. Then the algorithm finds the minimum number of removal points in order to have two disjoint convex hulls in $O(m + n)^2$.*

References

[1] M. I. Shamos. Geometric complexity. In Proceedings of the 7th Annual ACM Symposium on Theory of Computing, pages 224â233. ACM, 1975.723

[2] M. I. Shamos and D. Hoey. Geometric intersection problems. In Proceedings of the 17th Annual Symposium on Foundations of Computer Science, pages 208â215. IEEE, 1976.

[3] Toth, Csaba D., Joseph O'Rourke, and Jacob E. Goodman. Handbook of discrete and computational geometry. Chapman and Hall/CRC, 2017.

[4] Jimã©nez, Pablo, Federico Thomas, and Carme Torras. "3D collision detection: a survey." Computers and Graphics 25.2 (2001): 269-285.

[5] Lin, Ming, and Stefan Gottschalk. "Collision detection between geometric models: A survey." Proc. of IMA conference on mathematics of surfaces. Vol. 1. 1998.

[6] O'Rourke, Joseph, et al. "A new linear algorithm for intersecting convex polygons." Computer Graphics and Image Processing 19.4 (1982): 384-391.

[7] Barba, Luis, and Stefan Langerman. "Optimal detection of intersections between convex polyhedra." Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms. Society for Industrial and Applied Mathematics, 2015.

Uncertainty Quantification in M/G/1/N Queue with Dependent Breakdowns

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Abstract. We propose a new methodology for incorporating imprecision in structural parameters to compute performance measures of queueing models with breakdowns and repairs. An efficient implementation of computational algorithm is investigated to analyze the M/G/1/N queueing model with dependent breakdowns by using theory of interval-valued probabilities. Several numerical examples are presented to illustrate the performance of the proposed method.

Keywords. queues with breakdowns and repairs; epistemic uncertainty; Markov chain; imprecise transition probabilities; algorithm.

MSC2010. 60K25; 68T37; 60J10; 60J22.

1   Introduction

The analysis of an important class of stochastic systems involves imprecision or incompleteness, uncertainty in parameters. Frequently little data is available concerning the variability of the key input parameters required for a predictive analysis. This has led to widespread use of several uncertainty descriptions. Many authors have proposed frameworks to properly model such uncertainty. The recent theory of imprecise probability provides a framework that is more general than conventional probability theory. It allows to obtain lower and upper probabilities from partial information on the occurrence of events and provides more general and robust models of uncertainty. A comprehensive presentation of the theory is given in [1, 2].

In this paper we attempt to apply the theory of imprecise probability for computing performance measures of the M/G/1/N queueing model with dependent breakdowns. More formally, let θ denote the probability of a server breakdown; then, we compute the stationary distribution of the queue-length process, denoted by π_θ, and other performances under the assumption that the probability θ is imprecise or uncertain. Indeed, data on the period characterizing the interval of time between occurrence of breakdowns of the server is only available in censored form and thereby estimating mean times of such periods is a statistical challenge. For this reason, we are typically confronted with uncertainty concerning the true value of the parameters defining the distributions of the time between breakdowns, in our case the probability of a server breakdown θ. We develop efficient algorithm based on the use of the theory of imprecise probability to analyze the considered queueing model. A key feature of this approach is the ability of obtaining the performance measures of the studied queueing model under interval's forms. The obtained numerical results are non-trivial and intuitively explainable.

2 The M/G/1/N Queue with Dependent Breakdowns

Consider a M/G/1/N queue with server's breakdowns and repairs. Customers arrive according to a Poisson process with rate λ. Arriving customers that find the queue full are assumed to be blocked and lost. The service time provided by a single server is an independent and identically distributed random variable with a general distribution function $S(x)$, having finite mean $1/\mu$. The server can serve only one customer at a time. Customers are served according to the first-come, first-served (FCFS) discipline. Assume that the probability of a server breakdown increases with the number of successful service completions since the last breakdown (i.e., the servers wears down).

Let (i, j) denote a state of the system, where i is the number of customers in the system right after either a service completion or a repair completion, and j is the number of successful service completions since the last breakdown. At the end of each service, there is a probability $\theta(j)$ that the server breaks down and enters a repair state, the length of which is exponentially distributed with rate r and which is independent of everything else, and with probability $(1 - \theta(j))$ the server is operationally and serves the customer. The only points in time where a possible server breakdown can occur is right at the beginning of a service. As model for $\theta(j)$ we study the following decay models.

Example 1 *We identify three models for the dependence of the break down probability on the number of successive services.*

Geometric Decay: *For $\theta \in (0, 1)$ we let*

$$\theta(j) = \theta^j.$$

Linear Decay: *For $\theta \in (0, 1)$ we let*

$$\theta(j) = \left(1 - \frac{1}{(1 + j)^b}\right) \theta,$$

for some constant $0 < b \leq 1$.

Stepwise Decay: *Let $M_k \in \mathbb{N}$, $1 \leq k \leq K$, such that*

$$0 = M_0 < M_1 < M_2 < \cdots < M_K < M_{K+1} = \infty.$$

For $\theta \in (0, 1)$ we let $\theta(j) = \theta^k$ for $M_{k-1} \leq j < M_k$.

In this paper we propose an efficient computational algorithm to the M/G/1/N queue with dependent breakdowns and repairs with the purpose of obtaining intervals for the uncertain performance measures, where we consider that the probability of a server breakdown θ is imprecise.

References

[1] P. Walley, *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, New York, 1991.

[2] D. Skulj, Discrete time Markov chains with interval probabilities, International Journal of Approximate Reasoning **50** (2009), 1314–1329.

Existence and Stability of the Solution of the Two Stationary Diffusion Equation System With Internal Transition Layer in a Medium with Discontinuous Characteristics

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Abstract. We consider a reaction-diffusion system of equations at a segment with reaction terms undergoing a first kind discontinuity in some internal point of a segment. The main result is a proven theorem on existence, local uniqueness and asymptotic stability of the solution to this system having large gradient in the vicinity of the discontinuity point.

Keywords. asymptotic approximation; discontinuous terms; lower and upper solutions; reaction-diffusion problem; small parameter.

MSC2010. 34B16.

1   Introduction

We consider a boundary value problem for the system of equations with reaction terms undergoing a first kind discontinuity. This problem arose while creation of biophysical models on the autowave fronts propagation in media with barriers (the areas preventing the autowave propagation). Of main interest in this models are the steady state solutions having large gradients in the vicinity of barriers. The area of large gradient is called the internal transition layer. The presence of a small parameter multiplying the differential operator is characteristic for the problems with an internal transition layer. As a small parameter the ratio of the settlement region width to the transition layer width can be considered.

2   Problem statement

We consider the following boundary value problem:

$$\varepsilon^4 u'' = f(u,v,x,\varepsilon), \;\; \varepsilon^2 v'' = g(u,v,x,\varepsilon), \;\; 0 < x < 1,$$
$$u'(0) = u'(1) = 0, \; v'(0) = v'(1) = 0, \tag{1}$$

where $\varepsilon \in (0, \varepsilon_0]$ is a small parameter, functions $f(u,v,x,\varepsilon)$ and $g(u,v,x,\varepsilon)$ have a first kind discontinuity across the surface $\{u \in I_u, v \in I_v, x = x_0 \in (0,1)\}$:

$$f(u,v,x,\varepsilon) = \begin{cases} f^{(-)}(u,v,x,\varepsilon), & u \in I_u, v \in I_v, 0 < x \le x_0, \\ f^{(+)}(u,v,x,\varepsilon), & u \in I_u, v \in I_v, x_0 < x \le 1, \end{cases}$$

$$g(u,v,x,\varepsilon) = \begin{cases} g^{(-)}(u,v,x,\varepsilon), & u \in I_u, v \in I_v, 0 < x \le x_0, \\ g^{(+)}(u,v,x,\varepsilon), & u \in I_u, v \in I_v, x_0 < x \le 1, \end{cases} \tag{2}$$

$f^{(-)}(u, v, x, \varepsilon)$ and $g^{(-)}(u, v, x, \varepsilon)$ are of class $C^4(I_u \times I_v \times [0, x_0] \times [0, \varepsilon_0])$, $f^{(+)}(u, v, x, \varepsilon)$ and $g^{(+)}(u, v, x, \varepsilon)$ are of class $C^4(I_u \times I_v \times [x_0, 1] \times [0, \varepsilon_0])$.

(H$_1$) Each of the equations $f^{(\mp)}(u, v, x, 0) = 0$ is solvable with respect to u and the functions $u = \varphi^{(\mp)}(v, x)$ are the isolated solutions to these equations, respectively, in the rectangles $I_v \times [0, x_0]$ and $I_v \times [x_0, 1]$, and in the respective rectangles the inequalities $f_u^{(\mp)}(\varphi^{(\mp)}(v, x), x) > 0$ hold.

We shall denote $h^{(\mp)}(v, x) := g^{(\mp)}(\varphi^{(\mp)}(v, x), v, x, 0)$.

(H$_2$) Each of the equations $h^{(\mp)}(v, x) = 0$ is solvable with respect to v and the functions $v = \psi^{(\mp)}(x)$ are the isolated solutions to these equations, respectively, in the segments $[0, x_0]$ and $[x_0, 1]$, and the inequalities $h_v^{(\mp)}(\psi^{(\mp)}(x), x) > 0$ hold.

We investigate the solution $(u_\varepsilon(x), v_\varepsilon(x))$ to the problem of class $C^1[0, 1] \cap C^2((0, 1) \backslash x_0)$ that is close to $(\varphi^{(-)}(\psi^{(-)}(x), x), \psi^{(-)}(x))$ at the left side of x_0 neighbourhood and to $(\varphi^{(+)}(\psi^{(+)}(x), x), \psi^{(+)}(x))$ at the right side. In the vicinity of x_0 the solution undergoes fast change having the area of large gradient.

3 Main result

Theorem 1 *For sufficiently small $\varepsilon > 0$ there exists locally unique and asymptotically stable in the sense of Lyapunov solution $(u_\varepsilon(x), v_\varepsilon(x))$ to the problem* (1) *having the internal transition layer in the vicinity of the point x_0.*

The theorem is proved using the method of upper and lower solutions [2] and its modification for the problems with internal transition layers called asymptotical method of differential inequalities [3].

Acknowledgments

References

[1] A.E. Sidorova, N.T. Levashova, A.E. Semina, A.A. Melnikova. The Application of a Distributed Model of Active Media for the Analysis of Urban Ecosystems Development. Mathematical Biology and Bioinformatics **13**(2) (2018), 454–465.

[2] V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[3] N.N. Nefedov, The method of differential inequalities for some classes of nonlinear singularly perturbed problems with internal layers, Differential Equations **31** (7) (1996), 1077–1085.

Detecting Malware Software using Artificial Neural Network based on Adaptive Resonance Theory with a Hierarchical Structure of Memory

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Abstract. The process of detecting malicious software code by antivirus systems is considered. To analyze the executable code, a control flow graph is used. It is proposed to use artificial neural networks based on adaptive resonance theory with a hierarchical memory structure as a classifier. For convenient presentation of the control flow graph for classification it is proposed to use the graph2vec algorithm. Experiments were carried out on model examples when testing the proposed approach. The experimentresults show good accuracy and speed of determining the type of software.

Keywords. malware; analysis of portable executable files; control flow graph; vectorization; deobfuscation; artificial neural networks based on Adaptive Resonance Theory; clustering.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

The most effective and often applicable software for detecting malicious software are anti-virus programs. The main module of the antivirus is the file and process analysis module. These modules are based on signature and heuristic attack detection. The disadvantages of signature analysis [1] in detecting malicious software are the impossibility of detecting disguised or new type of software. Malware developers often resort to the following masking methods: malware encryption [2], self-modification [3], packaging [4], code obfuscation. Heuristic analysis is used to detect unknown malware. The basic principle is to find the deviation of the behavior of the signature from the normal state. For this, systems using artificial neural networks (ANN) have proven themselves [5]. To determine the type of software, we represent the computer process as a control flow graph (CFG). To compare the graph structures obtained, the graph2vec algorithm [6] was used, which allows the graph to be described by a vector of numbers.

2   Main results

The malware detection algorithm consists of the following actions: removal of insignificant commands; the construction of the CFG; vectorization of the CFG; development of a classifier based on ANN adaptive-resonance theory with multi-level memory (ART-2m); classification of the resulting vector of the CFG. After deobfuscation, the transformed code is analyzed and divided into basic blocks, i.e. the top of the CFG. A vertex consists of a set of commands that

do not result in the transfer of flow control. The transfer of flow control and the end of the vertex will be marked by control transfer commands (jmp, call, loop, etc.). Each vertex can be described by a vector of repetitions of the unique machine instructions contained in it. The resulting numerical vector representing the CFG is fed to the classifier based on ART-2m. The data for the classification of software are presented in table 1.

Table 1 - Baseline data for classification

Software	Size of executable code, KB.	The total number of vertices in the CFG	Number of unique vertices
Malicious software	202	1596	443
Original PE file	160	1256	417
Infected file	363	2851	558

When constructing the ART-2m classifier, the following tree-structure memory figure 1 was obtained.

Figure 1: The memory structure of ART-2m

The figure shows that the infected file looks like the original virus more than the source file. Since legitimate software was used as ansource file and malware, they have some common similarities.

References

[1] S. S. Harchenko, E. M. Davydova, S. V. Timchenko, Signature analysis of program code, Polzunovskijvestnik, no.3. (2012), 60–64.

[2] B. R. Babak, M. Maslin, I. Suhaimi, Camouflage in Malware: from Encryption to Metamorphism, IJCSNS International Journal of Computer Science and Network 74 Security, vol.12, no.8, (August 2012), 74–83.

[3] C. Hongxu, S. Zhong, V. Alexander, Certified Self-Modifying Code (extended version & coq implementation), Technical Report YALEU/DCS/TR-1379, Yale Univ., Dept. of Computer Science, (Mar. 2007).

[4] Y. Wei, Z. Zheng, A. Nirwan, Revealing Packed Malware, IEEE Security & Privacy, vol. 6, no. 5, (Sept.-Oct. 2008), 65–69.

[5] I. A. Solomon, A. Jatain, S. B. Bajaj, Neural Network Based Intrusion Detection: State of the Art, (20 Mar. 2019).

[6] A. Narayanan, M. Chandramohan, L. Chen, Y. Liu, S. Saminathan, Subgraph2vec: Learning distributed representations of rooted sub-graphs from large graphs, arXiv:1606.08928, (2016).

Autoimmune diabetes in HIV infected patients: insights from a mathematical model

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Abstract. Diabetes Mellitus (DM) is a disease that affects the levels of glucose in our body. It is a common condition with noteworthy related morbidity and mortality. DM diagnosis and management among patients infected with the human immunodeficiency virus (HIV) is extremely important. These patients live longer lives due to the anti-retroviral therapy (ART) and have significant chronic medical comorbidities. DM has higher prevalence among pre-ART HIV-infected patients. Moreover, more educated, hypertensive and obese HIV-infected adults are also more prone to have DM as comorbidity [1]. In this paper, we propose a within-host model for the dynamics of HIV in an infected person who has developed type 1 diabetes (T1D) [2]. The model also includes macrophages, and cytokines. T1D is an autoimmune disease characterized by the destruction of β-cells, which are responsible for the production of insulin. T1D develops from an abnormal immune response, where specific clones of cytotoxic T-cells invade the pancreatic islets of Langerhans. We compute the reproduction number of the model and the disease-free equilibrium. Numerical simulations reveal interesting dynamics and contribute to a better understanding of the role of macrophages.

Keywords. Diabetes Mellitus; HIV; autoimmune disease.

MSC2010. 26A33; 34A60; 34G25; 93B05.

Acknowledgments

References

[1] S.M. Abebe, A. Getachew, S. Fasika, M. Bayisa, A.G. Demisse, N. Mesfin (2016). *Diabetes mellitus among HIV-infected individuals in follow-up care at University of Gondar Hospital, Northwest Ethiopia*, British Medical Journal Publishing Group, 6(8), e011175.

[2] D. Takarabe, Y. Rokukawa, Y. Takahashi, A. Goto *et al.* (2010). *Autoimmune Diabetes in HIV-Infected Patients on Highly Active Antiretroviral Therapy*, The Journal of Clinical Endocrinology & Metabolism, 95(8), 4056–4060.

Existence and Uniqueness of the Weak Solution for an Antiplane Contact Problem with Friction

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Abstract. In this work we consider a mathematical model which describes the antiplane shear deformation of a cylinder in frictional contact with a rigid foundation. First, we describe the classical formulation for the antiplane problem and we give the corresponding variational formulation which is given by a system coupling an evolutionary variational equality for the displacement field and a time-dependent variational equation for the electric potential field. Then we prove the existence of a unique weak solution to the model. The proof is based on arguments of variational inequalities and by using the Banach fixed-point Theorem.

Keywords. electro-viscoelastic law; fixed point; weak solution; variational inequality; Tresca's friction law.

MSC2010. 74M10; 74F15; 74G25; 49J40.

References

[1] M. Sofonea, M. Dalah; *Antiplane Frictional Contact of Electro-Viscoelastic Cylinders*, Electronic Journal of Differential Equations. **no. 161** (2007), 1–14.

[2] K.T . Andrews, K. L. Kuttler and M. Shillor; *On the dynamic behaviour of a thermoviscoelastic body in frictional contact with a rigid obstacle*, Euro. Jnl. Appl. Math., **8** (1997), 417-436.

[3] O. Chau, W. Han, M. Sofonea; *Analysis and approximation of a viscoelastic contact problem with slip dependent friction*, Dynamics of Continuous, Discrete and Impulsive Systems, **8** (2001), 153-174.

Methods for calculating the quality criteria of medical images

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Abstract. This abstract is about evaluating image quality for further ranking. It's considering quality as an image-specific characteristic perceived by an average human observer. A couple of these metrics are blurriness, image entropy, sharpness. Further use of these metrics is try to calculate better values based on human eye perception.

Keywords. medical images; image analisys; image quality evaluation.

MSC2010. 65D18, 68U10, 94A08

1   Blurriness

This metrics described at [1]. It computes absolute vertical and horizontal difference D for neighboring pixels in original and blurred images (1,2):

$$D_{ver(x,y)} = |I(x,y) - I(x-1,y)|, x = 2...w, y = 1...h, \tag{1}$$

$$D_{hor(x,y)} = |I(x,y) - I(x,y-1)|, x = 1...w, y = 2...h, \tag{2}$$

where $I(x,y)$ is the intensity value at the (x,y) pixel, h and w are height and width of image. After that, variation of neighboring pixels before and after blurring needs to be analyzed (3):

$$V_{ver(x,y)} = max(0, D_{ver(x,y)} - D_{B_{ver}(x,y)})), x = 1...w-1, y = 0...h-1, \tag{3}$$

where $D_{B_{ver}(x,y)}$ is the absolute difference for blurred image B. Finally, blurriness for vertical direction is computed (4):

$$F_{blur_{ver}} = \frac{\sum_{x,y=1}^{w-1,h-1} D_{ver(x,y)} - \sum_{x,y=1}^{w-1,h-1} V_{ver(x,y)}}{\sum_{x,y=1}^{w-1,h-1} D_{ver(x,y)}}, \tag{4}$$

Finally, maximum of two is selected as the final blurriness measure (5)

$$F_{blur} = max(F_{blur_{hor}}, F_{blur_{ver}}), \tag{5}$$

2   Entropy

The basic idea behind entropy is to measure the uncertainty of the image (5). So higher entropy should mean that more signal is contained in the image. For example, if there are less details and mode plain surfaces, entropy would be less.

$$F_{ent} = -\sum_{k=1}^{n} p(I_k) * \log_2 p(I_k), \tag{6}$$

where $p(I_k)$ is the probability of the particular intensity value I_k.

3 Sharpness

This measure [2] is based on assumption that differences of neighboring pixels change more in the areas with sharp edges. Therefore the authors compute second-order difference for the neighboring pixels as a discrete analog of second derivative for the image passed through denoising median filter:

$$\Delta D_2(x, y) = \big[I_m(x+2, y) - I_m(x, y) \big] - \big[I_m(x, y) - I_m(x-2, y) \big] \tag{7}$$

where I_m the original image passed through median filter Then authors define vertical sharpness for each pixel S_{ver} as shown below:

$$S_{ver}(x, y) = \frac{\sum_{x-t<=k<=x+t} |\Delta D_2(k, y)|}{\sum_{x-t<=k<=x+t} |I(k, y) - I(k-1, y)|} \tag{8}$$

and each pixel is treated as sharp if its sharpness exceeds 0.0001. Then sharp to edge pixels ratio for vertical and horizontal directions is computed as:

$$F_{sharp} = \sqrt{ \big(\frac{N_{S_{hor}}}{N_{E_{hor}}} \big)^2 + \big(\frac{N_{S_{ver}}}{N_{E_{ver}}} \big)^2 } \tag{9}$$

Acknowledgments

References

[1] Dolmiere T., Ladret P. Crete F., *The Blur Effect: Perception and Estimation with a New No-Reference Perceptual Blur Metric*, Grenoble: SPIE Electronic Imaging Symposium Conf Human Vision and Electronic Imaging, Etats-Unis d'Amerique, 2007.

[2] Chen F., Doermann D. Kumar J., *Sharpness estimation for Document and Scene Images, in Pattern Recognition (ICPR)*, 2012 21st International Conference on, Tsukuba, 2012, pp. 3292 - 3295

Learning in the Hidden Markov Chain

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Abstract. The theory of learning addresses the problems of regression, of density estimation, but it is first defined around the problems of data classification, which consists in determining the parameters of the adaptable, probabilistic or deterministic system, starting from sets of examples. in order to generalize and correctly classify new data, these problems are usually addressed as a minimization problem, the intended perspective is most often minimax, we use hidden Markov chains (HMC) for evaluation and search of 'optimum. In this study we present the different procedures used, firstly the forward and backward procedure for calculating the likelihood of the study model and secondly the application of the Viterbi algorithm for the determination of the optimal path. we finish with a numirical application of these algorithms.

Keywords. Hidden Markov Chain; classification; forward, backward and Viterbi algorithm.

MSC2010. 62M05; 37A30; 37A50.

1   Introduction

In the hidden Markov chains, which are random functions of Markov chains, which were introduced by Baum, Petrie and Eagon in the late sixties [2]. It was necessary to wait a fortnight of year to find their first application. Currently, these models are widely applied in several fields such as artificial intelligence, speech recognition and cited form [4], signal, handwriting recognition [5], signal processing, molecular biology [1, 3], economics (the financial market), medical diagnostics, robotics, ... etc. learning is the transformation of the input space (observations) which are independent, to the output space (which represents the objectives) which also represents independent variables. in this work we have developed the forward and backward algorithm easier to apply and less complex in storage space. Moreover, the CMCs are known by the lack of data concerning the states and consequently the estimation of the parameters of the model by the likelihood method becomes impossible. Several methods of estimating these models have been developed: it is about the itirative algorithms which consists of recursively estimating the parameters of the model up to a predetermined stopping criterion. For our work we have developed the Baum-welch algorithm which is a variant of the EM algorithm in Hidden Markov chains.

References

[1] M. ASKAR, H. DERIN. *A Recursive Algorithme of Bayes Solution of the Sommoothing Problem* , IEEE Transaction on Automatic Control (1981), 26(2), 161–175.

[2] C. BARRET, R. HUGHER and K. KARPLUS.*Scoring Hidden Markov Models*, Reprint from CABIOS (1997), 13(2),191–199.

[3] LE. BAUM, JA. *An Equality with Application to Statistical Estimation for Probabilistic Function of Markov Processes and to a Model for Ecology communicated by RC back*, Bull See (1966), 73, 360–363.

[4] BG. LEROUX, JA. *Maximum Lekelihood Estimation for Hidden Markov Models*, Stochastique Processesand thier Applications (1992), 40, 127–143.

[5] V. KRISHNAMURTHY , JB. MORE and S. CHUNG . *Hidden Markov Models SignalProcessing in Presence of Unknouwn Deterministic Inference* , IEEE Transaction on Automatic Control (1981), 38, 146–152.

On non-steady solutions of ideal fibre-reinforced fluids on the plane

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Abstract. We show that the system describing planar *non-steady* motions of fibre-reinforced fluids can be reduced, under certain conditions, to a single third order PDE in two dimensions. Remarkably, this equation has the same form as the equation that governs the *steady* motions.

Keywords. kinematics, fiber-reinforced materials.

MSC2010. 76B99; 35C05; 35C09.

1 Introduction

A kinematic study of the motion of ideal fibre-reinforced fluids has been conducted by Spencer [1]. The ideal model describes an incompressible viscous liquid which contains inextensible fibre lines occupying the volume of the fluid by which they are convected [2].

Let $\mathbf{t} = \mathbf{t}(\mathbf{r}, t)$ denote the *unit* vector tangential to a generic fibre and $\mathbf{q} = \mathbf{q}(\mathbf{r}, t)$ be the fluid velocity. Then, the kinematic condition which encodes both the convection requirement and inextensibility of the fibres is given by

$$(\partial_t + \mathbf{q} \cdot \nabla)\, \mathbf{t} = (\mathbf{t} \cdot \nabla)\, \mathbf{q}. \tag{1}$$

The latter equation is also augmented by the continuity condition:

$$\operatorname{div} \mathbf{q} = 0. \tag{2}$$

2 Main result

We restrict our attention to the flows that are planar and whose fibre-divergence is not constant, i.e. $\theta \neq \text{const}$. These flows can be determined essentially from kinematic considerations. Planar steady flows have been studied in [3, 4, 5].

In the planar case it is convenient to parametrise the unit vector $\mathbf{t}$ as

$$\mathbf{t} = \mathbf{i} \cos \varphi + \mathbf{j} \sin \varphi,$$

hence the unit normal to the fibres adopts the form

$$\mathbf{n} = -\mathbf{i} \sin \varphi + \mathbf{j} \cos \varphi,$$

where $\mathbf{i}$ and $\mathbf{j}$ are the vectors of Cartesian orthonormal basis. Hence the velocity can be decomposed as follows

$$\mathbf{q} = v\mathbf{t} + w\mathbf{n}.$$

Theorem. Given a solution $(\rho(n,s,\tau), N(\rho(n,s,\tau),n))$ of the equation

$$\left(\frac{\rho_{sn}}{N}\right)_n + N_{\rho\rho}\rho_s = 0, \tag{3}$$

a solution of the governing equations (1) and (2) is given by

$$\mathbf{t} = (\cos\varphi, \sin\varphi), \quad \mathbf{q} = v(\cos\varphi, \sin\varphi) + w(-\sin\varphi, \cos\varphi),$$

where the angle φ is given by the integral

$$\varphi = \int^{(n,s)} N_\rho dn - \frac{\rho_{ns}}{N} ds$$

and the components of velocity v and w by formula

$$v = x_\tau \cos\varphi + y_\tau \sin\varphi - \rho_\tau, \quad w = -x_\tau \sin\varphi + y_\tau \cos\varphi. \tag{4}$$

The Eulerian coordinates are parametrised by the solution of the compatible system

$$x_s = \rho_s \cos\varphi, \quad x_n = -N \sin\varphi, \quad y_s = \rho_s \sin\varphi, \quad y_n = N \cos\varphi. \tag{5}$$

and $t = \tau$.

Remark. The equation that governs steady motions of planar fibre-reinforced fluids coincides with (3), where $N = N(\rho)$, $\rho = \rho(n,s)$.

References

[1] A. J. M. Spencer, Fibre-streamline flows of fibre-reinforced viscous fluids, *European J. Appl. Math.* **8** (1997) 209-215.

[2] B. D. Hull, T. D. Rogers and A. J. M. Spencer, Theoretical analysis of forming flows of continuous fibre-resin systems, Flow and Rheology in Polymer Composites Manufacturing (ed.S. G. Advani; Elsevier, Amsterdam 1994) 203-256.

[3] Schief, W. K.; Rogers, C. The kinematics of fibre-reinforced fluids. An integrable reduction. *Quart. J. Mech. Appl. Math.* **56** (2003), no. 4, 493-512. 76T99 (37K10 37N10)

[4] Rogers, C.; Schief, W. K. Novel integrable reductions in nonlinear continuum mechanics via geometric constraints. Integrability, topological solitons and beyond. *J. Math. Phys.* **44** (2003), no. 8, 3341-3369

[5] Rogers, C.; Schief, W. K. The kinematics of the plane motion of a fiber-reinforced ideal fluid. Integrable reduction and the Bäcklund transformation. *Theoret. and Math. Phys.* **137** (2003), no. 2, 1598-1608

Study of Option Pricing Models with Insufficient Liquidity and Transaction Costs

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Abstract. For option pricing models with the insufficient liquidity and transaction costs, a numerical solution is given to the price of call options and the price sensitivity factors from the market parameters (simply sensitivities, or Greeks), as well as a comparison with actual data on deals on Moscow Exchange is made.

Keywords. option pricing; insufficient liquidity; transaction costs; sensivities; numerical solution.

MSC2010. 91B25; 91G60; 91G80.

The models of option pricing of futures-styles options [1, 2, 3] with the insufficient liquidity can be written as

$$u_t + \frac{\sigma^2 x^2 u_{xx}}{(1 - xv(u_x)u_{xx})^2} = 0, \tag{1}$$

where $v(u_x)$ takes the form determined by the model. The results of the group classification of such class of models are presented in [4].

Leland's model [5], which takes into account the proportional transaction costs, has the form

$$u_t + \sigma^2 \left[1 + \sqrt{2/\pi} \, k/\sigma \, \sqrt{\Delta t} \right] x^2 u_{xx} = 0, \tag{2}$$

where k is the coefficient of the transaction costs, Δt is a small, but not infinitesimal time interval that determines the frequency of the portfolio rebalancing.

With the modifications of Y. M. Kabanov and M. M. Safarian [6], the Leland's model can be represented as

$$u_t + \sigma^2 \left[1 + \sqrt{8/\pi} \, k_n/\sigma \, \sqrt{n} \right] x^2 u_{xx} = 0. \tag{3}$$

where n is the number of the revisions, and k_n are transaction costs for each revision of the portfolio.

For these models, a numerical solution is given to the price of call options and the price sensitivity factors from the market parameters (simply sensitivities, or Greeks), as well as a comparison with actual data on deals on Moscow Exchange using the method from [7] is maid.

Acknowledgments

References

[1] R. Frey and A. Stremme, Market volatility and feedback effects from dynamic hedging, Math. Finance **7** (1997), 351–374.

[2] R. K. Sircar and G. Papanicolaou, General Black — Scholes models accounting for increased market volatility from hedging strategies, Appl. Math. Finance **5** (1998), 45–82.

[3] P. J. Schönbucher and P. Wilmott, The feedback effect of hedging in illiquid markets, SIAM J. Appl. Math. **61** (2000), 232–272.

[4] V. E. Fedorov and M. M. Dyshaev, Invariant solutions for nonlinear models of illiquid markets, Math. Meth. Appl. Sci. **41** (2018), 8963–8972.

[5] H. E. Leland, Option pricing and replication with transactions costs, J. of Finance **40** (1985), 1283–1301.

[6] Y. M. Kabanov and M. M. Safarian, On Leland's strategy of option pricing with transactions costs, Finance and Stochastics **1** (1997), 239–250.

[7] M. M. Dyshaev, V. E. Fedorov, A. S. Avilovich and D. A. Pletnev, Simulation of feedback effects for futures-style options pricing on Moscow Exchange, Chel. Phys. Math. J. **3** (2018), 379–394. (In Russ.).

Hyperbolic potentials generated by the Bessel operator

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Abstract. The report paper is devoted to the study of the fractional integral operator which is a negative real power of the singular wave operator generated by Bessel operator and its inverse using weighted generalized functions.

Keywords. hyperbolic Riesz B-potential; fractional power of singular hyperbolic operator; Lorentz distance; singular Bessel differential operator; generalized translation; multidimensional Hankel transform.

MSC2010. 42C15, 46E30, 46E35.

In recent years, the interest to the Fractional Calculus has been increasing due to its applications in many fields. As for multidimensional case the most developed type of fractional integrals are Riesz potentials which are generalized both Newton potential to the fractional case and Riemann-Liouville fractional integral to the multidimensional case.

Let us start from the classical mechanic Newton potential. If f is integrable function with compact support then the Newton potential of f is the convolution product

$$V_N f(x) = \int\limits_{\mathbb{R}^n} v(x-y)f(y)dy,$$

where

$$v(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & n=2; \\ \frac{1}{n(2-n)\omega_n}|x|^{2-n}, & n\neq 2, \end{cases} \qquad \omega_n \quad \text{is a volume of unit ball } \mathbb{R}^n.$$

Newton potential V_N of f is the solution to the Poisson equation

$$\Delta V_N = f, \qquad \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \qquad x = (x_1, ..., x_n) \in \mathbb{R}^n,$$

therefore, it can be considered as a negative degree of the Laplace operator:

$$V_N f = \Delta^{-1} f.$$

Along with the Newtonian potential, the wave potential of the function f has found wide applications

$$V_W f(x) = \int\limits_{\mathbb{R}^n} \varepsilon(x-y)f(y)dy,$$

where ε is a fundamental solution of the wave operator. For the wave potential V_W the next equality is true

$$\Box V_W = f,$$

therefore, it can be considered as a negative degree of the D'Alembert operator: $V_W f = \Box^{-1} f$.

Marsel Riesz was a Hungarian mathematician who first established the fractional powers of the Laplace and D'Alembert operators. Such potentials are called the **Riesz potentials** now and have the forms

$$I_\Delta^\alpha f(P) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} f(Q) r^{\alpha-n} dQ \qquad \text{and} \qquad I_\Box^\alpha f(P) = \frac{1}{H_n(\alpha)} \int\limits_{D} f(Q) r_{PQ}^{\alpha-n} dQ,$$

where $P = (x_1, ..., x_n)$, $Q = (\xi_1, ..., \xi_n)$, $\gamma_n(\alpha)$, $H_n(\alpha)$ is normalizing constant,

$$r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + ... + (x_n - \xi_n)^2}$$

is the Euclidean distance,

$$r_{PQ} = \sqrt{(x_1 - \xi_1)^2 - (x_2 - \xi_2)^2 - ... - (x_n - \xi_n)^2}$$

is the Lorentz distance, $D = \{x : x_1^2 \geq x_2^2 + ... + x_n^2\}$ is the positive cone.

It is not to see that

$$\Delta I_\Delta^{\alpha+2} f(P) = -I_\Delta^\alpha f(P) \qquad \text{and} \qquad \Box I_\Box^{\alpha+2} f(P) = I_\Box^\alpha f(P).$$

More attention was paid to Riesz potentials with Euclidean distance by S.G. Samko, L. N. Lyakhov, M. D. Ortigueira, B. Rubin and other. M. L. Gol'dman studied kernels of fractional powers of operators which are the set of all positive powers of the operator generated by the Green function for the Laplace equation.

Fractional powers of hyperbolic operators, with Bessel operators instead of all or some second derivatives, are much less studied. Such operators have wide areas of application such as singular differential equations, differential geometry and random walks.

In this report we study real powers of

$$\Box_\gamma = B_{\gamma_1} - B_{\gamma_2} - ... - B_{\gamma_n}, \qquad B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \qquad i = 1, ..., n.$$

Composition method developed by S. M. Sitnik was used for construction of $(\Box_\gamma)^{-\frac{\alpha}{2}}$, $\alpha > 0$. We present embedding theorems and inverse operator to $\Box_\gamma^{-\alpha}$, $\alpha > 0$.

Fractional derivative of the Lerch zeta function and Banach algebras

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Abstract. This paper deals with the fractional derivative of the Lerch zeta function. The main properties of this fractional derivative are presented and discussed. In particular, the study concerns the link with the Banach algebra and some new results are given. As a consequence, the class of zeta functions shows that the fractional calculus of complex functions can be extended in suitable Banach algebras.

Keywords. fractional derivative; Lerch zeta function; Banach algebra; continuous linear functional.

MSC2010. 11M35; 26A33; 46J15.

Numerical Solution for a Fuzzy Fractional Population Growth

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Abstract

In this paper, fuzzy fractional population with Caputo generalized Hukuhara differentiability is introduced. Some numerical examples are provided to confirm the accuracy of the proposed method.

Keywords: Population dynamics; Fractional derivative; Numerical solution

References

[1] J.H.He, Some applications of nonlinear fractional equations and their approximations, Bull. Sci. Technol. 15(2)1999.

[2] O. Fard, M. Salehi A survey on fuzzy fractional variational problems, J. Comput. Appl. Math. 271 2014.

On Volterra-Hammerstein Integral Inclusions and Their Applications

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Abstract. We consider a delayed Volterra-Hammerstein integral inclusion with a parameter. We obtain conditions that guarantee its solvability and continuous dependence of the solutions set on the parameter. Based on these results we investigate the well-posedness property of a wide class of mathematical models describing the electrical activity in the neocortex.

Keywords. Volterra integral inclusion; solvability; well-posedness; neural field models.

MSC2010. 34G25; 47H30; 92B99.

We denote R^m to be a m-dimensional real vector space with the norm $|\cdot|$, μ – to be a Lebesgue measure on R^m. Let Ω be some compact subset of R^m, $T > 0$. For any $t \in (0, T]$, we denote the direct product $[0, t] \times \Omega$ by $\widetilde{\Omega}_t$. Denote by $C(\widetilde{\Omega}_T, R^n)$, $L(\widetilde{\Omega}_T, \mu, R^n)$, and $L_\infty(\widetilde{\Omega}_T, \mu, R^n)$ the Banach spaces of continuous, Lebesgue integrable, and essentially bounded functions, respectively with the standard norms. Let $\mathcal{B}$ be some Banach space with the norm $\|\cdot\|_\mathcal{B}$. Denote by $\mathbf{2}^\mathcal{B}$ the set of all nonempty subsets of $\mathcal{B}$. For any $M \subset \mathcal{B}$, $r > 0$, we define the set $B_\mathcal{B}(M, r) = \{\mathrm{b} \in \mathcal{B}, \|\mathrm{b} - \mathrm{m}\|_\mathcal{B} < r, \mathrm{m} \in M\}$.

We consider the following delayed Volterra-Hammerstein integral inclusion

$$u(t, x) \in \int_0^t \int_\Omega W(t, s, x, y, \lambda) \mathcal{H}(s, y, u(s - \tau(x, y, \lambda), y), \lambda) dy ds, \qquad (t, x) \in \widetilde{\Omega}_T, \qquad (1)$$

$$u(t, x) = \varphi(t, x, \lambda), \quad t \in (-\infty, 0), \ x \in \Omega \qquad (2)$$

parameterized by $\lambda \in \Lambda$, where Λ is some metric space.

For any $\lambda \in \Lambda$, we assume the following conditions to be satisfied:

$(A1)$ For any $(t, x) \in \widetilde{\Omega}_T$, the kernel $W(t, \cdot, x, \cdot, \lambda) \in L_\infty(\widetilde{\Omega}_t, \mu, R^{n \times k})$ and the function $(t, x) \mapsto \|W(t, \cdot, x, \cdot)\|_{L_\infty(\widetilde{\Omega}_t, \mu, R^{n \times k})}$ is bounded; for any measurable set $\mathbb{I} \subset \widetilde{\Omega}_T$ and any $(t_0, x_0) \in \widetilde{\Omega}_T$, it holds true that $\lim\limits_{(t,x) \to (t_0, x_0)} \iint\limits_{\mathbb{I} \cap \widetilde{\Omega}_t} W(t, s, x, y, \lambda) dy ds = \int\int\limits_{\mathbb{I} \cap \widetilde{\Omega}_{t_0}} W(t_0, s, x_0, y, \lambda) dy ds.$

$(A2)$ For almost all $(t, x) \in \Omega_T$, the nonlinearity $\mathcal{H}(t, x, \cdot, \lambda) : R^n \to \mathbf{2}^{R^k}$ is upper semi-continuous and has convex compact values; for any $r > 0$, the mapping $\mathcal{H}(\cdot, \cdot, u, \lambda)$ is integrally bounded by some $\eta_{(r, \lambda)} \in L(\widetilde{\Omega}_T, \mu, [0, \infty))$ for all $u \in B_{R^n}(0, r)$; and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(t, x), (t_0, x_0) \in \widetilde{\Omega}_T$, the inequalities $0 \leq t_0 - t < \delta$ and $|x_0 - x| < \delta$ imply the relation $\int_0^t \int_\Omega |W(t, s, x, y, \lambda) - W(t_0, s, x_0, y, \lambda)| \eta_{(r, \lambda)}(s, y) dy ds < \varepsilon.$

$(A3)$ The delay $\tau(\cdot, \cdot, \lambda) : \Omega \times \Omega \to [0, \infty)$ is continuous.

$(A4)$ The prehistory function $\varphi(\cdot,\cdot,\lambda) : (-\infty,0) \times \Omega \to R^n$ is continuous and satisfies the conditions: $\forall x \in \Omega \lim\limits_{t\to-\infty} \varphi(t,x,\lambda) = 0$ and $\lim\limits_{t\to 0} \varphi(t,\cdot) \equiv 0$.

For any $\lambda \in \Lambda$, we understand the solutions to the problem $(1),(2)$ in the following sense.

Definition 1 *Let $\lambda \in \Lambda$ be fixed. Let $\xi, \zeta \in (0,T]$. We define a ξ-local solution to the problem $(1),(2)$ to be a function $u_\xi \in C(\widetilde{\Omega}_\xi, R^n)$ satisfying the inclusion (1) on $\widetilde{\Omega}_\xi$ and the condition (2). In the case $\xi = T$, the corresponding T-local solution $u_T \in C(\widetilde{\Omega}_\xi, R^n)$ we call a global solution to the problem $(1),(2)$. We define a ζ-maximally extended solution to the problem $(1),(2)$ to be a mapping $\widetilde{u}_\zeta : \widetilde{\Omega}_\zeta \to R^n$ such that: for any $\xi \in (0,\zeta)$, the restriction $u_\xi \in C(\widetilde{\Omega}_\xi, R^n)$ of $\widetilde{u}_\zeta$ on $\widetilde{\Omega}_\xi$ is a local solution to $(1),(2)$; and $\lim\limits_{\xi\to\zeta-0} \|u_\xi\|_{C(\widetilde{\Omega}_\xi,R^n)} = \infty$.*

Theorem 1 *Let for any $\lambda \in \Lambda$, the conditions $(A1)$ – $(A4)$ be satisfied. Then for any $\lambda \in \Lambda$, the problem $(1),(2)$ has a local solution. Any local solution can be extended to a global solution or to a maximally extended solution.*

If for some $\lambda_0 \in \Lambda$ and $\sigma > 0$, the condition $(A2)$ is fulfilled with the function $\eta_{(r,\lambda)} = \eta_r \in L(\widetilde{\Omega}_T, \mu, [0,\infty))$ that is independent of $\lambda \in B_\Lambda(\lambda_0,\sigma)$, then there exists $\widehat{\xi} > 0$ such that for any $\lambda \in B_\Lambda(\lambda_0,\sigma)$ and any maximally extended solution $\widetilde{u}_\zeta(\lambda)$ to the problem $(1),(2)$, it holds true that $\zeta > \widehat{\xi}$. Moreover, the set of all $\widehat{\xi}$-local solutions to $(1),(2)$ corresponding to $\lambda \in (B_\Lambda(\lambda_0,\sigma))$ is relatively compact in $C(\widetilde{\Omega}_{\widehat{\xi}}, R^n)$.

We also obtain conditions that guarantee upper semi-continuous dependence of the solution set of $(1),(2)$ on the parameter $\lambda \in \Lambda$.

We consider the Amari neural field equation (see [1]) with a Heaviside-type firing rate function that generates ill-posed initial value problems in the general setting (see e.g. [2]). We also introduce a generalization of the Amari neural field model, which takes into account such physiological factors as the variation of the firing threshold from one neuron to other and also the variation in time for each taken neuron, the non-homogeneity of the cerebral cortex, the memory effects, and the finite speed of signal propagation in the neocortex media. Based on the idea of the work [3] on reduction of an integral equation with discontinuous right-hand side to the corresponding integral inclusion, we exploit Theorem 1 and the results on continuous dependence of solutions to $(1),(2)$ on the parameter to derive the conditions that guarantee the well-posedness of both the Amari neural field equation and its aforementioned generalization.

Acknowledgments

References

[1] S. Amari, Dynamics of pattern formation in lateral-inhibition type neural fields, Biol. Cybern. **27** (1977), 77–87.

[2] B. F. Nielsen and J. A. Wyller, Ill-posed point neuron models, J. Math. Neurosci. **6(1)** (2016), 7.

[3] E. Burlakov, On inclusions arising in neural field modeling, Differ. Equ. Dyn. Syst. (2018), DOI: 10.1007/s12591-018-0443-5.

Optimization of a Telecommunications Network

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Abstract. This paper presents the flow and multi-flow models related to optimization problems in a telecommunications network. These issues are of strategic importance in the telecommunications industry. The network will have to allow the routing of all traffic requests even in case of failure.

Keywords. Optimization; Flow; Network; Telecommunications.

MSC2010.

Periodic Solutions for aThird-Order Nonlinear Delay Differential Equation with Variable Coefficients

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Abstract

In this paper, the following third-order nonlinear delay differential equation with periodic coefficients

$$\frac{d^3}{dt^3}x\left(t\right) + p\left(t\right)\frac{d^2}{dt^2}x\left(t\right) + q\left(t\right)\frac{d}{dt}x\left(t\right) + r\left(t\right)x\left(t\right) =$$

$$f\left(t, x(t), x\left(t - \tau\left(t\right)\right)\right) + \frac{d}{dt}g(t, x\left(t - \tau\left(t\right)\right)), \ t \in \mathbb{R}.$$

is considered. By employing Green's function, Krasnoselskii's fixed point theorem, and the contraction mapping principle, we state and prove the existence and uniqueness of periodic solutions to the third-order nonlinear delay differential equation.

Keywords: fixed point, positive periodic solutions, third-order differntial equations.

References

[1] T. A.Burton, *Liapunov functionals, fixed points and stability by Krasnoselskii's theorem, NonlinearStud* .9(2002),No. 2,181-190.

[2] A. Ardjouni and A. Djoudi, *Existence of periodic solutions for a second-order nonlinear neutral differential equations with variable delay,* Palestine J. Math., 3 (2014), no. 2, 191–197.

[3] A. Ardjouni, A. Djoudi and A. Rezaiguia, *Existence of positive periodic solutions for twotypes of third-order nonlinear neutral differential equations with variable delay,* Appl. Math.E-Notes, 14 (2014), 86–96.

[4] Z. Cheng and J. Ren, *Existence of positive periodic solution for variable-coefficient thirdorder differential equation with singularity,* Math. Meth. Appl. Sci., (2014), no. 37, 2281–2289.

[5] H. Deham and A. Djoudi, *Periodic solutions for nonlinear differential equation with functional delay,* Georgian Math. J., 15 (2008), no. 4, 635–642.

P-boxes for uncertainty analysis of classic risk model

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Abstract. This paper introduces sensitivity analysis of the ruin probability in the classical risk model, where we assume that the parameters involved in the definition of the ruin probability are modelled by probability-boxes, which account for both aleatory and epistemic uncertainty. For assessing the ruin probability, we have used the Taylor series expansion methodology, and obtain this one under boxes form. Several numerical examples are presented to illustrate the performance of the proposed method and are compared to the corresponding Monte Carlo simulations ones.

Keywords. Ruin theory; probability-boxes; Epistemic uncertainty; Multivariate Taylor-series expansions; Sensitivity analysis; Monte Carlo simulation.

MSC2010. 58J37; 62G05.

1   Introduction

In risk theory, computational methods of ruin probabilities have received significant attention in the last decade. The reader is referred to the books by Asmussen [1], Grandell [4], and Kaas et al. [3]. A central model in insurance risk theory is the compound Poisson model. Since the pioneering works of Lundberg and Cramï¿½$\frac{1}{2}$r [2]. It has been the object of a number of theoretical studies and practical applications. In this paper, we consider the classical compound Poisson model of risk theory, where the surplus process $R(t)$ of an insurance portfolio at time t is given by

$$R_t = u + c\,t - \sum_{k=1}^{N_t} U_k, \tag{1}$$

with $N(t)$ denoting a homogeneous Poisson process with intensity λ, and $u = R_0$ represents the non-negative initial reserve. The claim sizes $\{U_k, k = 1, 2, \ldots\}$ are positive independent and identically distributed random variables, independent of $\{N_t\}$, and c is a constant premium intensity. Assume that the net profit condition $c > \lambda\mu$ holds; thus the relative security loading $\rho = \lambda\mu/c < 1$.

This model is characterized by the exponential distribution of claim amounts. In this case, the ruin probability $\psi(u; \lambda, \mu)$ can be obtained in computationally tractable exact form

$$\psi(u; \lambda, \mu) = \frac{\lambda\,\mu}{c} e^{-u\left(\frac{c-\lambda\,\mu}{c\,\mu}\right)}.$$

2 Multivariate Taylor-series expansions

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space describing the underlying randomness of the uncertain parameter, where Ω denotes the event space equipped with σ-algebra $\mathcal{F}$ and probability measure $\mathbf{P}$. Consider that a model output $\mathcal{M}(\omega)$ depending on m parameters, $\theta_1, \theta_2, \ldots, \theta_m$, in writing $\mathcal{M}_\theta = \mathcal{M}(\theta_1, \theta_2, \ldots, \theta_m)$. We also assume that $\mathcal{M}(\omega)$ is infinitely differentiable function with respect to each parameter θ_i, $1 \leq i \leq m$. In the sequel, we are typically interested in providing a compact way to establish the Taylor-series expansion of the model output to multiple-parameter. For that, we introduce the following notation of the multi-index $\imath = (\imath_1, \imath_2, \ldots, \imath_m)$, which is an m-tuple of nonnegative integers

$$|\imath| = \imath_1 + \imath_2 + \ldots + \imath_m, \ \imath! = \imath_1! \times \imath_2! \times \ldots \times \imath_m!, \ \theta^\imath = \theta_1^{\imath_1} \times \theta_2^{\imath_2} \times \ldots \times \theta_m^{\imath_m}$$

for $\imath \in \mathbb{N}^m$ and $\theta \in \mathbb{R}^m$.

The Taylor series expansion of the model output $\mathcal{M}(\omega)$ of multiple-parameter $\theta(\omega) = (\theta_1(\omega), \theta_2(\omega), \ldots, \theta_m(\omega))$, is given as follows

$$\mathcal{M}(\omega) \triangleq \mathcal{M}_{\bar{\theta} + \sigma\,\varepsilon(\omega)} = \sum_{|\imath| \geq 0} \frac{(\theta(\omega) - \bar{\theta})^\imath}{\imath!} D^\imath \mathcal{M}_\theta, \tag{2}$$

where $D^\imath \mathcal{M}_\theta$ is the $|\imath|$-th order partial derivative of $\mathcal{M}_\theta$, defined in some neighborhood of θ by

$$D^\imath \mathcal{M}_\theta = \frac{\partial^{|\imath|} \mathcal{M}_\theta}{\partial \theta_1^{\imath_1} \partial \theta_2^{\imath_2} \ldots \partial \theta_m^{\imath_m}}.$$

3 Propagation of uncertainty exponential claims model

In this investigation, we are interested in the analysis of uncertainty in the classical risk model of the Lundberg model, where we consider that the parameters of the model are obtained under epistemic uncertainties. This uncertainty is due to a lack of knowledge about the exact value of the parameter. So, in our analysis, we assume that the parameters model are modelled by intervals [5]. This construction is defined as a Bayesian hierarchical model, where the distribution assigned to the parameters model is replaced by an interval. In this sense, we will use the Taylor series expansions method to assess the ruin probability. The resulting numerical values will be compared to the Monte Carlo corresponding ones.

References

[1] S. Asmussen, *Ruin Probabilities*, World Scientific, Singapore, 2000.

[2] F. Lundberg, *Approximerad Framställning av Sannolikhetsfunktionen.* Aterförsäkring av Kollektivrisker. II. Uppsala: Almqvist & Wiksell, 1903.

[3] R. Kaas, M.J. Goovaerts, J. Dhaene, and M. Denuit. *Modern Actuarial Risk Theory.* Kluwer Acad. Publ., Dordrecht, 2001.

[4] J. Grandell.*Aspects of Risk theory.* S. S. Huebner Foundation Monograph. Springer-Verlag, New York, 1990.

[5] S. Ferson and L. R. Ginzburg, Different methods are needed to propagate ignorance and variability, Reliab. Eng. Sys. Safety **54** (1996), 133–144.

On the finite orthogonality of q-classical polynomials

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Abstract. In this review work I discuss the finite orthogonality of three classes of q-polynomial whose weight functions correspond to Inverse Gamma, Fisher and T-student distributions as $q \to 1$, baised on the polynomial solutions of q-Sturm-Liouville problem.

Keywords. q-orthogonality; q-Sturm-Liouville problems; q-orthogonal polynomials; q-difference equations; Finite sequences of orthogonal polynomials.

MSC2010. 34B24; 39A13; 33C47; 05E05.

1 Introduction

We can consider regular or singular q-Sturm-Liouville problem in the form [3]

$$D_q \left(K(x;q) D_q y_n(x;q) \right) + \lambda_{n,q} w(x;q) y_n(x;q) = 0, \tag{1}$$

where $K(x;q) > 0$, $w(x;q) > 0$ and the q-difference operator is defined by $D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$ where $x \neq 0, q \neq 1$ and $D_q f(0) := f'(0)$. The solutions of the above equation are known as q-orthogonal functions. Therefore, for $n \neq m$, if we have two eigenfunctions of (1), denoted by $y_n(x;q)$ and $y_m(x;q)$, then these functions are orthogonal with respect to a weight function $w(x;q)$. Let $\varphi(x)$ and $\psi(x)$ be two polynomials of degree at most 2 and 1 respectively as

$$\varphi(x) = ax^2 + bx + c \quad and \quad \psi(x) = dx + e \quad (a,b,c,d,e \in \mathbb{C}, \ d \neq 0).$$

If $\{y_n(x;q)\}_n$ is a sequence of polynomials that satisfies the q-difference equation

$$\varphi(x) D_q^2 y_n(x;q) + \psi(x) D_q y_n(x;q) + \lambda_{n,q} y_n(qx;q) = 0, \tag{2}$$

$\lambda_{n,q} \in \mathbb{C}$, $n \in \{0,1,2,\dots\}$, $q \in \mathbb{R} \setminus \{-1,0,1\}$, then the following orthogonality relation holds

$$\int_a^b w(x;q) y_n(x;q) y_m(x;q) d_q x = \left(\int_a^b w(x;q) y_n^2(x;q) d_q x \right) \delta_{n,m},$$

in which $w(x;q)$ is the solution of q-Pearson equation

$$D_q(w(x;q)\varphi(q^{-1}x)) = w(qx;q)\psi(x). \tag{3}$$

Note that $w(x;q)$ is assumed to be positive and $w(q^{-1}x;q)\varphi(q^{-2}x)x^k$ for $k \in \mathbb{N}_0$ must vanish at $x = a, b$. Also if $P_n(x) = x^n + \cdots$ is a monic solution of equation (2), then by equating the coefficients of x^n in (2) we obtain the eigenvalue as

$$\lambda_{n,q} = -\frac{[n]_q}{q^n}(a[n-1]_q + d) \text{ where } [n]_q := \frac{q^n - 1}{q - 1}, \ [0]_q := 0. \tag{4}$$

where the q-number $[n]_q$ is defined by

$$[n]_q := \frac{q^n - 1}{q - 1} \quad and \quad [0]_q := 0.$$

2 Main results

In order to obtain q-analogues of three finite orthogonal polynomials $\{N_n^{(p)}(x)\}_n$, $\{M_n^{(s,t)}(x)\}_n$ and $\{I_n^{(r)}(x)\}_n$ [1], let us consider following specific q-difference equations of type (2)

$$x(q^2x+1)D_q^2y_n(x;q) - ((1-q)^{-1}((q^2-q^p)x+1))D_qy_n(x;q) + \lambda_{n,q}y_n(qx;q) = 0, \quad (5)$$

$$x(qx+1)D_q^2y_n(x;q) - (q\,[s-2]_q\,x + [-t-1]_q)D_qy_n(x;q) + \lambda_{n,q}y_n(qx;q) = 0, \quad (6)$$

$$(q^2x^2+1)D_q^2y_n(x;q) - q^2\,[2p-3]_q\,xD_qy_n(x;q) + \lambda_{n,q}y_n(qx;q) = 0, \quad (7)$$

where $\lambda_{n,q}$ are eigenvalues that can be directly obtained from (4).

Now by applying the q-Sturm-Liouville theorem [3] we obtain that polynomial solutions of these equations are finitely orthogonal with respect to three weight functions which correspond to Inverse Gamma, Fisher and T-student distributions as $q \to 1$.

Theorem 1 *Let $\{N_n^{(p)}(x;q)\}_n$, $\{M_n^{(s,t)}(x;q)\}_n$ and $\{I_n^{(r)}(x;q)\}_n$ be sequences of polynomials that satisfies the q-difference equation (5), (6) and (7) respectively. For $n = 0, 1, \ldots, N$ we have*

$$\int_0^\infty w_1(x;q)N_n^{(p)}(x;q)N_m^{(p)}(x;q)d_qx = \left(\int_0^\infty w_1(x;q)\left(N_n^{(p)}(x;q)\right)^2 d_qx\right)\delta_{n,m},$$

where $q > 1$, $N = \max\{n,m\}$, $N < \frac{1}{2}(p-1)$, $w_1(x;q) = \frac{x^{-p}}{(-q^{-1}x^{-1};\,q^{-1})_\infty}$ and

$(a;q)_\infty = \prod_{k=0}^\infty (1 - aq^k), \quad \text{for} \quad 0 < |q| < 1,$

$$\int_0^\infty w_2(x;q)M_n^{(s,t)}(x;q)M_m^{(s,t)}(x;q)d_qx = \left(\int_0^\infty w_2(x;q)\left(M_n^{(s,t)}(x;q)\right)^2 d_qx\right)\delta_{n,m},$$

where $0 < q < 1$, $t > -1$, $N = \max\{n,m\}$, $N < \frac{1}{2}(s-1)$ and $w_2(x;q) = \frac{x^t}{(-x;q)_{s+t}}$,

$$\int_{-\infty}^\infty w_3(x;q)I_n^{(r)}(x;q)I_m^{(r)}(x;q)d_qx = \left(\int_{-\infty}^\infty w_3(x;q)\left(I_n^{(r)}(x;q)\right)^2 d_qx\right)\delta_{n,m},$$

where $q > 1$, $N = \max\{n,m\}$, $N < r-1$, $(-1)^{2r} = -1$ and $w_3(x;q) = \frac{x^{1-2r}}{(-x^{-2};q^{-2})_{r-\frac{1}{2}}}$.

Proof. First, the weight functions are obtained by solving the q-Pearson equation (3) for (5), (6), (7). Then all results are proved by applying the q-Sturm-Liouville theorem on these equations. ∎

Corollary 2 *The finite set $\{N_n^{(p)}(x;q)\}_{n=0}^{N<\frac{1}{2}(p-1)}$ and $\{M_n^{(s,t)}(x;q)\}_{n=0}^{N<\frac{1}{2}(s-1)}$ are orthogonal with respect to their weight fuctions on $[0,\infty)$. Additionally, $\{I_n^{(r)}(x;q)\}_{n=0}^{N<r-1}$ is a symmetric finite polynomial set which is orthogonal with respect to its weight fuction on $(-\infty,\infty)$. It is also straigh to show that all general properties of these polynomial solutions in the continuous case can be recovered as $q \to 1$.*

References

[1] M. Masjed-jamei, Three finite classes of hypergeometric orthogonal polynomials and their application in functions approximation, Integral Transforms Special Functions, 13(2) (2002) 169–191.

[2] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

[3] M. Masjed-Jamei, F. Soleyman, I. Area and J. J. Nieto, Two finite q-Sturm-Liouville problems and their orthogonal polynomial solutions, Analysis and Mathematical Physics, 32 (2018), 231–244.

Existence and uniqueness of asymptotically w-periodic solutions of recurrent neural networks with time-varying coefficients and mixed delays

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Abstract. In this work, we give sufficient conditions for the existence and uniqueness of asymptotically w-periodic solution of recurrent neural networks with time-varying coefficients and mixed delays. This is done using the w-periodic limit functions and the Banach fixed point theorem.

Keywords. Asymptotically w-periodic functions, limit periodic functions, recurrent neural network, fixed point theorem.

MSC2010. 34K05, 92B20, 34K40, 47H10.

1   Introduction

Neural Networks have been widely studied due to their practical applications in lots of areas of mathematics such as signal and image processing, optimization problems etc ([**?**], [**?**]).

The notion of asymptotically w-periodic was introduced by Henriquez [**?**]. Dimbour in [**?**] has investigated the existence of solution for an evolution differential equation.

Our purpose, in this work, is to apply results of [**?**] for the following integro-differential equations

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} \left(c_{ij}(t) f_j(x_j(t)) + d_{ij}(t) g_j(x_j(t-\tau)) \right.$$

$$+ \sum_{j=1}^{n} p_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s) h_j(x_j(s)) ds + J_i(t), \quad 1 \le i \le n, \tag{1}$$

with initial condition $x_i(s) = \phi(s)$ with $\phi(t) = (\phi_1(t), \phi_2(t), ..., \phi_n(t))^t \in C\left(]-\infty, 0], \mathbb{R}\right)$,

where n corresponds to the number of neuron in a neural network, $x_j(t)$ is the state of the ith neuron at time t. f_j, g_j and h_j are the activation functions of jth neuron. $c_{ij}(t), d_{ij}(t), p_{ij}(t)$ denote respectively the connection weights, the discretely delayed connection weights, and the distributively delayed connection weights between ith neuron and jth neuron at time t, $J_i(t)$ is the external bias on ith neuron, a_i $(a_i > 0)$ is the rate with which the ith neuron will reset its potential to the resting state and τ is the constant discrete time delay.

2   Main results

We make the following assumptions.

(H1) For $j = 1, 2, \ldots, n$, f_j, g_j and $h_j : \mathbb{R} \to \mathbb{R}$, are asymptotically w-periodic functions, and Hï¿½lder continuous i.e there exists constants L_j^f, L_j^g and $L_j^h > 0$ such that for all $x, y \in \mathbb{R}$,

$$|f_j(x) - f_j(y)| < L_j^f |x-y|^{\alpha_j}, \quad |g_j(x) - g_j(y)| < L_j^g |x-y|^{\alpha_j}, |h_j(x) - h_j(y)| < L_j^h |x-y|^{\alpha_j} \ with \ 0 < \alpha_j < 1.$$

Furthermore, we suppose that $f_j(0) = g_j(0) = h_j(0) = 0$ and $\|h_j\|_\infty < +\infty$, $\forall 1 \le j \le n$.

(H2) For all $1 \le i, j \le n$ the functions: $t \longrightarrow c_{ij}(t)$, $t \longrightarrow d_{ij}(t)$ $t \longrightarrow p_{ij}(t)$ $t \longrightarrow J_i(t)$ are asymptotically w-periodic on $\mathbb{R}$.

(H3) For all $1 \le i, j \le n$, the delayed kernels $k_{i,j} : [0, +\infty[\longrightarrow [0, +\infty[$ are asymptotically w-periodic and there exist nonnegative constants k_{ij}^+ and σ such that

$$|k_{ij}(s)| \le k_{ij}^+ e^{-\sigma s}.$$

(H4) For all $1 \le i, j \le n$, we shall use the following notations: $c_{ij}^+ = \sup\limits_{t \in \mathbb{R}}|c_{ij}(t)|$; $d_{ij}^+ = \sup\limits_{t \in \mathbb{R}}|d_{ij}(t)|$; $p_{ij}^+ = \sup\limits_{t \in \mathbb{R}}|p_{ij}(t)|$; $J_i^+ = \sup\limits_{t \in \mathbb{R}}|J_i(t)|$ and we suppose that

$$r = \max_{1 \le i \le n} \left(\frac{\sum\limits_{j=1}^{n}(c_{ij}L_j^f + d_{ij}L_j^g + p_{ij}L_j^h)}{a_i} \right) < 1.$$

Theorem 1 *If (H_1)–(H_4) are satisfied then the delayed RNNs (??) has a unique asymptotically w-periodic in the region*

$$B = B(\varphi_0, r) = \left\{ \varphi \in AP_w(\mathbb{R}^+, \mathbb{R}^n) \ : |\varphi - \varphi_0| \le \frac{r\beta}{1-r} \right\},$$

where

$$\varphi_0(t) = \left(\begin{array}{c} \int_{-\infty}^{t} e^{-(t-s)a_1} J_1(s)ds \\ \vdots \\ \int_{-\infty}^{t} e^{-(t-s)a_n} J_n(s)ds \end{array} \right)^t$$

References

[1] A. Boudour, F. Chï¿½rif, and A.M. Alimi. *Existence and Uniqueness of Pseudo Almost-Periodic Solutions of Recurrent Neural Networks with Time-Varying Coefficients and Mixed Delays.* IEEE Transactions on neural networks and learning systems. 1 (2012).

[2] M. Dimbour and S.Mawaki. *Asymptotically w-periodic solution for an evolution differential equation via w-periodic limit functions.* International Journal of Pure and Applied Mathematics, Volume 113 No. 1 (2017).

[3] H. Henriquez and M. Pierri, P.Tï¿½boas. *On S-asymptotically w-periodic functions on Banach spaces and Applications.* J. Math. Anal. Appl. 343 (2008).

[4] M.S. M'Hamdi, C. Aouiti and A. Touati. *Weighted pseudo almost periodic solutions of shunting inhibitory cellular neural networks with mixed delays.* Acta Math. Sci. 36 (2016).

Study of an evolution problem in a thin domain with Fourier and Tresca boundary conditions

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Abstract. Asymptotic analysis of an incompressible Stokes fuid in a dynamic regime in a three dimensional thin domain Ω^ε with mixed boundary conditions and Tresca friction law is studied in this paper. The problem statement and variational formulation of the problem are reformulated in a fixed domain. In which case, the estimates on velocity and pressure are proved. These estimates will be useful in order to give a specific Reynolds equation associated with variational inequalities and prove the uniqueness.

Keywords. Asymptotic approach, Free boundary problems, Friction law, Reynolds equation, Stokes fluid.

MSC2010. 35R35, 76F10, 78M35, 35B40, 35J85, 49J40.

1 Introduction

The subject of this work is the study of the asymptotic analysis of an incompressible Stokes fluid in a dynamic regime in a three dimensional thin domain Ω^ε. The contact boundary conditions considered here are the mixed boundary conditions (Dirichlet-Fourier) and the friction law is the Tresca type.

We consider a mathematical problem governed to the dynamical fluid equations in a three dimensional bounded domain $\Omega^\varepsilon \subset \mathbb{R}^3$ defined by $\Omega^\varepsilon = \{(x', x_3) \in \mathbb{R}^3; x \in \omega, \ 0 < x_3 < \varepsilon h(x)\}$ and whose boundary Γ^ε, assumed to be sufficiently smooth is partitioned into three disjoint measurable parts. Let ω be a fixed bounded domain of $\mathbb{R}^3$ with equation $x_3 = 0$. We suppose that ω has a Lipschitz continuous boundary and is the bottom of the fluid. The upper surface $\overline{\Gamma}_1^\varepsilon$ is defined by $x_3 = \varepsilon h(x')$, $(x' = (x_1, x_2))$. Let meas $\overline{\Gamma}_1^\varepsilon > 0$, and let $T > 0$ be a time interval. We introduce a small parameter ε, that will tend to zero, and h a smooth bounded function such that $0 < \underline{h} < h(x) < \overline{h}$, for all $(x', 0) \in \omega$. We assume that the Dirichlet boundary conditions on $\overline{\Gamma}_L^\varepsilon$, where $\overline{\Gamma}_L^\varepsilon$ is the lateral surface, the Fourier boundary condition on $\overline{\Gamma}_1^\varepsilon$, the no-flux condition on ω. We also suppose that the non-linear Tresca interface condition on the bottom surface ω and that body forces f act in Ω^ε. With these assumptions, the complete problem we study may be formulated as follows.

$$\frac{\partial u^\varepsilon}{\partial t} - Div\sigma^\varepsilon\left(u^\varepsilon, p^\varepsilon\right) = f^\varepsilon \quad \text{in } \Omega^\varepsilon \times [0, T], \tag{1.1}$$

$$\sigma_{ij}^\varepsilon\left(u^\varepsilon, p^\varepsilon\right) = -p^\varepsilon \delta_{ij} + 2\mu d_{ij}\left(u^\varepsilon\right), \quad \text{in } \Omega^\varepsilon \times [0, T], \tag{1.2}$$

$$divu^\varepsilon = 0, \quad \text{in } \Omega^\varepsilon \times [0, T], \tag{1.3}$$

$$u^\varepsilon = g \quad \text{on } \Gamma_L^\varepsilon \times \,]0, T[, \tag{1.4}$$

$$\left.\begin{array}{l} \sigma_\tau(u^\varepsilon) + l^\varepsilon u^\varepsilon = 0, \\ u^\varepsilon \cdot \nu = 0; \end{array}\right\}, \text{ on } \Gamma_1^\varepsilon \times \,]0, T[, \tag{1.5}$$

$$u^\varepsilon \cdot \nu = 0 \quad \text{on } \omega \times \,]0, T[, \tag{1.6}$$

$$\left\{\begin{array}{l} |\sigma_\tau^\varepsilon| < k^\varepsilon \implies u_\tau^\varepsilon(t) = s \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \implies \exists \beta \geq 0 : u_\tau^\varepsilon(t) = s - \beta\sigma_\tau^\varepsilon \end{array}\right. \quad \text{on } \omega \times \,]0, T[\tag{1.7}$$

$$u^\varepsilon(x, 0) = 0, \quad \forall x \in \Omega^\varepsilon, \tag{1.8}$$

where $g = (g_i)_{1 \leq i \leq 3}$ is the vector function independent of t such that $\int_{\Gamma^\varepsilon} g.\nu d\sigma = 0$, $l^\varepsilon > 0$ is a given constant. The law of conservation of momentum is given by the equation (1.1). Equation (1.2) represents the constitutive law of a Stokes fluid. The incompressibility equation given by (1.3). The condition (1.4) is the Dirichlet boundary. (1.5) represent the Fourier boundary condition. Since there is no-flux condition across ω then we have equation (1.6). Condition (1.7) represents a Tresca thermal friction law on ω where k^ε is the friction coefficient, and finally, the initial data is given by (1.8).

2 Main results

This work is a companion of the result in [1, 5]. The novelty here consist in the fact that we study the asymptotic behavior of the same problem with Tresca free boundary friction conditions but this time in a dynamic regime occupying a bounded homogeneous domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with the mixed boundary conditions (Dirichlet-Fourier). To this aim we use the approach which consists in transposing the problem initially posed in the domain Ω^ε which depend on a small parameter ε in an equivalent problem posed in the fixed domain Ω which is independent of ε. We prove that the limit solution satisfies also a variational inequality. Furthermore, we obtain a weak form of the Reynolds equation and give a lower-dimensional fluid-solid law, prevalent in engineering literature.

References

[1] G. Bayada and M. Boukrouche, *On a free boundary problem for Reynolds equation derived from the Stokes system with Tresca boundary conditions*, J. Math. Anal. Appl., 382 (2003), pp.212–231.

[2] A. Benseghir, H. Benseridi and M. Dilmi, *On the asymptotic study of transmission problem a thin domain*, accepted in J. Inv. Ill-Posed. Prob., Vol. x, No. x, pp. xx-xx 2019.

[3] H. Benseridi and M. Dilmi, *Some inequalities and asymptotic behaviour of dynamic problem of linear elasticity*, Georgian Math. J., 20(1) (2013), pp. 25–41.

[4] M. Boukrouche and R. El mir, *Asymptotic analysis of a non-Newtonian fluid in a thin domain with Tresca law*, Nonlinear Anal. 59(2004), no. 1–2, pp. 85–105.

[5] M. Dilmi, M. Dilmi and H. Benseridi, *Study of generalized Stokes operator in a thin domain with friction law (case p < 2)*, J. Math. Meth. Appl. Sci. 2018;1-10.

On Properties of Mean Values of Solutions to Linear Hyperbolic Equations

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Abstract. We obtained a mean-value formula for a two-dimensional linear homogeneous hyperbolic equation with a factorized symbol. The proven formula can be interpreted as the expansion into the case of an arbitrary order of the well-known mean-value theorem (Asgeirsson principle) for the string vibration equation. In addition, this formula is an exact difference scheme for the specified equations.

Keywords. mean-value formula, accompanying distribution, difference scheme.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1 Introduction

An approach to the proof of mean-value theorems, based on analysis of a differential operator symbol, was proposed in [1], [2] and generalized in [3] by the methods developed in [4]. We use this approach to derive a mean-value formula for the equation

$$\prod_{j=1}^{m} \left(a_j \partial/\partial x + b_j \partial/\partial t + c_j \right) u = 0 \tag{1}$$

with constant coefficients $a_j, j = 1, \ldots, m$.

2 Main result

We consider Eq. (1) with simple characteristics. The line given by the equation

$$b_j x - a_j t = const \tag{2}$$

is called the $j-$th type characteristic of Equation (1), j $= 1, \ldots,$ m. Assume that all characteristics of Eq. (1) are simple. Denote by Θ the set of all ordered collections $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_j \in \{0; 1\}$, $j = 1, \ldots, m$. Let two points $A_l^{\alpha} = (\eta_l^{\alpha}, \omega_l^{\alpha})$ and $A_s^{\gamma} = (\eta_s^{\gamma}, \omega_s^{\gamma})$ lie on the j-th type characteristic iff their upper indexes $\alpha = (\alpha_1, \ldots, \alpha_m), \gamma = (\gamma_1, \ldots, \gamma_m)$, $\alpha_j, \gamma_j \in \{0; 1\}$, $j = 1, \ldots, m$,

are connected with relations $\alpha_k = \gamma_k$, $k = 1, \ldots, j-1, j+1, \ldots, m$, $\gamma_j = \neg \alpha_j$, $s = \neg l = l \oplus 1$, where "$\neg$" is the negation operation. Then for any solution $u(x, t)$ to Eq. (1) the following mean-value formula holds:

$$\sum_{\alpha \in \Theta} (-1)^l u(A_l^\alpha) \prod_{j=1}^m \exp(-\mu_j \xi_j^{\alpha_j} - \nu_j \tau_j^{\alpha_j}) = 0. \tag{3}$$

Let us now waive the requirement of simple characteristics. Then technically the mean-value formula will look like (3), but we have to take $2k$ points on every characteristic instead of two ones, where k is the multiplicity of the characteristic. Furthermore, let us assume that we deal with one m-multiple characteristic and $c_j = 0, j = 1, \ldots m$ in Eq. (1). In this case formula (3) becomes the following "inclusions-exceptions formula" (see [5]) for the polynomial of degree $m - 1$:

$$u(0) = \sum_{k=1}^m (-1)^{k-1} \sum_{i_1 < \cdots < i_k} u(x_{i_1} + \cdots + x_{i_k}), \tag{4}$$

where $x_1, \ldots, x_m$ are points of the real line.

We hope that these results will be useful for many applications. An interesting example of one was considered in [6], where similar ideas were explored.

References

[1] L. Zalcman, Mean values and differential equations, Israel J.Math. **14**(1973), $339 - 352$.

[2] A.V. Pokrovskii, Mean value theorems for solutions of linear partial differential equations, Mathematical Notes (Mat. Zametki), **64**(1998), Issue 2, $260 - 272$ ($220 - 229$).

[3] V.Z. Meshkov and I.P. Polovinkin, On the Derivation of New Mean-Value Formulas for Linear Differential Equations, Differential Equations (Differ. Uravn.), **47**(2011) No 12, $1724 - 1731$.

[4] L. Hörmander, *The Analysis of Linear Differential Operators*, Springer-Verlag, Heidelberg, 1983.

[5] V. Z. Meshkov, I. P. Polovinkin, M. V. Polovinkina, Yu. D. Ermakova, S.A. Rabeeakh, Mean-Value Formula for a Two-Dimensional Linear Hyperbolic Equation, Proceeding of Voronezh State University. Series: Physics. Mathematics, 2015, No 3, $138 - 145$.

[6] V.I. Korzyuk, E.S. Cheb, Le Thi Thu, Solution of the mixed problem for the biwave equation by the method of characteristics. (Russian. English summary), Tr. Inst. Mat., Minsk **18**(2010), No. 2, 36-54 .

A primal-dual interior point algorithm for semidefinite programming based on a new barrier function

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Abstract. Kernel functions play an important role in design and complexity analysis of interior-point algorithms for solving convex optimization problems. In each iteration of these methods, the search direction is determined and the distance between the current point and the central path is measured by a proximity function which is typically induced from a kernel function. In this paper, we present a new parametric kernel function, with an hyperbolic barrier term. A class of large- and small-update primal-dual interior-point methods for semidefinite programming based on this parametric kernel function is proposed. By simple tools, we derive that the iteration bounds for large- and small-update methods are, $\mathcal{O}\left(\sqrt{n}\ln n \ln \frac{n}{\epsilon}\right)$, and $\mathcal{O}\left(\sqrt{n}\ln \frac{n}{\epsilon}\right)$, respectively. The results obtained in this paper are the first for large- and small-update methods based on the hyperbolic kernel functions and coincide with the currently best known iteration bounds.

Keywords. linear semidefinite programming; primal-dual interior point methods; hyperbolic kernel function; complexity analysis; large and small-update methods.

MSC2010. 90C22; 90C51; 90C31.

References

[1] I. Touil, D. Benterki, A. Yassine, A feasible primal-dual interior point method for linear semidefinite programming, J. Comput. Appl. Math. **312** (2017) 216–230.

[2] G.Q. Wang, Y.Q. Bai, C. Roos, Primal-dual interior-point algorithms for semidefinite optimization based on a simple kernel function. Journal Math. Model. Algorithms. 4 (2005) 409-433.

[3] H. Wolkowicz, R. Saigal, L. Vandenberghe, Handbook of Semidefinite Programming, Theory, Algorithms, and Applications, Kluwer Academic Publishers, 2000.

A generalized operational matrix: An application to fractional Differential Equations

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Abstract. The aim of my talk is to study the approximate solution of multi-order fractional differential equations. For this purpose, the operational matrices of fractional order derivatives and integrals in the Caputo and Riemann-Liouville senses is generalized for the shifted Jacobi polynomials (JPs). By means of operational matrices, the underlying problem is reduced to a system of easily solvable algebraic equations which is indeed a more simplified way to deal with the approximate solution of fractional differential equations. The convergence analysis of the proposed numerical method is also a part of my talk. The accuracy of the proposed method is checked by applying the results on some examples. It is observed that the simulated results with literature solutions obtained otherwise, yielding negligible errors. Furthermore, as a result of the comparative study, some results presented in the literature are extended and improved in the investigation herein.

Keywords. Jacobi polynomials; Operational matrices; Generalized fractional coupled systems; Riemann-Liouville integral; Caputo derivative.

MSC2010. 65M99,26A33,35R11,45K05.

1   Introduction

For the last few decades, the subject fractional calculus (FC) has gotten a valuable attention of the researchers due to the generalization of integer order calculus. Special attention has been drawn to solve fractional differential equations (FDEs) analytically [1, 2] but the exact analytical solution does not exist for most nonlinear FDEs. Therefore, several new numerical methods were developed to find their approximate solutions. Among of several, few methods have attained the special attention, like finite difference [7], variational iteration [3], piecewise constant orthogonal functions [8], and orthogonal polynomials [4, 5, 6].

Recently, operational matrices approach was frequently utilized to find the approximate solution of various types of ordinary and partial FDEs. The use of the numerical techniques in conjunction with the operational matrices of some orthogonal polynomials for obtaining the solution of FDEs provide highly accurate solutions for such problems [9, 4, 6, 5]. The idea behind the approach is to reduce the underlying problems into a system of easily solvable algebraic equations, thus greatly simplifying the problems and easy to handle with any computational software.

The aim of this talk is to study the numerical scheme based on the operational matrices of fractional derivatives and integrals of shifted JPs to find the approximate solution of multi order FDEs. Unlike the Lagendre polynomials, the JPs have ability to approximate the solution of problems with the aid of two parameters, which is indeed a more generalized way of approximation. It is worth to mention that the method based on the operational matrices of JPs is computer oriented.

Acknowledgments

References

[1] H. Jiang, F.Liu, I. Turner, K. Burrage, *Analytical solutions for the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain*, Journal of Mathematical Analysis and Applications, 389 (2012), 1117-1127.

[2] H. Jiang, F.Liu, I. Turner, K. Burrage, *Analytical solutions for the generalized multi-term time-fractional diffusion-wave-diffusion equation in a finite domain, Computer and Mathematics with Application, 64 (2012), 3377-3388.*

[3] Z. Odibat, S. Momani, *Application of variational iteration method to nonlinear differential equations of fractional order*, Int. J. Nonlinear Sci. Numer. Simul. 7(2006), 27–34.

[4] A. Saadatmandi, M. Dehghan,*A new operational matrix for solving fractional–order differential equations*, Computers and Mathematics with Applications, 59(3) (2010), 1326–1336.

[5] I. Talib, F.B.M. Belgacem, N.A. Asif, H. Khalil, *On mixed derivatives type high dimensional multi-term fractional partial differential equations approximate solutions*, American Institute of Physics, 1798, 020024 (2017); doi: 10.1063/1.4972616.

[6] I. Talib, C. Tunc, Z.A. Noor, *New operational matrices of orthogonal Legendre polynomials and their operational*, Journal of Taibah University for Science, 13(1), (2019), 377-389.

[7] M. Meerschaert, C. Tadjeran, *Finite difference approximations for two–sided space fractional partial differential equations*, Appl. Numer. Math. 56 (2006), 80–90.

[8] Y. Li, N. Sun, *Numerical solution of fractional differential equations using the generalized block pulse operational matrix*, Computers and Mathematics with Applications, 62(3) (2011), 1046–1054.

[9] E.H. Doha, A.H. Bhrawy, S.S. Ezz–Eldien, *A new Jacobi Operational matrix: an application for solving fractional differential equations.* Applied Mathematical Modeling, 36(10) (2012), 4931–4943.

Generalized Stability for Linear Time-Varying Discrete-time Systems

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Abstract. This article focuses on the problem of generalized exponential stability for linear time-varying discrete-time systems in Banach spaces. Characterizations for this concept in terms of Lyapunov sequences are presented and connections with the classical concept of exponential stability existent in the literature is pointed out. An illustrative example clarifies the implications between these concepts. Finally, certain applications for adjoint system are suggested.

Keywords. LTV systems, exponential stability, difference equations

MSC2010. 34D05, 39A05.

1 Classical definition of exponential stability

In the sequel, the symbol X denotes a real or complex Banach space, $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators from X into itself. The norms of both these spaces will be denoted by $\| . \|$. I is the identity operator on X. $\mathbb{N}$ indicates the set of all positive integers, Δ the set of all pairs (m, n) of integers satisfying the inequality $m > n$ and T the set of all triplets (m, n, p) of positive integers with (m, n) and $(n, p) \in \Delta$.

In this work we deal with LTV discrete-time systems (also called one side system), described by

$$x_{n+1} = A(n)x_n. \tag{$\mathfrak{A}$}$$

where $A : \mathbb{N} \longrightarrow \mathcal{B}(X)$ is a sequence in $\mathcal{B}(X)$. Every solution $x = (x_n)$ of the LTV system ($\mathfrak{A}$) is given by

$$x_m = \mathcal{A}(m, n)x_n, \quad \text{for all} \quad (m, n) \in \Delta,$$

where $A : \Delta \longrightarrow \mathcal{B}(X)$, the so called Cauchy evolution map is given by

$$\mathcal{A}(m, n) = \begin{cases} A(m-1) \cdot \ldots \cdot A(n), & m > n \\ I & , \quad m = n. \end{cases}$$

As well known, this verifies the propagator property

$$\mathcal{A}(m, n)\mathcal{A}(n, p) = \mathcal{A}(m, p), \quad \text{for all} \quad (m, n, p) \in T.$$

In order to be self contain we first recall the notion of uniform exponential stability.

Definition 1 *(see, for example [5]) The LTV system ($\mathfrak{A}$) is said to admit a uniform exponential stability on $\mathbb{N}$ if there are some constants $N \geq 1$ and $\alpha > 0$ such that*

$$\|\mathcal{A}(m, p)x\| \leq Ne^{-\alpha(m-n)}\|\mathcal{A}(n, p)x\|, \tag{1}$$

for all $(m, n, p, x) \in T \times X$.

2   Proposed approach

Further on, we shall consider a strictly positive sequence $(a_n)_{n \in \mathbb{N}}$ satisfying the property

$$\sum_{j=p}^{q} a_j \to +\infty \ \text{ as } \ q \to +\infty \ \text{ for fixed } \ p \in \mathbb{N}. \tag{2}$$

We set $s_n = a_0 + a_1 + \ldots + a_n$, for all $n \in \mathbb{N}$.

Definition 2 *The LTV system ($\mathfrak{A}$) is said to admit a generalized exponential stability on $\mathbb{N}$ if there exists $K \geq 1$ and a strictly positive sequence $(a_n)_{n \in \mathbb{N}}$ satisfying (2) such that*

$$\|\mathcal{A}(m,p)x\| \leq Ke^{-(s_m - s_n)}\|\mathcal{A}(n,p)x\|, \tag{3}$$

for all $(m, n, p, x) \in T \times X$.

Note that: For $a_j = \alpha > 0$, for any $j \in \mathbb{N}$, we obtain the notion of uniform exponential stability.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] S. Elaydi, *An Introduction to Difference Equations*, third ed., in: Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 2005.

[3] L. Lakshmikantham, D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press, New York, 1991.

[4] I.-L. Popa, L. E. Biris, T. Ceauşu, T. Li, *Remarks on Generalized Stability for Difference Equations in Banach Spaces*, Electron. Notes Discrete Math. 70(2018), 77-82.

[5] I.-L. Popa, T. Ceauşu, M. Megan, *On exponential stability for linear discrete-time systems in Banach spaces*, Comput. Math. Appl. 63 (2012), 1497–1503.

[6] I.-L. Popa, T. Ceauşu, O. Bagdasar and R. P. Agarwal, *Characterizations of Generalized Exponential Trichotomies for Linear Discrete-Time Systems*, accepted for publication in An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.

Methods in system theory, differential equations & networks for mathematical modelling

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Abstract. In this talk, firstly I will present my latest results on areas that I currently work on such as singular systems of differential & difference equations, mathematics of networks, optimization, and fractional calculus. Then we will discuss how these results can be applied into mathematical models related to electrical power systems, materials, gas networks, macroeconomics etc.

Keywords. differential; difference; networks; optimization; fractional calculus; power systems; materials; gas.

MSC2010. 26A33; 34A60; 34G25; 93B05.

Acknowledgments

References

[1] I. Dassios, *Stability of Bounded Dynamical Networks with Symmetry*, Symmetry, MDPI, Volume 10, Issue 4, 121 (2018).

[2] I. Dassios, D. Baleanu, *Caputo and related fractional derivatives in singular systems*, Applied Mathematics and Computation, Elsevier, Volume 337, pp. 591–606 (2018).

[3] I. Dassios, *A practical formula of solutions for a family of linear non-autonomous fractional nabla difference equations*, Journal of Computational and Applied Mathematics, Elsevier, Volume 339, Pages 317–328 (2018).

[4] I. Dassios, G. O'Keeffe, A. Jivkov, *A mathematical model for elasticity using calculus on discrete manifolds*. Mathematical Methods in the Applied Sciences, Wiley, Volume: 41, Issue 18, pp. 9057–9070 (2018).

[5] I. Dassios, K. Fountoulakis, J. Gondzio *Preconditioner for a Primal-Dual Newton Conjugate Gradients Method for Compressed Sensing Problems*. SIAM Journal on Scientific Computing, Volume 37, Issue 6, pp. A2783–A2812 (2016).

[6] A. Ekhtiari, I. Dassios, M. Liu, E. Syron *A Novel Approach to Model a Gas Network System.* Applied Sciences (2019).

[7] M. Liu, I. Dassios, F. Milano, *On the Stability Analysis of Systems of Neutral Delay Differential Equations*, Circuits, Systems and Signal Processing (2019).

[8] F. Milano, I. Dassios, *Primal and Dual Generalized Eigenvalue Problems for Power Systems Small-Signal Stability Analysis.* IEEE Transactions on Power Systems, Volume: 32, Issue 6, pp. 4626–4635 (2017).

[9] J. O'Brien, I. Dassios, J. Gleeson, *Spreading of Memes on Multiplex Networks*, New Journal of Physics, IOP Institute of Physics, Volume 21 (2019).

Some open problems on multivariate orthogonal polynomials on nonuniform lattices

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Abstract. In this talk we shall review some recent results of classical multivariate orthogonal polynomials on nonuniform lattices. These polynomials have been presented and deeply studied by Tratnik and more recently by Geronimo and Iliev. Some open problems will be presented.

Keywords. Classical orthogonal polynomials; Multivariate orthogonal polynomials; Multivariate Wilson polynomials; Multivariate q-Racah polynomials.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

Classical orthogonal polynomials have been studied for several differential, difference or divided-difference operators. As recalled in [1, p. 189], in [2] the following definition of univariate classical orthogonal polynomials was suggested:

An orthogonal polynomial sequence is classical if it is a special case or a limiting case of the $_4\phi_3$ polynomials given by the q-Racah polynomials or the Askey-Wilson polynomials.

In all these cases, the corresponding orthogonal polynomials are solution of a second order linear differential (difference or divided-difference operator) of hypergeometric type: the derivatives (differences, or divided-differences) of a solution of the equation are solution of an equation of the same type.

For instance, univariate Racah polynomials can be defined in terms of hypergeometric series as [4, page 190]

$$r_n(\alpha, \beta, \gamma, \delta; s) = r_n(s) = (\alpha + 1)_n \, (\beta + \delta + 1)_n \, (\gamma + 1)_n$$
$$\times \, _4F_3 \left(\begin{array}{c} -n, n + \alpha + \beta + 1, -s, s + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{array} \middle| \, 1 \right), \quad n = 0, 1, \ldots, N, \quad (1)$$

where $r_n(\alpha, \beta, \gamma, \delta; s)$ is a polynomial of degree $2n$ in s and of degree n in the quadratic lattice [1, 6]

$$x(s) = s(s + \gamma + \delta + 1), \tag{2}$$

Univariate Racah polynomials are solution of the second-order linear divided-difference equation [3]

$$\phi(x(s))\mathbb{D}^2 r_n(s) + \tau(x(s))\mathbb{S}\mathbb{D} r_n(s) + \lambda_n r_n(s) = 0, \tag{3}$$

where ϕ is a polynomial of degree two in the lattice $x(s)$, τ is a polynomial of degree one in the lattice $x(s)$, and λ_n are constants, and the difference operators $\mathbb{D}$ and $\mathbb{S}$ are defined as [3, 5, **?**]

$$\mathbb{D}f(s) = \frac{f(s+1/2) - f(s-1/2)}{x(s+1/2) - x(s-1/2)}, \quad \mathbb{S}f(s) = \frac{f(s+1/2) + f(s-1/2)}{2}. \tag{4}$$

If we try to extend this approach to multivariate orthogonal polynomials, it can be used the following definition of classical orthogonal polynomials

A multivariate orthogonal polynomial sequence is classical if it is a special case or a limiting case of the multivariate Racah polynomials or the multivariate Askey-Wilson polynomials.

It is interesting to notice that for continuous case, discrete case and their q-analogues, the corresponding partial differential (difference or q-difference) equation is of second order (hypergeometric, potentially self-adjoint and admissible). On the other hand, for nonuniform lattices the corresponding families are solution of a fourth order divided-difference equation of hypergeometric type.

2 Open problems

The following questions, and many other, arise in a natural way. Is it possible to

1. State a general fourth-order divided-difference equation of hypergeometric type and study its polynomial solutions, both for quadratic and q-quadratic lattices?

2. Study if the fourth-order divided-difference equation is admissible?

3. Study if the fourth-order divided-difference equation is potentially self-adjoint?

Acknowledgments

References

[1] N. M. Atakishiyev, M. Rahman, and S. K. Suslov, *On classical orthogonal polynomials.* Constr. Approx. **11** (1995) 181–223.

[2] G. E. Andrews and R. Askey, *Classical orthogonal polynomials.* In: C. Brezinski et al. (Eds.), Polynômes orthogonaux et applications. Lecture Notes in Mathematics, vol. 1171. Berlin: Springer-Verlag (1985) pp. 36–62.

[3] M. Foupouagnigni, *On difference equations for orthogonal polynomials on nonuniform lattices.* J. Difference Equ. Appl. **14** (2008) 127–174.

[4] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and their q-analogues.* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

[5] A. P. Magnus, *Associated Askey-Wilson polynomials as Laguerre-Hahn orthogonal polynomials.* In: M. Alfaro et al (Eds.), Orthogonal polynomials and their applications (Segovia, 1986), volume 1329 of Lecture Notes in Math. Springer, Berlin (1988) pp. 261–278.

[6] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable.* Springer, Berlin, 1991.

Controllability of Conformable Fractional Differential Systems

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Abstract. This paper deals with complete controllability of systems governed by linear and semilinear conformable fractional differential equations. By establishing conformable fractional Gram criterion and rank criterion, we give sufficient and necessary conditions to examine that a linear conformable fractional system is null completely controllable. Further, we apply Krasnoselskii's fixed point theorem to derive a completely controllability result for a semilinear conformable fractional system. Finally, three numerical examples are given to illustrate our theoretical results.

Keywords. complete controllability; conformable fractional differential systems; gram and rank criterion.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1 Introduction

We study controllability of linear and semilinear conformable fractional control systems governed by

$$\mathfrak{D}_\alpha^0 x(t) = Mx(t) + Qu_1(t), \ t \in J := [0, t_1], \ t_1 > 0, \ x(0) = x_0, \tag{1}$$

and

$$\mathfrak{D}_\alpha^0 x(t) = Mx(t) + f(t, x(t)) + Qu(t), \ t \in J, \ x(0) = x_0. \tag{2}$$

where $\mathfrak{D}_\alpha^0(0 < \alpha < 1)$ denotes the conformable fractional derivative [1] with lower index zero, $M \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times r}$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$. The state $x(\cdot)$ take values from $\mathbb{R}^n$, the control functions $u_1(\cdot)$ and $u(\cdot)$ belong to $L^2(J, \mathbb{R}^r)$.

2 Main results

2.1 Linear systems

We introduce a notation of a conformable fractional Gram matrix as follows:

$$W_c[0, t_1] := \int_0^{t_1} e^{-M\frac{\tau^\alpha}{\alpha}} QQ^\top e^{-M^\top \frac{\tau^\alpha}{\alpha}} d\frac{\tau^\alpha}{\alpha}, \tag{3}$$

where the $\top$ denotes the transpose of the matrix. Then, we give two controllability results.

Theorem 1 *System (1) is null completely controllable if and only if $W_c[0, t_1]$ defined in (3) is nonsingular.*

Theorem 2 *The necessary and sufficient condition for null complete controllability of (1) is*

$$\operatorname{rank} \Gamma_c = n.$$

2.2 Semilinear systems

We introduce the following assumptions:

(A_1) The operator $W : L^2(J, \mathbb{R}^r) \longrightarrow \mathbb{R}^n$ defined by

$$Wu = \int_0^{t_1} e^{M(\frac{t_1^\alpha}{\alpha} - \frac{\tau^\alpha}{\alpha})} Qu(\tau) d\frac{\tau^\alpha}{\alpha}$$

has an inverse operator W^{-1} which takes values in $L^2(J, \mathbb{R}^r) \backslash \ker W$.

Then we set

$$H = \|W^{-1}\|_{L_b(\mathbb{R}^n, L^2(J, \mathbb{R}^r) \backslash \ker W)}.$$

(A_2) The function $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists $L_f(\cdot) \in L_\alpha^q(J, \mathbb{R}^+)$, $q > 1$, i.e., $\int_0^t L_f^q(\tau) d\frac{\tau^\alpha}{\alpha} < \infty$, such that

$$\|f(t, x_1) - f(t, x_2)\| \le L_f(t) \|x_1 - x_2\|, \ x_i \in \mathbb{R}^n, \ t \in J, \ i = 1, 2.$$

Theorem 3 *Assumptions (A_1) and (A_2) are satisfied. Then system (2) is completely controllable provided that*

$$H_2[1 + \frac{H}{2N}(e^{2N\frac{t_1^\alpha}{\alpha}} - 1)^{\frac{1}{2}} \|Q\|] < 1, \tag{4}$$

where $H_2 = [\frac{1}{Np}(e^{Np\frac{t_1^\alpha}{\alpha}} - 1)]^{\frac{1}{p}} \|L_f\|_{L_\alpha^q(J, \mathbb{R}^+)}, \ \frac{1}{p} + \frac{1}{q} = 1, \ p, \ q > 1.$

Acknowledgments

References

[1] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., **264** (2014), 65–70.

On Nonlinear Vibrations of an Elastic Plate on a Fractional Viscoelastic Foundation

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Abstract. In the present paper, the dynamic response of a nonlinear Kirchhoff–Love plate resting on a viscoelastic foundation in a viscoelastic medium, damping features of which are described by the Kelvin–Voigt fractional derivative model, is studied by the generalized method of multiple time scales.

Keywords. fractional viscoelastic foundation; nonlinear vibrations; Kirchoff–Love plate; generalized multiple time scales method.

MSC2010. 74B20; 74H45; 74K20; 74J30.

1   Introduction

The interaction between a loaded plate and the soil foundation is a typical problem in civil engineering. To solve the plate-foundation interaction problem, different viscoelastic models of foundations are used, among them fractional derivative Winkler-type or Pasternak-type models [1], since during last decades the fractional calculus plays an important role in dynamic problems of structural mechanics [2].

In the present paper, the dynamic responce of a nonlinear plate on a viscoelastic fractional Winkler-type foundation is studied using the fractional derivative expansion method [3, 4].

2   Problem formulation and method of solution

Consider a rectangular plate of sides a and b, with mass density ρ, thickness h, and flexural stiffness D resting on a fractional viscoelastic foundation and subjected to a transverse load $q = q(x, y, t)$ dependent on the space coordinates (x, y) and on the time variable t. Then, its transverse displacement $w = w(x, y, t)$ is governed by the following equation of motion:

$$D\nabla^4 w + \rho h \ddot{w} - w_x \phi_y - w_y \phi_x + 2 w_{xy} \phi_{xy} + F_1 + F_2 = q, \tag{1}$$

where ∇^4 is the biharmonic operator and $\phi = \phi(x, y, t)$ is the Airy stress function that is defined by the equation

$$\nabla^4 \phi = Eh \left(w_{xy}^2 - w_{xx} w_{yy} \right), \tag{2}$$

E is plate's Young modulus, $F_1 = \mu \tau_1^{\gamma_1} D^{\gamma_1} w$ is the reaction force of the surrounding medium, $F_2 = \tilde{\lambda} w$ is the reaction of the viscoelastic foundation with the fractional derivative Kelvin-Voigt operator of rigidity $\tilde{\lambda} = \lambda_0 \left(1 + \tau_2^{\gamma_2} D^{\gamma_2} \right)$, λ_0 is the prolonged magnitude of the rigidity coefficient

of the foundation, τ_1, γ_1 and τ_2, γ_2 are the retardation time and fractional parameter for the medium and foundation, respectively, D^γ is the Riemann-Liouville fractional derivative [2] of the order $0 < \gamma \leq 1$, lower indices label the derivatives with respect to the corresponding spaces coordinates, and over dots denote time derivatives.

Governing equation (1) is the fractional derivative type of partial differential equation with two fractional parameters γ_1 and γ_2, which could be solved numerically or analytically utilyzing the generalized multiple scales perturbation method proposed in [3]. In using this method, first we employ the Galerkin discretization method and take the main mode $W_{11}(x,y)$ into account, resulting in the following equation:

$$\ddot{x}_1 + \Omega_{11}^2 x_1 + k_1 \tau_1^{\gamma_1} D^{\gamma_1} x_1 + k_2 \left(1 + \tau_2^{\gamma_2} D^{\gamma_2}\right) x_1 + k_3 x_1^3 = P_1(t), \qquad (3)$$

where $k_j (j = 1, 2, 3)$ are rigidity coefficients, Ω_{11} is the fundamental natural frequency

$$\Omega_{11}^2 = \frac{D\left((\pi/a)^2 + (\pi/b)^2\right)^2}{\rho h}, \quad P_1(t) = \frac{\int\limits_0^a \int\limits_0^b q(x,y,t) W_{11}(x,y) dx dy}{\rho h \int\limits_0^a \int\limits_0^b [W_{11}(x,y)]^2 dx dy}.$$

Equation (3) is the Duffing-type equation with two fractional derivatives, and it could be solved via the perturbation technique proposed in [4]. In developing the multiple scales solution usual assumptions for the first and second time derivatives are adopted while for the fractional derivative expansion we have [2, 3]

$$D^\gamma = \left(\frac{d}{dt}\right)^\gamma = \left(D_0 + \varepsilon D_1 + \varepsilon^2 D_2\right)^\gamma = D_+^\gamma + \varepsilon \gamma D_+^{\gamma-1} + \frac{1}{2}\varepsilon^2 \gamma \left[(\gamma - 1) D_+^{\gamma-2} D_1^2 + 2 D_+^{\gamma-1} D_2\right] \quad (4)$$

with $D_n = \partial/\partial T_n$ $(n = 0, 1, 2...)$ and D_0^γ, $D_0^{\gamma-1}$, $D_0^{\gamma-2}$ denote the Riemann-Liouville fractional time drivatives with respect to $T_0 = t$, $T_n = \varepsilon^n t$, where ε is the small parameter.

3 Conclusion

In the present paper, nonlinear vibrations of a thin elastic plate on a fractional viscoelastic foundation in a viscoelastic medium have been studied, when the motion of the plate is described by a nonlinear differential equation involving two fractional derivatives of different order.

Acknowledgments

References

[1] W. Cai, W. Chen, W. Xu, Fractional modeling of Pasternak-type viscoelastic foundation, Mech. Time-Dependent Mat. **21** (2017), 119–131.

[2] Yu. A. Rossikhin and M. V. Shitikova, Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results, Appl. Mech. Rev. **63** (2010), 010801.

[3] Yu. A. Rossikhin and M. V. Shitikova, Application of fractional calculus for analysis of nonlinear damped vibrations of suspension bridges, J. Eng. Mech. **124** (1998), 1029–1036.

[4] Yu. A. Rossikhin and M. V. Shitikova, New approach for the analysis of damped vibrations of fractional oscillators, Shock Vibr. **16** (2009), 365–387.

Econometric modeling of Bejaia's citrus production from 1983 to 2018

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Abstract. The objective of this work is to model the production of Bejaia citrus fruits according to the different factors that come from macroeconomic policies, commercial policies, technological changes, climatic hazards ... etc. This is why we used an econometric analysis, of which we divided this paper into two sections, in the first we will build the regression equation of the independent variables by estimating the coefficients of the model by the method. ordinary least squares. The second section, it will be devoted to the tests of meaning of the deferent estimated parameters of the model and the validation of our model, using Eviews software.

Keywords. multiple linear regression, parametric estimation, statistical tests.

MSC2010. 60Gxx ; 62Jxx; 62G05; 62J05; 62J12.

1   Introduction

In this paper, we consider a multiple regression model whose equation is given by:

$$LPROD = B_1 * LPRECP + B_2 * LTEMP + B_3 * LSUPER + B_4 * LVEN + B_5 * LHUMD$$

where we used the logarith of each series:

*PROD: The production of citrus fruits from Bejaia (the endogenous variable)

*SUPER: The area: expressed in hectare.

*PRECP: Precipitation: represents the total annual rainfall and / or melted snow expressed in millimeters per square meter.

*TEMP: Temperature: represent average annual temperatures.

* WIND: The wind is the average annual speed expressed in km / h.

*HUMID: Moisture: expressed in %.

The absence of the constant in this model is interpreted economically by the fact that it is impossible to obtain production without a minimum of area, so if all the variables of this study are in the state of nullity the production of citrus fruits it would be too.

References

[1] R.W.M. Johnson,*the aggregate supply of new Zealand farm products*, Economics records, Mai 1955, vol: 31, 60-61.

[2] R.Bourbonnais,*econometrics, Ed Dunod, 2000 in Helene Hamisultane*, econometrics P11.

[3] D.N.GUJARAT, *econometrics of boeck university*, Paris, 2004.

[4] A.W.ASHBY, J.M.JONES, *The relation of price production of pigs.* Journal of the royal agricultural society, 1926; vol: 87, England.

Evaluating the effectiveness of text analyzers

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Abstract. The purpose of the study outlined in the article is to ensure the choice of text analyzers that allow processing large natural-language arrays for making effective control decisions. In this paper an indicator for evaluating the effectiveness of text analyzers is proposed.

Keywords. natural language big data arrays, text analyzer effectiveness, fuzzy inference, integral indicator.

MSC2010. 68T01; 68T50.

1   Introduction

In large organizations, institutions and departments huge amounts of diverse information in the form of texts in natural language are accumulated for many years. Such information resources can be classified as big data containing information relating to a particular company, its goals, objectives, structure, personnel, activities, implemented projects, financial turnovers, reports, future plans, partners, etc. These large data files are stored electronically in various formats (MSWord, MS Excel, txt, pdf, djvu, HTML, etc.) in numerous corporate systems, archives, portals, databases of various departments, electronic document management systems, electronic mail, file directories, etc. In the process of making certain management decisions, it is very important for the manager to consider the information contained in all sources. However, due to the fact that these data volume is extremely large, heterogeneous, not systematized and distributed to different repositories, the manager does not have the ability to use the full information necessary for this to make decisions. According to IBM research, executives have no more than 7 percents of the information required to select the best solution [1]. As a result, the quality of management suffers, effciency and competitiveness of the company decrease. The above problem can be solved if the large and poorly structured natural-language arrays highlight the necessary information, giving the opportunity to the head in a timely manner to get reliable answers to specific questions. For this purpose, special software is used to allow the semantic analysis of texts receiving a request and giving response information in natural language. The basis of such tools is computer technology linguistic processor, aimed at extracting meaning from large arrays of natural language data. To solve this problem, various software solutions, called text analyzers, were developed: for example, Google Desktop, Yandex.Server. For now, AskNet's semantic question-answer search engines, the Russian Context Optimizer software package for the analysis of Russian texts, Ontos, information and analytical system ARION, etc. can be used.

The purpose of the study outlined in the article is to ensure the choice of text analyzers that allow processing large natural-language arrays for making effective control decisions. To achieve this goal is required to solve the task: to offer analytical expressions for evaluating the software text analyzers effectiveness.

2 Main results

The user, trying to understand the details of the problem, forms a question in natural language and sends it to the program text analyzer. The analyzer performs semantic processing of existing text arrays and gives the user response in natural language. It is proposed to evaluate the effectiveness of a software text analyzer based on the calculation of the values of two indicators:

1) Q is quality of the answers given by the analyzer;

2) D is average response time.

The indicator Q should consider the veracity and completeness of the issued response. The indicator Q can take values from 0 to 1. The larger the value Q, the higher the efficiency of the software text analyzer. The indicator D value characterizes the ability of the analyzer to promptly provide answers to user questions. The value D indicates how much time the analyzer takes on average to generate and issue a response. The larger the value, the higher the efficiency of the software text analyzer. Values Q and D are particular indicators of a software text analyzer effectiveness. On their basis, we can propose an integral index for making decisions on the choice of means for processing large arrays of natural language data. The quality levels of the answers given by the analyzer and the promptness of their issuance are difficult to determine by strict numerical criteria. In this case, the fuzzy sets apparatus can be used. Then the quality of the analyzer's answers can be assessed using fuzzy sets "High quality answers" and "Low quality answers", and the promptness of giving answers to user questions may correspond to fuzzy sets "High efficiency answers" and "Low responsiveness answers". It is possible to find the resulting evaluation of a software text analyzer effectiveness by calculating a certain integral index E with the help of a fuzzy inference [2-4]. The value of the integral index is proposed to be calculated using the model corresponding to the zero-order Sugeno algorithm of fuzzy inference [5]. The calculation of the indicator E is the result of fuzzy inference system operation designed to evaluate the effectiveness of a software text analyzer. The higher the value of the integral index, the more efficient the software text analyzer is.

References

[1] SyTech. http://www.sytech.ru/about.php?id=60. Accessed 2019.

[2] N. Rvachova, G. Sokol, K. Polschykov, and J. Davies, Selecting the intersegment interval for TCP in Telecomms networks using fuzzy inference system, Proceedings of the Sixth International Conference 2015 Internet Technologies and Applications (ITA). Glyndwr University, Wrexham, 256-260 (2015).

[3] I. Konstantinov, K. Polshchykov, and S. Lazarev, Model of Neuro-Fuzzy Prediction of Confirmation Timeout in a Mobile Ad Hoc Network, CEUR Workshop Proceedings. Mathematical and Information Technologies **1839**, 174-186 (2017).

[4] I. Konstantinov, K. Polshchykov, and S. Lazarev, The Algorithm for Neuro-Fuzzy Controlling the Intensity of Retransmission in a Mobile Ad-Hoc Network, Int. J. of Applied Mathematics and Statistics **56** (2), 85-90 (2017).

[5] T. Takagi and M. Sugeno, Fuzzy Identification of Systems and Its Applications to Modeling and Control, IEEE Transactions on Systems, Man, and Cybernetics **15**, 116-132 (1985).

The Dirichlet problem on two-dimensional stratified sets.

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Abstract. We consider the Dirichlet problem for harmonic functions on two-dimensional stratified sets, which are assumed for simplicity to be complexes. We show that under certain conditions this problem is Fredholm in the Hölder space and in weighted Hölder spaces of functions satisfying the Hölder condition outside any neighbourhood of the vertex set of the complex and admitting power singularities. We also study the power-logarithmic asymptotics of solutions at these vertices.

Keywords. Dirichlet problem, Hölder class, Fredholm solvability.

MSC2010. 35J56.

1 Introduction

Let us consider a finite number $\mathcal{M}$ of flat convex polygons M that can pairwise intersect only along their sides in space $\mathbb{R}^3$. The set of segments L that are sides of one or more polygons is denoted by $\mathcal{L}$ and the set of their ends by F.

Let each segment $L \in \mathcal{L}$ be the boundary of at most two polygons. Then, the union K of polygons $M \in \mathcal{M}$ considered as closed subsets $\mathbb{R}^3$ is called a two-dimensional complex or network. With respect to the K, elements $M \in \mathbb{M}$ are called facets. The elements $L \in \mathcal{L}$ will be called side if L enters the boundary of one facet, or rib if L enters the boundary of several facets.

Let subset $K^1 \subseteq \mathbb{R}^3$ means the union of all segments $L \in \mathcal{L}$ taken without their ends, so that a closed set $F \cup K^1$ is broken in $\mathbb{R}^3$. Similarly, under the K^2, we agree to understand the union of all facets taken without its boundary.

Let the set $\mathcal{L}$ of all edges consists of two disjoint subsets $\mathcal{L}_D$ and $\mathcal{L}_H$, the first of which consists of some subset of sides. We denote K^1_D and K^1_H the corresponding unions of these edges taken without their ends. Similarly, let F_D consists of points which are the ends of segments $L \in \mathcal{L}_D$. Then $F_H = F \setminus F_D$ consists of points for which all segments $L \in \mathcal{L}$ with this end belong $\mathcal{L}_H$.

The function $u(x) \in C(K \setminus F)$ is called harmonic on the set $K^2 \cup K^1_H$ [1], if inside each facet $M \in \mathcal{M}_L$ it is harmonic, continuously differentiable up to the inner points of the segment L and satisfies the condition

$$\sum_{M \in \mathcal{M}_L} \frac{\partial u}{\partial \nu_M}\bigg|_L = 0.$$

The Dirichlet problem is to find a function $u \in C(K \setminus F)$, that is harmonic in $K^2 \cup K_H^1$ and satisfy the boundary condition

$$u\big|_{K_D^1} = f,$$

where $f \in C(K_D^1)$ is given.

2 Main results

This boundary problem is reduced to nonlocal Riemann boundary value problem, which were studied in detail in [2]. For this problem, we obtained the Fredholm solvability criterion in the Hölder classes with weight $C_{+0}^\mu(K, F)$, $C_{-0}^\mu(K, F)$ and $C_{(+0)}^\mu(K, F)$. We also obtained formula for index, and described the asymptotic behavior of the solution at the vertices of the complex.

Theorem 1 *Let the complex K be a two-dimensional network. Then the index of Dirichlet problem in space $C_{-0}^\mu(K, F)$ and $C_{+0}^\mu(K, F)$ is given by the formulas $\text{æ}_{-0} = n_H$ and $\text{æ}_{+0} = n_H$, respectively. In particular, the index $\text{æ}_{(+0)}$ of this problem in $C_{(+0)}^\mu(K, F)$ is zero.*

Let us consider the asymptotic of the solution near the vertices $\tau \in F$ for a two-dimensional complex K. Let function $f \in C_{+0}^\mu(K_D^1, F_D)$ and the sector with the vertex $\tau \in F_H$ be obtained by intersection of the facet M and the ball $B_\tau = \{|x - \tau| \leq \rho\}$, where $\rho > 0$ is small enough. Then, according [1] and Theorem 1, the function $u(x)$ can be represented as

$$u(x) = a + b \ln|x - \tau| + cA(x - \tau) + u_0(x), \ u_0(x) \in C_{+0}^\mu(M \cap B_\tau, \tau)$$

in this sector.

Other vertices τ belong F_D and the function $u(x)$ can be represented as $u \in C_{+0}^\mu(K \cap B_\tau, \tau)$.

References

[1] Kovaleva L.A., Soldatov A.P. The Dirichlet problem on two-dimensional stratified sets.Izvestiya: Mathematics. **79(1)** (2015), 74–108.

[2] Soldatov. A. P. A function theory method in boundary value problems in the plane. i: the smooth case. Mathematics of the USSR-Izvestiya **39(2):1033** (1992), 1033–1061

A logarithmic barrier method for linear programming using a new minorant function

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Abstract. In this paper We presents a logarithmic barrier method without line search for solving linear programming problem. The descent direction is the classical Newton's one. However, the displacement step is determined by a simple and efficient technique based on the notion of the minorant function approximating the barrier function.

Keywords. Linear Programming; Logarithmic Barrier methods.

MSC2010. 90C22; 90C51.

References

[1] J.P. Crouzeix and B. Merikhi, Alogarithm barrier method for semidefiniteprogramming, RAIRO-Operations Research. **42** (2008), 123–139.

[2] L. Menniche and D. Benterki, A logarithmic barrier apptoach for linear programming, Journal of computational and Applied Mathematics, Appl. Math. Comput. **312** (2017), 267–275.

Stochastic Analysis of M/G/1/1 retrial queue with impatient customers in the orbit

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Abstract. This paper deals with a stochastic analysis of M/G/1/1 retrial queue with impatient customers in the orbit. The steady state of the system has been established by using the embedded Markov chain at departure epochs. Some performance measures have been derived and the existing particular cases have been deduced. Finally, some numerical resuls are presented.

Keywords. Retrial queue; impatience; Imbedded Markov chain; performance measures.

MSC2010. 60K25; 60K30; 90B22.

1   Introduction

Retrial queueing systems are appropriate for modeling the processes in communication network, where a customer finding the server busy tries his demand again after a random time. This increasing interest in this topic is mainly motivated by the development of a new facilities in telecommunication technology. The detailed overviews of the related papers can be found in Falin (1997)[3]. Accessible bibliography is given by Artalejo (2010) [1]. A recent review of this topic can be found also in Kim and Kim (2016) [4] and in Shekhar et al. (2016) [5].

Impatience is a very natural and important concept in queueing models. There is wide range of situations in which customers may become impatient when they not receive service fast enough, for example customers at call centers. Customers waiting in the orbit lose their patience due to the long waiting time for the service. If the secondary customer finds the server idle, before the expiry of timer, he gets the chance to be served and leaves the system after the service completion. If the timer expires before he is getting a chance to be served, he leaves the system without service. Suganthi (2015) [6] analyzed the M/M/1 retrial queue with impatience in the orbit and obtained the steady state probability by using the probability generating function. Azhagappan et al. (2018) [2] analyzed the transient behavior of an M/M/1 retrial queue with regening from orbit.

In this paper we consider an M/G/1/1 retrial queue in which primary customers arrive according to a Poisson process of parameter $\lambda > 0$. There is one server. If the server is busy during a primary arrival, the arriving customer enters the the orbit and becomes a secondary customer. Each secondary customer retries according to a Poisson process of intensity $\theta > 0$. If the server is free when an arrival occurs (either from outside of queueing or from orbit), the arriving customer begins to be served immediately and leaves the system after service completion. The service time distribution function is $B(\cdot)$ (not exponential) for both primary and repeated customers. Let

$B^*(\cdot)$ be its Laplace Stieltjes transform and β_k is the kth moment of the service. The secondary customers in the orbit, activate individually an independent impatience timer which follows an exponential distribution of parameter $\alpha > 0$. If the timer expires, before getting the service, the customer abandons the orbit without receiving service. The inter-arrival durations, inter-retrials times, service times and impatience durations are mutually independent.

2 Main results

Let $N_i = N(t_i)$ be the number of customers in orbit at time t_i of the ith departure (due to the end of service or impatience). We have

$$N_i = N_{i-1} - B_i - L_i + A_i \tag{1}$$

where B_i is a Bernoulli random variable: $B_i = \begin{cases} 1, & \text{if the ith customer is from orbit;} \\ 0, & \text{if the ith customer is primary.} \end{cases}$

L_i is the number of customers from orbit which are lost during the ith service and A_i is the number of primary customers which arrive during the ith service.

Because of the recursive structure of the equation (1), to investigate the ergodicity of the embedded Markov chain under consideration, we use the Foster's criterion, we obtain the chain (N_i) is always ergodic. By taking expectations of both sides (1) and using Jensen's enequality we have

$$N^2 - \left(\frac{\lambda}{\theta\alpha} - \frac{\lambda}{\theta} - \frac{B^*(\theta\alpha)}{1 - B^*(\theta\alpha)} \right) N - \frac{\lambda^2}{\theta^2\alpha} \geq 0, \tag{2}$$

where $N = \mathbb{E}(N_i) = L_0$ is the mean number of customers in the orbit. Let N_+ be the positive root of this equation. Since for $N = 0$, the left hand of this equation is negative, it implies that $N \geq N_+$. We have the mean length of the idle period is $\sum\limits_0^\infty \pi_n \frac{1}{\lambda + n\theta} = \mathbb{E}(\frac{1}{\lambda + \theta N_i})$, where π_n is the stationary distribution of the chain. Then the fraction of the time that the server is busy is $p_1 = \frac{\beta_1}{\beta_1 + \mathbb{E}(\frac{1}{\lambda + \theta N_i})} \leq \frac{\beta_1 + \theta\beta_1 N_+}{1 + \beta_1 + \theta\beta_1 N_+}$. The mean number of the customers in the system is $L = L_0 + p_1$. The mean waiting time in the orbit W_0 and the mean sojourn time in the system W can be determined from L_0 and L using Little's formula: $W_0 = \frac{L_0}{\lambda}$ et $W = \frac{L}{\lambda}$.

References

[1] J. R. Artalejo, Accessible bibliography on retrial queues: Progress in 2000-2009, Mathematical and computer Modelling, **51** (2010), 1071–1081.

[2] A. Azhagappan, E. Veeramari, W. Monica and K. Sonabharathi,Transient solution of an M/M/1 retial queue with regining from orbit, Application and Applied Mathematics, **13**, no.2 (2018), 628–638.

[3] J. I. Falin and J. G. C. Templeton, *Retrial Queues*. Chapmann and Hall.(1997).

[4] J. Kim, and B. Kim, A survey on retrial queueing systems, Annals of Operations Research, **247**, no. 1 (2016), 3–36.

[5] C. Shekhar, A. A. Raina and A. Kumar, A brief review on retrial queue: Progress in 2010-2015, Int. Journal of Pure Applied Sciences and Engineering Research, **5** (2016), 324–336.

[6] P. Suganthi, Retrial queueing system with customer impatience, Global Journal on Pure and Applied Mathematics, **11**, no. 5 (2015), 3177–3188.

Generalized Hankel convolution with two parameters and related shift operator

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Abstract. We introduce the generalized convolution related to the Hankel integral transform. The existance conditions of this convolution are found and the corresponding generalized shift operator is studied. The results are obtained by using the Kakichev approach. The properties and applications of generalized convolution and the corresponding shift operator are considered.

Keywords. Hankel transform; convolutions for integral transforms; generalized convolutions; generalized shift operators.

MSC2010. 44A05; 44A35; 45P05; 47G10.

1 Introduction

In 1967 Valentin Kakichev proposed a method for constructing convolutions for various integral transforms [1]. This method is based on factorization equality. Later, he generalized his approach and introduced the concept of polyconvolution or generalized convolution [2]. Using this concept we can construct the generalized convolutions generated by various linear operators. In particular, the convolutions for the Hankel transform defined by [3]

$$h_\nu[f](s) = \tilde{f}_\nu(s) = \int_0^\infty f(t) j_\nu(st) t^{2\nu+1} dt, \quad \nu > -1/2 \tag{1}$$

can be found. Here the function

$$j_\nu(st) = \frac{2^\nu \Gamma(\nu+1)}{(st)^\nu} J_\nu(st) = \sum_{m=0}^\infty \frac{(-1)^m \Gamma(\nu+1)(st)^{2m}}{2^{2m} m! \Gamma(m+\nu+1)}$$

is associated with the Bessel function J_ν of the first kind of order ν.

The classical convolution for the Hankel transform (1) is well studied and is defined by [3, 4]

$$(f \overset{0}{*} g)(t) = \int_0^\infty f(\tau) \, _0T_t^\tau g \, \tau^{2\nu+1} d\tau \tag{2}$$

if we introduce the shift operator

$$_0T_t^\tau g = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi g(\sqrt{t^2+\tau^2-2t\tau\cos\varphi}) \sin^{2\nu}\varphi \, d\varphi, \tag{3}$$

$$_0T_t^0 g = g(t), \ _0T_0^\tau g = g(\tau).$$

The operator (3) is the generalized shift operator (also called generalized translation operator, or generalized displacement operator) of the Levitan's type [5].

2 Main results

In this work we study one of generalized convolutions for the Hankel transform (1) which depends on two parameters $\nu \in \mathbf{R}, \nu > 0$ and $n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and is definded by

$$(f \overset{n}{*} g)(t) = \frac{\Gamma^2(\nu+1)}{2^{2n}\Gamma^2(\nu+n+1)} \int\limits_0^\infty f(\tau) \, {}_nT_t^\tau g \, \tau^{2(\nu+n)+1} d\tau, \tag{4}$$

where the shift operator

$$_nT_t^\tau g = \frac{c_{\nu,n}}{(t\tau)^n} \int\limits_0^\pi g(\sqrt{t^2+\tau^2-2t\tau\cos\varphi})C_n^\nu(\cos\varphi)\sin^{2\nu}\varphi d\varphi, \tag{5}$$

$$c_{\nu,n} = \frac{2^{2(\nu+n)-1}n!\,\Gamma^2(\nu+n+1)\Gamma(\nu)}{\pi\Gamma(2\nu+n)\Gamma(\nu+1)};$$

$$\tau^n {}_nT_t^\tau g \bigg|_{\tau=0} = 0, \ t^n {}_nT_t^\tau g \bigg|_{t=0} = 0, \ \forall n > 0.$$

Here $C_n^\nu(z)$ is the n-th degree Gegenbauer polynomial with parameter ν.

If $n = 0$ then we get the classical convolution (2) and the corresponding shift operator (3).

The factorization equality for the generalized convolution (4) is written as

$$h_{\nu+n}[f \overset{n}{*} g](s) = h_{\nu+n}[f](s)h_\nu[g](s). \tag{6}$$

The problem of the convolution existence and the validity of factorization equality we consider in weighted Lebesgue spaces. For example, the following theorem is proved.

Theorem 1 *If* $f(t) \in L_2\left(\mathbf{R}_+; t^{2(\nu+n)+1}dt\right)$ *and* $g(t) \in L_1\left(\mathbf{R}_+; t^{\nu+1/2}dt\right)$, $\nu > 0$, $n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ *then the convolution* (4) *exists and the following estimation is valid.*

$$t^{2n+1}|(f \overset{n}{*} g)(t)|^2 \leq \hat{c}_{\nu,n} \, ||f||^2_{L_2\left(\mathbf{R}_+; t^{2(\nu+n)+1}dt\right)}||g||^2_{L_1\left(\mathbf{R}_+; t^{\nu+1/2}dt\right)},$$

where

$$\hat{c}_{\nu,n} = Const \cdot \frac{2^{2\nu-1}\Gamma(\nu)\Gamma^2(\nu+1)\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(\nu+1/2)\Gamma(2\nu+n+1/2)}.$$

We also consider the properties of the shift operator (5), its action at particular values ν and n, and some applications of the generalized convolution (4) and the corresponding shift operator.

References

[1] V.A. Kakichev, On the convolutions for integral transforms, Izv. Acad. Nauk BSSR. Ser. Fiz.-Mat. Nauk, No. 22 (1967), 48 – 57 (in Russian).

[2] V.A. Kakichev, *Polyconvolution. Definition, examples, convolution equations: lecture notes*, Taganrog State University of Radio Engineering, Taganrog, 1997 (in Russian).

[3] I.A. Kipriyanov, *Singular Elliptic Boundary Value Problems*, Nauka, Moscow, 1997 (in Russian).

[4] Ya.I. Zhitomirskii, Cauchy's problem for systems of linear partial differential equations with differential operators of Bessel type, Mat. Sb. (N.S.) **36** (1955), 299–310 (in Russian).

[5] B.M. Levitan, The application of generalized displacement operators to linear differential equations of the second order, Uspekhi Mat. Nauk. **4**:1(29) (1949), 3–112 (in Russian).

On Nano Topological Model Induced by New Class of Generalized Closed sets

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Abstract. Topology is a branch of Mathematics, grew out of geometry, expanding and loosening some of the ideas and structures appearing therein. General topology or point set topology is one of the most basic and traditional division within topology which studies the topological properties along with its structures. In 1970, Levine introduced generalized closed sets (briefly g-closed sets) in a topological space in order to extend many of the important of closed sets to a larger family. In the recent past, there has been considerable interest in the study of various forms of generalized closed sets. Ever since, general topologists enhanced the study of generalized closed sets on the basis of generalized open sets. By considering other generalized closure operators or classes of generalized open sets, different notions analogous to Levine's g- closed sets have been analyzed. Lellis Thivagar et al introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower and upper approximations of X. The main objective of this article is to derive and establish the concept of nano generalized closed sets by comparing the nano closure of a set with its nano open supersets. Further the characterizations of nano g-open sets are discussed like. Also we study the behaviour of nano continuous and nano closed (nano-open) functions on nano generalized closed sets.

Keywords. Nano topology, nano closed sets, nano generalized closed sets, nano continuous functions, nano closed functions.

MSC2010. 54B05; 54A05.

References

[1] Caldas, *On g-closed sets and g-continuous mappings*, Kyung pook Math.jr., **33**, (1993), 205-209.

[2] J. Cao, M. Gangster and I.L. Reiley, *On generalized closed sets*, Topology & Appl., **123**, (2001), 37-46.

[3] M. Lellis Thivagar and Carmel Richard, *On Nano Forms of weakly open sets*, International Journal of Mathematics and Statistics Invention, **1 (1)**, (2013), 31-37.

[4] M. Lellis Thivagar and Carmel Richard, *On Nano Continuity in a Strong Form*, International Journal of Pure and Applied Mathematics, **101(8)**, (2015), 893-904.

[5] M. Lellis Thivagar and J. Kavitha, *On Nano Resolvable spaces*, Missouri. Journal of Math. Sci., **29**, (2017), 80-91.

[6] N. Levine, *Generalized closed sets in topology*, Rend.Circ.Mat. Palermo, **19**, (1990),410-415.

Approximating the Solution of Some Advanced-Retarded Differential Equations Using a Technique Based on HAM

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Abstract. The mixed type functional differential equations (MTFDEs), equations with both delayed and advanced arguments, appear in many mathematical models in several contexts of applied sciences such as biology, quantum physics, economy, control, acoustics, aerospace engineering. This paper provides a technique to solve a functional nonlinear mixed differential equation. The proposed technique is based on homotopy analysis method. We analyze the performance and accuracy of the proposed method and compare with the results obtained previously using other numerical methods.

Keywords. mixed-type functional differential equations; non-linear equations; numerical approximation; homotopy analysis method.

MSC2010. 34K06; 34K10; 34K28; 65Q05.

1   Introduction

In applied sciences, a wide number of mathematical models show up functional differential equations with delayed and advanced arguments, the mixed type functional differential equations.

Functional differential equations with delay-advanced argument appear in a wide array of different areas of knowledge such as optimal control [9], economic dynamics [10], nerve conduction [3, 4, 7], traveling waves in a spatial lattice [1, 8], quantum photonic systems [2], acoustics [14, 15, 13], aerospace engineering. We are particularly interested in the numerical approximation of the a multi-delay-advance differential equation form (1)

$$x'(t) = F(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_n)), \tag{1}$$

where the shifts τ_i may take negative or positive values.

Some recent numerical methods to approximate the solution of a linear MTFDE with form (2) were improved in [5, 11, 6]. More recently, these methods were adapted and used to solve numerically a nonlinear MTFDE [7, 12], the FitzHugh-Nagumo equation (2)

$$x'(t) = F(t, x(t), x(t - \tau), x(t + \tau)), \quad \tau > 0. \tag{2}$$

This paper provides a technique based on HAM to solve the FitzHugh-Nagumo equation. The results obtained by HAM have enough accuracy when compared with the results in [12]. This work is still ongoing, the analysis of actual simulations shows that the results are promising, when compared with the others methods.

Acknowledgments

References

[1] K. A. Abell, C. E. Elmer, . A. R. Humphries, E. S. Van Vleck, Computation of mixed type functional differential boundary value problems , *S*IADS - Siam Journal on Applied Dynamical Systems, **4**(3) (2005), 755.

[2] U. Alvarez-Rodriguez and et al., Advanced-Retarded Differential Equations in Quantum Photonic Systems, Scientific Reports (2017). doi: 10.1038/srep42933

[3] J. Bell, Behaviour of some models of myelinated axons, IMA journal of mathematics applied in medicine and biology **1** (1984), 149.

[4] H. Chi, J. Bell, B. Hassard, Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction, *J*ournal of Mathematical Biology **2**4 (1986), 583.

[5] P. M. Lima, M. F. Teodoro, N. J. Ford, P. M. Lumb, Analytical and Numerical Investigation of Mixed Type Functional Differential Equations, Journal of Computational and Applied Mathematics **2**34(9) (2010), 2732.

[6] P. M. Lima, M. F. Teodoro, N. J. Ford, P. M. Lumb, Finite Element Solution of a Linear Mixed-Type Functional Differential Equation, Numerical Algorithms **5**5 (2010), 301.

[7] P. M. Lima, M. F. Teodoro, N. J. Ford, P. M. Lumb , in: S. Pinelas et al. (Ed.) Analysis and Computational Approximation of a Forward-Backward Equation Arising in Nerve Conduction, ICDEA 2011, Azores, Springer Proc in Mathematics & Statistics, NY (2013), **47**, 475.

[8] J. Mallet-Paret, The Global Structure of Traveling Waves in Spattially Discrete Dynamical Systems, Journal of Dynamics and Differential Equations **1**1(1) (1999), 49.

[9] L. S. Pontryagin, R. V. Gamkreledze, E. F. Mischenko, *T*he mathematical Theory of Optimal Process, Interscience, New York, 1962.

[10] A. Rustichini, Hopf bifurcation for functional differential equations of mixed type , Journal of Dynamics and Differential Equations, **1**(2) (1989), 145.

[11] M. F. Teodoro, N. J. Ford, P. M. Lima, P. M. Lumb, New approach to the numerical solution of forward-backward equations , Frontiers of Math on China, **4(1)** (2009), 155.

[12] M. F. Teodoro, Numerical Solution of a Delay-Advanced Differential Equation from Physiology, Applied Mathematics & Information Sciences **11(5)** (2017), 1287-1297.

[13] M. F. Teodoro, Numerical Solution of a Delay-Advanced Equation from Acoustics , International Journal of Mechanics **11** (2017), 107-114.

[14] I. R. Titze, *P*rinciples of Voice Production, Prentice-Hall, Englewood Cliffs, 1994.

[15] I. R. Titze, The Physics of Small Amplitude Oscillation of the Vocal Folds , The Journal of the Acoustical Society of America **83** (1988), 1536.

A Computational Technique of Nano Topological Model via Graphs

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Abstract. The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together.The antiquity of graph theory may be traced back to the year 1735, when the Swiss mathematician Leonhard Euler solved the Konigsberg bridge problem which is believed to be one of the first academic treatises in modern topology.The history of graph theory as well as topology are closely related. These two concepts share many common techniques and have common problems.Euler, in his work on the Konigsberg bridge problem referred as geometria situs the "geometry of position" whereas in the development of topological concept in the second half of the 19th century it is known as analysis situs the "analysis of position".Basing on the topological ideas, Lellis Thivagar[3,4] developed a new topology called "Nano Topology" in 2012. The word 'nano' is used simply to mean 'very small', for example Nano car. The word nano is prefixed to topology because of its size. Whatever may be the size of the universe it will have atmost five elements only in it. The use of nano topological ideas to explore various aspects of graph theory and viceversa, is a fruitful area of research. This paper proposes a new approach for nano topological space via closure and interior operator in simple digraphs.The basic motto of this paper is to impart the importance of Nano topology induced by graph which is taken as catalyst to find the domination number as well as all the possible dominating sets of a graph. Furthermore the computational and algorithmic aspects of graph is emphasized. To simplifying the problem I have used a Java programme code that suits even for a large graph of finite number of vertices. In order to enlighten the nano topological model which is helpful for applying this theory in practical issues.

Keywords. Nano topology, lower and upper approximation spaces, dominating sets, minimum dominating set.

MSC2010. 54A05; 54C08.

References

[1] R. Belohlavek, *Fuzzy closure operator*, Journal of math.Analysis and Application, **262**, (2001), 473-491.

[2] J. Bondy and D.S. Murty, *Graph theory with Applications*, North-Holland, (1992).

[3] M. Lellis Thivagar and V. Sutha Devi, *On Multi-granular nano topology*, South East Asian Bulletin of Mathematics, **40**, (2016), 875-885.

[4] M. Lellis Thivagar, Paul Manuel and V. Sutha Devi , *A Detection for Patent infringement Suit Via Nano Topology induced by Graph*, Cogent Mathematics, (Taylor and Francis), **3** (2016),1161129.

[5] Z. Pawlak, *Rough sets*, International Journal of Information and computer sciences, **11**, (1982), 341-356.

[6] D. Pei, *On definable concepts of Rough set models*, Information Sciences, **177**,(2007), 4230-4239.

Optimal control of a nonlocal thermistor problem with nonlocal and nonsingular Mittag-Leffler kernel

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Abstract. We study an optimal control problem associated to a fractional nonlocal thermistor problem involving the ABC (Atangana–Baleanu–Caputo) fractional time derivative. We first prove the existence and uniqueness of solution. Then, we show that an optimal control exists. Moreover, we obtain the optimality system that characterizes the control.

Keywords. fractional derivatives; optimal control.

MSC2010. 26A33, 35A01, 35R11, 49J20

Fractional calculus is a powerful mathematical tool to describe real-world phenomena with memory effects, being used in many scientific fields. Many published works in fractional calculus put emphasis on the Riemann–Liouville power-law differential operator; others suggest different fractional approaches of mathematical modeling to represent physical problems, calling attention that a singularity on the power law leads to models that are singular, which is not convenient for those with no sign of singularity. In particular, several applications of the exponential kernel suggested by Caputo and Fabrizio can be found in chemical reactions, electrostatics, fluid dynamics, geophysics and heat transfer.

The Riemann–Liouville fractional derivative seems not the most appropriate to describe diffusion at different scales. Thanks to the non-obedience of commutativity and associativity criteria, and due to Mittag–Leffler memory, the ABC fractional derivative promises to be a powerful mathematical tool, allowing to describe heterogeneity and diffusion at different scales, distinguishing between dynamical systems taking place at different scales without steady state. Here, we are interested to study an optimal control problem to a nonlocal parabolic boundary value problem.

Optimal control of problems governed by partial differential equations occurs more and more frequently in different research areas. Researchers are interested, essentially, to existence, regularity, and uniqueness of the optimal control problem, as well as necessary optimality conditions. The optimal control theory for systems of thermistor problems with integer-order derivatives on time has been developed. Works on control theory applied to fractional differential equations, where the fractional time derivative is considered in Riemann–Liouville and Caputo senses, also have been already studied. However, to the best of our knowledge, the use of the Atangana–Baleanu derivative is underdeveloped in this area. Particularly, we are not aware of any paper investigating the optimal control of our problem. In our work, we choose the heat transfer coefficient as a control, because it plays a crucial role in the temperature variations of a thermistor.

We are essentially interested to the existence and uniqueness results for the problem in consideration. we after investigate the corresponding control problem. Main results characterize, explicitly, the optimal control, extending those existing in the literature.

Acknowledgments

References

[1] A. Atangana and D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, Thermal Science **20** (2016), no. 2, 763–769.

[2] A. Atangana and J. F. Gòmez-Aguilar, Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena, Eur. Phys. J. Plus (2018) 133: 166.

[3] D. Baleanu, A. Mousalou and S. Rezapour, The extended fractional Caputo–Fabrizio derivative of order $0 \leq \sigma < 1$ on $C_{\mathbb{R}}[0,1]$ and the existence of solutions for two higher-order series-type differential equations, Adv. Difference Equ. **2018**, 2018:255.

[4] V. Hrynkiv, S. Lenhart and V. Protopopescu, Optimal control of a convective boundary condition in a thermistor problem, SIAM J. Control Optim. **47** (2008), no. 1, 20–39.

[5] H.-C. Lee and T. Shilkin, Analysis of optimal control problems for the two-dimensional thermistor system, SIAM J. Control Optim. **44** (2005), no. 1, 268–282.

[6] M. R. Sidi Ammi, Optimal control for a nonlocal parabolic problem resulting from thermistor system, Int. J. Ecol. Econ. Stat. **9** (2007), no. F07, 116–122.

[7] M. R. Sidi Ammi and D. F. M. Torres, Optimal control of nonlocal thermistor equations, Internat. J. Control **85** (2012), no. 11, 1789–1801.

Control of some ferromagnetic structures at elevated temperature

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Abstract. The Landau-Lifshitz-Bloch equation describes the dynamics of magnetization inside a ferromagnet at elevated temperature. This equation is nonlinear and has an infinite number of stable equilibria. It is desirable to control the system from one equilibrium to another. A control theory will be presented and the results will illustrated with some simulations.

Keywords. control theory; ferromagnetic microstructures; Landau-Lifshitz-Bloch equation.

MSC2010. 78A25. 93C10. 82D40

The influence of thermal excitations on magnetic materials is a topic of increasing relevance in the theory of magnetism. The Landau-Lifshitz-Bloch (LLB) dynamical equation of motion for macroscopic magnetization vector [7] has recommended itself as a valid micromagnetic approach at elevated temperatures [5], especially useful for temperatures θ close to the Curie temperature θ_c ($\theta > 3\theta_c/4$) and ultrafast timescales. In several exciting novel magnetic phenomena this approach has been shown to be a necessary tool. These phenomena include laser-induced ultrafast demagnetization, thermally driven domain wall motion, spin-torque effect at elevated temperatures or heat-assisted magnetic recording, we refer to [1], [4] for physical issues and derivation of the LLB model. The LLB equation has an infinite number of stable equilibria. Steering the system from one equilibrium to another is a problem of both theoretical and practical interest. Since the objective is to steer between equilibria, approaches based on linearization are not appropriate. It is proven that affine proportional control can be used to steer the system from an arbitrary initial state, including an equilibrium point, to a specified equilibrium point.

Acknowledgments

References

[1] U. Atxitia, D. Hinzke and U. Nowak. *Fundamentals and applications of the Landau-Lifshitz-Bloch equation. Journal of Physics D: Applied Physics,* 50, 3. - 033003, 2017.

[2] U. Atxitia, P. Nieves, O. Chubykalo-Fesenko. *The Landau-Lifshitz-Bloch equation for ferri-magnetic materials. Phys. Rev. B,* 86, 104414, 2012.

[3] C. Ayouch, El-H. Essoufi, and M. Tilioua. *Global weak solutions to a spatio-temporal fractional Landau-Lifshitz-Bloch equation. Computers & Mathematics with Applications,* 77(5):1347-1357, 2019.

[4] A. Berti and C. Giorgi. *Derivation of the Landau-Lifshitz-Bloch equation from continuum thermodynamics. Physica B: Condensed Matter,* 500:142–153, 2016.

[5] O. Chubykalo-Fesenko, U. Nowak, R. W. Chantrell and D. Garanin *Dynamic approach for micromagnetics close to the Curie temperature. Phys. Rev. B,* 74, 094436, 2006.

[6] D. A. Garanin. *Generalized equation of motion for a ferromagnet. Physica A: Statistical Mechanics and its Applications,* 172(3):470–491, 1991.

[7] D. A. Garanin. *Fokker-Planck and Landau-Lifshitz-Bloch equations for classical ferromagnets. Physical Review B - Condensed Matter and Materials Physics,* 55(5):3050–3057, 1997.

Uniform attractors for weak solutions of nonlinear viscous fluid motion model

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Abstract. In this paper, based on the theory of uniform attractors of non-invariant trajectory spaces the existence of weak solutions attractors for non-autonomous nonlinear viscous motion model is studied. The existence of a minimal uniform trajectory attractor as well as a uniform global attractor of the family of trajectory spaces is proved for the model under consideration.

Keywords. nonlinear viscous fluid; attractor; uniform attractor; non-invariant trajectory spaces; weak solutions.

MSC2010. 34A34; 34A45; 47J05.

1   Introduction

The following initial boundary value problem (see [1]) is considered, which describes the motion of a nonlinear viscous fluid in a bounded region $\Omega \in \mathbb{R}^n$ ($n = 2, 3$)

$$\frac{\partial v(t,x)}{\partial t} + \sum_{i=1}^{n} v_i(t,x) \frac{\partial v(t,x)}{\partial x_i} - \text{Div}\{2\phi(I_2(v(t,x)))\varepsilon(v(t,x))\} + \text{grad}\, p(t,x) = f(t,x); \quad (1)$$

$$\text{div}\, v = 0, \ (t,x) \subset (0; +\infty) \times \Omega; \quad v(0,x) = v^0, \ x \in \Omega; \quad (2)$$

$$v(x,t) = 0, \ (t,x) \subset (0; +\infty) \times \partial\Omega. \quad (3)$$

Here $v = (v_1(t,x), \ldots, v_n(t,x))$, $n = 2, 3$, and $p(t,x)$ is the velocity vector function and fluid pressure, respectively, $f(t,x)$ is the density of external forces. Given a continuously differentiable scalar function ϕ, defined on the interval $[0, \infty)$, characterizes viscosity.

Consider the set $\mathcal{X} = L_2^{loc}(\mathbb{R}_+; V^*)$. Space $\mathcal{X}$ is Banach with norm $\|\varphi\|_{\mathcal{X}} = \sup_{t \geq 0} \|\varphi\|_{L_2(t,t+1;V^*)}$. Choose an arbitrary set $\Sigma \subset \mathcal{X}$ so that $f \in \Sigma$ and for any element $\sigma \in \Sigma$, the inequality $\|\sigma\|_{\mathcal{X}} \leq \|f\|_{\mathcal{X}}$ holds. Define family of trajectory spaces $\{\mathcal{H}_\sigma^+ : \sigma \in \Sigma\}$.

Definition 1 *The space of trajectories $\mathcal{H}_\sigma^+$ of the system (1)-(3), corresponding to the symbol $\sigma \in \Sigma$, we call the set of functions v from $L_2^{loc}(\mathbb{R}_+, V) \cap L_\infty(\mathbb{R}_+, H)$ such that the restriction of v on arbitrary interval $[0, T]$ is the weak solution of this problem with some initial condition $v(0) = v^0$, $v^0 \in H$ and satisfy for almost all $t \geq 0$ assessment*

$$\frac{1}{2}\|v(t)\|_{L_\infty(t,t+1;H)}^2 + \frac{C_1}{2}\|v(t)\|_{L_2(t,t+1;V)}^2 \leq$$

$$\leq e^{-2\gamma t}\|v\|_{L_\infty(\mathbb{R}_+;H)}^2 + \frac{5e^{2\gamma} - 1}{2C_1(e^{2\gamma} - 1)} \sup_{s \in [0,T-1]} \|f\|_{L_2(s,s+1;V^*)}^2,$$

where γ is some constant greater than zero.

Note that each of the spaces $\mathcal{H}_\sigma^+$ belongs to $C(\mathbb{R}_+; V^*) \cap L_\infty(\mathbb{R}_+; H)$, moreover, for any $v^0 \in H$ and $\sigma \in \Sigma$, there exists a trajectory $v \in \mathcal{H}_\sigma^+$ such that $v(0) = v^0$.

2   Main results

Theorem 1 *Let the function f be an element of the space $\mathcal{X}$, and let the set of symbols $\Sigma \subset \mathcal{X}$ containing f be such that for every $\sigma \in \Sigma$ the condition $\|\sigma\|_\mathcal{X} \leq \|f\|_\mathcal{X}$ is satisfied. Then there exists a minimal uniform trajectory attractor $\mathcal{U}$ of the family of trajectory spaces $\{\mathcal{H}_\sigma^+ : \sigma \in \Sigma\}$.*

Theorem 2 *Under the conditions of Theorem 1, in the space H there exists a uniform global attractor of the family of trajectory spaces $\{\mathcal{H}_\sigma^+ : \sigma \in \Sigma\}$.*

Proof. For the existence of a uniform global attractor of the family of the trajectory space, the existence of a minimal uniform trajectory attractor is sufficient (see [2]). However, the existence of the latter is established by Theorem 1. ∎

Acknowledgments

References

[1] Litvinov V. G. Motion of a nonlinear viscous fluid / V. G. Litvinov - Moscow: Science. The main edition of the physical and mathematical literature, 1982. - 376 p.

[2] D. A. Vorotnikov, V. G. Zvyagin. Uniform attractors for non-automous motion equations of viscoelastic medium // J. Math. Anal. Appl., 325(2007), 438-458.

Strong Solutions for a Class of Semilinear Equations with Lower Fractional Derivatives

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Abstract. We find conditions of a unique strong solution existence for the Cauchy problem to solved with respect to the highest fractional Gerasimov — Caputo derivative semilinear fractional order equation in a Banach space with nonlinear operator, depending on the lower Gerasimov — Caputo derivatives. Then the generalized Showalter — Sidorov problem for semilinear fractional order equation in a Banach space with a degenerate linear operator at the highest order fractional derivative is researched in the sense of strong solution. The nonlinear operator in this equation depends on time and on lower fractional derivatives. The corresponding unique solvability theorem was applied to study of linear degenerate fractional order equation with depending on time linear operators at lower fractional derivatives. Applications of the abstract results are demonstrated on examples of initial-boundary value problems to partial differential equations with time-fractional derivatives.

Keywords. fractional order differential equation, fractional Gerasimov — Caputo derivative, degenerate evolution equation, Cauchy problem, generalized Showalter — Sidorov problem, initial boundary value problem.

MSC2010. 34G20; 35R11.

Consider the semilinear equation of fractional order

$$D_t^\alpha Lx(t) = Mx(t) + N(t, D_t^{\alpha_1}x(t), D_t^{\alpha_2}x(t), \dots D_t^{\alpha_n}x(t)), \quad t \in (t_0, T), \tag{1}$$

where $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ (linear and continuous operator from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$), $M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$ (linear closed operator with dense domain D_M in the space $\mathcal{X}$ and with the image in $\mathcal{Y}$), $n \in \mathbb{N}$, $N : \mathbb{R} \times \mathcal{X}^n \to \mathcal{Y}$ is a nonlinear operator, $D_t^\alpha, D_t^{\alpha_1}, D_t^{\alpha_2}, \dots, D_t^{\alpha_n}$ are the fractional Gerasimov — Caputo derivatives, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le m - 1 \le \alpha \le m \in \mathbb{N}$. The equation is supposed to be degenerate, i. e. $\ker L \ne \{0\}$.

Denote $g_\delta(t) = \Gamma(\delta)^{-1}t^{\delta-1}$, $\tilde{g}_\delta(t) = \Gamma(\delta)^{-1}(t - t_0)^{\delta-1}$, $J_t^\delta h(t) = (g_\delta * h)(t) = \int\limits_{t_0}^{t} g_\delta(t - s)h(s)ds$

for $\delta > 0$, $t > 0$. Let $m - 1 < \alpha \le m \in \mathbb{N}$, D_t^m is the usual derivative of the orderï¿½$\frac{1}{2}$ $m \in \mathbb{N}$, J_t^0 is the identical operator. The Gerasimov — Caputo derivative of a function h is

$$D_t^\alpha h(t) = D_t^m J_t^{m-\alpha}\left(h(t) - \sum_{k=0}^{m-1} h^{(k)}(t_0)\tilde{g}_{k+1}(t)\right), \quad t \ge t_0.$$

Define L-resolvent set $\rho^L(M) := \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})\}$ of an operator M and its L-spectrum $\sigma^L(M) := \mathbb{C}\backslash\rho^L(M)$, and denote $R_\mu^L(M) := (\mu L - M)^{-1}L$, $L_\mu^L := L(\mu L - M)^{-1}$.

An operator M is called (L, σ)-*bounded*, if

$$\exists a > 0 \quad \forall \mu \in \mathbf{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

Under the condition of (L, σ)-boundedness of operator M we have the projections

$$P := \frac{1}{2\pi i} \int\limits_{\gamma} R_\mu^L(M) \, d\mu \in \mathcal{L}(\mathcal{X}), \ \ Q := \frac{1}{2\pi i} \int\limits_{\gamma} L_\mu^L(M) \, d\mu \in \mathcal{L}(\mathcal{Y}),$$

where $\gamma = \{\mu \in \mathbf{C} : |\mu| = r > a\}$. Put $\mathcal{X}^0 := \ker P$, $\mathcal{X}^1 := \operatorname{im} P$, $\mathcal{Y}^0 := \ker Q$, $\mathcal{Y}^1 := \operatorname{im} Q$. Denote by L_k (M_k) the restriction of the operator L (M) on $\mathcal{X}^k$ ($D_{M_k} := D_M \cap \mathcal{X}^k$), $k = 0, 1$. It is known that, if an operator M is (L, σ)-bounded, then $M_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, $M_0 \in Cl(\mathcal{X}^0; \mathcal{Y}^0)$, $L_k \in \mathcal{L}(\mathcal{X}^k; \mathcal{Y}^k)$, $k = 0, 1$; there exist operators $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$.

Denote $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $G := M_0^{-1} L_0$. For $p \in \mathbb{N}_0$ operator M is called (L, p)-bounded, if it is (L, σ)-bounded, $G^p \neq 0$, $G^{p+1} = 0$.

Let $\alpha_n \leq r \leq m - 1$. A strong solution of equation (1) on (t_0, T) is a function $x \in C^r([t_0, T]; \mathcal{X}) \cap L_q(t_0, T; D_M)$, such that $Lx \in C^{m-1}([t_0, T]; \mathcal{Y})$,

$$J_t^{m-\alpha} \left(Lx - \sum_{k=0}^{m-1} (Lx)^{(k)}(t_0) \tilde{g}_{k+1} \right) \in W_q^m(t_0, T; \mathcal{Y}), \quad q \in (1, \infty),$$

and almost everywhere on (t_0, T) equality (1) holds.

A solution of the generalized Showalter — Sidorov problem

$$(Px)^{(k)}(t_0) = x_k, \quad k = 0, 1, \ldots, m - 1, \tag{2}$$

to equation (1) is a solution of the equation, such that conditions (2) are true. Note here that $Px = L_1^{-1} L_1 Px = L_1^{-1} QLx$, and the smoothness of Px is not less the smoothness of Lx, since $L_1^{-1}Q \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$.

Theorem 1 *Let $p \in \mathbf{N}_0$, an operator M be (L, p)-bounded, $N : (t_0, T) \times \mathcal{X}^n \to \mathcal{Y}$ be Caratheodory mapping, uniformly Lipschitz continuous in $\overline{x} \in \mathcal{X}^n$, at all $y_1, y_2, \ldots, y_n \in \mathcal{Z}$ and almost everywhere on (t_0, T) the inequality*

$$\|N(t, y_1, y_2, \ldots, y_n)\|_{\mathcal{Z}} \leq a(t) + c \sum_{k=1}^n \|y_k\|_{\mathcal{Z}}$$

be true for some $a \in L_q(t_0, T; \mathbf{R})$, $c > 0$; $N(t, y_1, y_2, \ldots, y_n) \subset \mathcal{Y}^1$. Then problem (1), (2) has a unique strong solution.

Acknowledgments

Fuzzy inference methods for flexible neuro-fuzzy systems

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Abstract. The paper considers inference methods for systems of the Mamdani type and a logical type system with many fuzzy inputs based on a fuzzy truth value. Proposed approach allows reducing the exponential complexity of a fuzzy inference to a polynomial one. The possibility of using Q-implications for constructing flexible fuzzy production systems, whose structure can be changed from a Mamdani type to a logical type system depending on the parameter, is considered.

Keywords. fuzzy truth value, fuzzy inference, flexible fuzzy systems, logical type systems, Mamdani type systems

MSC2010.

1 Problem statement

Considered system is the interrelation of inputs and output in which is described using following N fuzzy rules:

$$\widetilde{R}_k : \text{If } \langle x_1 \text{ if } \widetilde{A}_{ik} \rangle \text{ and ... and } \langle x_n \text{ if } \widetilde{A}_{nk} \rangle, \text{ then } \langle y \text{ is } \widetilde{B}_k \rangle \tag{1}$$

where $(k = \overline{1,N})$ $\widetilde{A}_{ik} \in Fuzzy(X_1)$, ..., $\widetilde{A}_{nk} \in Fuzzy(X_1)$ – terms of input linguistic variables $x_1, \ldots, x_n$, of which the antecedent k-rule is formed; $\widetilde{B}_k \in Fuzzy(Y)$ – term of output linguistic variable y, with which the consequent of k-rule is formed. Let given fuzzy sets $\widetilde{A}'_i \in Fuzzy(X_i)$, $(i = \overline{1,n})$, representing the actual values of the inputs $x_1, \ldots, x_n$. The task is to determine system output $\widetilde{B} \in Fuzzy(Y)$.

2 Main results

Fuzzy Mamdani and logical type systems can be represented as:

$$\overline{y} = f(\mathbf{A}') = \frac{\sum\limits_{i=1}^{N} \overline{y}_l \cdot agr_l(\widetilde{\mathbf{A}}', \overline{y}_l)}{\sum\limits_{i=1}^{N} agr_l(\widetilde{\mathbf{A}}', \overline{y}_l)} \tag{2}$$

where

$$agr_l = f(\mathbf{A}', \overline{y}_l) = \begin{cases} \underset{k=\overline{1,N}}{\mathbf{S}} \{F_{kl}(\tau_k(\sigma), \overline{y}_l)\}, & \text{for Mamdani type;} \\ \underset{k=\overline{1,N}}{\mathbf{T}} \{F_{kl}(\tau_k(\sigma), \overline{y}_l)\}, & \text{for Logical–type;} \end{cases} \tag{3}$$

and

$$F_{kl}(\tau_k(\sigma), \overline{y}_l) = \begin{cases} \sup\limits_{\sigma \in [0,1]} \left\{ \tau_k(\sigma) \overset{T}{*} T\left(\sigma, \mu_{\widetilde{B}_k}(\overline{y}_l)\right) \right\}, & \text{for Mamdani type;} \\ \sup\limits_{\sigma \in [0,1]} \left\{ \tau_k(\sigma) \overset{T}{*} S\left(1 - \sigma, \mu_{\widetilde{B}_k}(\overline{y}_l)\right) \right\} & \text{for Logical–type;} \end{cases} \quad (4)$$

where $\tau_k(\sigma) = \tau_{\widetilde{A}, \widetilde{A}'}(\sigma)$ – fuzzy truth value of a fuzzy set $\widetilde{A}$, about $\widetilde{A}'$ representing compatibility $\text{CP}(\widetilde{A}, \widetilde{A}')$ of term $\widetilde{A}'$ in relation to the input value $\widetilde{A}'$: $\tau_{\widetilde{A}, \widetilde{A}'}(\sigma) = \mu_{CP(\widetilde{A}, \widetilde{A}')}(\sigma) = \sup\limits_{x \in X, \mu_{\widetilde{A}}(x) = \sigma} \left\{ \mu_{\widetilde{A}'}(x) \right\}$, $\sigma \in [0,1]$.[1]

Q-implication, according to [2], is defined as $I_\nu(a, b) = H(N_{1-\nu}(a), b, \nu)$ at $\nu \in [0,1]$. $N_\nu(a) = (1 - \lambda) \cdot (1 - a) + \nu \cdot a$ – compromise operator, $\nu \in [0,1]$, and $N_0(a) = 1 - a$, $N_1(a) = a$

$$H(\mathbf{a}, \nu) = N_\nu\left(\underset{i=\overline{1,n}}{\mathbf{S}} N_\nu(a_i) \right) = N_{1-\nu}\left(\underset{i=\overline{1,n}}{\mathbf{T}} N_{1-\nu}(a_i) \right) \quad (5)$$

at $\nu \in [0,1]$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ When changing a parameter ν from 0 to 1 Q–implication is changing from t–norm $I_0(a, b) = T(a, b)$ to fuzzy implication $I_1(a, b) = S(1 - a, b)$ accordingly.

References

[1] Dubois D., Prade H. Abe Mamdani: A Pioneer of Soft Artificial Intelligence // Combining Experimentation and Theory. Heidelberg. Springer. 2012. P. 49–60.

[2] Rutkowski L. Methods and technologies of artificial intelligence. M. Hot line–Telecom. 2010. P. 520.

Study of Option Pricing Models with Insufficient Liquidity and Transaction Costs

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Abstract. For option pricing models with the insufficient liquidity and transaction costs, a numerical solution is given to the price of call options and the price sensitivity factors from the market parameters (simply sensitivities, or Greeks), as well as a comparison with actual data on deals on Moscow Exchange is made.

Keywords. option pricing; insufficient liquidity; transaction costs; sensivities; numerical solution.

MSC2010. 26A33; 34A60; 34G25; 93B05.

The models of option pricing of futures-styles options [1, 2, 3] with the insufficient liquidity can be written as

$$u_t + \frac{\sigma^2 x^2 u_{xx}}{(1 - xv(u_x)u_{xx})^2} = 0, \tag{1}$$

where $v(u_x)$ takes the form determined by the model. The results of the group classification of such class of models are presented in [4].

Leland's model [5], which takes into account the proportional transaction costs, has the form

$$u_t + \sigma^2 \left[1 + \sqrt{2/\pi}\, k/\sigma\, \sqrt{\Delta t}\right] x^2 u_{xx} = 0, \tag{2}$$

where k is the coefficient of the transaction costs, Δt is a small, but not infinitesimal time interval that determines the frequency of the portfolio rebalancing.

With the modifications of Y. M. Kabanov and M. M. Safarian [6], the Leland's model can be represented as

$$u_t + \sigma^2 \left[1 + \sqrt{8/\pi}\, k_n/\sigma\, \sqrt{n}\right] x^2 u_{xx} = 0. \tag{3}$$

where n is the number of the revisions, and k_n are transaction costs for each revision of the portfolio.

For these models, a numerical solution is given to the price of call options and the price sensitivity factors from the market parameters (simply sensitivities, or Greeks), as well as a comparison with actual data on deals on Moscow Exchange using the method from [7] is maid.

Acknowledgments

References

[1] R. Frey and A. Stremme, Market volatility and feedback effects from dynamic hedging, Math. Finance **7** (1997), 351–374.

[2] R. K. Sircar and G. Papanicolaou, General Black — Scholes models accounting for increased market volatility from hedging strategies, Appl. Math. Finance **5** (1998), 45–82.

[3] P. J. Schönbucher and P. Wilmott, The feedback effect of hedging in illiquid markets, SIAM J. Appl. Math. **61** (2000), 232–272.

[4] V. E. Fedorov and M. M. Dyshaev, Invariant solutions for nonlinear models of illiquid markets, Math. Meth. Appl. Sci. **41** (2018), 8963–8972.

[5] H. E. Leland, Option pricing and replication with transactions costs, J. of Finance **40** (1985), 1283–1301.

[6] Y. M. Kabanov and M. M. Safarian, On Leland's strategy of option pricing with transactions costs, Finance and Stochastics **1** (1997), 239–250.

[7] M. M. Dyshaev, V. E. Fedorov, A. S. Avilovich and D. A. Pletnev, Simulation of feedback effects for futures-style options pricing on Moscow Exchange, Chel. Phys. Math. J. **3** (2018), 379–394. (In Russ.).

Nonlinear type contraction in G-metric spaces

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Abstract. The metric fixed point theory and the Banach contraction principle [1] are very popular tools for solving problems in many branches of applied sciences. For instances, they are used in order to get sufficient conditions under which Newton's method of successive approximations is guaranteed to work and similarly for Chebyshev's third order method. They are also used to prove existence and uniqueness of an equilibrium in dynamic economic models; prove the Picard-Lindelöf theorem about the existence and uniqueness of solutions to certain ordinary differential equations; prove existence and uniqueness of solutions to integral equations. Generalizations of metric spaces and extensions of the Banach contraction principle were proposed over the past couple of years which lead to more general and extended fixed point results. We introduce in this paper a definition of a new contraction which will be called the G-cyclic $(\phi - \psi)$-Kannan type contraction and we establish new fixed point results for this type of contraction in the new generalized metric spaces known as G-metric spaces that were introduced by Mustafa and Sims [2].

1   Introduction

In 2006, Mustafa and Sims [2] were able to introduce a new generalization called, G-metric space, in which all flaws of previous generalizations were amended. They were also able to prove that every G-metric space is topologically equivalent to a usual metric space, which means that it is a straightforward task to transform concepts from usual metric spaces to G-metric spaces. Furthermore, one can obtain similar results to those in usual metric spaces straightforwardly but in a more general setting.

Next, we introduce the definition of generalized metric space, as introduced by Mustafa and Sims in [2].

Definition 1 (See [2]) *Let X be a nonempty set and let $\mathbb{R}^+$ denote the set of all positive real numbers. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies*

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ whenever $x \neq y$, for all $x, y \in X$,

(G3) $G(x, x, y) \leq G(x, y, z)$ whenever $y \neq z$, for all $x, y, z \in X$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \dots$. *(Symmetry in all of the three variables),*

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$. *(Rectangle inequality).*

Then G is called a generalized metric, G-metric on X, and (X, G) is called a generalized metric space, G-metric space.

As an example we include the following from [2].

Example 1 (See [2]) *Let (X, d) be any metric space. Define G_s and G_m on $X \times X \times X$ to $\mathbb{R}^+$ by*

$$\begin{cases} G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z), \\ G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \forall x, y, z \in X. \end{cases}$$

Then (X, G_s) and (X, G_m) are G-metric spaces.

2 Main results

We consider the G-metric space as our generalization of the usual metric space, and we consider generalizations of Banach contractions to introduce a new fixed point theorem for cyclic non-linear $(\phi - \psi)$ contractive mappings that extend some previously proved theorems in G-metric spaces.

Theorem 1 *Let $\{A_i\}_{i=1}^p$ be non-empty closed subsets of a complete G-metric space (X, G) and $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ satisfies the following condition:*
There exists constants α, γ with $0 \leq \gamma < 1$ and $0 < \alpha + \gamma \leq 1$, such that for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \dots, p$, we have

$$\begin{aligned} \phi(G(Tx, Ty, Ty)) &\leq \phi(\alpha G(x, Tx, Tx) \\ &+ \gamma G(y, Ty, Ty)) - \psi(G(x, Tx, Tx), G(y, Ty, Ty), \\ &G(y, Ty, Ty)). \end{aligned}$$

Then T has a unique fixed point $u \in \bigcap_{i=1}^p A_i$.

References

[1] S. Banach, Surles operations dans les ensembles et leur application aux equation sitegrales, *Fund. Math.*, 3:133-181, 1922.

[2] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *J. Nonlinear and Convex Anal.*, 7(2):289–297, 2006.

Improvement combined with (analysis, CSP, Arithmetic analysis and Interval) of Simulations for efficient combination of two lower bound functions in univariate global optimization .

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Abstract. Univariate global optimization problems attract attention of researchers.Several methods [5] have been studied in the literature for univariate global optimization problems . Optimization in R presents the same difficulty as in R^n. Many algorithms are directed in this direction. For cutting methods in Global optimization or Optimsation gradient method in general . In this work, we propose to improve: The article submitted: (Simulations for efficient combination of two lower bound functions in univariate global optimization. AIP Conference Proceedings 1863, 250004 (2017) ; https ://doi.org/10.1063/1.4992412, (2017).) In this context too, we will accelerate the speed of the Algorithm for better complexity with technics (CSP, Arithmetic analysis and Interval and another). It should be noted that, we have made conclusive simulations in this direction . .

Keywords. Global optimization, αBB method, quadratic lower bound function, Branch and Bound, pruning method.

MSC2010. MSC2010 : 90-08 / 90C26.

1 Introduction

We consider the following problem

$$(P) \begin{cases} \min f(x) \\ x \in [x^0, x^1] \subset R \end{cases}$$

Main Improvements (Main Contributions)

Improvements are (Types C.S.P)

1 / Extra rapid convexity test (test A_1)

2 / Computation of bounds, to inhibit intervals by anlyse intervals or affine arithmetic (test A_2)

3 / The derivative and its bound, to inhibit intervals by anlyse intervals or affine arithmetic.(test A_3) . 4/ if / Smooth Form involved Direct Execution.

These 04 procedures will be integrated in the process of the Algorithm, to accelerate the speed of convergence towards the optimal solution.

2 Branch and Bound Algorithm and its convergence

Algorithm Branch and Bound (BB)

Step 1 : Initialization

Step 2 : Iteration

a) Selection step

- Select $T^k = [a_k, b_k] \in M$, the interval such that $LB_k = \min LB(T^k)$

b) Bisection step

- Bisect T^k into two sub-rectangles $T_1^k = [a_k^1, b_k^1], T_2^k = [a_k^2, b_k^2]$ by w-subdivision procedure via $\mathbf{s}*^k$

c) Computing step

- **For** $i = 1, 2$ **do**
  1. Compute K_α^{ki} and K_q^{ki} on the interval T_i^k
  2. Convex test : if $K_\alpha^{ki} = 0$ then update $LB(T_i^k)$ and $UB(T_i^k)$ and go to **step d**
  3. Concave test: if $K_q^{ki} = 0$ then update $LB(T_i^k)$ and $UB(T_i^k)$ and go to **step d**
  31) Test C.S.P A_1, Test C.S.P A_2 , Test C.S.P A_3 on the interval T_i^k
  4. Pruning test : Compute LB_q^{ki} and solve $LB_q^{ki} = UB_k$ to reduce the searching interval $[a_k^i, b_k^i]$
  5. Compute $LB_\alpha^{ki}(x)$. Let z^{ki} and s_{ki}^* be the solution of the convex problem

  $$\min \left\{ z : LB_\alpha^{ki}(x) \leq z, LB_q^{ki}(x) \leq z, z \in R, x \in T_i^k \right\} \tag{1}$$

  and $LB(T_i^k) = z^{ki}$
  6. Set $M \leftarrow M \bigcup \{T_i^k : UB_k - LB(T_i^k) \geq \varepsilon, i = 1, 2\} \setminus \{T^k\}$

d) Updating step

- Update the lower bound: $LB_k = \min\{LB(T) : T \in M\}$.
- Delete from M all the intervals T such that $LB(T) > UB_k - \varepsilon$.

e) Stopping step

- **If** $M = \emptyset$ **then Output** $\overline{s}^k$ as an optimal solution and exit algorithm
- **else** set $k \leftarrow k + 1$, and return to **Step 2a)**.

References

[1] I.P. Androulakis, C.D. Marinas, C.A. Floudas, *αBB: A global optimization method for general constrained nonconvex problems* J. Glob. Optim. (1995), 7, 337-363.

[2] C de Boor, *A practical Guide to Splines Applied Mathematical Sciences.* Springer-Verlag (1978).

[3] C. A. Floudas and C. E. Gounaris, *A review of recent advances in global optimization* , J Glob Optim. (2008), DOI 10.1007/s10898-008-9332-8

[4] Chrysanthos E. Gounaris ï¿$\frac{1}{2}$ Christodoulos A. Floudas. *Tight convex underestimators for C2-continuous problems: I. univariate functions* J Glob Optim (2008) 42:51 - 67 DOI 10.1007/s10898-008-9287-9 Received: 14 December 2006 / Accepted: 6 February 2008 / Published online: 13 March 2008 ï¿$\frac{1}{2}$ Springer Science+Business Media, LLC. 2008

[5] Le Thi Hoai An and Ouanes Mohand, *Convex quadratic underestimation and Branch and Bound for univariate global optimization with one nonconvex constraint,* RAIRO Oper. Res. (2006) 40: 285-302.

[6] D.G. Sotiropoulos and T.N. Grapsa, *Optimal centers in branch-and-prune algorithms for univariate global optimization,* Applied Mathematics and Computation, 169 (2005), pp.247-277.

[7] Mohand Ouanes, Hoai An Le Thi, Trong Phuc Nguyen, Ahmed Zidna, *New quadratic lower bound for multivariate functions in global optimization* Mathematics and Computers in Simulation, (2015) 109, 197-211.

Asymptotic analysis of viscoelastic problems with nonlinear dissipative terms

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Abstract. The purpose of this article is to study the asymptotic analysis of the solutions of a linear viscoelastic problem with a dissipative terms in a three-dimensional thin domain Ω^ε. Firstly, we give the strong formulation of the problem and the existence and uniqueness theorem of the weak solution. Then we establish some estimates independent of the parameter ε. These last will be useful to obtain the limit problem with a specific weak form of the Reynolds equation.

Keywords. A priori estimates; Dissipative term; Reynolds equation; Tresca law; Viscoelastic.

MSC2010. 35R35, 76F10, 78M35.

1 Introduction

In recent years, much attention has been paid to the study of mathematical models of viscoelastic problems. These types of problems are not interesting only from the mathematical point of view, but arise in many practical problems such as engineering, mechanical applications and physics, where the dynamics of viscoelastic materials are used in many of applications, including uses in civil engineering, the food industry, ultrasonic imaging, metals, polymers and rocks. Our work in this paper concerns the asymptotic analysis of the solutions of a linear viscoelastic problem with a dissipative in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$, with ε a small parameter which will tend to zero. The boundary of this thin domain consists of three parts: the bottom, the lateral part and the top surface.

We Assume that on the bottom, the normal velocity is equal to zero but the tangential velocity is unknown and satisfies friction of Tresca boundary conditions, with friction coefficient a given positive function. On the lateral and the top parts, we have Dirichlet boundary conditions. The complete problem is therefore to find the displacement field $u^\varepsilon : \Omega^\varepsilon \times]0, T[\rightarrow \mathbb{R}^3$ satisfies the following equations, boundary conditions and initial conditions:

$$\frac{\partial^2 u_i^\varepsilon}{\partial t^2} - \sigma_{ij,j}^\varepsilon (u^\varepsilon) + \alpha^\varepsilon \left(1 + \left|\frac{\partial u_i^\varepsilon}{\partial t}\right|\right) \frac{\partial u_i^\varepsilon}{\partial t} = f_i^\varepsilon, \, i, j = 1, 2, 3 \; in \; \Omega^\varepsilon \times]0, T[, \tag{1}$$

$$\sigma_{ij}^\varepsilon (u^\varepsilon) = 2\mu d_{ij}(u^\varepsilon) + 2\lambda d_{ij}\left(\frac{\partial u^\varepsilon}{\partial t}\right), \, i, j = 1, 2, 3 \; in \; \Omega^\varepsilon \times]0, T[, \tag{2}$$

$$u^\varepsilon = 0 \; on \; \Gamma_1^\varepsilon \times]0, T[, \tag{3}$$

$$u^\varepsilon = 0 \; on \; \Gamma_L^\varepsilon \times]0, T[, \tag{4}$$

$$\frac{\partial u^\varepsilon}{\partial t} . n = 0 \ on \ \omega \times]0, T[, \tag{5}$$

$$\left. \begin{array}{l} |\sigma_\tau^\varepsilon| < k^\varepsilon \Rightarrow \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau = 0, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \Rightarrow \exists \beta > 0, \ such \ that \ \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau = -\beta \sigma_\tau^\varepsilon, \end{array} \right\} \ on \ \omega \times]0, T[, \tag{6}$$

$$u^\varepsilon(x, 0) = u_0(x), \quad \frac{\partial u^\varepsilon}{\partial t}(x, 0) = u_1(x), \ \forall x \in \Omega^\varepsilon. \tag{7}$$

2   Main results

We give the related weak formulation of the problem $(1) - (7)$, then we discuss the existence and uniqueness theorem of the weak solution.

We study the asymptotic analysis according to the change of variables $z = \frac{x_3}{\varepsilon}$. First of all, we transform the initial problem posed in the domain Ω^ε into a new problem posed on a fixed domain Ω independent of the parameter ε, then we establish some estimates independent of ε. These estimates will be useful to obtain the limit problem with a specific weak form of the Reynolds equation.

References

[1] G. Bayada and M. Boukrouche, *On a free boundary problem for Reynolds equation derived from the Stokes system with Tresca boundary conditions*, J. Math. Anal. Appl., 382 (2003), pp.212–231.

[2] H. Benseridi and M. Dilmi, Some inequalities and asymptotic behaviour of dynamic problem of linear elasticity, Georgian Math. J., 20(1) (2013), pp. 25–41.

[3] D. Benterki, H. Benseridi and M. Dilmi, Asymptotic behavior of solutions to a boundary value problem with mixed boundary conditions and friction law. Boundary Value Problems. 2017;1(2017):55.

[4] M. Dilmi, M. Dilmi and H. Benseridi, Study of generalized Stokes operator in a thin domain with friction law (case $p < 2$), J. Math. Meth. Appl. Sci. 2018;1-10.

The general solution of a fractional Bessel equation

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Abstract. A fractional generalization of the Bessel equation is considered. Two partial solutions are constructed for this equation using the Mellin transform method and the general solution of the considered fractional Bessel equation can be represented as their linear conbination. In the limiting case, when the fractional integral order is $\alpha = 0$, these two solutions coincide with the linearly independent solutions of the classical Bessel equation.

Keywords. fractional differential equations; bessel equation; mellin transform method; general solution.

MSC2010. 26A33; 65R10; 34A08; 44A20.

1 Introduction

It is well-known [1] that for the classical Bessel equation

$$\left(x\frac{d}{dx}\right)^2 y(x) + (x^2 - \nu^2)y(x) = 0 \tag{1}$$

the general solution can be written as

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x), \tag{2}$$

where C_1, C_2 are arbitrary constants and $J_\nu(x)$, $Y_\nu(x)$ are Bessel functions of the first and second kind respectively.

We consider a fractional generalization of the equation (1) in the form

$$\left(x\frac{d}{dx}\right)^2 K_\nu^\alpha g(x) - \nu^2 K_\nu^\alpha g(x) + x^2 g(x) = 0, \tag{3}$$

$$K_\nu^\alpha g(x) = 2^{-\alpha} x^{-\nu}\left[{}_0I_s^{\alpha/2} s^{\nu-\alpha/2} {}_sI_\infty^{\alpha/2} s^{-\nu/2} g(x)|_{x=\sqrt{s}}\right]_{s=x^2}, \quad \nu \geq 2.$$

where ${}_0I_s^{\alpha/2}f(s)$, ${}_sI_\infty^{\alpha/2}f(s)$ are the left-sided and right-sided Riemann-Liouville fractional integrals [2].

2 Main results

Denoting $x = \sqrt{t}$ and applying the Mellin transform $\mathcal{M}$ to the equation (3), we have

$$2^{-\alpha} \frac{(2s-\nu)(2s+\nu)\Gamma\left(1-s+\frac{\nu}{2}-\frac{\alpha}{2}\right)\Gamma\left(s+\frac{\nu}{2}\right)}{\Gamma\left(1-s+\frac{\nu}{2}\right)\Gamma\left(s+\frac{\nu}{2}+\frac{\alpha}{2}\right)} k\left(s+\frac{\alpha}{2}\right) = k(s+1), \qquad (4)$$

where $k(s) = \mathcal{M}[g(t)](s)$, $1 - \frac{m}{2} - \frac{\alpha}{2} < s < 1 + \frac{m}{2} - \frac{\alpha}{2}$.

Solutions of the equation (4) can be written in the form

$$k_1(s) = 2^{2s} \frac{\Gamma\left(\frac{\nu}{2}+s\right)}{\Gamma\left(1-s+\frac{\nu}{2}\right)}, \quad k_2(s) = 2^{2s} \frac{\Gamma\left(\frac{\nu}{2}+s\right)\Gamma\left(\frac{-2s+\nu+2}{2-\alpha}\right)\Gamma\left(\frac{2s-\alpha-\nu}{2-\alpha}\right)}{\Gamma\left(1-s+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{2s-\alpha-\nu}{2-\alpha}\right)\Gamma\left(\frac{1}{2}-\frac{2s-\alpha-\nu}{2-\alpha}\right)}. \qquad (5)$$

Applying the inverse Mellin transform to (5), we obtain:

$$g_1(x,\nu) = J_\nu(x), \qquad (6)$$

$$g_2(x,\nu) = H_{2\;4}^{2\;1}\left[\frac{x^2}{4} \,\middle|\, \begin{matrix} \left(\frac{-\alpha-\nu}{2-\alpha};\frac{2}{2-\alpha}\right),\; \left(\frac{1}{2}+\frac{-\alpha-\nu}{2-\alpha};\frac{2}{2-\alpha}\right) \\ \left(\frac{\nu}{2};1\right),\; \left(\frac{-\alpha-\nu}{2-\alpha};\frac{2}{2-\alpha}\right),\; \left(-\frac{\nu}{2};1\right),\; \left(\frac{1}{2}+\frac{-\alpha-\nu}{2-\alpha};\frac{2}{2-\alpha}\right) \end{matrix}\right], \qquad (7)$$

where $H_{p\;q}^{m\;n}\left[z \,\middle|\, \begin{matrix}(a_p;A_p)\\(b_q;B_q)\end{matrix}\right]$ is the Fox H–function [3].

Since (6) and (7) are linearly independent, the general solution of (3) has the form

$$y(x) = C_1 g_1(x,\nu) + C_2 g_2(x,\nu), \qquad (8)$$

where C_1, C_2 are arbitrary constants.

It is easy to verify that in a limiting case of $\alpha = 0$ the solution (8) concides with the general solution of the classical Bessel equation (2).

Acknowledgments

References

[1] Bateman H., Erdelyi A, *Higher transcendental functions*, New York: McGraw-Hill, 1953-1955.

[2] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, London: Gordon and Breach Science Publishers, 1993.

[3] A. A. Kilbas and M. Saigo, *H-Transforms: Theory and Application, vol. Analytical methods and special functions of An International Series of Monographs in Mathematics*, Chapman and Hall, 2004.

Propagation and Stabilizing of a Front in a Media with Discontinuous Characteristics

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Abstract. We investigate the stabilizing of the initial boundary value problem solution of a moving front form to the steady-state solution with an internal transition layer of a stationary boundary value problem. To obtain the stationary solution stability area we use the method of upper and lower solutions.

Keywords. moving front; discontinuous terms; upper and lower solutions; small parameter; asymptotic method of differential inequalities.

MSC2010. 34B16.

1   Introduction

We consider a front form solution to the initial boundary-value problem

$$\begin{cases} \varepsilon^2 \dfrac{\partial^2 u}{\partial x^2} - \varepsilon \dfrac{\partial u}{\partial t} = f(u), & -1 < x < 1,\ t > 0; \\ u(-1, t, \varepsilon) = \varphi_1,\ u(1, t, \varepsilon) = \varphi_3,\ t > 0; \quad u(x, 0, \varepsilon) = u_{init}(x, \varepsilon),\ -1 \le x \le 1, \end{cases} \tag{1}$$

where $\varepsilon \in (0; \varepsilon_0]$ is a small parameter, and the function at the right side of the equation undergoes a first kind discontinuity at some point $x_0 \in (-1, 1)$ and is expressed as

$$f(u) = \begin{cases} f^{(L)}(u) = u - \varphi_1, & -1 \le x < x_0; \\ f^{(R)}(u) = (u - \varphi_1)(u - \varphi_2)(u - \varphi_3), & x_0 \le x \le 1, \end{cases}$$

where $\varphi_i,\ i = \overline{1,3}$ are the constants satisfying the inequalities

$$\varphi_1 < \varphi_2 < 0.5\,(\varphi_1 + \varphi_3), \quad \int_{\varphi_3}^{p} (u - \varphi_1)\,(u - \varphi_2)\,(u - \varphi_3)\,du > 0, \ \text{ if } p \in [\varphi_1, \varphi_3).$$

The moving front is called the solution having the area of large gradient in the vicinity of some inner point $x^*(t) \in (-1, 1)$ the location of which changes in time, which is close to φ_1 at the left side of the x^* vicinity and to φ_3 at the right side. The area where function has large gradient is called the internal transition layer. The existence of a front form solution to problem (1) is proved in [1].

The aim of the study is to prove that the front form solution for a large enough time period stabilizes to the steady-state solution with internal transition layer localized in the vicinity of point x_0 of the problem

$$\varepsilon^2 \frac{d^2 u}{dx^2} = f(u), \quad -1 < x < 1; \quad u(-1, \varepsilon) = \varphi_1, \ u(1, \varepsilon) = \varphi_3. \tag{2}$$

The solution of problem (2) we consider in the sense of the following definition.

Definition 1 . *The function $u_\varepsilon(x) \in C[-1,1] \cap C^1(-1,1) \cap C^2\left((-1,x_0) \cup (x_0,1)\right)$ is called the solution of problem (2) if for all $x \in (-1, x_0) \cup (x_0, 1)$ it satisfies equation (2), and also the boundary conditions.*

The theorem was proved in [2] on existence, local uniqueness and asymptotic stability of the solution to problem (2).

2   Main results

To obtain the area of stability we use the method of upper and lower solutions [3] and its modification for the problems with internal transition layers called the asymptotic method of differential inequalities [4]. According to this method we construct the upper and the lower solutions as modifications of asymptotic approximation of problem (1) solution. The upper solution $(\beta(x, \varepsilon))$ is the same as for problem (2) [2]. The lower solution $(\alpha(x, \varepsilon))$ is constructed separately in each of the segments $[-1, x_0]$, $[x_0, x^*(t)]$, $[x * (t), 1]$.

The main result is the theorem

Theorem 1 *For every continuous initial function u_{init} satisfying inequalities $\alpha(x, \varepsilon) \le u_{init} \le \beta(x, \varepsilon)$ the moving front solution of problem (1) for a large enough time period falls into the stability domain of the steady-state solution with internal transition layer of problem (2), if ε is sufficiently small.*

Acknowledgments

References

[1] Yu.V. Bozhevol'nov, N.N. Nefedov, Front motion in a parabolic reaction-diffusion problem, Comput. Math. Math. Phys. **50** (2) (2010), 264–273.

[2] N.N. Nefedov, N.T. Levashova, A.O.Orlov, The asymptotic stability of a stationary solution with an internal transition layer to a reaction-diffusion problem with a discontinuous reactive term, Moscow University Physics Bulletin. **73** (6) (2018), 565–572.

[3] V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[4] N.N. Nefedov, The method of differential inequalities for some classes of nonlinear singularly perturbed problems with internal layers, Differential Equations **31** (7) (1996), 1077–1085.

On diagonalization of some class matrices

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Abstract. In this paper, we study a certain class operators, which given for there infinite matrices. We construct a similary transform which allows one to represent this operators in block-diagonal form. For such operators, asymptotic estimates of eigenvalues are obtained.

Keywords. infinite matrix; linear clozed operator; similar operators method; eigenvalue.

MSC2010. 39A70; 47A10; 47B37; 47B39.

1   Introduction

Let $\ell_2 = \ell_2(\mathbb{Z})$ be the complex Hilbert space of square summable sequence $x\colon \mathbb{Z} \to \mathbb{C}$ indexed by integers. The inner product and the norm in ℓ_2 are defined by $(x, y) = \sum_{n \in \mathbb{Z}} x(n)\overline{y(n)}$, $\|x\|^2 = \sum_{n \in \mathbb{Z}} |x(n)|^2$, $x, y \in \ell_2$.

We consider the operator A defined by the infinite dimensional matrix

$$\mathcal{A} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{-\beta_{-2}}{2} & -2a & -\beta'_{-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\beta_{-1} & -a & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & 0 & \beta'_1 \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \beta_1 & a & \dfrac{\beta'_2}{2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \dfrac{\beta_2}{2} & 2a & \dfrac{\beta'_3}{3} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $D(A) = \{x \in \ell_2 \colon \sum_{n \in \mathbb{Z}} |nx(n)|^2 < \infty\}$, $a \in \mathbb{C}, a \neq 0$ and sequences $\beta\colon \mathbb{Z} \to \mathbb{C}$, $\beta'\colon \mathbb{Z} \to \mathbb{C}$ are bounded, $\sup_{n \in \mathbb{Z}} |\beta_n| < c$, $\sup_{n \in \mathbb{Z}} |\beta'_n| < c, c > 0, c \in \mathbb{R}$.

In this paper, we develop a general approach called the method of similar operators to stady the operator A. The main idea of the method to construct a similar transform which would allow one to represent the operator as a diagonal or block-diagonal matrix (see [1], [2]). The matrices diagonalization of the linear operators was consider in the [3], [4].

2 Main results

Let $End\,\ell_2$ is the Banach space of all linear bounded operators, acting in ℓ_2 and $\mathfrak{S}_2(\ell_2)$ is the ideal of Hilbert-Schmidt operators in ℓ_2. Let $A = A_0 - B$, where $A_0 \colon D(A_0) = D(A) \subset \ell_2 \to \ell_2$, $(A_0 x)(n) = anx(n)$, $n \in \mathbb{Z}$ and $B = A_0 - A$.

Let $P_n = P(\{an\}, A_0), n \in \mathbb{Z}$ is the Riesz projector for the spectral set $\sigma_n = \{an\}, n \in \mathbb{Z}$, from the operator A_0 and $P_{(m)} = \sum_{|i| \leqslant m} P_i, m \in \mathbb{Z}_+$.

Theorem 1 *There is the number $k \in \mathbb{Z}_+$, such that the operator A is similar to the block-diagonal operator $A_0 - P_{(k)} X P_{(k)} - \sum_{|i|>k} P_i X P_i$, where $X \in \mathfrak{S}_2(\ell_2)$, $AU = U(A_0 - P_{(k)} X P_{(k)} - \sum_{|i|>m} P_i X P_i)$. The invertible operator of transformation U from $End\,\ell_2$ has the form $U = I + W$, where $W \in \mathfrak{S}_2(\ell_2)$. We have*

$$\sigma(A) = \sigma_{(k)} \cup (\cup_{|i|>k}\{\mu_i\}),$$

where $\sigma_{(k)} = \sigma\left((P_{(k)} A_0 - P_{(k)} X)|_{H_{(k)}}\right)$, $H_{(k)} = ImP_{(k)}, k \in \mathbb{Z}_+$, consists of no more then $2k+1$ eigenvalues and $\mu_i = ai + \eta_i, |i| > k$, where the sequence $\{\eta_i, |i| > k\}$ belong to ℓ_2. The corresponding eigenvectors $\widetilde{e}_i$ form Riesz basis in ℓ_2.

The operator W in Theorem 1 can be effectively calculated as a limit of a sequence of operators that emerge when applying the method of simple iterations to a Riccati-type eqoation (see [1], [2] for detail). The operator X in Theorem 1 is a solution of non-linear equation of the similar operator method [1], [2].

Corollary 1 If $\dfrac{c}{a}\pi^2 < \dfrac{3}{8}$, then the operator A is similar to the diagonal operator $A_0 - \sum_{i \in \mathbb{Z}} P_i X P_i$.

Let $\widetilde{P}_n, n \in \mathbb{Z}$ be the spectral projection corresponding to the one-point sets $\sigma_n = \{\mu_n\}, |n| > k$ of the operator A, $\widetilde{P}_{(k)} = P(\{\sigma_{(k)}\}, A)$.

Theorem 2 *We have $\widetilde{P}_n = UP_iU^{-1}$, $\widetilde{P}_{(k)} = UP_{(k)}U^{-1}$, $\widetilde{P}_n - P_n = (WP_i - P_iW)U^{-1}$, $\widetilde{P}_{(k)} - P_{(k)} = (WP_{(K)} - P_{(k)}W)U^{-1}$.*

Acknowledgments

References

[1] A. G. Baskakov, I. A. Krishtal and N. B. Uskova, Similarity techniques in the spectral analysis of perturbed operator matrices, J. Math. Anal. App. **477** (2019), 930–960.

[2] G. V. Garkavenko and N. B. Uskova, Method of similar operators in research of spectral properties of difference operators with growing potential, Siber. Electr. Math. Reports **14** (2017), 673–689.

[3] A. Hinkkanen, On the diagonalization of a certain class of operators, Michigan Math. J. **32** (1985), 349–359.

[4] G. V. Garkavenko, On diagonalization of certain classes of linear operators, Russian Math. Iz. VUZ., **38:11** (1994), 11–16.

On a higher-order nonautonomous rational difference equation with periodic coefficients

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Abstract. In this talk, we give some results on the qualitative behavior of the higher-order nonautonomous rational difference equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}}, \ n \in \mathbb{N}_0,$$

where $\{\alpha_n\}_{n\geq 0}$ is a periodic sequence of positive numbers with period T, with $T, k \geq 1$ are nonnegative integers and the initial values $y_{-k}, y_{-k+1}, \cdots, y_0$ are nonnegative real numbers. The obtained results give some answers to two open problems proposed by Camouzis and Ladas in their monograph (Dynamics of Third Order Rational Difference Equations With Open Problems and Conjectures, 2008).

Keywords. nonautonomous difference equation, boundedness, uniform asymptotic stability, global attractivity.

MSC2010. 39A10; 40A05.

In 2008, Camouzis and Ladas studied in their monograph [1] the third-order difference equation

$$y_{n+1} = \frac{\alpha + y_n}{\alpha + y_{n-2}}, \ n \in \mathbb{N}_0,$$

and they proposed in the same monograph the following two open problems:

Open Problem 5.26.1. Let $\{\alpha_n\}_{n\geq 0}$ be a periodic sequence with nonnegative values with period k. Investigate the global character of solutions of the equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-1}}, \ n \in \mathbb{N}_0, \tag{1}$$

Open Problem 5.27.1. Let $\{\alpha_n\}_{n\geq 0}$ be a periodic sequence of nonnegative real numbers with prime period $k \geq 2$. Investigate the global character of solutions of the equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-2}}, \ n \in \mathbb{N}_0, \tag{2}$$

Hu et al. in [4] (respectivly, Dekkar et al. in [3]) gave answer to the first open problem (respectivly to the second open problem) when the sequence $\{\alpha_n\}_{n\geq 0}$ is periodic of period two.

Here, we consider the more general difference equation

$$y_{n+1} = \frac{\alpha_n + y_n}{\alpha_n + y_{n-k}}, \ n \in \mathbb{N}_0, \tag{3}$$

where $\{\alpha_n\}_{n \geq 0}$ is a periodic sequence of positive numbers with period T, with $T, k \geq 1$ are nonnegative integers and the initial values $y_{-k}, y_{-k+1}, \cdots, y_0$ are nonnegative real numbers [2]. Clearly, Equation (3) has a unique positive equilibrium point $y^* = 1$. We will focus on this special solution of Equation (3) and the qualitative behavior of the other solutions around it.

References

[1] E. Camouzis and G. Ladas, *Dynamics of Third Order Rational Difference Equations With Open Problems and Conjectures*, Chapman & Hall/CRC, 2008.

[2] I. Dekkar and N. Touafek and Q. Din, On the global dynamics of a rational difference equation with periodic coefficients, J. Appl. Math. Comput., (2018), https://doi.org/10.1007/s12190-018-01227-w.

[3] I. Dekkar and N. Touafek and Y. Yazlik, Global stability of a third-order nonlinear system of difference equations with period-two coefficients, Rev. R. Acad. Cienc. Exactas Fï¿½½s. Nat., Ser. A Mat., RACSAM, **111(2)** (2017), 325-347.

[4] L. X. Hu and H. M. Xia, Global asymptotic stability of a second order rational difference equation, Appl Math Comput, **233** (2014), 377-382.

On some estimates for discrete boundary value problems

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Abstract. We consider some discrete pseudo-differential equations and discrete boundary value problems and give a comparison between discrete and continuous solutions. Also we present certain result on unique solvability of discrete boundary value problem.

Keywords. discrete boundary value problem, solvability, error estimate.

MSC2010. 35S15; 65T50.

1 Introduction

Given function u_d of a discrete variable $\tilde{x} \in h\mathbf{Z}^m$, $h > 0$, we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi) \equiv \widetilde{u}_d(\xi) = \sum_{\tilde{x} \in \mathbf{Z}^m} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}), \quad \xi \in \hbar\mathbf{T}^m,$$

where $\mathbf{T}^m = [-\pi, \pi]^m$, $\hbar = h^{-1}$.

We will remind here some definitions of functional spaces [1] and will consider discrete analogue of the Schwartz space $S(h\mathbf{Z}^m)$. Let us denote $\zeta^2 = h^{-2} \sum_{k=1}^{m} (e^{-ih\cdot\xi_k} - 1)^2$ and introduce the following

Definition 1 *The space $H^s(h\mathbf{Z}^m)$ is a closure of the space $S(h\mathbf{Z}^m)$ with respect to the norm*

$$||u_d||_s = \left(\int\limits_{\hbar\mathbf{T}^m} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}.$$

Fourier image of the space $H^s(h\mathbf{Z}^m)$ will be denoted by $\widetilde{H}^s(\hbar\mathbf{T}^m)$.

One can define some discrete operators for such functions u_d. If $\widetilde{A}_d(\xi)$ is a periodic function in $\mathbf{R}^m$ with the basic cube of periods $\hbar\mathbf{T}^m$ then we consider it as a symbol. We will introduce a digital pseudo-differential operator in the following way. Let $D \subset \mathbf{R}^m$ be a domain, $D_d \equiv D \cap h\mathbf{Z}^m$.

Definition 2 *A digital pseudo-differential operator A_d in a discrete domain D_d is called the operator*

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbf{Z}^m} \int\limits_{\hbar\mathbf{T}^m} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d,$$

We use the class $E_\alpha, \alpha \in \mathbf{R}$ [1] with the following condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2}$$

and universal positive constants c_1, c_2.

2 Main results

We study the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{1}$$

in the discrete domain $D_d \equiv D \cap h\mathbf{Z}^m$ and will seek a solution $u_d \in H^s(D_d)$, $v_d \in H_0^{s-\alpha}(D_d)$ [1]. Everywhere here $D = \mathbf{R}_+^m$. Solvability of the equation (1) depends on special number κ which is called an index of periodic factorization $\tilde{A}_d(\xi)$ [2,3]. We have non-uniqueness of a solution for the equation (4) for the case $\kappa - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$. We consider here the case $n = 1$. To obtain the unique solution one needs some additional conditions. Discrete analogues of Dirichlet or Neumann conditions give a very simple case. We will consider here the discrete Dirichlet condition.

$$u_d|_{\tilde{x}_m=0} = g_d(\tilde{x}'), \tag{2}$$

where g_d is a given function of a discrete variable in the discrete hyper-plane hZ^{m-1}.

Theorem 1 *Discrete boundary value problem* $(1), (2)$ *is uniquely solvable in the space* $H^s(h\mathbf{Z}_+^m)$ *for arbitrary right-hand side* $v_d \in H_0^{s-\alpha}(h\mathbf{Z}_+^m)$ *and arbitrary boundary function* $g_d \in H^{s-1/2}(h\mathbf{Z}^{m-1})$.

To obtain some comparison between discrete and continuous solutions we will remind how the continuous solution looks. The continuous analogue of the discrete boundary value problem is the following

$$(Au)(x) = 0, \quad x \in \mathbf{R}_+^m, \tag{3}$$

$$u(x', 0) = g(x'), \quad x' \in \mathbf{R}^{m-1}. \tag{4}$$

Theorem 2 *Let* $\kappa > 1$. *If* $\tilde{g}(\xi)$ *is a bounded function then a comparison between solutions of problems* $(1), (2)$ *and* $(3), (4)$ *is given in the following way*

$$|\tilde{u}(\xi) - \tilde{u}_d(\xi)| \leq Ch^{\kappa-1}, \quad \xi \in \hbar\mathbf{T}^m$$

Acknowledgments

References

[1] A. Vasilyev and V. Vasilyev, Pseudo-Differential Operators and Equations in a Discrete Half-Space, *Math. Model. Anal.* **23** (2018), pp. 492–506.

[2] V. Vasilyev, Digital Approximations for Pseudo-Differential Equations and Error Estimates, *Lect. Notes Comp. Sci.* **11189** (2019), pp. 483–490.

[3] V. Vasilyev, On a Digital Version of Pseudo-Differential Operators and Its Applications, *Lect. Notes Comp. Sci.* **11386** (2019), pp. 596–603.

Integral representation of solutions of an elliptic system

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Abstract. A general elliptic system with complex coefficients is considered. The matrix of the system is a constant complex triangular matrix with eigenvalues in upper half-plane. The paper shows that a solution of the system is uniquely represented by generalized Cauchy type integrals with real density and potential type integrals with complex density.

Keywords. Elliptic system, Hölder class, generalized Cauchy type integral, potential type integral.

MSC2010. 35J56.

1 Introduction

Let $J \in \mathbb{C}^{l \times l}$ be a constant complex triangular matrix with eigenvalues in upper half-plane, $\text{Im}\,\lambda > 0$. We denote by σ the set of all eigenvalues of the matrix J. Each nonzero complex number $x + iy \in \mathbb{C}$ generates $l \times l-$matrix $z_J = x \cdot 1 + y \cdot J$, $x, y \in \mathbb{R}$, whose proper eigenvalues are numbers $x + \lambda y$, where $\lambda \in \sigma(J)$, 1 is the identity matrix. According to [7] the matrix–function $E(z) = \dfrac{1}{2\pi i} z_J^{-1}$ is fundamental solution of the elliptic system of first order

$$\frac{\partial U(z)}{\partial y} - J\frac{\partial U(z)}{\partial x} = F(z). \tag{1}$$

A solution of this system is understood as a complex $l-$vector–function $U = (U_1, \ldots, U_l) \in C^1(D)$. Note that the homogeneous system (1) is an analogue of the Cauchy-Riemann system when the J-matrix takes a role of the imaginary unit. The basic information concerning the homogeneous system (1) is detailed in [6]. Under the assumption that the matrix J is Töplitz matrix this system was first studied by A. Douglis [4] in the framework of so-called hypercomplex numbers. Therefore, $J-$analytic functions are also called Douglis analytic functions. Later, elliptic systems of first order with constant coefficients were investigated by B. Boyarsky [1], R. P. Gilbert, J. L. Buchanan [5], and many other authors.

2 Main results

Let $D \subset \mathbb{C}$ be a finite domain with a smooth boundary $\Gamma = \bigcup_{i=1}^m \Gamma_i \in C^{1,\nu}$, where Γ_i are smooth simple contours and Γ_m covers the remaining contours.

Let

$$L_J = \frac{\partial}{\partial y} - J\frac{\partial}{\partial x}$$

be a differential operator which is defined by the matrix J and acts in the space $l-$ vector functions. Consider the first order elliptic system in the domain D

$$L_J U(z) + c(z)U(z) + d(z)\overline{U(z)} = F(z) \tag{2}$$

where $l \times l-$ matrix coefficients c, d and $l-$vector function $F(z)$ belong to $C^\mu(\overline{D})$, $\mu < \nu$. We introduce a class of functions $C_J^\mu(\overline{D}) = \{\phi \in C^1(D) \cap C^\mu(\overline{D}), L_J\phi \in C^\mu(\overline{D})\}$. According to [2] $(I_J^1\varphi)(z)$ is the generalized Cauchy type integral, $(I_J^2\psi)(z)$ is the generalized operator Vekua-Pompeiu [3].

Theorem 1 *Let $J \in \mathbb{C}^{l \times l}$ be a triangular matrix. Then any solution of the system (2) from class $C_J^\mu(\overline{D})$ can be uniquely represented as*

$$U(z) = (I_J^1\varphi)(z) + (I_J^2\psi)(z) + i\xi,$$

where $\psi \in C^\mu(\overline{D})$ is a complex $l-$vector–function, $\xi \in \mathbb{R}^l$ is a constant vector and $l-$ vector function $\varphi \in C_{\mathbb{R}}^\mu(\Gamma)$ satisfies the conditions

$$\int_{\Gamma_i} \varphi(t)d_1 t = 0, \quad 1 \le i \le m - 1.$$

Acknowledgments

References

[1] B. Boyarsky The theory of a generalized analytic vector. Annales Polon. Math. **17:3** (1966), 281–320.

[2] O. Chernova Generalized operator of Vekua-Pompeiu. Belgorod State University Scientific Bulletin **50** (2018), 40–46.

[3] O. Chernova Fredholm solvability of the linear conjugation problem for a first-order elliptic system with complex coefficients. Dynamical Systems. **8(36):4** (2018), 357–371.

[4] A. Douglis A function-theoretic approach to elliptic systems of equations in two variables, Comm. Pure Appl. Math. **6** (1953), 259–289.

[5] R. P. Gilbert, J. L. Buchanan First order elliptic systems. A function theoretic approach. Mathematics in Science and Engineering, 163. Academic Press Inc., Orlando, 1983.

[6] A. Soldatov Hyperanalytic functions and their applications. Modern Mathematics and its Applications. Function Theory. Tbilisi. **15** (2004), 142–199.

[7] O. Vashchenko Integral representation of solutions of first-order elliptic systems in Gelders classes, Proceedings of the III School of Young Scientists Non-local boundary value problems and problems of modern analysis and informatics, Nalchik-Elbrus. (2005), 11–14.

APPLICATION OF DISCRETE MODELS OF DYNAMIC SYSTEMS FOR REMOTE DETECTION OF CYANOTOXIN SOURCES ON AQUATORIES

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Abstract. We are considering the task of identifying sites for the elimination of accumulations of toxic cyanobacteria from a digital photograph of a reservoir. The paper proposes a solution to this problem. The photo is divided into a segment and using the methods of discrete models of dynamic systems, the color characteristics are selected. The selected characteristics form the description of the input data of the neural network, which determines the place for the elimination of toxic cyanobacterial clusters. The operation of the algorithm is successful in photographs of reservoirs with accumulations of toxic cyanobacteria.

Keywords. discrete models of dynamic systems; toxic cyanobacterial clusters; neural networks.

MSC2010. 68T20.

1   Introduction

Serious problems of biosafety of water consumption are associated with accumulations of toxic cyanobacteria (ATC) in the waters. They are sources of dangerous cyanotoxins [1]. Such clusters are observed, for example, in the Baltic region. Therefore, remote monitoring of the ATC becomes important: it becomes an aspect of national security. The international HELCOM structure has been created in the Baltic region . She is armed with modern means of satellite monitoring. These funds are used inefficiently due to the similarity of the spectral parameters of the pigments of the ATC and other phytoplankton communities [2]. In [3] , the solution of a part of these problems is proposed by using the original [4] class of mathematical models - discrete models of dynamic systems (DMDS). The DMDS methods by the form of the correlation matrix between the values of the system components form a structural model of intercomponent and intra-component relations. Based on this structure, for certain initial conditions, an idealized system trajectory (IST) can be constructed. It reflects the cycle of the dynamics of the change of system states.

2   Main results

In [5] , an interpretation of the ISï¿$\frac{1}{2}$ type is proposed. The presence in its cycle of a specific set of different combinations of system component values is understood as measures of biodiversity. It reflects the quantitative and qualitative parameters of a set of "strategy-combinations" of the functioning of a living system in certain situations. In [3], an analysis of a set of strategy combinations corresponding to the distribution of colorimetric parameters on the surface of the ATC was carried out and, based on such an analysis, the type of systemic colorimetric parameter was found. The essential point is a subtle analysis of the spatial distribution of colorimetric parameters. In some situations, we can talk about colorimetric parameters corresponding to the components of the RGB-model of digital photographs taken by the equipment supplied with relatively cheap drones. In a number of extreme situations, the need arises to eliminate ATC treatment from the air (by algaecides, using EM- or nano-technologies, specifically, depending on the type of bioproduction processes in this section of the ATC). Correspondingly, remote methods of diagnosing the influence of the type of bioproduction processes are required, which determine vulnerability to a specific type of elimination in certain areas of the ATC. In the present work, algorithms using neural networks are presented for such diagnostics . To select the characteristics of the description of the input data used methods DMDS. The digital image was broken up into segments. Correspondingly,, each segment was broken up into microsegments. For a variety of microsegments of each segment, the average, maximum, variational range of color characteristics are determined . Idealized trajectories are analyzed and color-rhythmic characteristics are selected, which are chosen as the description of the input data for the neural network. Note that the colorimetric characteristics themselves have an important biological interpretation. For example, when analyzing a digital image of a reservoir in Uruguay (Rio Santa Lucia), an indicator was selected $\frac{G}{R+G}/\frac{R}{G}$. Here, the numerator is an indicator of productivity (the ratio of live to the sum of the living and the dead), and the denominator is an indicator of pigment diversity and stability of the ATC system. Training and test samples to determine the site for elimination were selected from segments of a digital image of the reservoir. On test samples, the network demonstrated zero error level.

References

[1] Walls, J.T., K.H. Wyatt, J.C. Doll, E.M. Rubenstein, and A.R. Rober. 2018. Hot and toxic: Temperature regulates microcystin release from cyanobacteria. Science of the Total Environment 610-611: 786 ï¿$\frac{1}{2}$ 795.

[2] Karabashev, G. S., Evdoshenko, M. A. (2015). Spektralnyie priznaki tsveteniya tsianobakteriy v Baltiyskom more po dannyim skanera MODIS. Sovremennyie Problemyi Distantsionnogo Zondirovaniya Zemli iz Kosmosa, 12(3), 158ï¿$\frac{1}{2}$170.

[3] Bespalov, Yu. G., Nosov, K. V., Kabalyants, P. S. (2017). Modeling systemic colorimetric parameters as a tool for processing images of clumps of toxic cyanobacteria targeted at their boundaries detection. bioRxiv, doi: https://doi.org/10.1101/232413.

[4] 4. Zholtkevych, G. N., Bespalov, Y. G., Nosov, K. V., Abhishek, M. (2013). Discrete Modeling of Dynamics of Zooplankton Community at the Different Stages of an Antropogeneous Eutrophication. Acta Biotheoretica, 61(4), 449ï¿$\frac{1}{2}$465.

[5] Bespalov, Yu. G., Nosov, K. V., Kabalyants, P. S. (2018). A mathematical model of the effect of natural selection on adaptation forms that implemented by disruptive coloration of Taurotragus oryx. bioRxiv, doi: http://dx.doi.org/10.1101/368084.

On the Solution Tracking Problem with Incomplete Information for the System of Nonlinear Fractional Differential Equations

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Abstract. We consider the solution tracking problem for the system of nonlinear differential equations when the vector of coordinates is partly measured and unknown input control action. A solving algorithm resistant to information interference and computational errors, which is based on regularization methods and constructions of guaranteed control theory is proposed.

Keywords. control theory; fractional differential equations; incomplete information.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1 Introduction

We consider a dynamical system described by the system of nonlinear fractional differential equations

$$
\begin{aligned}
[{}^{\mathrm{C}}D^{\gamma}x_1](t) &= g_1(t, x_1) + Bx_2(t), \quad t \in T = [\sigma, \theta], \quad x_1(\sigma) = x_{1\sigma}, \quad \gamma \in (0, 1), \\
[{}^{\mathrm{C}}D^{\gamma}x_2](t) &= g_2(t, x_1, x_2) + Cu(t), \quad x_2(\sigma) = x_{2\sigma}.
\end{aligned}
\tag{1}
$$

Here $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $(x_{1\sigma}, x_{2\sigma})$ is the initial state, $g_1 \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g_1 \colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are given vector functions satisfying the jointly Lipschitz condition. A trajectory of the system $x(t) = (x_1(t), x_2(t))$ is unknown in advance and determined by some disturbance $u(\cdot)$, which is also not specified, but is subject to a prior restriction $u(t) \in P$, $t \in T$, where P is a given bounded and closed set in $\mathbb{R}^q$.

Definition 1 [1]. Fractional integral of an order γ starting at σ of an arbitrary function $f \in L_1(T, \mathbb{R}^n)$ is defined by the formula

$$
[I^{\gamma}f](t) = \frac{1}{\Gamma(\gamma)} \int_{\sigma}^{t} \frac{f(s)}{(t-s)^{1-\gamma}} \, ds, \quad \gamma \in (0, 1), \quad t \in T, \quad \theta < +\infty.
$$

Definition 2 [1, p. 91]. For a function $x \colon T \to \mathbb{R}^n$ and arbitrary real $\gamma \in (0, 1)$, the expression

$$
[{}^{\mathrm{C}}D^{\gamma}x](t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_{\sigma}^{t} \frac{x(s) - x(0)}{(t-s)^{\gamma}} \, ds, \quad t \in T,
$$

is called the left-side Caputo fractional derivative.

At discrete enough frequent points in time $\tau_i \in T$, $\tau_i < \tau_{i+1}$ the coordinates $x_1(\tau_i)$ are measured with some error $h \in (0,1)$. Measurement results $\xi_i^h \in \mathbb{R}^n$ such that $|x_1(\tau_i) - \xi_i^h|_n \leqslant h$, where $|\cdot|_n$ is a norm in Euclidean space $\mathbb{R}^n$. It is required to find the control $u(\cdot)$ generating $x_1(\cdot)$: $u(\cdot) = u(\cdot; x_1(\cdot))$. Since in the presence of the error h and possible non-uniqueness, an exact recovery $u(\cdot)$ is impossible,it is necessary to construct an algorithm for calculating some approximation of $u(\cdot)$. This approximation should be the better, the smaller the measurement error of $x_1(\tau_i)$ and the finer the grid $\{\tau_i\}$.

2   Main result

The algorithm for solving the problem is based on the dynamic inversion method, developed in [2], as well as the method of extremal shift known in the theory of positional control [3]. In connection with the incompleteness of information (namely, with the possibility of measuring at the moments of τ_i only part $x_1(\tau_i)$), along with the control unit, we will use an additional unit — a unit for dynamically reconstructing an unknown coordinate. The latter will play the role of a provider of missing information about the current phase state of the system (1). This information will be promptly transmitted to the control unit, which forms u according to the law of feedback. The case of measuring part of the coordinates was discussed in [5]. By using the works [4, 6] we prove the following statement.

Theorem 1 *Let the functions g_1 and g_2 satisfy the jointly Lipschitz condition with constant $L > 0$ with respect to variables x_1, x_2, t, the order $\gamma \in (1/2, 1)$, and the regularization parameter $\alpha(h)$ and the partition step $\delta(h)$ be chosen in such a way that $\alpha(h) \to 0$, $\delta(h) \to 0$, and $(h + \delta^\gamma(h))/\alpha(h) \to 0$ as $h \to 0$. Then the constants $d_i > 0$ and $\beta \in (0,1)$ those are independent of h, α, and δ can be explicitly specified such that the system satisfies the inequalities*

$$|w_2(t) - x_2(t)|_m^2 \leqslant d_1\delta^\gamma + d_2h^\beta + d_3h^{1-\beta} + d_4\frac{\alpha}{h^\beta} + d_5\frac{\delta^\gamma}{h^\beta} + d_6\alpha, \tag{2}$$

where w_2 is a coordinate of the auxiliary system of Krasovskii extremal shift method.

References

[1] A. Kilbas, H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

[2] Yu. S. Osipov, A. V. Kryazhimskii, *Inverse problems for ordinary differential equations: dynamical solutions*, Gordon & Breach, Basel, 1995.

[3] N. N. Krasovskii, A. I. Subbotin, *Game-theoretical control problems*, Springer, New York, 1988.

[4] P. G. Surkov, Dynamic right-hand side reconstruction problem for a system of fractional differential equations, Differential Equations, **55** 6 (2019), 849–858.

[5] M. S. Blizorukova, V. I. Maksimov, An algorithm for dynamic reconstruction of input disturbances from observations of some of the coordinates, Computational Mathematics and Mathematical Physics, **47** 3 (2011), 415–421.

[6] M. I.Gomoyunov, Fractional derivatives of convex Lyapunov functions and control problems in fractional order systems, Fractional Calculus and Applied Analysis. **21** 5 (2018), 1238–1261.

The Moisil–Theodorescu system in complex form

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Abstract. In this paper we discuss the conditions under which a problem (3), (5) has the Fredholm property.

Keywords. Moisil – Theodorescu system; Riemann-Hilbert problem; Pauli matrices; Fredholm property.

MSC2010. 35J25, 35J55; 45F15.

We consider the Moisil – Theodorescu system

$$M\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) u(x) = 0, \quad M(\zeta_1, \zeta_2, \zeta_3) = \begin{pmatrix} 0 & \zeta_1 & \zeta_2 & \zeta_3 \\ \zeta_1 & 0 & -\zeta_3 & \zeta_2 \\ \zeta_2 & \zeta_3 & 0 & -\zeta_1 \\ \zeta_3 & -\zeta_2 & \zeta_1 & 0 \end{pmatrix}. \tag{1}$$

For a complex-valued vector-function $\phi = (\phi_1, \phi_2)$

$$\phi_1 = u_2 - iu_1, \quad \phi_2 = u_4 + iu_3$$

we have complex form the Moisil – Theodorescu system

$$E_1 \frac{\partial \phi}{\partial x_1} + E_2 \frac{\partial \phi}{\partial x_2} + E_3 \frac{\partial \phi}{\partial x_3} = 0 \tag{2}$$

with Pauli matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $D \subseteq \mathbb{R}^3$ be a domain with smooth boundary Γ. The Riemann-Hilbert problem in D for the system (1)

$$Bu^+ = f \tag{3}$$

with 2×4 –matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix}$$

is equal to

$$Re\, G\phi^+ = g, \tag{4}$$

with matrix–function G, $G_{k1} = b_{k2} + ib_{k1}$, $G_{k2} = b_{k4} - ib_{k3}$, $k = 1, 2$.

Let $\Gamma \in C^{1,\nu}$, $B \in C^{\nu}(\Gamma)$, $0 < \nu < 1$ and B^{ij} be a minor defined by i – and j–th columns of the B. The criterion for Fredholm property of the problem (1), (2) is equal to the vector $l = (l_1, l_2, l_3)$

$$l_1 = B^{12} + B^{34}, \quad l_2 = B^{13} - B^{24}, \quad l_3 = B^{14} + B^{23}$$

is not tangent to Γ. Under this condition, the index of the problem (1), (2) is determined by the formula $\ae = m - s$, where s is the number of connected components of the boundary Γ, m is the order of the first de Rham cogomology group of a domain [1].

A special case of problem (4) for the system (3) is a $\mathbb{C}$ –linear problem

$$a\phi_1 + b\phi_2 = f, \quad |a|^2 + |b|^2 = 1, \tag{5}$$

with complex-valued functions $a = a_1 + ia_2$, $b = b_1 + ib_2 \in C^{\nu}(\Gamma)$. In this case

$$l_1 = |a|^2 - |b|^2, \ l_2 = 2(a_2 b_1 - a_1 b_2), \ l_3 = (a_1 b_1 + a_2 b_2) \tag{6}$$

and $|l| = 1$.

We discuss the conditions under which the problem (3), (5) has Fredholm property. The necessary and sufficient conditions on the coefficients a, b from the report are found found; it provides a positive solution to this problem.

Acknowledgments

References

[1] Soldatov A.P., The Riemann–Hilbert boundary value problem for the Moisil-Theodoresco system, *J. Math. Sci.*, 2019, Vol. 239, No. 3, P. 381-395.

Coexistence of algebraic and non-algebraic limit cycles of a family of polynomial differential systems

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Abstract. In this work we give an explicit expression of invariant algebraic curves of a multi-parameter planar polynomial differential systems of degree nine, then we prove that these systems are integrable and we introduce an explicit expression of a first integral. Moreover, we determine sufficient conditions for these systems to possess two limit cycles.

Keywords. Limit cycle; Riccati equation; Invariant algebraic curve; First Integral.

MSC2010. 34C05, 34C07, 37C27, 37K10

1   Introduction

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of the form

$$x' = \frac{dx}{dt} = P(x,y), y' = \frac{dy}{dt} = Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y. Here, the degree of system (1) is denoted by $n = \max\{\deg P, \deg Q\}$. In 1900 Hilbert [1] in the second part of his 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, and also to study their distribution or configuration in the plane $\mathbb{R}^2$. This has been one of the main problems in the qualitative theory of planar differential equations in the 20th century.

In this work we give an explicit expression of invariant algebraic curves, then we prove that these systems are integrable and we introduce an explicit expression of a first integral of a multi-parameter planar polynomial differential systems of degree nine of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = x + P_5(x,y) + xR_8(x,y), \\ y' = \dfrac{dy}{dt} = y + Q_5(x,y) + yR_8(x,y), \end{cases} \tag{2}$$

where

$$P_5(x,y) = -(a+2)\,x^5 + (4+4b)\,x^4y - (2a+4)\,x^3y^2 + (8+4b)\,x^2y^3 - (a+2)\,xy^4 + 4y^5,$$

$$Q_5(x,y) = -4x^5 - (a+2)\,x^4y + (4b-8)\,x^3y^2 - (2a+4)\,x^2y^3 + (4b-4)\,xy^4 - (a+2)\,y^5,$$

and

$$R_8(x, y) = (a+1)x^8 - 4bx^7y + (4a+4)x^6y^2 - 12bx^5y^3 + (6a+6)x^4y^4 - 12bx^3y^5$$
$$+ (4a+4)x^2y^6 - 4bxy^7 + (a+1)y^8,$$

in which a, b are real constants.

Moreover, we determine sufficient conditions for a polynomial differential system to possess two limit cycles : one of them is algebraic and the other one is shown to be non-algebraic, explicitly given. Concrete examples exhibiting the applicability of our result are introduced.

2   Main results

Our main result is contained in the following theorem.

Theorem 1 *Consider a multi-parameter planar polynomial differential systems (2), then the following statements hold.*

1) The origin of coordinates $O(0,0)$ is the unique critical point at finite distance.

2) The curve $U(x, y) = x^4 + y^4 + 2x^2y^2 - 1$, is an invariant algebraic curve of system (2) with cofactor

$$K(x, y) = (-4)\left(x^2 + y^2\right)^2 \left((-a-1)\left(x^2 + y^2\right)^2 + 4bxy\left(x^2 + y^2\right) + 1\right).$$

3) The system (2) has the first integral

$$H(x, y) = \frac{\left(x^2 + y^2\right)^2 + \left(1 - \left(x^2 + y^2\right)^2\right)\exp\left(a\arctan\frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right)f\left(\arctan\frac{y}{x}\right)}{\left(\left(x^2 + y^2\right)^2 - 1\right)\exp\left(a\arctan\frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right)},$$

where $f(\arctan\frac{y}{x}) = \int_0^{\arctan\frac{y}{x}} \exp(-as - b\cos 2s)ds$.

4) The system (2) has an explicit limit cycle, given in Cartesian coordinates by $(\Gamma_1) : x^4 + y^4 + 2x^2y^2 - 1 = 0$.

5) If $a > 0$ and $b \in \mathbb{R} - \{0\}$, then system (2) has non-algebraic limit cycle (Γ_2), explicitly given in polar coordinates (r, θ), by the equation

$$r(\theta, r_*) = \left(\frac{\exp\left(a\theta + b\cos 2\theta\right)\left(\dfrac{e^{2\pi a}}{1 - e^{2\pi a}}f(2\pi) + f(\theta)\right)}{-1 + \exp\left(a\theta + b\cos 2\theta\right)\left(\dfrac{e^{2\pi a}}{1 - e^{2\pi a}}f(2\pi) + f(\theta)\right)}\right)^{\frac{1}{4}},$$

where

$$f(\theta) = \int_0^\theta \exp(-as - b\cos 2s)ds .$$

Moreover, the non-algebraic limit cycle (Γ_2) lies inside the algebraic limit cycle (Γ_1).

References

[1] D. Hilbert; *Mathematische Probleme, Lecture, Second Internat. Congr. Math.* (Paris, 1900), Nachr. Ges. Wiss. Gttingen Math. Phys. Kl, 1900, pp. 253-297.

[2] R. Boukoucha; *Explicit limit cycles of a family of polynomial differential systems*, Elect. J. of Diff. Equ, vol. 2017, no. 217, pp. 1-7.

Existence, Stability and Controllability of Fractional Damped Dynamical Equation with Non-instantaneous Impulses

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Abstract. In this manuscript, we establish the existence, uniqueness, stability and controllability results for a fractional damped dynamical equation with non-instantaneous impulses. This manuscript mainly has three segments. First segment is dedicated to the study of existence of a unique solution. In the second segment, we establish the stability results. Finally, we discuss the controllability results of the taken problem. Controllability Grammian matrix, Mittag-Leffler matrix function, nonlinear functional analysis and Banach fixed-point technique have been used to establish these results. Also, with the help of MATLAB, some numerical examples with simulations are given to outline the effectiveness of the obtained results.

Keywords. Controllability; Non-instantaneous Impulses; Fractional dynamical equation; Mittag-Leffler function.

MSC2010. 93B05; 34A37; 34A08; 33E12.

Tempered fractional equation for electron transport in quantum wires with fractal disorder

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Abstract. New aspects of electron transport in quantum wires with Lévy type disorder are considered with the use of the tempered fractional equations and fractional stable statistics. We consider the weak scattering and sequential incoherent tunneling in quasi-1d quantum systems characterized by the tempered Lévy stable distribution of spacing between scatterers or tunneling barriers. We generalize the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation for distribution of transmission matrix eigenvalues. The solution describes the evolution from the anomalous conductance distribution to the Dorokhov function for a long wire. For the sequential tunneling through quasi-fractal wires, the case of barriers with widely distributed resistances is also considered. Neglecting Coulomb blockade of single-electron tunneling, we show that the relative fluctuations of the conductance do not vanish with the increase of L and the system does not possess the property of self-averaging.

Keywords. nanowire, quantum transport, fractional calculus, stochastic fractal

MSC2010. 35R11; 26A33; 60E07

1   Introduction

Several experiments and numerical simulations have shown that disorder in mesoscopic systems can be of fractal (self-similar) type. In Ref. [1] fractal disorder is observed in optical material, in which light waves perform a random walk characterized by path length distributions of power law type $p(s) \propto s^{-1-\alpha}$, if $s \to \infty$, where $0 < \alpha < 2$. This random walk is known as *Lévy flights*, and the system has been called "Lévy glass". Kohno and Yoshida [2] reported on the scaling behavior of diameter modulations in SiC quantum wires grown via a self-organization process. There is a series of theoretical studies devoted to electron and wave transport in quasi-one-dimensional wires with Lévy type disorder (see, e.g. [3, 4]). Here, we study some new aspects of electron transport in quantum wires with Lévy type disorder in terms of tempered fractional-order equations and tempered fractional stable statistics. We consider two regimes of transmission (weak scattering and incoherent sequential tunneling) in quasi-1d quantum systems characterized by the tempered Lévy stable distribution of spacing between scatterers.

2   Main results

The generalized DMPK-equation for quantum wire with quasi-fractal disorder is obtained in the form

$$l^{\alpha} \mathrm{D}_L^{\alpha,\gamma} \left[P(\{\lambda\}, L) - \delta(\lambda) \right] = K \Lambda_{\mathrm{DMPK}} P(\{\lambda\}, L), \tag{1}$$

where

$$_0\mathsf{D}_L^{\alpha,\gamma}P(\{\lambda\},L) = e^{-\gamma L} \, _0\mathsf{D}_L^\alpha e^{\gamma L} \, P(\{\lambda\},L) - \gamma^\alpha P(\{\lambda\},L)$$

is the tempered fractional derivative, $_0\mathsf{D}_L^\alpha$ is the Riemann-Liouville operator and

$$\Lambda_{\mathrm{DMPK}}P = \frac{2}{\beta N + 2 - \beta}\sum_{j=1}^N \frac{\partial}{\partial\lambda_j}\left[\lambda_j(1+\lambda_j)J(\lambda)\frac{\partial}{\partial\lambda_j}\frac{P}{J(\lambda)}\right]$$

is the DMPK operator, λ_j are eigenvalues of quantum-mechanical transfer matrix t, N number of transverse modes in the wire, $P(\{\lambda\},L)$ the N-dimensional probability distribution function of random vector $(\lambda_1,\lambda_2,\ldots,\lambda_N)$. The integer parameter β equals 1 in a zero magnetic field and 2 in a time-reversal-symmetry breaking magnetic field. In the case of zero field and strong spin-orbit scattering $\beta = 4$. The Jacobian $J = \prod_{i<j}|\lambda_j - \lambda_i|^\beta$ corresponds to the transformation from the transfer matrix space to the eigenvalue space. Interpreting the process described by equation (1) as a subordinated process, one can represent solution as the following integral $P(\{\lambda\},L) = \int_0^\infty P_1(\{\lambda\},\xi)\,g(\xi,L)d\xi$, where $g(\xi,L)$ is the inverse density to the tempered stable pdf (tempered subordinator) with the following Laplace transform $\tilde{g}(\xi,s) = l^\alpha s^{-1}[(s+\gamma)^\alpha - \gamma^\alpha]e^{-\xi l^\alpha[(s+\gamma)^\alpha - \gamma^\alpha]}$. Function $P_1(\{\lambda\},\xi)$ represents solution of the standard DMPK-equation, where ξ is associated with the dimensionless length L/l. For $L \gg \gamma^{-1}$ the standard DMPK-approach can be used. This solution describes evolution from the anomalous conductance distribution to the Dorokhov function for a long wire.

In regular quantum wires, $\langle -\ln G\rangle$ scales linearly with length ξ. For quasi-fractal wires, we derived $\langle -\ln G\rangle = l^{-\alpha}\int_0^L d\lambda\,\lambda^{\alpha-1}E_{\alpha,\alpha}(\gamma^\alpha\lambda^\alpha)\,\exp(-\gamma\lambda)$. Here, $E_{\alpha,\alpha}(x)$ is the two-parameter Mittag-Leffler function. The scaling of $\langle -\ln G\rangle$ for intermediate lengths is of the form $\langle -\ln G\rangle \approx \frac{1}{\Gamma(1+\alpha)}\frac{L^\alpha}{l^\alpha}$, $l \ll L \ll \gamma^{-1}$. For very long wires, we observe the normal scaling, $\langle -\ln G\rangle \approx \frac{(\gamma l)^{1-\alpha}}{\alpha}\frac{L}{l}$, $l \ll \gamma^{-1} \ll L$. The crossover is observed near value $L_{\mathrm{cr}} = \gamma^{-1}[\alpha/\Gamma(1+\alpha)]^{1/(1-\alpha)}$.

Also, we study the case of sequential incoherent tunneling through the system with fractal space disorder accompanied by wide (power law) distribution of individual barrier resistance, $\mathrm{Prob}\{R_i > r\} \sim \frac{(r/\rho_\beta)^{-\beta}}{\Gamma(1-\beta)}$, $0 < \beta < 1$. The Coulomb blockade is neglected. In this case, the asymptotic distribution of the resistance R of a long quantum wire with length L presents the pdf of a one-sided fractal random walk, which is expressed through the fractional stable density. We show that for $\beta < 1$ the relative fluctuations of the conductance do not vanish with the increase of L and the system does not possess the property of self-averaging even for the Poisson distribution of barriers along wire.

Acknowledgments

References

[1] P. Barthelemy, J. Bertolotti, and D. S. Wiersma. *Nature*, 453:495, 2008.

[2] H. Kohno and H. Yoshida. *Physical Review E*, 70(6), 2004.

[3] C.W.J. Beenakker, C.W. Groth, and A.R. Akhmerov. *Physical Review B*, 79(2), 2009.

[4] F. Falceto and V. A. Gopar. *EPL (Europhysics Letters)*, 92(5), 2011.

Positve iterative solutions for nonlinear fractional differential equations

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Abstract. We study the following nonlinear fractional differential equation with nonlocal boundary conditions

$$-D^\alpha u(t) + a(t)u(t) = p(t)f(t, u(t), u(t)) + q(t)g(t, u(t)) \text{ in } (0,1), \qquad (1)$$

with the boundary conditions

$$u(0) = u'(0) = u'(1) = 0, \qquad (2)$$

where $2 < \alpha < 3$, D^α denotes the Riemann-Liouville derivative of order α In addition suppose that the following assumptions are satisfied:

  (i) $a, p, q \in \mathcal{C}((0,1), [0, +\infty))$

  (ii) $f \in \mathcal{C}((0,1) \times (0, +\infty) \times (0, +\infty), [0, +\infty))$.

  (iii) $g \in \mathcal{C}((0,1) \times (0, +\infty), [0, +\infty))$.

An associated Green's function is constructed as a series of functions by the perturbation approach. By using the properties of the Green function and the fixed point theorem of mixed monotone operators in cones we obtain some results on the existence and uniqueness of positive solutions. Our results extend and improve many known results including singular and non-singular cases.

Keywords. Fractional differential equation, positive solution, Green's functions.

MSC2010. 34B16; 34B18.

References

[1] J. R. Graef, L. Kong, Q. Kong, and M. Wang, On a fractional boundary value problem with a perturbation term, Journal of Applied Analysis and Computation, vol. 7, no. 1, pp. 57-66, (2017).

[2] Liu, L, Zhang, X, Liu, L, Wu, Y: Iterative positive solutions for singular nonlinear fractional differential equation with integral boundary conditions. Adv. Differ. Equ. 2016, 154 (2016).

A Stable and Convergent Non-Standard Finite Difference Method for Fractional Black-Scholes Model of Digital Put Option Pricing

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Abstract. We introduce the mathematical model of digital put option pricing under the Fractional Black-Scholes (FBS) model. The digital put option pricing is option pricing where the payoff is non-smooth, the optimal exercise boundary is equal with exercise price. Following this, we introduce the non-standard Grünwald Letnikov approximation and use this approximation for time fraction term in FBS model to reach a fractional non-standard finite difference (NSFD) problem. We show that the proposed fractional NSFD scheme is stable and convergent. Uniqueness of the approximate solution is also proved. We also show that numerical results satisfy the physical conditions of digital put option pricing under the FBS model.

Keywords. Fractional Differential Equation; Digital Option Pricing; Non-Standard Finite Difference Method; Interpolation Method.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

The digital put option pricing under the FBS model are presented as

$$\frac{\partial^\alpha P}{\partial t^\alpha} = \left(rP - rS\frac{\partial P}{\partial S} \right) \frac{t^{1-\alpha}}{(1-\alpha)!} - \frac{\alpha!}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}, \quad S > E, \ \ 0 \le t < T, \tag{1}$$

$$P(S,T) = \begin{cases} 1 & S \le E, \\ 0 & S > E. \end{cases}, \quad P(E,t) = 1, \ \ \lim_{S\to\infty} P(S,t) = 0, \tag{2}$$

$$P(S,t) = 1, \quad 0 \le S \le E, \tag{3}$$

where $P(S,t)$ is the digital put option pricing. The parameters $\frac{3}{4}$, r and E show the volatility of the underlying asset, the interest rate and the exercise price of the digital option, respectively. The details of obtaining the FBS model are explained in [1].

2   Main results

We consider (1)-(3) and use a NSFD scheme for derivatives on the right side and a non-standard Grünwald Letnikov approximation for fractional derivative on the left side of (1). To do this Let $t_k = k\Delta t, k = 0,1,2\cdots,n-1$ and $S_j = j\Delta S, j = 1,2,\cdots,m-1$, where for $0 \le t \le T$,

$\Delta t = \frac{T}{n}$, and $\Delta S = \frac{S_{max}}{m}$ are time and stock price steps respectively. Then instead of step sizes Δt and ΔS, we use the functions

$$\varphi(\Delta t) = 1 - e^{-\Delta t}, \quad \psi(\Delta S) = e^{\Delta S} - 1, \quad \phi(\Delta S^2) = 4\sin^2(\frac{\Delta S}{2}), \tag{4}$$

to get

$$(1 - e^{-\Delta t})^{-\alpha} \sum_{i=0}^{k+1} g_i^\alpha P_j^{k+1-i} = \left(rP_j^k - rS_j \frac{P_j^k - P_{j-1}^k}{(e^{\Delta S} - 1)} \right) \frac{t_k^{1-\alpha}}{(1-\alpha)!} \tag{5}$$

$$-\frac{\alpha!}{2}\sigma^2 S_j S_{j+1} \frac{P_{j-1}^k - 2P_j^k + P_{j+1}^k}{(2\sin(\frac{\Delta S}{2}))^2}, \quad S_j > E, \quad 0 \le t_k < T,$$

subject to

$$P(S_j, T) = \left\{ \begin{array}{ll} 1 & S_j \le E, \\ 0 & S_j > E. \end{array} \right. , \quad P(E, t_k) = 1, \quad \lim_{S \to \infty} P(S_j, t_k) = 0, \tag{6}$$

$$P(S_j, t_k) = 1, \quad 0 \le S \le E. \tag{7}$$

If $j = 0$, then for all time values, it also follows from (3) that $\mathbf{P}_0 = 1$ (since $S_0 = 0$) and if $S_1 = \Delta S \le E$ then from Eq. (7) we have $P(S_1, t) = 1$ and so

$$C\mathbf{P}_{j+1} = A\mathbf{P}_j + B\mathbf{P}_{j-1}, \quad \mathbf{P}_0 = 1, \quad \mathbf{P}_1 = 1. \tag{8}$$

Theorem 1 (Stability) *If $\Delta S \le E$ then the NSFD scheme (8) is stable and we have $\| \mathbf{E}_{j+1} \|_\infty \le K \max\{\| \mathbf{E}_0 \|_\infty, \| \mathbf{E}_1 \|_\infty\}$, $j = 2, 3, \cdots, m-1$, where K is a positive constant independent of Δt, ΔS and j.*

Theorem 2 (Convergence) *Let (5) have the smooth solution $P(S,t) \in C_{S,t}^{2,\alpha}(\Omega)$. Let P_j^k be the numerical solution computed by use of (8). Then P_j^k converges to $P(S_j, t_k)$, if $\Delta S \le E$.*

We can obtain

$$\mathbf{P}^k = (\mathbf{I} - D\mathbf{A})^{-1} \left(-\sum_{i=1}^{k-1} a_{i,k}^{(\alpha)}(\mathbf{P}^{k-i}) + \mathbf{P}^0 \sum_{i=0}^{k-1} a_{i,k}^{(\alpha)} + \mathbf{P}^0 \right). \tag{9}$$

Theorem 3 (Uniqueness) *If the matrix $\mathbf{I} - D\mathbf{A}$ is invertible, then, Eq (9) has a unique approximate solution.*

To solve (5)-(7), we use a higher order Grünwald Letnikov approximation. Since the digital put option pricing is only known at the end point (exercise time), then in order to use a higher order GrÃ¼nwald Letnikov approximation, we need some other points of digital put option pricing. To get these intermediate values, we use a suitable interpolation method, that is, for the points $P_j^0 = P(S_j, 0)$ and $P_j^T = P(S_j, T)$.

References

[1] G. Jumarie, Stock exchange fractional dynamics defined as fractional exponential growth driven by (usual) gaussian white noise. application to fractional black–scholes equations, Insurance: Mathematics and Economics 42 (1) (2008) 271–287.

[2] R. Kalantari, S. Shahmorad, A Stable and Convergent Finite Difference Method for Fractional Black–Scholes Model of American Put Option Pricing, Computational Economics, 53, 1, 191–205, 2019,Springer

Second order differential inclusion governed by sweeping process in Hilbert spaces

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Abstract. We study a differential inclusion known as second order sweeping process for a class of prox-regular non-convex sets. Assuming that such sets depend continuously on time and state, we give a new proof of the existence of solutions via Schauderâs fixed point theorem and the well-posedness for the perturbed first-order sweeping process in Hilbert spaces.

Keywords. differential inclusion ; prox-regular sets, normal cones ; evolution quasi-variational inequalities.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

Differential inclusion $\dot{u}(t) \in \mathrm{N}_{C(t)}(u(t))$ a.e. on $[T_0, T]$, C is a set-valued mapping defined from $[T_0, T]$ to Hilbert space H with closed convex values have been extensively studied over the past by several authors. This differential inclusion known as the sweeping process problem, this problem is equivalent to the following variational inequality: find $u(t) \in C(t)$ a.e. on I such that

$$\langle \dot{u}(t), v - u \rangle \geq 0, \, \forall v \in C(t).$$

In this work, we study the existence of solutions for non convex second order sweeping process

$$\begin{cases} -\ddot{u}(t) \in \mathrm{N}_{C(t,u(t))}(\dot{u}(t)) + F(t, u(t), \dot{u}(t)) & \text{a.e. on } [T_0, T], \\ \quad \dot{u}(t) \in C(t, u(t)), \text{ for all } t \in [T_0, T], \\ \quad u(0) = u_0, \dot{u}(0) = v_0 \in C(T_0, u_0), \end{cases} \tag{1}$$

where F is a Carathéodory unbounded function and $C(t, u)$ is uniformly prox regular set moving in an absolutely continuous way with respect to t and Lipschitz continuous with respect to the state u

2   Main results

We obtain existence of solution of the differential inclusion (**??**). Let T_0 and T be two non negative real numbers with $T_0 < T$. Throughout $C : [T_0, T] \times H \rightrightarrows H$ will be a multimapping with nonempty closed values. We will consider the following assumptions on the multimapping $C(\cdot, \cdot)$:

($\mathbf{HC}_1$): there exists constant $\rho \in]0, +\infty]$ such that, for each $t \in [T_0, T]$ and $u \in H$, the sets $C(t, u)$ are uniformly ρ-prox-regular;

($\mathbf{HC}_2$): the set $C(t, u)$ varies in an absolutely continuous way with respect to t and lipschitz continuous with respect to the state u, that is, there exists an absolutely continuous non negative function $\zeta : [T_0, T] \to \mathbb{R}_+$ and $L > 0$ such that

$$|d(C(t, u), x) - d(C(s, v), y)| \leq \|x - y\| + |\zeta(t) - \zeta(s)| + L\|u - v\|$$

for all $x, y, u, v \in H$ and all $s, t \in [T_0, T]$;

($\mathbf{HC}_3$): for all $(t, u) \in [T_0, T] \times H$, $C(t, u) \subset \mathcal{A} \subset \lambda \mathbb{B}_H$ for some compact set $\mathcal{A}$ in H and some $\lambda > 0$.

We also suppose that $F(\cdot, \cdot, \cdot)$ satisfies the next assumptions:

($\mathbf{HF}_1$):$F : [T_0, T] \times H \times H \to H$ be a function, Lebesgue measurable on $[T_0, T]$ such that for all $\eta > 0$ there exists a non-negative function $\beta_\eta(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that for all $(t, h) \in [T_0, T] \times H$ and any $(x, y) \in B[0, \eta] \times B[0, \eta]$

$$\|F(t, h, x) - F(t, h, y)\| \leq \beta_\eta(t)\|x - y\|; \tag{2}$$

($\mathbf{HF}_2$): there exists a non-negative function $\gamma(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that, for all $(t, h) \in [T_0, T] \times H$ and any $x \in \cup_{s \in [T_0, T]} C(s, h)$, one has

$$\|F(t, h, x)\| \leq \gamma(t)(1 + \|h\| + \|x\|). \tag{3}$$

Theorem 1 *Let H be a separable Hilbert space, $C : [T_0, T] \times H \rightrightarrows H$ be a multimapping satisfying ($\mathbf{HC}_1$), ($\mathbf{HC}_2$) and ($\mathbf{HC}_3$). Let $F : [T_0, T] \times H \times H \to H$ be a function satisfying ($\mathbf{HF}_1$) and ($\mathbf{HF}_2$). Then, for any $u_0 \in H$ and $v_0 \in C(T_0, u_0)$ there exist two absolutely continuous mappings $u, v : [T_0, T] \to H$ such that*

$$\begin{cases} u(t) = u_0 + \int_{T_0}^t v(s)\, ds, & \forall t \in [T_0, T]; \\ -\dot{v}(t) \in N_{C(t, u(t))}(v(t)) + F(t, u(t), v(t)) \ a.e. \ on \ [T_0, T]; \\ v(t) \in C(t, u(t)), \ a.e. \ on \ [T_0, T]; \\ u(T_0) = u_0, v(T_0) = v_0. \end{cases} \tag{4}$$

In other words, there is a absolutely solution $u : [T_0, T] \to H$ to the Cauchy problem (??).

Proof. We will use the existence and uniqueness of solution for first-order sweeping process proved in [?] and the Schauder's fixed point theorem for continuous mappings to prove our main result ∎

References

[1] N. Chemetov and M. D. P. Monteiro Marques, *Non-convex quasi-variational differential inclusions. Set-Valued Anal. 15, 209-221 2007.*

[2] *J. F. Edmond, A. and L. Thibault, Relaxiation and optimal control problem involving a perturbed sweeping process, Math. program, Ser. B, 104, pp. 347–373, 2005.*

Integral representation of the Mittag-Leffler function

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Abstract. The integral representation of the Mittag-Leffler function is considered in the report. The integral representation of the Mittag-Leffler function is obtained which is expressed through a contour integral. An investigation of the integrand has allow to define the singular points this function and values of the parameters when the contour integral is possible to compute. In these cases the Mittag-Leffler function is expressed through the elementary functions.

Keywords. Gamma-function, Mittag-Leffler function

MSC2010. 33B15, 33E12

The two-parametric Mittag-Leffler function is considered

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu + n/\rho)}, \quad \rho > 0, \quad \mu \in \mathbb{C}, \quad z \in \mathbb{C}$$

introduced in the work [4]. In the book [1] the integral representation of this function had been obtained which is expressed through a contour integral. Later basing on this integral representation the transition from contour integration to integration on real variable has been performed in the works [3, 2]. In these works were obtained the integral representation of the Mittag-Leffler function through improper and definite integrals. However detail investigation of the integral representation obtained in the book [1] (see Lemma 3.2.1 in [1]) has shown that in the proof the mistake was permitted. As a result of the mistake the case $z \in G^{(+)}(\varepsilon, \beta)$ has been considered which will not be implemented under any circumstances. This fact forced reconsider the proof of the lemma. As a result the following lemma is true for the integral representation of the Mittag-Leffler function

Lemma 1 *For any real $\rho, \delta_{1\rho}, \delta_{2\rho}, \epsilon$ such that $\rho > 1/2$, $\frac{\pi}{2\rho} < \delta_{1\rho} \leqslant \min(\pi, \pi/\rho)$, $\frac{\pi}{2\rho} < \delta_{2\rho} \leqslant \min(\pi, \pi/\rho)$, $\epsilon > 0$, any $\mu \in \mathbb{C}$ and any $z \in \mathbb{C}$ such that $\frac{\pi}{2\rho} - \delta_{2\rho} + \pi < \arg z < -\frac{\pi}{2\rho} + \delta_{1\rho} + \pi$ the Mittag-Leffler function can be represented in the form*

$$E_{\rho,\mu}(z) = \frac{\rho}{2\pi i} \int_{\gamma_\zeta} \frac{\exp\left\{(z\zeta)^\rho\right\}(z\zeta)^{\rho(1-\mu)}}{\zeta - 1} d\zeta. \tag{1}$$

Here the integration contour γ_ζ has the form

$$\gamma_\zeta = \begin{cases} S_1 = & \{\zeta : \arg \zeta = -\delta_{1\rho} - \pi, \quad |\zeta| \geqslant 1 + \epsilon\}, \\ C_\epsilon = & \{\zeta : -\delta_{1\rho} - \pi \leqslant \arg \zeta \leqslant \delta_{2\rho} - \pi, \quad |\zeta| = 1 + \epsilon\}, \\ S_2 = & \{\zeta : \arg \zeta = \delta_{2\rho} - \pi, \quad |\zeta| \geqslant 1 + \epsilon\}. \end{cases} \tag{2}$$

An investigation of the integrand in (1) has shown that this function has one or two singular points in dependence of values of the parameters ρ and μ. As a result the following lemma is true.

Lemma 2 *For any real $\rho > 1/2$ and any complex values of the parameter $\mu = \mu_R + i\mu_I$ the function*

$$\Phi_{\rho,\mu}(\zeta, z) = \frac{\rho}{2\pi i} \frac{\exp\{(\zeta z)^\rho\}(\zeta z)^{\rho(1-\mu)}}{\zeta - 1}.$$

has two singularities $\zeta = 1$ and $\zeta = 0$ according to the variable ζ. The point $\zeta = 1$ is pole of 1st order. The point $\zeta = 0$ is

1. *the regular point of the function $\Phi_{\rho,\mu}(\zeta, z)$ at the values of the parameters $\rho = n$ where $n = 1, 2, 3, \ldots$ (integer positive number) $\mu_I = 0$ and $\mu_R = 1 - m_1/\rho$, where $m_1 = 0, 1, 2, 3, \ldots$ (integer positive number);*

2. *the pole of the order m_2 if $\rho = n$, where $n = 1, 2, 3, \ldots$ (integer positive number), $\mu_I = 0$, and $\mu_R = 1 + m_2/\rho$, where $m_2 = 1, 2, 3, \ldots$ (integer positive number);*

3. *the point of branching for the others values of the parameters ρ, μ_I, μ_R.*

As it follows from this lemma and from the definition of the integration contour γ_ζ (2) at value of the parameter $\rho = 1$, integer μ and at $\delta_{1\rho} = \delta_{2\rho} = \pi$ the integration contour γ_ζ closes and the integral (1) become possible compute analytically.

Corollary 3 *For values of the parameters $\rho = 1$, $\delta_{1\rho} = \delta_{2\rho} = \pi$, any complex z such that $\pi/2 < \arg z < 3\pi/2$ and for integer and real values of the parameter $\mu = n, n = 0, \pm 1, \pm 2, \pm 3, \ldots$ the Mittag-Leffler function has the form*

1. $E_{1,n}(z) = e^z z^{1-n}$, *if $n \leqslant 1$ ($n = 1, 0, -1, -2, -3, \ldots$).*

2. $E_{1,n}(z) = z^{1-n} \left(e^z - \sum_{k=0}^{n-2} \frac{z^k}{k!} \right)$, *if $n \geqslant 2$ ($n = 2, 3, 4, \ldots$).*

References

[1] M. M. Dzhrbashian. *Integral Transfororms and Representation of the functions in the Complex Domain*. Nauka. Glav. red. fiz.-mat. lit, Moscow, (in Russian), 1966.

[2] Rudolf Gorenflo, Anatoly A. Kilbas, Francesco Mainardi, and Sergei V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, Berlin, Heidelberg, 2014.

[3] Rudolf Gorenflo, Joulia Loutchko, and Yuri Luchko. Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivative. Fractional Calculus and Applied Analysis, **5** (2002), 491–518.

[4] P. Humbert and R. P Agarwal. Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations. Bull. Sci. Math., II. Sér., **77** (1953), 180–185.

The encapsulation of some derivatives of Naphthol sulfonate in β-Cyclodextrin: A DFT investigation

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Abstract. Different calculating methods of the molecular modeling have been applied to study the host/guest inclusion complexes of the host molecule β-cyclodextrin (β-CD) with the guest molecule 1-naphthol-2-sulfonate (1N2S); Using the following theories, semi-empirical PM3 and density functional (DFT) incorporating various hybrid exchange-correlation functionals: B3LYP, M06-2X and WB97X-D with the basis set 6-31G(d) in the gas and aqueous phases. NBO, QTAIM and NMR analyses are performed to understand the nature of various interactions between the 1-naphthol-2-sulfonate and β-CD. The results indicate that the orientation in which the guest molecule directed to the primary hydroxyls of the hydrophobic cavity of β-CD is the most stable.

Keywords. β-Cyclodextrin, 1-naphthol-2-sulfonate, DFT, NBO, QTAIM.

MSC2010. 74A25; 92C40

References

[1] S. Himri, I. Lafifi, A. Guendouzi, et al. Density functional theories study of the interactions between host β-Cyclodextrin and guest 8-Anilinonaphthalene-1-sulfonate: Molecular structure, HOMO, LUMO, NBO, QTAIM and NMR analyses. Journal of Molecular Liquids, Vol. 280 (2019) 218–229.

IMPULSIVE FRACTIONAL DIFFERENTIAL INCLUSIONS INVOLVING THE LIOUVILLE -CAPUTO-HADAMARD FRACTIONAL DERIVATIVE

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2000 MSC: 34C10

Abstract: T In this paper we establish existence results for a class of initial value problems for impulsive fractional differential inclusions involving the Liouville - Caputo - Hadamard fractional derivative of order $0 < \alpha < 2$, when the right hand side is convex, as well as nonconvex, valued. The topological structure of the set of solutions is also considered. This conference deals with the existence of solutions of the initial value problem (IVP for short), for the impulsive fractional order differential inclusion,

$$^{CH}D^\alpha y(t) \in F(t, y(t)), \text{ for a.e. } t \in J := [a, T], \, a > 0, \, t \neq t_k, \, k = 1, \ldots, m, 0 < \alpha < 1,$$

$$(1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{2}$$

$$y(a) = y_a, \tag{3}$$

where $^{CH}D^\alpha$ is the Caputo-Hadamard fractional derivative , $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I_K : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$ are continuous functions, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim\limits_{\varepsilon \to 0^+} y(t_k + \varepsilon)$ and $y(t_k^-) = \lim\limits_{\varepsilon \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1, \ldots, m$, and $y_a \in \mathbb{R}$.

and

$$^{CH}D^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [a, T], \, a > 0, \, t \neq t_k, \, k = 1, \ldots, m, 1 < r \leq 2,$$

$$(4)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{5}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{6}$$

$$y(a) = y_1, y'(a) = y_2, \tag{7}$$

where $^{CH}D^r$ is the Caputo-Hadamard fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}()$ is the family of all nonempty subsets of $\mathbb{R}$, $I_k, \overline{I}_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$, are continuous functions, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, $y(t_k^+) = \lim_{\varepsilon \to 0^+} y(t_k + \varepsilon)$ and $y(t_k^-) = \lim_{\varepsilon \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1, \ldots, m$, and $y_1, y_2 \in \mathbb{R}$.

References

[1] Y. Adjabi, F. Jarad, D. Baleanu and T. Abdeljawad, On Cauchy problems with Caputo Hadamard fractional derivatives, *J. Comput. Anal. Appl.* **21** (4) (2016), 661-681.

[2] E. Ait Dads, M. Benchohra and S. Hamani, Impulsive fractional differential inclusions involving the Caputo fractional derivative, *Fract. Calc. Appl. Anal.* **12** (1) (2009), 15-38.

[3] F. Belhannache, S. Hamani and J. Henderson, Impulsive fractional differential inclusions for the Liouville-Caputo-Hadamard fractional derivative, *Comm. Appl. Nonlinear Anal.* **25** (2018), No. 4, 52-67.

[4] A. Hammou, S. Hamani and J. Henderson, Initial Value Problems for Impulsive Caputo-Hadamard Fractional Differential Inclusions , *Comm. Appl. Nonlinear Anal.* **26** (2019), No. 2, 17-35.

Numerical path integration and probability modelling of chaotic systems

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Abstract. We propose an original framework to compute probability for some system to be in the appointed state once it was in some defined state. The system is considered as governed by normal system of ordinary differential equations. The initial conditions of the system are used to describe the initial state of the system. Then the target state is defined by the same way. The connection between initial and target states is described as some part of the curves bundle which went out from initial state and reach the target state. The situation is similar to the boundary problem, but the areas of the boundary values are of the same dimension as the phase space. We have defined the probabilities and have constructed the probability space according to the Kolmogorov's axioms. The path integral concept has the same idea: to consider the curves as elementary units and collect them into some value. We have not used Feynmann's approach but we have formulated some simplest principles and then found some relevant results. Based on the proposed conceptions we develop the numerical algorithm to compute those probabilities. Our concept could be useful for simulations of dynamical systems with chaos because the area around the starting point is taken into account.

Keywords. probability modelling; chaotic dynamical systems; computational forecasts; probability space; path integral.

MSC2010. 60D05;65C20;68Q10;81Q30.

1 Introduction

The dynamical system of general type is considered and it is expressed by the system of the ordinary differential equations

$$\dot{x} = f(x), \tag{1}$$

where x and $f(x)$ are some m-dimensional vector and vector field, respectively and the dot over means derivation on some parameter t. The functions $f^j(x^1, ..., x^m)$ are given and the system is presented as autonomous for simplicity. Instead of the initial condition

$$x(t_0) = x_0 \tag{2}$$

we will deal with the initial state which is determined by the set of points inside of the parallelepiped with the following vertices:

$$\left(x_0^1 + \epsilon_1 \xi^1, ..., x_0^m + \epsilon_1 \xi^m \right), \tag{3}$$

where $\epsilon_i \in \{0; 1\}$, ξ^i are some given real constants for any $i = 1, ..., m$. That is, any vertex of the state is in one-to-one correspondence with some binary number $\epsilon_m ... \epsilon_1$.

Our idea is to follow the integral curves of the system eq:1.1 which leave the initial state and scatter through the space. When we define some parallelepiped of the target state, in order to calculate the probability for the system to reach this state, we should find how many integral curves from the initial state we met in target state. Of course, one of the main question is what is "how many"? And the second question is how we can calculate the probability?

The first question is closed to the path integral idea and can be answered if we put some invariant for every curve. We use the vector field $\mu(x)$ of integral constants of the system eq:1.1 as such invariant, because this vector field is constant along the integral curves of the system.

2 Main results

Let us focus on any state of the system, for instance, on initial state. We introduce the *space of outcomes* Ω as a continuous set of vector fields μ of such kind that components μ^i belong to the intervals $[\min_\alpha \mu_i^\alpha, \max_\alpha \mu_i^\alpha]$, where $\alpha = 0, ..., 2^m - 1$ is the vertex number. The set of all subsets of the space Ω forms the $\sigma-$algebra.

For any $A_{l_1...l_m} \subset \Omega$ defined as

$$A_{l_1...l_m} = \left\{ \mu : [\mu_{il_i-1}, \mu_{il_i}], i = 1, ..., m, l_i = 1, ..., n \right\}, \tag{4}$$

where $\mu_{i0} = \min_\alpha \mu_i^\alpha$, $\mu_{in} = \max_\alpha \mu_i^\alpha$, the following measure

$$P(A_{l_1...l_m}) = \frac{(\mu_{1l_1} - \mu_{1l_1-1}) ... (\mu_{ml_m} - \mu_{ml_m-1})}{(\mu_{1n} - \mu_{10}) ... (\mu_{mn} - \mu_{m0})} \tag{5}$$

is probability, that can be directly checked. The space of outcomes Ω with such measure P over its subsets form probability space in Kolmogorov's sense. If we post that the distribution $l_i = 1, ..., n$ defines the intervals containing our integral curves' invariants components in target state, we can determine the target probability.

It is evident that in initial state $P = 1$ because all curves are inside initial state and every point of initial state is crossed by some curve. But in order to compute the values μ_{il_i} we should find the way to calculate them in neighbour state, near with initial state. Then we can move to the target state by the evident iterations.

In order to find the next generation of μ, we have explore the well-known fact that Lie derivative of the vector field, which is constant along the curve, is zero. That is,

$$[\mu, f] = 0. \tag{6}$$

This relation together with finite derivatives instead of derivations of μ leads us to the linear algebraic system and its solution gives us the new generation of μ. Making all necessary iterations, we will have all that needs to be put into formula eq:2.2 and we can compute the probability in such way.

Acknowledgments

Analysis of solutions of p-Laplacian parabolic system with integrable data

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Abstract

In this paper, we study the existence of weak-renormalized solutions for nonlinear parabolic system with p-Laplacian diffusion operator. This model consists of three unknown variables namely concentrations of microglia, chemoattractant and chemorepellent to understand the dynamics of attraction-repulsion system. First, we introduce suitable approximation problem of the given parabolic system and study the existence of a weak solution of the regularized problem using the Galerkin approximation method. Then using the weak solution of the regularized problem and truncation function, we prove the existence of weak-renormalized solutions of the given system under the assumption of integrable data.

1. Introduction

In this paper, we consider the following system of parabolic partial differential equation with three unknown variables:

$$
\left.
\begin{aligned}
&\frac{\partial u}{\partial t} - \nabla \cdot (|\nabla u|^{p-2}\nabla u) + \nabla \cdot (b(u)\nabla v) = \lambda u(1 - u - v) + f(x,t) && \text{in } Q_T, \\
&\frac{\partial v}{\partial t} - D\Delta v = -\eta vw + \rho v(1 - u - v) + g(x,t) && \text{in } Q_T, \\
&\frac{\partial w}{\partial t} - \nabla \cdot (d(\nabla w)) = \alpha u(1 - w) - \beta w + h(x,t) && \text{in } Q_T, \\
&u(x,t) = v(x,t) = w(x,t) = 0 && \text{on } \Sigma_T, \\
&u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x) && \text{in } \Omega,
\end{aligned}
\right\}
\tag{1.1}
$$

where $Q_T = \Omega \times [0, T]$, $\Sigma_T = \partial\Omega \times [0, T]$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ (no smoothness assumed on the boundary) and $p \geq 2$. The parameters $\lambda, \eta, \rho, \alpha, \beta$ are nonnegative constants. Further, we assume that initial conditions $u_0(x), v_0(x), w_0(x)$ and the source functions f, g, h are simple integrable functions. For simplicity, we assumed the Dirichlet boundary conditions for unknowns u, v and w on the boundary. This system (1.1) models the dynamics of microglia (u), chemoattractant (v) and chemorepellent (w) and it is a direct generalization of the classical Keller-Segel chemotaxis system. This system (1.1) is also called as attraction-repulsion chemotaxis system. In the literature, various qualitative anlaysis for related chemotaxis system studied by many researchers. We refer the interested readers [1, 2, 3, 4, 5, 6] and also the references there in. Further, we assume the following hypotheses throughout the article.

(H1) $d(\zeta)\zeta \geq \varphi|\zeta|^2$ for every $\zeta \in \mathbb{R}^n$, where $\varphi > 0$ is a real number.

(H2) For any $k > 0$ there exists $c_k > 0$ such that $|d(\zeta)| \leq c_k(1 + |\zeta|)$ for all $\zeta \in \mathbb{R}^n$.

(H3) $[|\xi|^{p-2}\xi - |\xi_1|^{p-2}\xi_1](\xi - \xi_1) \geq 0$ and $(d(s)\zeta - d(s_1)\zeta_1)(\zeta - \zeta_1) \geq 0$ for all $\xi, \xi_1, \zeta, \zeta_1 \in \mathbb{R}^n, (\xi \neq \xi_1, \zeta \neq \zeta_1), s, s_1 \in \mathbb{R}$.

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(H4) $u_0(x), v_0(x), w_0(x) \in L^1(\Omega)$.

(H5) $f, g, h \in L^1(Q_T)$ for every $(x, t) \in Q_T$.

In order to define the weak-renormalized solution, we consider the following truncation function $T_k(s) = \max\{-k, \min\{k, s\}\}$ at levels $\pm k$. Further, we denote $\tilde{T}_k(z) = \int_0^z T_k(s)ds$.

Definition: 1.1. *A renormalized solution of* (1.1) *is a measurable pair of function* $(u, w) : Q_T \times Q_T \to \mathbb{R}$ *and a weak solution* $v : Q_T \to \mathbb{R}$ *of* (1.1) *satisfies the following conditions,* $u(x, t) \geq 0, v(x, t) \geq 0, w(x, t) \geq 0$ *for a.e* $(x, t) \in Q_T$,

$$T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega)), \quad T_k(w) \in L^2(0, T; H_0^1(\Omega)),$$

$$v \in L^\infty(Q_T) \cap L^2(0, T; H_0^1(\Omega)) \cap C([0, T], L^2(\Omega)), \quad v_t \in L^2(Q_T),$$

$$\int_{(n \leq |u| \leq n+1)} |\nabla u|^p dx dt \to 0 \quad as \quad n \to \infty,$$

$$\int_{(n \leq |w| \leq n+1)} (d(\nabla w)\nabla w)\nabla w dx dt \to 0 \quad as \quad n \to \infty.$$

For all $S \in C^\infty(\mathbb{R})$ *with supp* S' *is compact and* $\phi \in L^2(0, T; H_0^1(\Omega))$,

$$S(u)_t - \nabla \cdot (S'(u)|\nabla u|^{p-2}\nabla u) + S''(u)|\nabla u|^p + \nabla \cdot (b(u)\nabla v S'(u)) - S''(u)b(u)\nabla v\nabla u$$

$$= \lambda u(1 - u - v)S'(u) + fS'(u) \text{ in } D'(Q_T),$$

$$\int_{Q_T} v_t \phi dx dt + D \int_{Q_T} \nabla v \nabla \phi dx dt = - \int_{Q_T} \eta v w \phi dx dt + \int_{Q_T} \rho v(1 - u - v)\phi dx dt + \int_{Q_T} g\phi dx dt,$$

$$S(w)_t - \nabla \cdot (d(\nabla w)S'(w)) + S''(w)d(\nabla w)\nabla w = \alpha u(1 - w)S'(w) - \beta w S'(w) + hS'(w) \text{ in } D'(Q_T),$$

$$S(u(x, 0)) = S(u_0(x)), \quad v(x, 0) = v_0(x) \text{ and } S(w(x, 0)) = S(w_0(x)) \text{ in } \Omega.$$

Here $D(Q_T)$ *represents the set of all infinitely differentiable functions on* Q_T *with compact support and* $D'(Q_T)$ *denotes the distributions on* Q_T.

Theorem: 1.1. *Under the hypothesis* $(H1) - (H5)$ *there exists atleast one weak-renormalized solutions for the system* (1.1) *in the sense of Definition 1.1.*

[1] D. Li, C. Mu, K. Lin and L. Wang, Global weak solutions for an attraction-repulsion system with nonlinear diffusion, Math Meth Appl Sci., 7 (2017), 1-28.

[2] K. Lin and C. Mu Global existence and convergence to steady states for an attraction-repulsion chemotaxis system, Nonlinear Anal. TMA, 31 (2016), 630-642.

[3] L. Shangerganesh, and K. Balachandran, Renormalized and entropy solution of nonlinear parabolic systems, Electron J. Differ. Eqns, 268 (2013), 1-24

[4] L. Shangerganesh, N. Barani Balan, and K. Balachandran, Weak Renormalized solution for predator-prey system, Appl. Anal., 92 (2013), 441-459.

[5] Y. Tao, Global existence of classical solutions to a predator-prey model with nonlinear prey-taxis, Nonlinear Anal. RWA, 11 (2010), 2056-2064.

[6] Y. Tao and J. I. Tello, Nonlinear stability of a heterogeneous state in a PDE-ODE model for acid-mediated tumor invasion, Math. Biosci. Eng., 13(2016), 193-207.

On Compact Fractional Semigroup of Bounded Linear Operators

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Abstract. Let X be a Banach space and $T(t), 0 \leq t < \infty$, be a one parameter a fractional semigroup of bounded linear operators on X. In this paper, we give sufficient conditions for a strongly continuous fractional semigroup of bounded linear operators to be compact. More over we prove that under certain conditions, the resolvent operator $R(\lambda, A)$, of the generator of an exponentially bounded fractional semigroup can be compact.

Keywords. conformable fractional derivative; semigroup of operators; compact operators.
MSC2010. 47D60, 47B10, 47B65.

1   Introduction

We are concerned with the fractional semigroups of bounded linear operators which defined by Alhorani, Khalil and abdeljawad in 2015 as follows: Let X be a Banach space, and $\mathcal{L}(X, X)$ be the space of all bounded linear operators on X. A family $\{T(t)\}_{t \geq 0} \subseteq \mathcal{L}(X, X)$ is called a fractional $\alpha-$semigroup (or $\alpha-$semigroup) of operators if:

$T(0) = I$,

$T(s + t)^{\frac{1}{\alpha}} = T(s^{\frac{1}{\alpha}})T(t^{\frac{1}{\alpha}})$ for all $s, \ t \in [0, \infty)$. This definition coincides with the usual semigroups of operators when $\alpha = 1$.

If for each fixed $x \in X$, $T(t)x \to x$ as $t \to 0^+$, then the semigroup $T(t)$ is called a $c_0 - \alpha-$semigroup or strongly continuous $\alpha-$semigroup. In this paper, we give sufficient conditions for a strongly continuous fractional semigroup of bounded linear operators to be compact. More over we prove that under certain conditions, the resolvent operator $R(\lambda, A)$, of the generator of an exponentially bounded fractional semigroup can be compact.

2   Main results

We obtain some results concerning compactness of fractional semigroup of operators in the sense of conformable derivatives

Theorem 1 *Let $T(t)$ be a strongly continuous exponentially bounded fractional-semigroup of bounded linear operators on a Banach space X. If $T(t)$ is compact for all $t > t_0$ for some $t_0 > 0$, then the map $t \to T(t)$ is continuous from the right in the uniform operator topology for all $t > t_0$.*

Theorem 2 *Let $T(t)$ be a strongly continuous F-semigroup of bounded linear operators on a Banach space X with generator A, such that $\|T(t)\| \leq Me^{\omega t}$. If $T(t)$ is compact for all $t > 0$, then $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$.*

References

[1]   A. Pazy, *Semigroups of linear operators and applications to partial differential equations, Volume 44 (Applied Mathematical Sciences)*. Springer Verlag Publishers, New York, (1993).

[2]   R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative. *J. Comput. Appl. Math.* **264**, (2014), 65–70.

[3]   A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations, Volume 204 (North-Holland Mathematics Studies)*. Elsevier (North-Holland) Science Publishers, New York, USA (2006).

[4]   I. Podlubny, *Fractional Differential Equations, Volume 198 (Mathematics in Science and Engineering)*. Academic Press, San Diego, California, USA, (1999).

[5]   K. Hosseini, R. Ansari, New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method. *Waves in Random and Complex Media.* **27**, (2017), 628–636.

[6]   Sh. Al-Sharif, A. Malkawi, Modification of conformable fractional derivative with classical properties. *Italian Journal of Pure and Applied Mathematics.* **44**, (2018) to appear.

[7]   R. Anderson, J. Ulness, Newly defined conformable derivatives. *Advances in Dynamical Systems and Applications.* **10**, (2015), 109–137.

[8]   A. Atangana, D. Baleanu, A. Alsaedi, New properties of conformable derivative. *Open Mathematics.* **13**, (2015), 889–898.

[9] S. Bochner, Diffusion equation and stochastic processes. *Proceedings of the National Academy of Sciences* **35**, (1949), 368–370.

[10] A. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them. *Pacific Journal of Mathematics.* **10**, (1960), 419–439.

[11] E. Popescu, On the fractional Cauchy problem associated with a Feller Semigroup. *Mathematical Reports.* **12**, (2010), 181–188.

[12] M. Al-Horani, R. Khalil, T. Abdeljawad, Conformable fractional semigroups of operators. *Journal of Semigroup Theory and Applications.* **2015**, (2015), Article-ID 7.

Geometric Interpretation of Solution of Differential Equation in Thermodynamics

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Abstract. We consider the problem of a wording of thermodynamic provisions for the spaces of ideal gas on the basis of analysis of solution of quasilinear partial differential equation of the first order using method of characteristics. Differential geometry tools and means of computer mathematics are also applied. The result is presented in the form of connection between physical content of thermodynamic values and their mathematical analogs. By numerical methods using the means of computer mathematics it is shown the possibility of establishing patterns of implementation of thermodynamic processes as functions of time.

Keywords. ideal gas; geometric interpretation; thermodynamic provisions.

MSC2010. 34M45; 34A26; 35R50; 35R15.

1 Introduction

As thermodynamics is a basis for many physical sciences its theory must be clear and logical. But nowdays the axiomatic creation of thermodynamics is not completed. The problem of entropy and its existence has various aspects of interpretation. And the acceptable solution is not found yet. The second problem is related with the proceeding of thermodynamic processes in time. Classical science does not give the answer to a quastion what is the place of time in the theory. So we try to establish the communication between the physical content of thermodynamic values and their mathematical analogs on the basis of application of methods of differential geometry and means of computer mathematics.

2 Main results

We consider a partial differential equation [2]:

$$\frac{v}{2c_p}\frac{\partial Q}{\partial v} + \frac{p}{2c_v}\frac{\partial Q}{\partial p} = T, \tag{1}$$

where Q - amount of heat is the physical quantity characterizing process of heat exchange between thermodynamic system and the environment. Temperature T is called the measure of a deviation of a condition of a studied thermodynamic system from a condition of a reference body in the standartized conditions. We assume that each state of ideal gas is unambiguously defined by values of specific volume v and pressure p which are presented by the parametrical equations concerning time τ: $v = v(\tau)$ and $p = p(\tau)$.

The solution $Q = Q(v, p)$ of equation (1) geometrically represents a surface in three-dimentional space (v, p, Q) which is called an integrated surface. We use method of characteristics which are defined by the system of ordinary differential equations [1]:

$$2c_p \frac{dv}{v} = 2c_v \frac{dp}{p} = \frac{dQ}{T} = ds, \qquad (2)$$

where s - any real parameter, changing along the characteristic curve.

From the first two equations (2) the dependence for the value ds turns out. It has a form of thermodynamic equation which is used in definition of entropy. Thus, in geometric representation entropy is an arch length for the characteristic curves corresponding to the field of directions defined by system (2).

The integrated solution of the equation (1) can be found in an analitical way. The general solution of equations (2) concerning entropy has a look:

$$v = v_l \exp\left(\frac{s}{2c_p}\right); p = p_l \exp\left(\frac{s}{2c_v}\right); Q = Q_l + c_p \beta_1 \frac{p_l v_l}{R_i}\left(\exp\left(\frac{s}{c_p \beta_1}\right) - 1\right); \beta_1 = \frac{2c_v}{c_p + c_v}.$$

We set a curve of any process l in a parametrical form concerning time τ: $v_l = v_l(\tau)$, $p_l = p_l(\tau)$, $Q_l = Q_l(\tau)$. Similar results can be received by numerical methods, using means of computer mathematics.

The integrated surface of the equation (1) can be covered by the collection of characteristics. The functions $f_1 = \frac{v}{2c_p}, f_2 = \frac{p}{2c_v}$, $f_3 = T$ define a field of directions in space (v, p, Q). In each point of this space there is a direction which directional cosines are propotional to f_1, f_2, f_3. So in each point of integrated surface vector determined by the field of directions stated above have to be in the tangent plane to this surface.

We find a projection l_0 of a curve of process l on vOp plane. For the equation (1) Cauchy problem in vOp plane is formulated in a form: to find the integrated surface passing through the curve l in the neighborhood of l_0. Geometric interpretation of Cauchy problem in a space (v, p, Q) assumes that through each point of process l it is necessary to carry out characteristic of the equation (1) and "to stick together" the integrated surface from them.

Let's assume on vOp plane the projection l_0 passes from the point A_0 to the point B_0. Through these points characteristics, isoterms and adiabatic curves pass. The collection of characteristics is described by the equation $p = C_1 v^k$ (k is adiabatic degree), the collection of isoterms – $pv = C_2$, the collection of adiabatic curves – $pv^k = C_3$. The entropy in such representation will be the characteristic and s – its arc length. Adiabatic curves will be the lines of level for the characteristics at $s = const$. Adiabatic curves, isoterms and characteristics form network of curvilinear not ortogonal coordinates on the vOp plane.

The geometric solution of Cauchy problem in (v, p, Q) space can be constructed as follows. Through any point on vOp plane characteristic is carried out until it crossing with projection l_0. Then it is necessary to put $Q = Q_l(\tau)$ taking into account the parametrical equations of process l. The integrated surface will characterize amount of heat for all set of conditions of ideal gas in the neighborhood of process l or its projection l_0.

References

[1] Koshlyakov I.S., *Partial Differential Equations of Mathematical Physics*, Moscow, High School, 1970.

[2] Gennadiy V. Averin, Anna V. Zviagintseva, Maria V. Shevtsova and Liliana N. Kurtova, Justification of Provisions of Thermodynamics by Methods of Differential Geometry of Multidimensional Spaces. Journal of Engineering and Applied Sciences. **12** (2017), 7830–7835.

Full Fuzzy function Approximation by using Radial Basis Function interpolation

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Abstract. In this paper radial basis function interpolation is used to solve fuzzy system of linear equations. We consider the method in spatial case when the coefficient matrix is symmetric positive definite. To this end, radial basis function is used for approximate the solution of fuzzy system and its lower and upper functions in the parametric forms are determined. The proposed method is illustrated with numerical examples.

Keywords. radial basis function interpolation; fully fuzzy systems; approximated solutions

MSC2010.

1 Introduction

Linear systems have important applications in many branches of science and engineering. Since many real-world engineering systems are too complex to be defined in precise terms, uncertainty is often involved in any engineering design process. In many applications, at least some of the parameters of the system should be represented by fuzzy rather than crisp numbers. So that, it is immensely important to develop numerical and analytical procedures that would appropriately treat general fuzzy linear systems and solve them. The system of linear equations $A\tilde{X} = \tilde{b}$ b where the elements, a_{ij}, of the matrix A are crisp values and the elements, $\tilde{b}_i$ of the vector b are fuzzy numbers, is called a fuzzy linear system (FSLE). Friedman et al. [**?**] investigated a general model to solve a FSLE by using an embedding approach. Buckley and Qu in their continuous work [**?**, **?**] proposed different solutions for FFLSs.

In this study we consider Radial Basis Function (RBF) to approximate the solutions of the fuzzy systems that resulted of full fuzzy function. Finally, concluding remarks is given in the last Section.

2 Main results

In this paper we approximate the Fuzzy function $\tilde{f} : \mathcal{TSF} \to \mathcal{TSF}$ on a discrete points set $\tilde{X} = \{x_{1,}x_2, \ldots, x_n\}$, We used RBF interpolation to find fuzzy values. A *step* fuzzy valued function on $\mathcal{TSF}$, is a function $\tilde{U} : \mathcal{TSF} \to \mathcal{TSF}$, such that, for all $\tilde{v} \in \mathcal{TSF}$,

$$\tilde{U} = (\underline{u}(r), \overline{u}(r)) = (\underline{v}(r) + l, \overline{v}(r) + u) \tag{1}$$

where, $r \in [0,1]$ and $\underline{v}(r) + l \leq \overline{v}(r) + u$ for all r. Using the Hasdorf metric for $\tilde{v}, \tilde{u} \in \mathcal{TSF}$, we define source distance of $\tilde{v}$ and $\tilde{u}$ by

$$D_s(\tilde{u}, \tilde{v}) = \frac{1}{2}\{|Val(\tilde{u}) - Val(\tilde{v})| + |Amb(\tilde{u}) - Amb(\tilde{v})| + d_H([\tilde{u}]^1, [\tilde{v}]^1)\} \tag{2}$$

.

Let $(\tilde{x}_i, \tilde{y}_i)$, for $i = 1, 2, \ldots, n$ be defined nods, when $\tilde{x}_i = [\underline{x}_i, \overline{x}_i]$ and $\tilde{y}_i = [\underline{y}_i, \overline{y}_i]$. We used RBF interpolation to find function approximation for this nods. So we let

$$\tilde{f}(\tilde{x}) = \sum_{j=1}^{n} \tilde{c}_j \phi_j(\tilde{x}) \qquad \tilde{f} = [\underline{f}, \overline{f}], \quad \phi_j(\tilde{x}) = \phi(D(\tilde{x}, \tilde{x}_j)). \tag{3}$$

Now we let $\tilde{y}_i = \tilde{f}(\tilde{x}_i)$ for $i = 1, 2, \ldots, n$, So we have $Y = \tilde{C}\Phi$, where the coefficient matrix $\Phi = [\phi_j(\tilde{x}_i)], i, j = 1, 2, \ldots, n$ is a crisp $n \times n$ matrix and $\tilde{Y} = [\tilde{y}_i]$, $\tilde{X} = [\tilde{x}_i]$ in which, $\tilde{y}_i, i = 1, \ldots, n$ are known fuzzy numbers and $\tilde{C}_i, i = 1, \ldots, n$ are unknown fuzzy numbers. This fuzzy linear system can be solved by the RBF method.

3   Example

Example 1 *For the following exponential function $\tilde{f}(\tilde{x}) = e^{\tilde{x}}$ and $\phi(x) = \sqrt{x^2 + 0.01}$, we interpolated by considering $n = 7$ for*

$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$
$(\frac{r}{8}, \frac{2}{8} - \frac{r}{8})$	$(\frac{1}{8} + \frac{r}{8}, \frac{3}{8} - \frac{r}{8})$	$(\frac{2}{8} + \frac{r}{8}, \frac{4}{8} - \frac{r}{8})$	$(\frac{3}{8} + \frac{r}{8}, \frac{5}{8} - \frac{r}{8})$	$(\frac{4}{8} + \frac{r}{8}, \frac{6}{8} - \frac{r}{8})$	$(\frac{5}{8} + \frac{r}{8}, \frac{7}{8} - \frac{r}{8})$	$(\frac{6}{8} + \frac{r}{8}, 1 - \frac{r}{8})$

There is has been set $e^{\tilde{x}} = [e^{\underline{x}}, e^{\overline{x}}]$. Function diagram for $\tilde{x}$ are have shown in the following figure. The function error for $\tilde{x} = (1.5 + r, 3.5 - r)$ are have been shown in the folowing figure.

Figure 1: Left figure shows function diagram for all points and function diagram for $\tilde{x}$ and the right one shows the function error for $\tilde{x} = (1.5 + r, 3.5 - r)$ in two and tree dimantions.

References

[1] M. Friedman, Ma. Ming, A. Kandel, Fuzzy linear system, Fuzzy Sets and Systems 96 (1998) 209-261.

[2] J.J. Buckley, Y. Qu, Solving linear and quadratic fuzzy equations, Fuzzy Sets and Systems 38 (1990) 43-59.

[3] R. Firouzdor, M. Amirfakhrian, Approximation of a Fuzzy Function by Using Radial BasisFunctions Interpolation,nternational Journal of Mathematical Modelling Computations, 03(2017)299- 307

P-box Uncertainty Quantification of Queues Using the Taylor Series Expansion

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Abstract. The main purpose of the current work is to use Taylor series expansion approach for analyzing the uncertainty of the M/G/1/N queue. Specifically, we propose a numerical approach based on Taylor series expansion with a statistical aspect for evaluating the stationary performances of the considered queueing model, when the inter-arrival rate is not assessed in perfect manner, in our analysis we introduce probability-boxes. In this case, we approximate the cumulative distribution function (cdf) of each components of the stationary distribution in a boxes. Several numerical examples are also presented for illustrative purposes.

Keywords. Taylor series expansion; Parametric uncertainty; probability-boxes; M/G/1/N queuing system.

MSC2010. 60K25; 68T37; 60J10; 60J22.

1   Introduction

Traditionally, uncertainty in engineering has been treated with probability theory which offers a single measure (i.e. the probability measure) to describe variability in variable X. In other words, we assume that the variability in X is known and quantifiable by the cumulative distribution function (cdf) and the corresponding density function (pdf). This describes the case where variability is treated as the only source of uncertainty.

The probability-boxes (p-boxes) define the cdf of a variable X by its lower and upper bound distributions. The idea is that due a lack of knowledge (epistemic uncertainty), the cdf cannot be given a precise formulation. Thus the probability-box framework accounts for aleatory as well as for epistemic uncertainty in the description of a variable X.

The main purpose of this paper is to provide a perturbation analysis of the $M/G/1/N$ queue, all model parameters are imprecisely known because they are determined through insufficient statistical data, leading to uncertainty in the assessment of their values. This parametric uncertainty induced from the incomplete information concerning the parameter is called "epistemic uncertainty".

2 The M/G/1/N queuing system

Consider the M/G/1/N queue, where customers arrive according to a Poisson process with rate λ and demand an independent and identically distributed service time with common distribution function B(t) with mean $1/\mu$, N denote the buffer capacity of the queue.

Let $X(t)$ denote the number of customers in the M/G/1/N queue at time t, for $t \geq 0$. Note that the queue-length processes $X(t) : t \geq 0$ of the M/G/1/N system fails to be a Markov process because the service time distribution does not have the memoryless property. Since we have assumed that customers that do not find an empty buffer place upon their arrival are lost, the stationary distribution of $\{X(t) : t \geq 0\}$, denoted by π, exists (independent of the traffic rate). Let $\{X_n : n \in \mathbb{N}\}$ denote the queue-length process embedded right after the departure of the nth customer. Note that X_n has state-space $E = \{0, 1, \ldots, N-1\}$ as after the departure of a customer the system cannot be full. Then $X_n : n \in \mathbb{N}$ is a Markov chain with transition matrix $(P = (p_{i,j})_{i,j \in E})$

3 The parametric uncertainty analysis method

Now we turn to consider the factor of the epistemic uncertainties in the prediction of the parameter θ. So, this parameter may be considered as random variable. For that, we assume that the model input parameter θ is of the form:

$$\theta(w) = \bar{\theta} + \sigma\epsilon(w), \tag{1}$$

where $\bar{\theta}$ is the estimated mean value provided by the statistic of θ, σ is the standard deviation of the same parameter, and $\epsilon(w)$ is a random variable modeling the epistemic distribution. Specifically $\epsilon(w)$ is zero mean random variable.

In [1] the authors propose numerical approach to compute the probability density functions (pdfs) of the stationary distribution π_θ, they consider it as a function the parameter θ, therefore, we will use the well-known random- variable-transformation method stated in the following Theorem to determine the pdfs of $\pi_i(\theta)$ for $i = 0, ..., N-1$.

For our studies, we estimate the commutative function distribution of each components of the stationary distribution of the considered model. More specifically, we estimate the (CFD) in a boxes of $\lambda_i \in [\lambda_{min}, \lambda_{max}]$ and $\sigma_i \in [\sigma_{min}, \sigma_{max}]$.

References

[1] S. Ouazine, K. Abbas. 2016. A functional approximation for retial queues with two way communication. Annals of Operations Research, **247** (2016) *211-227*.

[2] K. Abbas, B. Heidergott and D. Aï¿½ssani 2013. A functional approximation for the M/G/1/N queue. Discrete Event Dynamic System, **23** (2013) *93-104*.

Linear Conjugation Problem for Bi-Analytic Functions in the Weighted Holder Spaces

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Abstract. We consider the classical linear conjugation problem for bi-analytic functions in piecewise-smooth curve in the whole scale of weighted Holder spaces and describe its solvability in dependence on a weight order.

Keywords. linear conjugation problem; bi-analytic functions; weighted Holder spaces.

MSC2010. 35F15; 45E05.

We consider on the complex plane a piecewise-smooth contour Γ consisting of finite number of oriented smooth curves $\Gamma_j, 1 \le j \le m$. Let us denote by F the set of endpoints of these arcs. Then its complement consists of domains D_1, D_2 such that the domain D_1 is finite and D_2 infinite and contains a neighborhood of ∞. We consider in these domains a bi-analytic function u, i.e., a function $u \in C^2(D)$ satisfying the equation

$$u_{\bar{z}\bar{z}} = 0,$$

considered a problem of linear conjugation

$$u^+ - B_1 u^- - B_3 u_{\bar{z}}^- = f_1,$$

$$u_{\bar{z}}^+ - B_2 u_{\bar{z}}^- = f_2,$$

with piecewise continuous coefficients B_j, that allows discontinuities at corner points of the contour Γ. The function u satisfies the conditions

$$u = O(|z|^{-k+1}), \quad u_{\bar{z}} = O(|z|^{-k}) \quad \text{when } z \to \infty.$$

We obtained the necessary and sufficient conditions for the solvability of this problem in the weighted Holder spaces $C_\lambda^\mu(\overline{D}_j, F)$, $j = 1, 2$, introduced in [1]. Moreover, the solution was obtained explicitly.

References

[1] Soldatov A. P. Singular integral operators and elliptic boundary value problems. Sovrem. Mat. Fundam. Napravl. 2017; 63(1):1–189. (in Russian)

Bitsadze-Samarski Problem for Elliptic Systems of Second Order

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Abstract. Under general assumptions with respect to the shift Bitsadze-Samarski problem for elliptic systems of second order on the plane with constant and only leading coefficients is considered. The Fredholm theorem for this problem is proved and the index formula is obtained.

Keywords. elliptic system, Fredholm solvability, index.

MSC2010. 35J57, 45B05.

Existence, Permanence and Stability of Almost Automorphic Solution of a Non-autonomous Leslie-Gower Prey-Predator Model with Control Feedback Terms on Time Scales

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Abstract. The following modified non-autonomous Leslie-Gower prey predator model on time scales with control feedback terms is studied,

$$u_1^\Delta(t) = a_1(t) - b_1(t)\exp\{u_1(t)\} - \frac{c(t)\exp\{u_2(t)\}}{\mathfrak{p}_1(t) + \exp\{u_1(t)\}} - \frac{d(t)}{\mathfrak{h}(t) + \exp\{u_1(t)\}} - e_1(t)x_1(t)$$

$$x_1^\Delta(t) = -\alpha_1(t)x_1(t) + \beta_1(t)\exp\{u_1(t)\}$$

$$u_2^\Delta(t) = \rho(t)\left(a_2(t) - \frac{b_2(t)\exp\{u_2(t)\}}{\mathfrak{p}_2(t) + \exp\{u_1(t)\}}\right) - e_2(t)x_2(t)$$

$$x_2^\Delta(t) = -\alpha_2(t)x_2(t) + \beta_2(t)\exp\{u_2(t)\}.$$

The important property permanence is investigated along with existence of almost automorphic solution of the considered model. By constructing a appropriate Lyapunov function, existence of a unique globally attractive positive almost automorphic solution of the system are obtained. In last, two numerical examples are given to show the effectiveness of our theoretical results with simulation. Our results are existing new.

Keywords. Almost automorphy, Time scales, Leslie-Gower prey predator model.

MSC2010. 34N05; 34C25; 92A15; 47H11.

References

[1] Chen, Fengde, "Permanence and global stability of nonautonomous Lotka-Volterra system with predator-prey and deviating arguments." Applied Mathematics and Computation 173.2 (2006): 1082-1100.

[2] Bohner, M., Peterson, A, Dynamic equations on time scales: An Introduction with Applications. Birkauser Boston (2001)

Correct Solvability of Nonstationary Problems in Stepanov Spaces

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Abstract. The article deals with the correct solvability of some differential equations in Stepanov spaces. The solution is sought by the method of the theory of semigroups and related fractional powers of operators. The study of many mathematical models in the theory of heat and mass transfer often reduces to solving non-stationary problems for partial differential equations of parabolic type.

Keywords. nonstationary problem; semigroup theory; fractional degrees of operators; weighted Stepanov spaces.

MSC2010. 37L05

The establishment of the correct solvability of mathematical problems is one of the most important conditions for their numerical realization. Starting with the work of S.G. Krein's method of the theory of semigroups and associated fractional powers of operators has become one of the main topics in the study of the correct solvability of initial-boundary value problems for evolution equations and their applications. The study of many mathematical models in the theory of heat and mass transfer often reduces to solving non-stationary problems for partial differential equations of parabolic type [1]. At the same time, many of these problems can be reduced to the elliptic case when there is a solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = Au, 0 \leq t \leq T < \infty \tag{1}$$

with the corresponding initial conditions $u(0) = \phi, u(\infty) = 0$ and a linear operator A acting in some Banach space.

With this formulation of the problem it is natural to use the method developed by S.G. Krein in investigating the correct solvability of boundary value problems for differential equations of the form (1). We consider an assumption of the positivity of the operator A , which ensures the presence of the square root of $A^{1/2}$, in terms of which solutions are given, appropriate criteria for the correct solvability of these problems are formed and the representations of their solutions are derived . At the same time, the positivity condition for the operator A can be replaced by the requirement that the operator $-A$ be a C_0-semigroup generator with an estimate of

$$|U(t)| \leq Me^{-\omega t}, \omega \geq 0.$$

In the case of the Dirichlet problem, when the solution to equation (1) satisfies the boundedness conditions for $t \to 0$, it has the form

$$u(t) = U(t, -A^{1/2})g_0 \tag{2}$$

where $U(t, -A^{1/2})$ is a strongly continuous semigroup of class C_0 with estimate $U(t, -A^{1/2}) \leq Me^{-\omega^{1/2}t}$. It follows from (2) that

$$\left.\frac{\partial u(t)}{\partial t}\right|_{t=0} = \left.\frac{\partial}{\partial t}U(t, -A^{1/2})g_0\right|_{x=0} = -A^{1/2}g_0. \tag{3}$$

Thus, to determine the rate of heat and mass transfer at the interface between media, it is sufficient to know the operator $-A^{1/2}$.

Our main result is to obtain existence and approximate controllability properties for the fractional nonlocal control inclusion (1)–(2).

References

[1] V. A. Kostin and S. V. Pisareva, Evolutional equations with singularities in generalized Stepanov spaces, Russian Math. (Iz. VUZ). **51:6** (2007), 32–41.

The discrete counterpart of fractional operators with generalized Mittag-Leffler kernels

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Abstract. By always referring to the continuous counterpart, we define fractional difference operators with discrete generalized Mittag-Leffler kernels of the form $(E^{\gamma}_{\overline{\theta,\mu}}(\lambda, t - \rho(s))$ for both Riemann type (ABR) and Caputo type (ABC) cases, where AB stands for Atangana-Baleanu . Then, we employ the discrete Laplace transforms to formulate their corresponding $AB-$fractional sums, and prove useful and applicable versions of their semi-group properties. The action of fractional sums on the ABC type fractional differences is proved and used to solve the $ABC-$fractional difference initial value problems. The nonhomogeneous linear ABC fractional difference equation with constant coefficient is solved by both the discrete Laplace transforms and the successive approximation, and the Laplace transform method is remarked for the continuous counterpart. In fact, for the case $\mu \neq 1$, we obtain a nontrivial solution for the homogeneous linear $ABC-$ type initial value problem with constant coefficient. The relation between the ABC and ABR fractional differences are formulated by using the discrete Laplace transform. We iterate the fractional sums of order $-(\theta, \mu, 1)$ to generate fractional sum-differences for which a semigroup property is proved. The nabla discrete transforms for the $AB-$fractional sums and the $AB-$iterated fractional sum-differences are calculated. Examples and remarks are given to clarify and confirm the obtained results and some of their particular cases are highlighted. Finally, the discrete extension to the higher order case is discussed.

Keywords. Discrete generalized Mittag-Leffler function, discrete nabla Laplace transform, convolution; AB fractional sums, ABR fractional difference, ABC fractional difference, iterated AB sum-differences, higher order.

MSC2010. 26A33.

Acknowledgments

References

[1] T. Abdeljawad, D. Baleanu, On fractional derivatives with generalized Mittag-Leffler kernels, Advances in Difference Equations 2018, 2018:468., https://doi.org/10.1186/s13662-018-1914-2.

[2] T. Abdeljawad, Fractional operators with generalized Mittag-Leffler kernels and their differ-integrals, Chaos **29**, 023102 (2019) https://doi.org/10.1063/1.5085726.

[3] T. Abdeljawad, Fractional difference operators with discrete generalized Mittag-Leffler kernels, Chaos &Solitons and Fractals https://doi.org/10.1016/j.chaos.2019.06.012.

[4] A.Atagana, D. Baleanu, New fractional derivative with non-local and non-singular kernal, Thermal Science (2016), vol 20 (**2**), 757-763.

On a Condition for the Existence of Solutions to the Homogeneous Schwarz Problem

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Abstract. The homogeneous Schwarz problem is considered for n-vector functions that are analytic in Douglis sense with matrix J. It is assumed that the determinant of the complex part of the matrix J is non-zero. A homogeneous reduction of first order elliptic PDE system to an equivalent second order system. Using this transformation the necessary and sufficient condition is obtained under which there are solutions of the homogeneous Schwarz problem in the form of second-degree vector polynomials.

Keywords. J-analytic functions; matrix; polynomial vector; quadratic form; ellipse; system of algebraic equations.

MSC2010. 35J56; 30G20.

1 Introduction

Let us assume that all eigenvalues of $n \times n$-matrices J it has non-zero complex parts. Analytical after Douglis, or J-analytic function with matrix J [1, 2] is called the complex n-vector function $\phi = \phi(z) \in C^1(D)$, $D \subset \mathbb{R}^2$, which satisfies the equation

$$\frac{\partial \phi}{\partial y} - J \cdot \frac{\partial \phi}{\partial x} = 0. \tag{1}$$

We will say that the function $\phi(z)$ corresponds to the matrix J, if the equality (1) holds. We consider the following homogeneous Schwarz problem [1, 2] for the elliptic system (1).

Let the *finite* domain $D \subset \mathbb{R}^2$ be such that it is bounded by the contour Γ. WE seek an analytical after Douglis vector-function $\phi(z) \in C(\overline{D})$ for which the boundary condition

$$\operatorname{Re} \phi(z)\big|_\Gamma = 0. \tag{2}$$

is fulfilled.

If $n \geq 2$ then the problem (2) may have nontrivial solutions. We give an example.

Let

$$J = \begin{pmatrix} -i & 4 \\ 1 & 3i \end{pmatrix}, \quad \phi(z) = \begin{pmatrix} -4i\,(x^2+y^2) \\ x^2 + 3y^2 - 1 - 2xy\,i \end{pmatrix}. \tag{3}$$

The matrix J in (3) has a multiple eigenvalue $\lambda = i$. Here the vector-polynomial of the second degree $\phi(z)$ will be a function, J-analytic with the given matrix. Have: $\operatorname{Re} \phi(z)\big|_\Gamma = 0$ on the ellipse $x^2 + 3y^2 = 1$.

2 Main results

We describe a general method of constructing functions of the form (3). Write n-vector function $\phi(z)$ as $\phi(z) = u(x,y) + iv(x,y)$, where $u(x,y)$, $v(x,y)$ are real n-vector-functions. We also introduce the following notations: $J = A + Bi$, where $A, B \in \mathbb{R}^{n \times n}$, and $\det B \neq 0$. Substitute the matrix $J = A + Bi$ in (1). Then, after the transformations, we obtain that the function $u = u(x,y)$ will be the solution of the following system of second order:

$$(B^2 + BAB^{-1}A) \frac{\partial^2 u}{\partial x^2} - B \cdot (AB^{-1} + B^{-1}A) \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{4}$$

Regarding (4) the following statement holds.

Theorem 1 *If there is a solution to the Dirichlet problem $u(x,y)\big|_\Gamma = 0$ for the system (4) in the form of a vector-polynomial of $u = u(x,y)$, then it is possible to construct uniquely the function $\phi = u + iv$ as a solution of (2).*

Let $ab - c^2 > 0$. We will look for a solution to the problem $u(x,y)\big|_\Gamma = 0$ for the system (4) in the ellipse $\Gamma: ax^2 + 2cxy + by^2 = 1$ as follows:

$$u(x,y) = \begin{pmatrix} t_1 \cdot (ax^2 + 2cxy + by^2 - 1) \\ \dots\dots\dots\dots\dots\dots\dots \\ t_n \cdot (ax^2 + 2cxy + by^2 - 1) \end{pmatrix}, \quad ab - c^2 > 0. \tag{5}$$

In (5) numbers $a, c, b, t_1, \ldots, t_n$ are real parameters. Since $ab - c^2 > 0$, it is possible to take $a = 1$, $c = c$, $b = c^2 + \varepsilon^2$, where $\varepsilon \neq 0$. According to this observation substitution (5) in (4) leads to a homogeneous algebraic system with respect to variables $t_1, \ldots, t_n$. It is well known such a system has non-zero solutions only if its determinant equals to zero:

$$\det \left[B^2 + BAB^{-1}A - c \cdot B \cdot (AB^{-1} + B^{-1}A) + (c^2 + \varepsilon^2) \cdot E \right] = 0. \tag{6}$$

As a result taking into account Theorem 1 we obtain the following basic statement.

Theorem 2 *Existence of (arbitrary) real solutions (c, ε), where $\varepsilon \neq 0$ of the algebraic equation (6) is necessary and sufficient condition for existence of the solutions $\phi(z)$ of the homogeneous Schwarz problem (2) in the form of a vector polynomial of second degree with the matrix $J = A + Bi \in \mathbb{C}^{n \times n}$, $\det B \neq 0$.*

As direct calculations show for the matrix J (3) there are at least two solutions $(c, \varepsilon) = (0, 3)$ and $(c, \varepsilon) = (0, 1/3)$ of equations (6).

Acknowledgments

References

[1] V. B. Vasiliev and V. G. Nikolaev, Schwarz Problem for First-Order Elliptic Systems on the Plane, Differential Equations. **53** (2017), 1318–1328.

[2] V. G. Nikolaev, A Criterion for the Existence of Nontrivial Solutions to the Homogeneous Schwarz Problem, Journal of Mathematical Sciences. **219** (2016), 220–225.

THE CONSTRUCTIONS OF TRANSMUTATIONS FOR HYPERGEOMETRIC DIFFERENTIAL OPERATORS OF HIGHER ORDERS

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Abstract

Proposed to use the Euler transform for hypergeometric functions, as well as the Erdely-Kober transform, in order to construct transmutations of higher-order differential operators.

Introduction

The article deals with the transmutation operator, represented by Volterra integral operator of the first kind

$$Tu(x) = \int\limits_{\varepsilon}^{x-\delta} K(x,t)u(t)dt; \quad \varepsilon, \delta \to 0;$$

of intertwining differential expressions of a hypergeometric class of arbitrary order with polynomial coefficients on R^+.

$$Au(x) = \sum_{k=0}^{n} (-1)^k D^k(a_k(x)u(x)); \quad Bu(x) = \sum_{k=0}^{n} b_k(x)D^k u(x); \quad D = \frac{d}{dx};$$

under corresponding initial conditions at the point $x = \delta \to 0$. The types of kernals $K(x,t)$ and functions $u(x)$ are studied, for which the transmutation operator satisfies the defining identity

$$T(Au)(x) = B(Tu)(x); \quad \forall x \in (0; +\infty);$$

Main results

In the papers ([1], [2], [3]) much attention is paid to the existence theorems of the transmutation operator. We, on the contrary, are interested in options when such an operator obviously exists and the method of its construction is important. Two possible designs are proposed.

a) use of the modified Euler ratio

$$z^{2(d-1)}{}_{p+1}F_{q+1}\left(\begin{array}{cccc} a_1 & \ldots & a_p & c \\ b_1 & \ldots & b_q & d \end{array}; \kappa z^2\right) =$$

$$= \frac{2\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int\limits_{0}^{z}(z^2-t^2)^{d-c-1}t^{2c-1}{}_{p}F_{q}\left(\begin{array}{cccc} a_1 & \ldots & a_p & c \\ b_1 & \ldots & b_q & d \end{array}; \kappa t^2\right)d\xi$$

b) the use of Erdely-Koberr integrals ([4]) of the form

$$\int\limits_{0}^{x} f(t)(x^{\beta}-t^{\beta})^{\alpha}t^{\gamma}dt$$

especially when $f(t) = \cos(\omega t)$ or $f(t) = \sin(\omega t)$

Acknowledgments

References

[1] M. Yu. Ignatiev Linear equivalence of certain integrodifferential operators of higher orders. Abstract on cand. phys. - mat sciences, Saratov, 1998 (in Russ)

[2] M. K. Fage, N. I. Nagnibida The Equivalence Problem of Ordinary Linear Differential Operators, 1987, Novosibirsk, from 'Nauka', 278 pp. (in Russ)

[3] Yu. S. Linchuk On the equivalence of perturbations of a differential operator by functions of the Pommiez operator, Dopov. Nac. akad. nauk Ukr. 2016, 6:20-24

[4] V V.Katrahov, S. M.Sitnik Method of transformation operators and boundary value problems for singular elliptic equations, Modern Mathematics. Fundamental directions. Vol. 64, N 2, 2018, p. 497 211-428 (in Russ)

On strong solutions of a fractional nonlinear viscoelastic model of Voigt type

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Abstract. In the present paper we establish the existence and uniquecess of strong solutions to one fractional nonlinear Voigt type model of viscoelastic fluid in the planar case. The investigation is based on the fractional powers of operators, approximation of the problem under consideration by a sequence of regularized Navier-Stokes type systems and the following passage to the limit.

Keywords. viscoelastic fluid; a priori estimate; strong solution; fractional power.

MSC2010. 76A05; 35Q35.

1   Introduction

Let $Q = [0, T] \times \Omega$, $\Omega \in R^2$ is a bounded domain with a smooth boundary $\partial\Omega$. We consider the initial-boundary value problem

$$\partial v/\partial t + \sum_{i=1}^{N} v_i \partial v/\partial x_i - \mu_0 \Delta v - \operatorname{Div} \mu_1(S(v)) \, \mathcal{E}(v)(s, x) -$$

$$\mu_2 \operatorname{Div} \int_0^t (t-s)^{-\alpha} \, \mathcal{E}(v)(s, x) ds + \nabla p = f(t, x), \ (t, x) \in Q_T; \quad (1)$$

$$v(0, x) = v^0(x), \ x \in \Omega, \quad v|_{[0,T] \times \partial\Omega} = 0. \tag{2}$$

Here the velocity v and the pressure p are unknown, $\mathcal{E}(v) = \{\mathcal{E}_{ij}(v)\}_{i,j=1}^2$, $\mathcal{E}_{ij}(v) = \frac{1}{2}\left(\partial v_i/\partial x_j + \partial v_j/\partial x_i\right)$ is the strain rate tensor, $S(v) = \sum_{i,j=1}^2 (\partial v_i/\partial x_i)^2$, $0 < \alpha < 1$, $\mu_0 > 0$, $\mu_2 \geq 0$, $\mu_1(s) \geq 0$ is a differentiable function of $s \geq 0$.

For $\mu_1 \neq 0$, $\mu_2 = 0$ in [1] the existence and uniqueness of strong solutions to (1)-(2) was established. The weak solvability in the case $\mu_1 = 0$, $\mu_2 \geq 0$ was investigated in [2]. The strong solvability in the case $\mu_1 \geq 0$, $\mu_2 = 0$ was established for regularized system (1)-(2) in [3].

Our goal is the existence and uniqueness of strong solutions to (1)-(2) in the case $\mu_1 \neq 0$, $\mu_2 \geq 0$.

Rewrite problem (1)-(2) in the operator form. Let $W_2^{k,m}(Q)^2$ be usual Sobolev spaces of k times differentiable w.r.t. t and m times differentiable w.r.t. x R^2-valued functions. Let $\mathcal{V} = \{v = (v_1, v_2): v_i \in \mathcal{D}(\Omega)^2, i = 1, 2; \operatorname{div} v = 0\}$ and H be the closure of $\mathcal{V}$ in the norm of $L_2(\Omega)^2$ and V be the closure of $\mathcal{V}$ in the norm of $W_2^1(\Omega)^2$. Let $\mathcal{P}$ be the orthoprojector of $L_2(\Omega)^2$ on H. Define

in H operator A as $Av = -\mathcal{P}\triangle v$ on $D(A) = W_2^2(\Omega)^2 \cap H \cap \overset{\circ}{W}_2^1(\Omega)^2$. Operator A is selfadjoint positively defined operator.

Let $D_0(v) = \sum_{i=1}^2 v_i \frac{\partial v}{\partial x_j}$ for $v \in V$. Introduce operators

$B_0(v) = -\mathcal{P}\mathrm{Div}(\mu_1(S(v))\mathcal{E}(v))$, $C_0(v) = \int\limits_0^t (t-s)^{-\alpha} Av(s,x)\,ds$. Operators B_0 and C_0

are defined for a.a. t and $v(t,x) \in W_2^{0,2}(Q)^2$.

Introduce the space

$$W \equiv \{v: \ v \in L_2(0,T;W_2^2(\Omega)^2 \cap H) \cap L^\infty(0,T;H), \ v' \in L_2(0,T;H)\}.$$

Consider the problem

$$v' + \mathcal{P}D_0(v) + \mu_0 Av + B_0(v) + \mu_2 C_0(v) = f, \ t \in [0,T], \ v(0) = v^0. \tag{3}$$

Definition 1 *Strong solution to problem (3) is defined as a function $v \in W$ satisfying the initial condition and equation (3) by a.a. $t \in [0,T]$.*

The main results.

Theorem 1 *Let $f \in L_2(0,T;H)$, $v^0 \in V$. Let $\mu_1(s)$ satisfy conditions*

$$\mu_1(s) + 2\mu_1'(s) \geq 0, s \geq 0; \ \ s|\mu_1'(s)| \leq M, \ \ s \geq a > 0. \tag{4}$$

Then the problem (3) has a strong solution.

Theorem 2 *Under conditions of Theorem 1 the strong solution to problem (3) is unique.*

Acknowledgments

References

[1] V. P. Orlov and P. E. Sobolevskii, On smoothness of weak solutions to motion equations of almost Newtonian fluid, Chislenn. Metod. Mechan. Sploshn. Sredy, **16:1**(1985), 107-119 (in Russian).

[2] Victor Zvyagin and Vladimir Orlov, Weak solvability of fractional Voigt model of viscoelasticity, Discrete and Continuous Dynamical Systems, Series A, **38:12** (2018), 6327-6350.

[3] V. P. Orlov, On strong solutions to regularized model of non-linear viscoelastic medium, Matemat. Zametki, **84:2** (2008), 238-253 (in Russian).

[4] R. Temam; *Navier–Stokes equations. Theory and numerical analysis*, Stud. Math. Appl. **2**, North-Holland, Amsterdam–New York, 1979.

The Study Of The Correctness Of The Coupled Mathematical Models Of Sediment And Suspended Matter Transport

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Abstract. This paper is devoted to the study of coupled models of transport of sediment and suspended matter in the coastal zone under the influence of waves. The uniqueness of the solution of the initial-boundary value problem is proved and an a priori estimate of the norm of the solution is obtained depending on the integral estimates of the right-hand side, boundary conditions, and the initial condition. Numerical experiments for model problems of sediment transport and bottom relief transformation were performed, the results of which are consistent with actual physical experiments.

Keywords. sediment transport; coastal infrastructure facilities; suspended matter; coastal zone; uniqueness of the solution, estimation of the norm of the solution of the initial-boundary value problem.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

The sediment transport, sedimentation and deformation of the bottom in coastal and shallow water systems significantly affect the safety of navigation, the conditions for the reproduction of marine bioresources in shallow water bodies, changes in recreational areas, etc. The creation and use of precision models of these processes with predictive accuracy is an urgent problem of mathematical modeling of water systems, since it allows to predict both the results of anthropogenic impact associated with the construction and reconstruction of coastal infrastructure facilities, and the consequences of the evolution of weather and climate phenomena - an increase in frequency and intensity storms, extreme rainfall, etc. In connection with the foregoing, the creation of a complex of prognostic interconnected models of sediment transport, suspensions and bottom deformation, methods for their numerical implementation, which allow real-time predictive modeling of these phenomena, is relevant. This paper is devoted to the study of coupled models of transport of sediment and suspended matter in the coastal zone of the reservoir. The models developed take into account coastal currents and stress near the bottom, caused by wind waves, turbulent spatial three-dimensional movement of the aquatic environment, a complex shape of the coastline and bottom topography, and other factors.

2 Main results

The uniqueness of the solution of the initial-boundary value problem is proved and an a priori estimate of the norm of the solution is obtained depending on the integral estimates of the right-hand side, boundary conditions, and the initial condition. A conservative difference scheme with weights was constructed, approximating a continuous initial-boundary value problem. Sufficient conditions for the stability of a difference scheme are imposed, imposing restrictions on the time step of the scheme with weights. Numerical experiments for model problems of sediment transport and bottom relief transformation were performed, the results of which are consistent with actual physical experiments.

Acknowledgments

References

[1] V.V. Sidoryakina and A.I. Sukhinov, Well-posedness analysis and numerical implementation of a linearized two-dimensional bottom sediment transport problem Computational Mathematics and Mathematical Physics, 2017, 57:6, 978-994.

[2] A.A. Samarskiy and A.V.Gulin, Numerical methods. M.: Nauka, 1989.

Periodic solutions in some biochemical models

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We give an introduction to algorithms of the elimination theory and methods for solving of polynomial systems and show how they can be used for the qualitative investigation of autonomous systems of ordinary differential equations arising in modeling of biochemical networks. An application to the study of two polynomial systems of ODEs, which model some ecological and chemical processes, is presented. In particular some integrals and periodic solutions in the systems are found and limit cycle bifurcations are investigated.

Peculiarities of nonlinear system of two-dimensional singularly perturbed differential equations. Time-periodic problem.

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Abstract. Problems for nonlinear systems of differential equations have been investigated in the case of Dirichlet boundary condition for one function and Neyman boundary condition for the other. The solving process of such type time-periodic parabolic, elliptic problems contain significant features, difficulties. The new described method of equation formation for boundary layer series enables to eliminate boundary condition singularity. Asymptotics contains two different types of boundary series. First type coefficients of boundary series exponentially decrease, coefficients of the second type have a different structure, a special benchmark function emerges in order to evaluate them. The suggested approach allows to solve the problem. Such type problems are of importance, could have practical use. For example, they are in demand for studying the processes of biophysics.

Keywords. nonlinear differential equations; elimination of the boundary condition singularity; constructing of boundary layer function's equation; time-periodic problem; system of singularly perturbed elliptic equations; degenerate equation's multiple root; multi zoned boundary layer; solution's asymptotic expansion.

MSC2010. 35-06; 35G50; 35K55; 35B20; 35J47; 35Q92

1 Introduction

Consider the following system of parabolic equations with a small parameter ε

$$\varepsilon^2 \left(\Delta u - \frac{\partial u}{\partial t} \right) = F(u,v,x,t,\varepsilon), \quad \varepsilon \left(\Delta v - \frac{\partial v}{\partial t} \right) = f(u,v,x,t,\varepsilon) \Big|_{x \in \Omega; -\infty < t < \infty} \tag{1}$$

with boundary conditions different for each function

$$u(x,\varepsilon) = u^0(x,t), \quad \frac{\partial v}{\partial n}(x,\varepsilon) = v^0(x,t) \Big|_{x \in \partial\Omega; -\infty < t < \infty}. \tag{2}$$

Hereafter $\Delta = \partial^2/\partial x_1{}^2 + \partial^2/\partial x_2{}^2$- is the Laplace operator, x is two-dimensional variable $x = (x_1, x_2)$, $G = \Omega \times (\infty < t < \infty)$, function F has nonlinear dependence on u, functions F, f, u^0, v^0 are T-periodic in time. Also a condition of solution periodicity takes place

$$u(x,t+T,\varepsilon) = u(x,t,\varepsilon), \quad v(x,t+T,\varepsilon) = v(x,t,\varepsilon)|_{(x,t) \in \bar{G}}. \tag{3}$$

Solution asymptotics of problem (1)–(3) with rapid and slow equations has the following structure

$$u = \bar{u} + \Pi u + Pu, \quad v = \bar{v} + \Pi v + Pv.$$

Here $\bar{u}$, $\bar{v}$ are regular parts of asymptotics, Πu, Πv are simple boundary layer series, Pu, Pv are multizone boundary layer series.

Nonlinear system of elliptic equations of a such type $\varepsilon^2 \Delta u = F(u, v, x, \varepsilon)$, $\varepsilon \Delta v = f(u, v, x, \varepsilon)$, $x \in \Omega$ with different boundary conditions of type (2) has also solution with asymptotic expansion containing two different types of boundary functions.

Boundary layer series are expanded in powers of $\varepsilon^{1/4}$. Asymptotics contains two different types of boundary series. First type coefficients $\Pi_i u, \Pi_i v$ of boundary series exponentially decrease, coefficients of the second type $P_i u, P_i v$ have a different structure, a special benchmark function P_κ emerges in order to evaluate them. The boundary layer becomes multizoned.

The process of solving such type problems contains significant features, difficulties. One of them, that boundary conditions for functions $\Pi_i v$ contain singularity. A new method of forming boundary functions equation enabled to eliminate this obstacle:

$$\frac{\partial^2 P_i v}{\partial \zeta^2} = \sqrt{\varepsilon} s_i, \tag{4}$$

where s_i is expressed through previously found functions $P_j u, P_j v, P_i u$, $j < i$, and is estimated as $|s_i| \leq c \cdot P_\kappa(\zeta)$. ζ is boundary layer variable. The right hand side of equation (4) must have the power of small parameter ε two steps of expanding more then the left hand side (if Pv is expanded in powers of $\varepsilon^{1/4}$, then right hand side of equation for function $P_i v$ is multiplied by $\varepsilon^{1/4+1/4}$ more).

The theorem on the existence and asymptotic expansion of solutions:

Theorem 1 *For sufficiently small ε the original problem has solution u, v and corresponding partial sums are found to be uniform asymptotic approximation with the accuracy of the order $O(\varepsilon^{\frac{n}{2}+\frac{1}{4}})$.*

Acknowledgments

References

[1] V. A. Beloshapko, Special properties of the stationary solution for two-dimensional singularly perturbed parabolic problem, stability, and attraction domain *Math. Meth. Appl. Sci.* **41** (2018), 9264–9275, https://doi.org/10.1002/mma.5235

[2] V. A. Beloshapko, Singularly perturbed elliptic Dirichlet problem with three-band boundary layer *Model. and Analysis of Inf. Syst.* **24** No.3 (2017), 280–287.

[3] V. A. Beloshapko and V. F. Butuzov, Asymptotics of the solution of a singularly perturbed elliptic problem with three-band boundary layer *Comp. Math. and Math. Phys.* **56** No.8, (2016), 1414–1425.

[4] A. B. Vasil'eva, V. F. Butuzov and L. V. Kalachev, The boundary function method for singular perturbation problems *SIAM studies in applied mathematics* **14**, (1995).

Multi-scale modelling for wildfire propagation: interplay between meso- and macro-scale factors

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Abstract. Propagation of wildfire involves a large spectrum of spatial scales. We present a concurrent multi-scale modelling approach and we perform Uncertainty Quantification and Global Sensitivity Analysis of model's parameters for ranking their role and for analysing the interplay between macro-scale factors, such as the atmospheric stability, and meso-scale factors, such as the flame geometry.

Keywords. front propagation; multi-scale modelling; level-set method; sensitivity analysis; uncertainty quantification.

MSC2010. 65C20; 35K57; 70H20; 65Z05.

Wildfire propagation is studied in the literature by two alternative approaches: the reaction-diffusion equation and the front tracking level-set method. The solution of the reaction-diffusion equation is a smooth function in an infinite domain, while the level-set method generates a sharp function that is not zero inside a compact domain.

However, these two approaches can indeed be considered complementary and reconciled. Thus, a method based on the idea to split the motion of the front into a drifting part and a fluctuating part is proposed in [1]. The drifting part can be provided by chosen existing method (e.g. based on level-set method). The fluctuating part is the result of a comprehensive statistical description of the physics of the system and includes the random effects, e.g., turbulent hot-air transport and fire-spotting. As a consequence, the fluctuating part can have a non-zero mean (for example, due to ember jump lengths), which means that the drifting part does not correspond to the average motion. This last fact distinguishes between the present splitting and the well-known Reynolds decomposition adopted in turbulence studies.

The resulting formulation is independent of the method used for the fire-line propagation and the definition of the rate of spread and it is versatile enough to be utilized with any of existing operational fire spread model.

The physical parametrization of the proposed fire-spotting model is developed in [2, 3, 4]. The proposed parametrization allows the estimation of impact of various macro- and mesoscale parameters, such as the atmospheric stability of the flame geometry.

A global sensitivity analysis for the proposed fire-spotting sub-model is provided in [5]. The parameters are ordered by the means of the Sobol indices for what concerns their effect on topology and size of the burned area. The analysis demonstrates the importance of the wind for

the propagation of the main fire front, as well as for the generation of secondary fires through fire-spotting. An extensive work of Surrogate Modelling is performed to compare the performance of the surrogates for varying size and type of training sets as well as for varying parametrization and choice of algorithms. A meta-model is built using sparse generalized Polynomial Chaos Expansion and Gaussian Process (Kriging scheme), and Sobol' Indices are used to spot the most influential factors.

Results show the suitability of the approach for simulating random effects due to turbulent transport and fire-spotting, which are cases not resolved by standard operational codes. Moreover, the proposed formulation and parametrization can be extended to include further random parameters such as moisture, spatial distribution of combustible, orography and also atmospheric factors such as wind and pressure.

Acknowledgments

References

[1] G. Pagnini A. Mentrelli, Modelling wildland fire propagation by tracking random fronts, Nat. Hazards Earth Syst. Sci. **14** (2014), 2249–2263.

[2] A. Trucchia, V. Egorova, A. Butenko, I. Kaur, G. Pagnini, Randomfront 2.3: a physical parameterisation of fire spotting for operational fire spread models – implementation in WRF-Sfire and response analysis with LSFire+, Geosci. Model Dev. **12 (1)** (2019).

[3] V. N. Egorova, A. Trucchia, G. Pagnini, Fire-spotting generated fires. Part I: The role of atmospheric stability, Appl. Math. Mod. (in press, 2019).

[4] V. N. Egorova, A. Trucchia, G. Pagnini, Fire-spotting generated fires. Part II: The role of flame geometry, Appl. Math. Mod. - Submitted, (2019).

[5] A. Trucchia, V. Egorova, G. Pagnini, M. Rochoux, On the merits of sparse surrogates for global sensitivity analysis of multi-scale nonlinear problems: Application to turbulence and fire-spotting model in wildland fire simulators, Comm. Nonlin. Sci. Num. Simul. **73** (2019), 120–145.

The application of Lagrange's operational matrix method for two-dimensional hyperbolic telegraph equation

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Abstract

In this work, an efficient method is proposed to find the numerical solution of two dimensional hyperbolic telegraph equation. In this method, the roots of Legendre's polynomial are taken as the node points for the Lagrange's polynomial. First, we convert the main equation in to partial integro-differential equations(PIDEs) with the help of initial and boundary conditions. The operational matrices of differentiation and integration are then used to transform the PIDEs in to algebraic generalized Sylvester equations. We compared the results obtained by the proposed method with Bernoulli matrix method which shows that the proposed method is accurate for small number of basis function.

Controllability Results for Fractional Damped Dynamical Equation with Non-instantaneous Impulses

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Abstract. In this manuscript, we establish the controllability results for a fractional damped dynamical equation with non-instantaneous impulses. Controllability Grammian matrix, Mittag-Leffler matrix function and Banach fixed-point technique have been used to establish these results. Moreover, we establish the controllability results of the considered system with integro term. Also, some numerical examples with simulations are given to outline the effectiveness of the developed results.

Keywords. Controllability; Non-instantaneous Impulses; fractional dynamical equation; Mittag-Leffler function.

MSC2010. 93B05; 34A37; 34A08; 33E12.

1   Introduction

We consider the following system

$$
\begin{aligned}
{}^{C}D^{p}x(t) &= A\,{}^{C}D^{q}x(t) + Bu(t) + f(t, x(t)), \quad t \in \cup_{i=0}^{m}(s_i, t_{i+1}], \\
x(t) &= \mathcal{J}_i(t, x(t_i^{-})), \quad t \in (t_i, s_i], \ i = 1, 2, \cdots, m, \\
x'(t) &= \mathcal{G}_i(t, x(t_i^{-})), \quad t \in (t_i, s_i], \ i = 1, 2, \cdots, m, \\
x(0) &= x_0, \ x'(0) = y_0,
\end{aligned}
\tag{1}
$$

where ${}^{C}D^{p}x(t)$ and ${}^{C}D^{q}x(t)$ denote Caputo fractional derivatives of order $p \in (1, 2], q \in (0, 1]$. A and B are the $n \times n$ and $n \times m$ matrices respectively, $x \in \mathbb{R}^n, u \in \mathbb{R}^m, 0 = s_0 = t_0 < t_1 < s_1 < t_2 < \cdots s_m < t_{m+1} = T, \ x(t_i^{-}) = \lim_{k \to 0^{+}} x(t_i - k), \ x(t_i^{+}) = \lim_{k \to 0^{+}} x(t_i + k)$ denotes the left and right limits of $x(t)$ at $t = t_i$, the functions $f, \mathcal{J}_i$ and $\mathcal{G}_i, i = 1, 2, \cdots, m$, are satisfying certain suitable conditions to be stated later.

Degenerate Distributed Order Differential Equations in Banach Spaces and Applications

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Abstract. We consider a class of linear differential equations in Banach spaces with a degenerate operator at the distributed order Gerasimov — Caputo derivative. The Cauchy problem for such equations is studied. The results on the unique solvability of the problem are obtained by using the conditions on the operators in the equation, which guarantee the existence of invariant subspaces pairs. As an example, an initial-boundary value problem for a partial differential equation not solved with respect to the distributed in time derivative is considered.

Keywords. fractional derivative; distributed order derivative; degenerate evolution equation; initial-boundary value problem.

MSC2010. 34G10; 35R11.

Let $L, M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$ (linear closed and densely defined operators) have domains D_L, D_M, $\ker L \neq \{0\}$. Since L and M are closed operators, we can consider D_L and D_M as the Banach spaces with the graph norms of the operator L and M respectively. Let us consider the distributed order equation

$$\int_b^c \omega(\alpha) D_t^\alpha L x(t) d\alpha = M x(t) + f(t), \quad t > 0, \tag{1}$$

where D_t^α is the Gerasimov — Caputo fractional derivative, $0 \leq b < c \leq 1$, $\omega : (b, c) \to \mathbf{C}$, $f \in C(\overline{\mathbf{R}}_+; \mathcal{Y})$. Equation (1) is called degenerate, because it is supposed that $\ker L \neq \{0\}$.

A function $x : \mathbf{R}_+ \to D_L \cap D_M$ is called a solution of equation (1), if $Mx \in C(\mathbf{R}_+; \mathcal{Y})$, there exists $\int_a^b \omega(\alpha) D_t^\alpha L x(t) d\alpha \in C(\mathbf{R}_+; \mathcal{Y})$ and equality (1) is valid. A solution x of (1) is called a solution to the Cauchy problem

$$x(0) = x_0 \tag{2}$$

for equation (1), if $x \in C(\overline{\mathbf{R}}_+; \mathcal{X})$ satisfies condition (2).

By $\rho^L(M)$ the set of $\mu \in \mathbf{C}$ is denoted, for which the mapping $\mu L - M : D_L \cap D_M \to \mathcal{Y}$ is injective, and $R_\mu^L(M) := (\mu L - M)^{-1} L \in \mathcal{L}(\mathcal{X})$ (linear bounded operator), $L_\mu^L(M) := L(\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y})$.

Let $L, M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$. A pair of operators (L, M) belongs to the class $\mathcal{H}_\alpha(\theta_0, a_0)$, if the following two conditions are valid:

(i) there exist $\theta_0 \in (\pi/2, \pi)$ and $a_0 \geq 0$, such that for all $\lambda \in S_{\theta_0, a_0}$ we have $\lambda^\alpha \in \rho^L(M)$;

(ii) for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists a constant $K = K(\theta, a) > 0$, such that for all $\mu \in S_{\theta,a}$ $\max\{\|R_{\mu^\alpha}^L(M)\|_{\mathcal{L}(\mathcal{X})}, \|L_{\mu^\alpha}^L(M)\|_{\mathcal{L}(\mathcal{Y})}\} \leq \frac{K(\theta,a)}{|\mu^{\alpha-1}(\mu-a)|}$.

It is easy to show, that the subspaces $\ker R_\mu^L(M) = \ker L$, $\mathrm{im}R_\mu^L(M)$, $\ker L_\mu^L(M)$, $\mathrm{im}L_\mu^L(M)$ do not depend on the parameter $\mu \in \rho^L(M)$. Denote $\ker R_\mu^L(M) = \mathcal{X}^0$, $\ker L_\mu^L(M) = \mathcal{Y}^0$. By $\mathcal{X}^1$ ($\mathcal{Y}^1$) the closure of the subspace $\mathrm{im}R_\mu^L(M)$ ($\mathrm{im}L_\mu^L(M)$) in the norm of $\mathcal{X}$ ($\mathcal{Y}$) is denoted. The projection on the subspace $\mathcal{X}^1$ ($\mathcal{Y}^1$) along $\mathcal{X}^0$ ($\mathcal{Y}^0$) is denoted by P (Q). By L_k (M_k) we denote the restriction L (M) on $D_{L_k} := D_L \cap \mathcal{X}^k$ ($D_{M_k} := D_M \cap \mathcal{X}^k$), $k = 0, 1$. Introduce also the denotations $S = L_1^{-1}M_1 : D_S \to \mathcal{X}^1$, $D_S = \{x \in D_{M_1} : M_1 x \in \mathrm{im}L_1\}$; $T = M_1 L_1^{-1} : D_T \to \mathcal{Y}^1$, $D_T = \{y \in \mathrm{im}L_1 : L_1^{-1}y \in D_{M_1}\}$.

Theorem 1 [1]. *Let Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ be reflexive, $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$. Then*

(i) $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, $\mathcal{Y} = \mathcal{Y}^0 \oplus \mathcal{Y}^1$; $L_0 = 0$, $M_0 \in Cl(\mathcal{X}^0; \mathcal{Y}^0)$, $L_1, M_1 \in Cl(\mathcal{X}^1; \mathcal{Y}^1)$;

(iii) *there exist inverse operators* $L_1^{-1} \in Cl(\mathcal{Y}^1; \mathcal{X}^1)$, $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$;

(iv) $\forall x \in D_L$ $Px \in D_L$ *and* $LPx = QLx$; $\forall x \in D_M$ $Px \in D_M$ *and* $MPx = QMx$;

(v) D_S *is dense in the space* $\mathcal{X}$, D_T *is dense in* $\mathcal{Y}$;

(vi) *if* $L_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, *or* $M_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, *then the operator* $S \in Cl(\mathcal{X}^1)$;

(vii) *if* $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, *or* $M_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, *then* $T \in Cl(\mathcal{Y}^1)$.

Define the contour $\Gamma = \partial S_{\theta_1, a_1}$ with $\theta_1 \in (\pi/2, \theta_0)$, $a_1 > a_0$, and operators

$$X_0(t) := \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t}}{\lambda}W_b^c(\lambda)R_{W_b^c(\lambda)}^L(M)d\lambda, \quad X(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R_{W_b^c(\lambda)}^L(M)d\lambda,$$

where $W_b^c(\lambda) := \int_b^c \omega(\alpha)\lambda^\alpha d\alpha$, Denote by $E(\mathcal{X}; P)$ the set of all functions $x : \overline{\mathbf{R}}_+ \to \mathcal{X}$, such that Px is exponentially bounded.

Theorem 2 *Let Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ be reflexive, $c \in (0, 1]$, a pair $(L, M) \in \mathcal{H}_c(\theta_0, a_0)$, $L_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$ or $M_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, $W_b^c(\lambda)$ be a holomorphic function on S_{θ_1, a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, such that for all $\lambda \in S_{\theta_1, a_1}$ $(W_b^c(\lambda))^{1/c} \in S_{\theta_0, a_0}$, and*

$$\exists C_1, C_2 > 0 \quad \exists \varepsilon \in (0, c) \quad \forall \lambda \in S_{\theta_1, a_1} \quad C_1|\lambda|^\varepsilon \leq |W_b^c(\lambda)| \leq C_2|\lambda|^c,$$

$f \in C(\overline{\mathbf{R}}_+; \mathcal{Y})$, $L_1^{-1}Qf \in C(\overline{\mathbf{R}}_+; D_S) \cap E(D_S)$, $x_0 \in \mathcal{X}$, *such that* $Px_0 \in D_S$, *and* $(I - P)x_0 = -M_0^{-1}(I - Q)f(0)$. *Then the function*

$$x(t) = X_0(t)x_0 + \int_0^t X(t-s)L_1^{-1}Qf(s)ds - M_0^{-1}(I - Q)f(t)$$

is a unique solution to Cauchy problem (1), (2) *from the class* $E(\mathcal{X}; P)$.

Acknowledgments

References

[1] V. E. Fedorov, E. A. Romanova and A. Debbouche, Analytic in a sector resolving families of operators for degenerate evolution fractional equations, J. Math. Sc. **228** (2018), 380–394.

The Study of the Statistical Characteristics of the Text Based on the Graph Model of the Linguistic Corpus

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Abstract. We study the statistical characteristics of the text, which are calculated on the basis of the graph model of the text from the linguistic corpus. The introduction describes the relevance of the statistical analysis of the texts and some of the tasks solved using such an analysis. The graph model of the text proposed in the article is constructed as a graph in the vertices of which the words of the text are located, and the edges of the graph reflect the fact that two words fall into any part of the text, for example, in - a sentence. For the vertices and edges of the graph, the article introduces the concept of weight as a value from some additive semigroup. Formulas for calculating a graph and its weights are proved for text concatenation. Based on the proposed model, calculations are implemented in the Python programming language. For an experimental study of statistical characteristics, 24 values are distinguished, which are expressed in terms of the weights of the vertices, edges of the graph, as well as other characteristics of the graph, for example, the degrees of its vertices. It should be noted that the purpose of numerical experiments is to squeak in the characteristics of the text, with which you can determine whether the text is man-made or randomly generated. The article proposes one of the possible such algorithms, which generates random text using some other text created by man as a template. In this case, the random text is preserved the sequence of alternation of parts of speech auxiliary text. It turns out that the required conditions are satisfied by the median value of the ratio of the value of the weight of the edge of the text graph to the number of sentences in the text.

Keywords. text, graph, linguistic corpus, automatic text processing

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

We name alphabet an arbitrary finite set Σ. Elements of the set Σ are called symbols. Ordered set of symbols is called a word or a chain of symbols. The set of all chains of symbols in alphabet Σ we denote by Σ^*. The text is an ordered set of symbols

$$T = a_1 a_2 ... a_n, \quad a_i \in \Sigma, n = |T|.$$

Here and further by $|X|$ we denote the number elementsof the set X. If T_1, T_2 are two texts then by $T_1 \cdot T_2$ we denote the text which can be constructed by concatenation. Also, by 2^X we denote the set of all subsets of the given set X. Some collection of texts we name *corpus*. In order to construct graph model we need notion of partition of text. By partition of text we

mean some subset $P \subset 2^N$ of the ordered subsets of the set $N = \{1, 2, 3, ..., |T|\}$. Every such subset is defined by set of tuples of numbers $(i_1, ..., i_k) \in P$ including symbols. The examples of partions are words of text or m-grams.

We suppose that $I \in P$. Consider the mapping $\omega : P \to \Sigma^*$

$$\omega(I) = a_{i_1} a_{i_2} ... a_{i_k}, \quad i_1 < i_2 < ... < i_k, \quad i_j \in I, j = 1, 2, ..., k.$$

By definition the partition P' is smaller then P'' , (in this case we write $P' \subset P''$) if for every $I' \in P'$ there is $I'' \in P''$ such that $I' \subset I''$. For two given texts T_1, T_2 and corresponding partitions P_1, P_2 we define partition $P = P_1 \circ P_2$ for text $T_1 \cdot T_2$ by the following way

$$P_1 \circ P_2 = \{I \subset (1, 2, ..., |T_1| + |T_2|) : I = I_1 \subset P_1 \text{ or } I = \{i + |T_1| : i \in I_2 \in P_2\}\}.$$

For the given text T and it's partition P we denote by $U(T, P) = \{\omega(I) : I \in P\}$ the set of unique parts of partition. Consider the text T and some it's tow partitions $P' \subset P''$. We construct the graph $G = G(T, P', P'')$ with vertices $VG = U(T, P')$ and edges

$$EG = \{(a, b), a, b \in U(T, P') : \exists I \in P'', I_a, I_b \subset I, \text{ where } \omega(I_a) = a, \omega(I_b) = b\}.$$

This graph represents the relationship of a pair of parts of the text of one partition of one part of another partition. For example, the edge of a graph with vertices in the form of unique words of a text corresponds to a pair of words that occur in any one sentence. You can directly check the following key formula for graph construction when pasting two texts.

2 Main results

We introduce the following notation. Let $L(v), v \in VG$ denote the sequence of the weights of the edges incident to the vertex v, denote the degree of the vertex v by $deg(v)$, and denote the edge weight $\theta(e), e \in EG$, finally, by $\sigma(T)$ we denote the number of sentences of the text T - or, in other words, the number of elements of the partitioning P''. Further, for any finite sequence $x = \{x_1, x_2, ..., x_n\}$ we introduce the notation $\max(x), \text{mean}(x), \text{median}(x$ for maximum, average and median value of x. For text T we denote the corresponding sequences.

For statistical study of texts we choose the following features of graph $G(T, P', P'')$

- maximum, median, average degree of graph vertices, $\max(d(T)), \text{median}(d(T)), \text{mean}(d(T))$,

- maximum, median, average weights of edges of the graph, $\max(\theta(T)), \text{median}(\theta(T)), \text{mean}(\theta(T))$,

Also we investigate the following values

$$d(T) = \{d(v), v \in VG\}, \quad \theta(T) = \{\theta(e), e \in EG\},$$

$$\theta_s(T) = \{\theta(e)/\sigma(T), e \in EG\}, \quad d_{mx}(T) = \{\max(L(v)), v \in VG\},$$

$$d_{mn}(T) = \{\text{mean}(L(v)), v \in VG\}, \quad d_{mdn}(T) = \{\text{median}(L(v)), v \in VG\}.$$

The main result of experiences is that the value $\text{median}(\theta_s(T))$ is characteristic which can be used as a determinant either the given text is human written text or not.

Acknowledgments

Fredholm Properties of Elliptic Operators and Boundary Value Problems

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Abstract. We study elliptic pseudo-differential equations on a manifold with a non-smooth boundary. These operators and equations arise in various problems of mathematical physics. Using the local principle we first consider a local situation for each point of a manifold, and add some boundary conditions if it is necessary. Such boundary conditions appear if an index of special symbol factorization is not vanishing. For the latter case we consider simplest non-smooth domain like a multidimensional cone and describe some boundary conditions for which we can construct the solution of the boundary value problem in Sobolev–Slobodetskii spaces.

Keywords. elliptic operator; symbol; boundary value problem; local representative; invertibility; Fredholm property.

MSC2010. 35S15; 47G30; 58J05.

1 General concept

We study some special operators on compact manifold M which are generated by operato family $\{A_x\}_{x \in M}$. Such an operator A will be considered in Sobolev–Slobodetskii spaces $H^s(M)$, and local variants of such spaces will be spaces $H^s(D_{x_0})$. An operator A_x we call a local representative of the operator A at point $x \in M$.

Definition 1. *The symbol of an operator A is called the operator-function $A(x) : M \to \{A_x\}_{x \in M}$ which is defined by local representatives of the operator A. An operator A is called an elliptic operator if its symbol is composed by invertible operators.*

Some details of this operator approach are given in [2, 3].

2 Applications

For simplicity we consider here the case when M is a bounded domain in $\mathbb{R}^m$ and its classical symbol of pseudo-differential operator A looks as $A(x, \xi)$ and this function $A(x, \xi), (x, \xi) \in M \times \mathbb{R}^m$ is continuous differentiable up to boundary. We assume that there are sub-manifolds $M_k \subset \partial M$ of dimension $k = 0, 1, \cdots, m - 2$, for which each point $x \in M_k$ has a neighborhood diffeomorphic to the set $\mathbb{R}^k \times C_x^{m-k}$; C_x^{m-k} is a convex cone in $\mathbb{R}^{m-k}$. By definition $M_m \equiv M, M_{m-1} \equiv \partial M, M_0 \equiv C_m$.

Let $\kappa_{n-1}(x)$ be the index of factorization of the function $A(x, \xi)$ in the point $x \in \partial M \setminus \cup_{k=0}^{m-2} M_k$, $\kappa_k(x)$ be indices of k-wave factorization [1] with respect to the cone C_x^{m-k} in points $x \in M_k, k =$

$0, 1, \cdots, n-2$ and we assume that the functions $\kappa_k(x), k = 0, 1, \cdots, n-1$, are continuously continued in $\overline{M_k}$.

Using uniqueness result for the wave factorization [1] one can verify that the functions $\kappa_k(x), k = 0, 1, \cdots, m-1$, don't depend on local coordinates

Theorem 1 *If the classical elliptic symbol $A(x, \xi)$ admits k-wave factorization with respect to the cones C^{m-k} with indices $\kappa_k(x), k = 0, 1, \cdots, m-2$ satisfying the condition*

$$|\kappa_k(x) - s| < 1/2, \quad \forall x \in M_k, \quad k = 0, 1, \cdots, m-1, \tag{1}$$

then the operator $A : H^s(M) \to H^{s-\alpha}(M)$ has a Fredholm property.

Remark 1 *If the ellipticity does not hold on sub-manifolds M_k then we can modify the operator A using boundary or co-boundary operators [1]. Particularly we need such constructions if one of conditions (1) does not hold.*

Acknowledgments

References

[1] V. Vasil'ev, *Wave Factorization of Elliptic Symbols: Theory and Applications. Introduction to the Theory of Boundary Value Problems in Non-Smooth Domains*, Kluwer Academic Publishers, Dordrecht–Boston–London, 2000.

[2] V. Vasilyev, Pseudo-differential operators, equations and boundary value problems, AIP Conf. Proc. **2037** (2018), 020028.

[3] V. Vasilyev, Operator symbols, ArXiv: FA/1901.06630

The transmission problem for the Laplace operator on a domain with a cuspidal point

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Abstract. We shall study the transmission problem for the Laplace operator subject to Dirichlet boundary conditions in a plane domain Ω with an external cusp, Ω being divided into two subdomains Ω_1 and Ω_2 separated by a straight interface. We look for a solution in the framework of L^p-Sobolev spaces, $1 < p < +\infty$. We prove that $u \in H_0^1(\Omega)$ solution of $-\mathrm{div}(p\nabla u) = f$ with f in $L^p(\Omega)$ and $p = p_i$ on Ω_i, $i = 1, 2$, (p_1, p_2 are two positive real numbers, supposed to be different), is picewise in $W^{2,p}$.

Keywords. transmission problem; regularity; cuspidal domains.

MSC2010. 35J25; 35R05; 35A20.

References

[1] B. Chaouchi, Solvability of second-order boundary-value problems on non-smooth cylindrical domains. Electronic Journal of Differential Equations. **199** (2013), 1–7.

[2] W. Chikouche and S. Nicaise, Regularity of the solution of some transmission problems in domains with cuspidal points. Ann. Fac. Sc. Toulouse XVI. **3** (2007), 529–560.

[3] P. Grisvard, Problèmes aux limites dans des domaines avec points de rebroussement. Ann. Fac. Sc. Toulouse **II**. **3** (1995), 561–578.

Divergent second order evolution equations of solenoidal vector field on $\mathbb{R}^3$

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Abstract. It is found the description of the class $\mathfrak{K}_2^{(0)}(\mathbb{R}^3)$ of divergent second order differential operators that act in the $\mathfrak{L}_2(\mathbb{R}^3)$ vector fields space. These operators are translational invariant in $\mathbb{R}^3$, they are transformed by covariant way under rotations $\mathbb{R}^3$ and each of them conserves the solenoidal property when generates the infinitesimal time shift of of the field.

Keywords. divergent differential operator, translational invariance, vector field, covariance, unimodality, solenoidality.

MSC2010. 35Q60; 35K10.

1   Introduction

We study the mathematical physics problem concerned the construction of an adequate evolutionary equation of he special vector field $\boldsymbol{P}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^3$ that takes values in $\mathbb{R}^3$

$$\dot{\boldsymbol{P}}(\boldsymbol{x}, t) = \big(\mathsf{L}[\boldsymbol{P}]\big)(\boldsymbol{x}, t) \ \ t \in \mathbb{R}_+ . \tag{1}$$

The field describes the dynamics of the spontaneous electric polarization field in ferroelectric medium. In contrast to the approaches used in theoretical physics concerned with the construction of such evolutionary equations (see, for example, [1]), the problem that is solved is the description of a subclass $\mathfrak{K}_2^{(0)}(\mathbb{R}^3)$ contained in the class $\mathfrak{K}_2(\mathbb{R}^3)$ of all operators $\mathsf{L}[\cdot]$ which are differential on the variable $\boldsymbol{x} \in \mathbb{R}^3$ such that they have the second order and the divergent type. They act in the space $\mathfrak{L}_2(\mathbb{R}^3)$ of locally twice continuously differentiable vector fields. Besides, these operators are not explicitly depend on the parameter $t \in \mathbb{R}_+$ and the radius-vector $\boldsymbol{x}$ and they are *covariant* under rotations of the space $\mathbb{R}^3$. The subclass $\mathfrak{K}_2^{(0)}(\mathbb{R}^3)$ highlighted among all operators $\mathsf{L}[\cdot]$ of the class $\mathfrak{K}_2(\mathbb{R}^3)$ by the fact that, for each of them, each solution $\boldsymbol{P}(\boldsymbol{x}, t)$ of the correspondent equation (1) has an invariant $(\nabla, \boldsymbol{P}) = \text{const}$. In particular, it conserves the property of solenoidality, i.e. the equality $(\nabla, \boldsymbol{P})(\boldsymbol{x}, t) = 0$ if it takes place for the field $\boldsymbol{P}(\boldsymbol{x}, t)$. The solution of the problem is based on the following statement.

Theorem 1 *The class $\mathfrak{K}_2(\mathbb{R}^3)$ of divergent differential operators $\mathsf{L}[\boldsymbol{P}] \equiv \nabla_k S_{j,k}[\boldsymbol{P}]$ that act in the space $\mathfrak{L}_2(\mathbb{R}^3)$ consists of all operators for which the tensor field $\big(S[\boldsymbol{V}]\big)_{j,k}(\boldsymbol{x}, t)$ looks like*

$$S_{j,k}[\boldsymbol{P}] = p^{(1)}\delta_{jk} + p^{(2)}P_j P_k + f^{(1)}\delta_{jk}(\nabla, \boldsymbol{P}) + f^{(2)}\nabla_k P_j + f^{(3)}\nabla_j P_k +$$

$$+ g^{(1)}\delta_{jk}(\boldsymbol{P}, \nabla)\boldsymbol{P}^2/2 + g^{(2)}P_k(\boldsymbol{P}, \nabla)P_j + g^{(3)}P_k\nabla_j\boldsymbol{P}^2/2 +$$

$$+ g^{(4)}P_j(\boldsymbol{P}, \nabla)P_k + g^{(5)}P_j\nabla_k\boldsymbol{P}^2/2 + g^{(6)}P_j P_k(\nabla, \boldsymbol{P}) + h P_j P_k(\boldsymbol{P}, \nabla)\boldsymbol{P}^2/2 \tag{2}$$

where the coefficients $p^{(1)}$, $p^{(2)}$, $f^{(\alpha)}$, $\alpha = 1, 2, 3$; $g^{(\beta)}$, $\beta = 1 \div 6$, h are continuously differentiable functions on $\boldsymbol{V}^2 \in [0, \infty)$.

2 Main result

The class $\mathfrak{K}_2^{(0)}(\mathbb{R}^3)$ is described by the following way.

Theorem 2 *The operator* $\mathsf{L}[\boldsymbol{P}]$ *belongs to the class* $\mathfrak{K}_2^{(0)}(\mathbb{R}^3) \subset \mathfrak{K}_2(\mathbb{R}^3)$ *if its action on any vector field* $\boldsymbol{P}(\boldsymbol{x}, t) = \langle P_j(\boldsymbol{x}, t); j = 1, 2, 3 \rangle$ *in the space* $\mathfrak{L}(\mathbb{R}^3)$ *is defined by the following formula*

$$\left(\mathsf{L}[\boldsymbol{P}]\right)_j = \nabla_k \Big[f_+ \left(\nabla_j P_k + \nabla_k P_j\right) + f_- \left(\nabla_j P_k - \nabla_k P_j\right) + g_- \left(P_j(\boldsymbol{P}, \nabla)P_k - P_k(\boldsymbol{P}, \nabla)P_j\right) +$$

$$+\bar{g}_- \left(P_j \nabla_k - P_k \nabla_j\right)\boldsymbol{P}^2 + f'_+ \left(P_j \nabla_k + P_k \nabla_j - 2\delta_{jk}(\boldsymbol{P}, \nabla)\boldsymbol{P}^2\right) \Big]. \tag{3}$$

where the coefficients $f_\pm = (f^{(2)} \pm f^{(3)})/2$, $g_= (g^{(2)} - g^{(4)})/2$, $\bar{g}_- = (g^{(3)} - g^{(5)})/2$ *are continuously differentiable functions on* $\boldsymbol{V}^2 \in [0, \infty)$ *and only at this case.*

The proof of the statement is based on the analysis of the functional equation for functions $p^{(1)}$, $p^{(2)}$, $f^{(\alpha)}$, $\alpha = 2, 3$; $g^{(\beta)}$, $\beta = 1 \div 5$, h which follows from the condition $(\nabla, L[\boldsymbol{P}])(\boldsymbol{x}, t) = 0$ that provides an invariant $(\nabla, \boldsymbol{P})(\boldsymbol{x}, t) = \text{const}$ in the evolutionary equation (1). It has the form

$$\Delta p^{(1)} + \nabla_j \nabla_k p^{(2)} P_j P_k + \nabla_j \nabla_k \left[f^{(2)} \nabla_k P_j + f^{(3)} \nabla_j P_k\right] + \Delta g^{(1)}(\boldsymbol{P}, \nabla)\boldsymbol{P}^2/2+$$

$$+\nabla_j \nabla_k \left[g^{(2)} P_k(\boldsymbol{P}, \nabla)P_j + g^{(3)} P_k \nabla_j \boldsymbol{P}^2/2 + g^{(4)} P_j(\boldsymbol{P}, \nabla)P_k + g^{(5)} P_j \nabla_k \boldsymbol{P}^2/2\right] +$$

$$+\nabla_j \nabla_k h P_j P_k(\boldsymbol{P}, \nabla)\boldsymbol{P}^2/2 = 0 \tag{4}$$

where the equality should be taken place for all fields $\boldsymbol{P}(\boldsymbol{x}, t) \in \mathfrak{L}(\mathbb{R}^3)$. The process of the proof consists of the sequential separation of such linearly independent combinations of the summands presented in (4).

References

[1] Isayev A.A., Kovalevskii M.Yu., Peletmiskii S.V. Hamiltonian approach in the theory of condensed media with spontanuously destructed symmetry. Physics of elementary particles and nucleus.— 1996.— 27.— 2.— P.431–492.

[2] Subbotin A.V. Description of the class of devergent evolutionary equations. Belgorod State University Scientific Bulletin.— 2018.— 50;4.— P.492–497.

A Chebyshev based numerical methods for solving fractional fuzzy differential equations

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Abstract. The dynamics of many complex systems can be modeled by a system of fractional differential equations, However, usually, there is much uncertainty due to any source of errors in modeling. The fuzzy concept can help us to consider this type of uncertainty in some sense. Therefore, we study systems of linear fractional fuzzy differential equations to consider uncertainty in our modeling. We propose a Chebyshev based method for solving these systems numerically. We provide numerical examples to show the efficiency and effectiveness of the proposed methods.

Keywords. Fuzzy fractional differential equation; Chebyshev polynomials; Fractional dynamic inclusions.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

Fuzzy differential equations are a suitable tool to model the dynamical systems under uncertainty. Among which, the first order linear fuzzy differential equations are the simplest fuzzy differential equations which appear in many real world problems. The first and foremost approach to model the uncertainty of dynamical systems is the usage of Hukuhara derivative in 1967 [2]. From then, fuzzy differential equations were started as a separate field in mathematics in combination with set differential equations and fuzzy analysis. Later in 1983, Puri and Ralescu [3] defined the derivative for fuzzy mappings. The first theorem on existence and uniqueness for fuzzy differential equations under Hukuhara derivative was proposed by Kaleva [4] in 1987. Parallel to this development, fuzzy fractional equations is also studied by many authors [1, 5]. In this paper, we study a system of fuzzy fractional differential equation of the form

$$
\begin{aligned}
{}^{C}D^{\beta}y(t) &= Ay(t) + f(t), \quad t \in [0, T], \\
y(0) &= y_0
\end{aligned}
\tag{1}
$$

where ${}^{C}D^{\beta}$ Caputo-type fuzzy fractional derivatives of order $0 < \beta \le 1$, A is a $\nu \times \nu$ matrices of real numbers, f is an uncertain source function (here, a fuzzy function), $y = [y_1, \cdots, y_\nu]^T$ is a vector of fuzzy functions, $y_0 = [y_{01}, \cdots, y_{0\nu}]^T$ is a vector of fuzzy numbers and ν is the dimension of system 1.

2 Main results

Theorem 1 *Let A^+ be a matrix such that the negative elements of A is replaced by zero and A^- be a matrix such that positive elements of A is replaced by zero. Suppose that the perturbation transform $\sim: (\mathbb{R}_{\mathcal{F}})^\nu \to (\mathbb{R})^{2\nu}$ be defined by $\tilde{Y} = [\underline{Y}, \overline{Y}]^T$. Let $C = AY$. Then, $\underline{C} = [A^+, A^-]\tilde{Y}$ and $\overline{C} = [A^-, A^+]\tilde{Y}$. We define $A^\pm$ as*

$$A^\pm = \begin{pmatrix} A^+ & A^- \\ A^- & A^+ \end{pmatrix}.$$

Then, $\tilde{C} = A^\pm \tilde{Y}$.

Theorem 2 *Let A^+ be a matrix which its negative elements is replaced by zero and A^- be a matrix which its positive elements is replaced by zero. Also, the solution of the system (1) be Hukuhara differentiable of type (1). Then,*

$$\begin{aligned}
{}^C D^\beta \underline{y}(t) &= \underline{Ay(t) + f(t)}, \quad t \in [0, T], \\
{}^C D^\beta \overline{y}(t) &= \overline{Ay(t) + f(t)}, \\
\underline{y}(0) &= \underline{y_0}, \\
\overline{y}(0) &= \overline{y_0}.
\end{aligned} \tag{2}$$

This system can be written by $\sim$ transform as ${}^C D^\beta \tilde{y}(t) = A^\pm \tilde{y}(t) + \tilde{f}(t)$, with initial value $\tilde{y}(t) = \tilde{y_0}$.

Theorem 3 *Let the solution of the system (1) be Hukuhara differentiable of type (2). Then,*

$$\begin{aligned}
{}^C D^\beta \overline{y}(t) &= \underline{Ay(t) + f(t)}, \quad t \in [0, T], \\
{}^C D^\beta \underline{y}(t) &= \overline{Ay(t) + f(t)}, \\
\underline{y}(0) &= \underline{y_0}, \\
\overline{y}(0) &= \overline{y_0}.
\end{aligned} \tag{3}$$

Let

$$A^\mp = \begin{pmatrix} A^- & A^+ \\ A^+ & A^- \end{pmatrix}.$$

Then, the system (3) is equivalent with the system ${}^C D^\beta \tilde{y}(t) = A^\mp \tilde{y}(t) + \tilde{f}(t)$, with initial value $\tilde{y}(t) = \tilde{y_0}$.

References

[1] Arshad, S and Lupulescu, V. On the fractional differential equations with uncertainty. Non-linear Anal, 74(2011) 3685-3693.

[2] Hukuhara, M. Integration des applications measurables dont la valeur est un compact convex, Funkcial. Ekvacioj, 10, 1967, pp. 205-229.

[3] Puri, M. L. and Ralescu, D. A. Differentials of fuzzy functions. Journal of Mathematical Analysis and Applications, 91(2), 1983, pp. 552-558.

[4] Kaleva, O. Fuzzy differential equations, Fuzzy Sets and Systems, 24, 1987, pp. 301-317.

[5] Carpenter GA, Grossberg S, Rosen DB. Fuzzy ART: Fast stable learning and categorization of analog patterns by an adaptive resonance system. Neural networks. 1991 Jan 1;4(6):759-71.

Mathematical modeling and analysis of some reaction-diffusion problems with nonsingular-kernel derivatives

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Abstract. The aim of this talk is to apply the newly trending Atangana-Baluanu derivative operator for modeling some reaction-diffusion models. The choice of AB derative is justified by the fact that it combines nonlocal and nonsingular properties in its formulation, which are the essential ingredients when dealing with models of real-life applications. Mathematical analysis of these dynamical models is considered to guide in the correct use of parameters therein, with chaotic and spatiotemporal results reported for some instances of fractional power α. The numerical simulations were conducted via a new predictor corrector method.

References

[1] Owolabi, Kolade M., and Zakia Hammouch. "Mathematical modeling and analysis of two-variable system with noninteger-order derivative." Chaos: An Interdisciplinary Journal of Nonlinear Science 29.1 (2019): 013145.

[2] Owolabi, Kolade M., and Zakia Hammouch. "Spatiotemporal patterns in the Belouso-Zhabotinskii reaction systems with Atangana-Baleanu fractional order derivative." Physica A: Statistical Mechanics and its Applications (2019).

Evaluation of the Integrals of the Green's Function of the Lamb's Model Entering in the Contact Problems

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Abstract. The dynamic analysis of the contact problems is a hot topic and certainly of a scientific interest. However, this problem is related to a great mathematical difficulties and has to date not been completely solved. In this work, we present a semi-analytical method to evaluate some integrals of the Green's function used in the dynamic analysis of a rectangular plate resting on the surface of an elastic foundation of inertial properties (Lamb's model). The great challenge of this study is to overcome the problem of singularity present in the study of the Green's function related to this problem. The proposed solution to solve this task is the discretization of the studied system (rectangular plate resting on the surface of elastic foundation of inertial properties), which finally leads to a numerical solution in the matrix form. All the terms of the matrix are doubly indexed and the singularity is present in the terms having the same indices. Therefore, a special attention and an important efforts have been made to calculate the terms of the matrix with the same indices in order to get rid of the singularity. This imposed us to solve the integrals of the terms of the matrix with same indices analytically and the integrals of the terms of the matrix of different indices by numerical methods leading to approximate solutions. Finally, the analysis of the free vibration of the studied system is successfully accomplished by a semi-analytical method leading to determine the values of the Eigen-frequencies and the Eigen-shapes of the plate.

Keywords. Dynamic analysis; contact problems; rectangular plate; elastic foundation of Lamb's model; Green's function.

Integrable Dissipative Dynamical Systems: Approach and Applications

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Abstract. We study nonconservative systems for which the usual methods of the study, e.g., Hamiltonian systems, are inapplicable. Thus, for such systems, we must "directly" integrate the main equation of dynamics. We generalize previously known cases and obtain new cases of the complete integrability in transcendental functions of the equation of dynamics of a rigid body of different dimensions in a nonconservative force field.

Keywords. dissipative dynamical system, integrability, transcendental first integral.

MSC2010. 37C; 37E; 37L; 37N.

We obtain a series of complete integrable nonconservative dynamical systems with nontrivial symmetries. Moreover, in almost all cases, all first integrals are expressed through finite combinations of elementary functions; these first integrals are transcendental functions of their variables. In this case, the transcendence is understood in the sense of complex analysis, when the analytic continuation of a function into the complex plane has essentially singular points. This fact is caused by the existence of attracting and repelling limit sets in the system (for example, attracting and repelling focuses). We detect new integrable cases of the motion of a rigid body, including the classical problem of the motion of a multi-dimensional spherical pendulum in a flowing medium.

This activity is devoted to general aspects of the integrability of dynamical systems with variable dissipation. First, we propose a descriptive characteristic of such systems. The term "variable dissipation" refers to the possibility of alternation of its sign rather than to the value of the dissipation coefficient (therefore, it is more reasonable to use the term "sign-alternating") [1, 2].

We introduce a class of autonomous dynamical systems with one periodic phase coordinate possessing certain symmetries that are typical for pendulum-type systems. We show that this class of systems can be naturally embedded in the class of systems with variable dissipation with zero mean, i.e., on the average for the period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either energy pumping or dissipation can occur, but they balance to each other in a certain sense. We present some examples of pendulum-type systems on lower-dimension manifolds from dynamics of a rigid body in a nonconservative field.

Then we study certain general conditions of the integrability in elementary functions for systems on the two-dimensional plane and the tangent bundles of a one-dimensional sphere (i.e., the two-dimensional cylinder) and a two-dimensional sphere (a four-dimensional manifold). Therefore, we propose an interesting example of a three-dimensional phase portrait of a pendulum-like

system which describes the motion of a spherical pendulum in a flowing medium (see also [2, 3]).

To understand the difficulty of problem resolved, for instance, let us consider the spherical pendulum (ψ and θ — the coordinates of point on the sphere where the pendulum is defined) in a jet flow. Then the equations of its motion are

$$\ddot{\theta} + (b_* - H_1^*)\dot{\theta}\cos\theta + \sin\theta\cos\theta - \dot{\psi}^2\frac{\sin\theta}{\cos\theta} = 0, \tag{1}$$

$$\ddot{\psi} + (b_* - H_1^*)\dot{\psi}\cos\theta + \dot{\theta}\dot{\psi}\frac{1+\cos^2\theta}{\cos\theta\sin\theta} = 0, \ b_* > 0, \ H_1^* > 0, \tag{2}$$

and the phase pattern of the eqs. (1), (2) is on the Fig. 1.

Figure 1: Phase pattern of spherical pendulum in a jet flow

The assertions obtained in the work for variable dissipation system are a continuation of the Poincare–Bendixon theory for systems on closed two-dimensional manifolds and the topological classification of such systems.

The problems considered in the work stimulate the development of qualitative tools of studying, and, therefore, in a natural way, there arises a qualitative variable dissipation system theory.

Acknowledgments

References

[1] Shamolin M. V. *Dynamical Systems with Variable Dissipation: Approaches, Methods, and Applications*// Journal of Mathematical Sciences, 2009, vol. 162, no. 6, pp. 741–908.

[2] Trofimov V. V., Shamolin M. V. *Geometric and Dynamical Invariants of Integrable Hamiltonian and Dissipative Systems*// Journal of Mathematical Sciences, 2012, vol. 180, no. 4, pp. 365–530.

[3] Shamolin M. V. *Variety of Integrable Cases in Dynamics of Low- and Multi-Dimensional Rigid Bodies in Nonconservative Force Fields*// Journal of Mathematical Sciences, 2015, vol. 203, no. 1, pp. 379–530.

Solution of a multipoint control problem for a dynamical system in partial derivatives

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Abstract. The problem of building a control function for a linear dynamic system (to transfer it from any initial state to an arbitrarily given final state through any finite number of arbitrarily given control points) is solved. The cascade decomposition method is used. The criteria for complete controllability of such system are derived. The control and state functions are constructed in an analytical form with any desired degree of smoothness.

Keywords. Dynamic system with partial derivatives, multipoint conditions, complete controllability, control and state construction, analytical solution, cascade decomposition method.

MSC2010. 34H05; 49K15; 93C155.

1 Introduction

Here considered the system

$$\frac{\partial x}{\partial t} = B\frac{\partial x}{\partial s} + Du, \quad t \in T, \quad s \in S, \tag{1}$$

where $x = x(t,s) \in \mathbb{R}^n$, $u = u(t,s) \in \mathbb{R}^m$; $B: n \times n$; $D: n \times m$; $T = [t_0, t_k]$, $S = [0, \infty)$.

Special properties B and D are established, that such control $u(t,s)$ exist , under the influence of which, the trajectory $x(t,s)$ of system (1) (with these coefficients) leaves the arbitrary initial point $(t_0, \alpha_0(s))$, and arrives at an arbitrarily given point $(t_k, \alpha_k(s))$, are established. The trajectory is passing through an arbitrary set points $(t_i, \alpha_i(s))$ at arbitrary time moments t_i, $i = \overline{1, k-1}$, $t_0 < t_1 < ... < t_{k-1} < t_k$:

$$x(t_i, s) = \alpha_i(s), \ \forall \alpha_i(s) \in \mathbb{R}^n, \ i = \overline{0, k}. \tag{2}$$

In the case of the existence of the suitable $u = u(t,s)$, the system (1) is called completely controllable (c. controllable).

The problem of constructing the control $u(t,s) \in C^j(T \times S)$ and the suitable state $x(t,s) \in C^{j+1}(T \times S)$ in an analytical form with any desired $j \geq 0$ is formulated. Control of system (1) assumed to be programmed. The problem has of no unic solution. Here is determined which of the parameters $u(t,s)$ and $x(t,s)$, (or the combinations of parameters), can be set arbitrarily; that makes it possible to solve a wider range of problems (for example, the problem of suboptimal control).

2 Main results

One of the criteria for the controllability of the system (1) is established:

Theorem 1 *Theorem 1. The system (1) is completely controllable if and only if the system*

$$\frac{d\bar{x}}{dt} = B\bar{x}(t) + D\bar{u}(t), \quad t \in T,\ \bar{x}(t) \in \mathbb{R}^n,\ \bar{u}(t) \in \mathbb{R}^m,$$

is controllable (with the same B and D and with arbitrary conditions at the initial and final times t_0, t_k).

In particular, the system (1) with conditions (2) is c.controllable if and only if

$$\text{rank}(D\,BD\ldots B^p D) = n$$

with some $p \in \mathbb{N}$, $p \le n$.

In the proof of the theorem, the cascade decomposition method is used [1] - [3]. At each step of the decomposition the system

$$\frac{\partial x_j}{\partial t} = B\frac{\partial x_j}{\partial s} + D\frac{\partial u_j}{\partial s},\ j = \overline{1, p}, \tag{3}$$

is formed. Here $x = x_j(t, s)$, $u = u_j(t, s)$ — some parts of $x(t, s)$; B_j, D_j — some matrices. The additional requirements for the derivatives of the functions of pseudo-states $x_j(t, s)$ at the points t_i are arised. (This requirements follow from (2)). As a result at the last p-th decompositions step, the problem of finding pseudo-control $u_p(t, s)$ of system (3) with $j = p$, here $D_p = (0)$ or D_p — is a surjection, and $x_p = x_p(t, s)$ satisfies the conditions

$$\frac{\partial x^j}{\partial t^j}\Big|_{t=t_i} = \alpha^j_{p\,i}(s),\ j = \overline{0, p};\ i = \overline{0, k}, \tag{4}$$

with some $\alpha^j_{p\,i}(s)$. In the case of $D_p = (0)$ the system (3) with $j = p$ and conditions (4) has no solution. If D_p is surjective, then for any smooth $x_p(t, s)$,), satisfying conditions (4), there exists $u_p(t, s)$, satisfying equation (3) with $j = p$. Wherein $x_p(t, s) + u_p(t, s) = x_{p-1}(t, s)$. The $x(t, s)$ and $u(t, s)$ are found (in the reverse direction of decomposition) for system (1) with conditions (2). The smoothness of $x(t, s)$ and $u(t, s)$ depends only on the smoothness of an arbitrarily chosen $x_p(t, s)$, satisfying only conditions (4). Thus proved

Theorem 2 *Theorem 2. The system (1) is c. controllable if and only if $\exists p \in N$ is such that D_p - is surjection.*

Note that when solving practical problems there is no need to build subspaces; one can makes some substitutions for the variables, (that is described step by step in [3]).

References

[1] S.P. Zubova, E.V. Raetskaya, Le Hai Trung, On polynomial solutions of the linear stationary control system, Automation and Remote Control. **69**, No **11** (2008), 1852–1858.

[2] S.P. Zubova, Solution of Inverse Problems for Linear Dynamical Systems by the Cascade Method, Doklady Mathematics. **447**, No. **6** (2012), 599–602.

[3] S.P. Zubova and E.V.Raetskaya, Algorithm to solve linear multipoint problems of control by the method of cascade decomposition, Automation and Remote Control. **78**, No. **7** (2017), 1189–1202 .

Numerical Simulation of Electromagnetic Scattering Using Boundary Hypersingular Integral Equations Method

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Abstract. The problem of monochromatic electromagnetic waves scattering by a piecewise homogeneous bodies that can consist of domains with different dielectric properties and can contain ideally conducting inclusions in the form of solid objects and screens is considered. This problem is reduced to a system of boundary integral equations with hypersingular integrals. This equations are writing on surfaces separating the media with different dielectric properties and over surfaces of the ideally conducting objects. To solve the resulting boundary integral equations, a numerical scheme was constructed based on the methods of piecewise constant approximations and collocations.

Keywords. numerical methods, boundary integral equations; hypersingular integrals; electromagnetic scattering.

MSC2010. 65R20; 65N38; 78A45; 78M15.

The problem of monochromatic electromagnetic wave scattering by the system of dielectric medias and perfectly conducting bodies and screens is considered (see figure , at left). It is assumed that the all dielectric medias and external environment is homogeneous, isotropic and free of electric charges. Let Ω_1 is a domain occupied by the external environment, $\Omega_2, ..., \Omega_K$ – are domains occupied by the dielectrics medias. Each of the domains $\Omega_1, ..., \Omega_K$ can include ideally perfectly conducting bodies or screens.

The intensities of electric and magnetic fields are sought in the following form respectively: $\mathbf{E}(x)e^{-i\omega t}$, $\mathbf{H}(x)e^{-i\omega t}$, where ω is the angular frequency of electromagnetic field, t — time, $x = (x_1, x_2, x_3) \in R^3$ — points of space. It is assumed that the total field is induced by the given incident field so that the following representation for total electric field is satisfied in the external domain Ω_1: $\mathbf{E} = \mathbf{E}_{inc} + \mathbf{E}_{sc}$, $\mathbf{E}_{inc}$ – incident electric field, $\mathbf{E}_{sc}$ – unknown scattered electric field. The electric and magnetic fields should satisfy Maxwell equations (see [1]):

$$rot\,\mathbf{E} = i\mu_m\mu_0\omega\mathbf{H}, \quad rot\,\mathbf{H} = -i\varepsilon_m\varepsilon_0\omega\mathbf{E} \quad in\ \Omega_m, \quad m = 1, ..., K,$$

here μ_0 end ε_0 – the electric constant and magnetic constant, ε_m and μ_m – the relative permittivity end the relative permeability of the medium in domain Ω_m, respectively. The following boundary conditions must be satisfied: $\mathbf{E} \times \mathbf{n} = 0$ on surfaces of perfectly conducting bodies or screens, as well as on ideally-conducting parts of boundaries between medias(an ideally conductive screen is enclosed between different medias); $[\mathbf{E}^+ - \mathbf{E}^-] \times \mathbf{n} = 0$, $[\mathbf{H}^+ - \mathbf{H}^-] \times \mathbf{n} = 0$ on dielectric boundaries between medias. We pose the conditions of radiation at infinity in the Sommerfeld form for scattered field.

Let Ω_m be one of the domains occupied by the dielectric, $\partial\Omega_m$ — the boundary of Ω_m. The boundary $\partial\Omega_m$ is some total surface, which may contain surface $\Sigma_m^{(c)}$ – the total surface of perfectly conducting screens lying in this domain. Let $\partial\Omega_m'$ be the part of the surface $\partial\Omega_m$ without the surface $\Sigma^{(c)}$ (see figure at right). In each of the domains Ω_m, $m = 1, ..., K$, the electric field is sought as

$$\mathbf{E}(x) = \frac{i}{\omega\varepsilon_m}\mathbf{K}_m[\partial\Omega_m', \mathbf{j}_E^m](x) - \mathbf{R}_m[\partial\Omega_m', \mathbf{j}_M^m](x) + \frac{i}{\omega\varepsilon_m}\mathbf{K}_m[\Sigma_m^{(c)}, \mathbf{j}_E](x) + \delta_m^1\mathbf{E}_{inc}(x), x \in \Omega_m,$$

where $\mathbf{j}_E^m$ and $\mathbf{j}_M^m$ – is unknown tangent vector fields placed on the surface $\partial\Omega_m'$ on the side facing the region Ω_m (equivalent electric and magnetic currents, respectively), $\mathbf{j}_E$ – is unknown tangent vector field placed on the surface $\Sigma_m^{(c)}$, $\delta_m^j = 1$ for $m = j$, $\delta_m^j = 0$ for $m \neq j$,

$$\mathbf{K}_m[\Sigma, \mathbf{j}] = grad\, div \int_\Sigma \mathbf{j}(y)F_m(x-y)d\sigma_y + k^2 \int_\Sigma \mathbf{j}(\mathbf{y})F_m(x-y)d\sigma_y,$$

$$\mathbf{R}_m[\Sigma, \mathbf{j}](x) = \int_\Sigma rot_x[\mathbf{j}(y)F_m(x-y)]d\sigma_y, \quad F_m(x) = \frac{e^{ik|x|}}{|x|}, \quad k = \frac{\omega}{c_m}, \quad c_m = \frac{1}{\sqrt{\varepsilon_m\varepsilon_0\mu_m\mu_0}}.$$

Note that if surface Σ is the boundary between domains Ω_m and Ω_n, then two pairs of unknown currents are placed on it: $\{\mathbf{j}_E^m, \mathbf{j}_M^m\}$ and $\{\mathbf{j}_E^n, \mathbf{j}_M^n\}$.

The problem had reduced to the system of integral equations, which were written on each closed surface $\partial\Omega_m'$, and on each perfectly conducting surface $\Sigma_m^{(c)}$:

$$\frac{1}{2}\mathbf{j}_M + \mathbf{n} \times \left\{ \frac{i}{\omega\varepsilon_m}\mathbf{K}_m[\partial\Omega_m', \mathbf{j}_E] - \mathbf{R}_m[\partial\Omega_m', \mathbf{j}_M] + \frac{i}{\omega\varepsilon_m}\mathbf{K}_m[\Sigma^{(c)}, \mathbf{j}_E] \right\} = -\delta_1^m\mathbf{n} \times \mathbf{E}_{inc} \text{ on } \partial\Omega_m',$$

$$\mathbf{n} \times \left\{ \frac{i}{\omega\varepsilon_m}\mathbf{K}_m[\partial\Omega_m', \mathbf{j}_E] - \mathbf{R}_m[\partial\Omega_m', \mathbf{j}_M] + \frac{i}{\omega\varepsilon_m}\mathbf{K}_m[\Sigma^{(c)}, \mathbf{j}_E] \right\} = -\delta_1^m\mathbf{n} \times \mathbf{E}_{inc} \text{ on } \Sigma^{(c)} \subset \bar{\Omega}_m,$$

in the operator $\mathbf{K}_m$ integral should be understood in the sense of Hadamard finite value.

A numerical scheme for solving these integral equations based on the methods of piecewise constant approximations and collocations was constructed. For particular cases, the system of integral equations and the numerical method were constructed and described in detail in papers [2]-[3]. In the present work, the results for the general case are presented.

References

[1] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, SIAM, 2013.

[2] E.V. Zakharov, A.V. Setukha and E.N. Bezobrazova, Method of Hypersingular Integral Equations in a Three-Dimensional Problem of Diffraction of Electromagnetic Waves on a Piecewise Homogeneous Dielectric Body. Differential Equations, **51** (9), (2015), 1197–1210.

[3] A.V. Setukha, E.N. Bezobrazova, The method of hypersingular integral equations in the problem of electromagnetic wave diffraction by a dielectric body with a partial perfectly conducting coating. Russian Journal of Numerical Analysis and Mathematical Modelling **32**(6), (2017), 371–380.

Hyperbolic spherically symmetric first order vector equation of divergent type for a vector field in $\mathbb{R}^3$

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Abstract. It is described the class of evolutionary equations of the first order of divergent type for a vector field $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$, which are invariant relative to time $t \in \mathbb{R}$ and spatial translations. Moreover they are covariant relative to all group $\mathbb{O}_3$ transformations and hyperbolic according Fridrichs.

Keywords. quasilinear systems, hyperbolicity, vector field, covariance, symmetric tensors.

MSC2010. 35L40, 35L60.

1 Introduction

The report investigates the Friedrichs hyperbolicity (see, for example, [1]) of the class $\mathfrak{K}_1(\mathbb{R}^3)$ of vector quasi-linear equations of divergent type of the first order

$$\dot{v}_j(\boldsymbol{x}, t) = (\nabla_k S_{jk}[\boldsymbol{v}])(\boldsymbol{x}, t), \quad j = 1, 2, 3 \tag{1}$$

$\nabla_j \equiv \partial/\partial x_j, j = 1, 2, 3)$ that describes the evolution of the vector field $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$. It is supposed that each equation is invariant with respect to time and space $\mathbb{R}^3$ translations and it is covariantly transformed with the group $\mathbb{O}_3$ transformations. Here, the function $S_j k[\boldsymbol{v}]$ is a tensor function on the vector of $\mathbb{R}^3$ and it is used the rule of summing of all values of the repeated vector indices. The requirement of the equation (1) covariance with respect to the transformations of the group $\mathbb{O}_3$ means that the values of the function $S_{jk}[\boldsymbol{v}])(\boldsymbol{x}, t)$ are tensors of second rank in $\mathbb{R}^3$ at each point $\langle \boldsymbol{x}, t \rangle$.

The class $\mathfrak{K}_1(\mathbb{R}^3)$ is fully described by the formula

$$S_{jk}[\boldsymbol{v}] = f(\zeta)\delta_{jk} + g(\zeta)v_j v_k, \qquad \zeta = \boldsymbol{v}^2. \tag{2}$$

Let

$$\dot{u}_j(\boldsymbol{x}, t) = U_{jm}^{(k)}(\boldsymbol{v})\nabla_k u_m,$$

be the linearized vector equation for the variation $\boldsymbol{u}(\boldsymbol{x}, t) = \delta \boldsymbol{v}(\boldsymbol{x}, t)$ at the fixed vector $\boldsymbol{v}(\boldsymbol{x}, t)$ where According to the definition of the Fridrichs hyperbolicity (see, for example, [1], [2]) of quasi-linear systems, it is a positively definite symmetric matrix $U_{jk}^{(0)}$ for each of them such that all matrices $(U^{(0)}U^{(p)})_{jk}$ are symmetric.

2　Main result

We have proved the following statement

Theorem 1 *In order the quasi-linear equations* (1) *of the class* $\mathfrak{K}_1(\mathbb{R}^3)$ *have the Fridrichs hyperbolicity property where the class is described by the equation* (2) *with the functions* $f(\zeta)$ *and* $g(\zeta)$ *are arbitrary continuously differentiable,* $\zeta = \boldsymbol{v}^2$, *it is necessary and sufficient that* $g = hf'$ *where* h *is an arbitrary continuously differentiable, strictly positive function on* $\zeta \in [0, \infty)$ *and* f *is twice continuously differentiable.*

The proof consists of the construction a symmetric strictly positive matrix $U_{jl}^{(0)}$ such that the matrix $U_{jk} = \sum_{p=1}^{3} k_p U_{jl}^{(0)} U_{lk}^{(p)}$, $\boldsymbol{k} = \langle k_1, k_2, k_3 \rangle$ is the symmetric one. The matrix $U_{jl}^{(0)}$ is found in the form $U_{jl}^{(0)} = f^{(0)}(\zeta)\delta_{jl} + g^{(0)}(\zeta)v_j v_l$ with some continuously differentiable functions $f^{(0)}$ and $g^{(0)}$ on $\zeta = \mathbf{v}^2$. By the calculation of the expression $U_{jl}^{(0)} U_{lk}^{(p)}$ we find that it is symmetric only in the case $2f^{(0)}f' = g(f^{(0)} + \zeta g^{(0)})$. Further, it is found that the matrix $U_{jl}^{(0)} = f^{(0)}(\zeta)\delta_{jl} + g^{(0)}(\zeta)v_j v_l$ is strictly positive when the inequalities $f^{(0)} > 0$, $f^{(0)} + \zeta g^{(0)} > 0$ are fulfilled. It is done by the analysis of the correspondent spectral equation. Finally, we put $g = 2f'(1 + \zeta g^{(0)}/f^{(0)})^{-1}$. Since the functions $g^{(0)}$ and $f^{(0)}$ are arbitrary if the above pointed restrictions are taken in account then we introduce the positive function $h(\zeta) = 2(1 + \zeta g^{(0)}/f^{(0)})^{-1}$ on the basis of the functions $g^{(0)}$ and $f^{(0)}$.

References

[1] Majda A. The existence of multi-dimensional shock fronts // Memoirs of the American Mathematical Society. - 1983.- 43; 281.- P.1-94.

[2] Kulikovskii A.G., Slobodkina F.A. On the stability of one-dimensional stationary solutions of hyperbolic systems of differential equations containing points at which one characteristic velocity becomes zero // J. Appl. Math. Mech.- 1984.- 48;3.- P.414-419.

[3] Gantmakher F.R. Matrix theory / Moscow: Fizmatlit, 2004.- 560 p.

Bimodality of the probability distribution of the maximum of independent random variables

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Abstract. It is proven the violation of probability distribution unimodality of maximum samples of independent identically distributed random variables that have an arbitrary fixed volume and that have the Erlang probability of arbitrary order being common for them.

Keywords. equivalent independent random values, sample maximums, probability distribution, unimodality.

MSC2010. 60E05, 62E10.

1   Introduction

There are many problems of mathematical modeling when it is necessary to use a sample of independent equivalently distributed random values $\tilde{r}_1, ..., \tilde{r}_N$ in the model construction and in these cases it is necessary to analyze the probability distribution of sample extremums (see, for example, [1]). In particular, such a problem arises at the study of the probabilistic model of dielectric strength of many-layers enamel-lacquer ï¿½covers. In this model the random values $\tilde{r}_j$, $j = 1 \div N$ is some sizes of aerial inclusions into the cover. The local dielectric strength is a decreasing function of the maximum $\tilde{r} = \max\{\tilde{r}_j; j = 1 \div N\}$ of random sizes. In such a case it is necessary to study the probability distribution of the random value $\tilde{r}$ when $\tilde{r}_j$, $j = 1 \div N$ are some statistically independent and equivalently distributed random values with an arbitrary sample volume $N \geq 2$ that is not large. It is needed to find some important qualitative properties of the probability distribution $F_N(x) = \Pr\{\tilde{r} < x\}$ which one may watch at the statistical analysis. At this case the required properties of the $F_N(x)$ distribution should be investigated at wide mathematical suppositions about the common probability distribution $Q(x) = \Pr\{\tilde{r}_j < x\}$, $j = 1 \div N$ of random values. Moreover, since the sample volume is not definite, these qualitative properties do not depend on it.

One of the most important qualitative property of probability distributions which are watched in mathematical statistics is their unimodality. To the unimodality property of distributions in general case was devoted a considerable attention (see, for example, [2]). However, at some physical situations when one may not expect any surprises one may find the violation of the unimodality property. Just such a situation occurs in the case of enamel-lacquer covers statistics. Namely, the statistical analysis points out that there is the distribution bimodality of random dielectric strength. So, in this communication we set the problem of explanation of this effect from the view point of the probabilistic model based on sample maximums of $\tilde{r}_1, ..., \tilde{r}_N$.

2 Main result

Let $Q(x)$ be the common probability distribution of random positive values $\tilde{r}_1, ..., \tilde{r}_N$. We suppose that $Q(x)$ has the continuous density $q(x)$, $x \geq 2$, i.e. it is unimodal. Since $\tilde{r}_1, ..., \tilde{r}_N$ are statistically independent, $F_N(x) = \prod_{j=1}^{N} \Pr\{\tilde{r}_j < x\} = [Q(x)]^N$. In this case the distribution $F_N(x)$ has similarly continuous density $f_N(x)$, $x \geq 0$ which is given by the following formula

$$f_N(x) = Nq(x)Q^{N-1}(x).$$

It is necessary to set when the density $f_N(x)$ has more than one maximum. Suppose that the density $q(x)$ is continuously differentiable. Then the density $f_N(x)$ is also continuously differentiable and its derivative has the form $\dot{f}_N(x) = NQ^{N-2}(x)\big(\dot{q}(x)Q(x) + (N-1)q^2(x)\big)$, $N \geq 2$. The problem is reduced to determination those cases when the equation

$$\dot{q}(x)Q(x) + (N-1)q^2(x) = 0, \quad x > 0 \tag{1}$$

has more than one solution. We investigated the case when the distribution $Q(x)$ is the Erlang one with an arbitrary order $n \in \mathbb{N}$, i.e. it is the gamma-distribution with number order and it has been proved the statement.

Theorem 1 *If the probability distribution $Q(x)$ has the density*

$$q(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x), \qquad \lambda > 0$$

then the density $f_N(x)$ has two maximums.

References

[1] Gumbel E. Statistics of Extrems / New York: Columbia University Press, 1962.

[2] Lukacs E. Characteristic Functions, 2nd ed. / London: Griffin, 1970.

Stationary solutions of spherically symmetric Landau-Lifshitz ferrodynamics equation

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Abstract. The basic equation of ferrodynamics for pseudovector field $\boldsymbol{M}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$ is studied in spherically symmetric case. It is described the class of all one-dimensional solutions $\boldsymbol{M}(x)$, $x \in \mathbb{R}$. It is shown that this class consists of spiral structures and at the limit case it contains the constant ferromagnetic field $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{M}$.

Keywords. pseudovector field, Landau-Lifshitz' equation, stationary solutions, spiral structures.

MSC2010. 35K10, 35Q60.

1 Introduction

In the report it is analyzed stationary solutions of the Landau-Lifshitz equation (1)

$$\dot{\boldsymbol{M}}(\boldsymbol{x},t) = \gamma[\boldsymbol{M}, \Delta \boldsymbol{M}](\boldsymbol{x},t), \quad \gamma \in \mathbb{R}$$

which is the basic equation of ferrodynamics in spherically symmetric case. Here, $\boldsymbol{M}(\boldsymbol{x},t)$, $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$ is the pseudovector field that is the magnetic moment density from the physical viewpoint. This equation corresponds to the energy density $\big(\partial M_j/\partial x_k\big)\big(\partial M_j/\partial x_k\big)$ where we use the agreement about summing over all values of repeated indexes. We study the solutions of the equation $[\boldsymbol{M}, \Delta \boldsymbol{M}] = 0$ with supplement condition $\boldsymbol{M}^2(\boldsymbol{x},t) = M^2 = $ const because of all solutions of (1) possesses by such property if it is at the time moment $t = 0$. Thus, it is necessary to find all solutions of the equation $\Delta \boldsymbol{M} = \lambda \boldsymbol{M}$ for the field pseudovector $\boldsymbol{M}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$ where λ is a scalar field $\lambda(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$ and each solution under consideration satisfy the condition $\boldsymbol{M}^2(\boldsymbol{x},t) = M^2$. Only one of them satisfies the minimality energy condition, namely $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{M} = $ const.

2 Main result

We has been proved that all solutions which are satisfied conditions in one-dimensional case exhausts the spiral fields, i.e. they have the form $\boldsymbol{M}(\boldsymbol{x}) = M\big(\boldsymbol{n}_1 \cos(\boldsymbol{k}, \boldsymbol{x}) + \boldsymbol{n}_2 \cos(\boldsymbol{k}, \boldsymbol{x})\big)$ where $\boldsymbol{k}, \boldsymbol{n}_1, \boldsymbol{n}_2$ are three arbitrary mutually orthogonal vectors with $\boldsymbol{n}_1^2 = \boldsymbol{n}_2^2 = 1$.

References

[1] Landau L.D., Lifshitz E.M. To the theory of magnetic permeability of ferromagnetic media. // Collected works, v.1.- Moscow: Nauka, 1969.- P.128-135.

Second generation unit root tests for linear Panel data using Stata

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Abstract. Panel data refer to data containing time series observations of a number of individuals. However, panel data have a more complicated clustering structure than cross-sectional or time series data. Since the appearance of the paper by Levin and Lin (1992), the use of panel data unit root tests have become very popular among researchers. In this paper, we have tried to provide a summary of Second generation unit root tests for linear Panel data, including Pesaran (2007) and Pesaran-Smith-Yamagata (2009) and its application with stata.

Keywords. Data statistics.

MSC2010. 35R05.

References

[1] C. Hurlin, V. Mignon, Second Generation Panel Unit Root Tests, Working Paper, Hal, 2006.

[2] G. S. Maddala, S. Wu, A Comparative Study of Unit Root Tests with Panel Data and a New Simple Test, Oxford bulletin of economics and statistics, special issue, 1999.

Solving heat equiation using lattice neural network

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Abstract. Considered the solving of heat equation with neural network with the lattice architecture. Developed neuroequations, proved assertions about the output values of the neurons.

Keywords. heat equation; neural network.

MSC2010. 39A99

1 Introduction

It should be noted the special role of differential equations in solving many problems of mathematics, physics and engineering, because they do not always manage to establish a functional relationship between the unknown and the data variables, but it is often possible to derive a differential equation to accurately predict the course of a specific process under certain conditions. Differential equations are of great practical importance, being a powerful tool for exploring the many natural science and engineering tasks: they are widely used in mechanics, astronomy, physics, many problems of chemistry and biology. This is due to the fact that very often the laws that govern the various processes are written in the form of differential equations, and these equations, so are the means to quantify these laws. Of course, there is also a classical numerical methods. But there are situations where these methods may not lead to a solution, or it can be obtained for a very large number of iterations. Neural networks are much more flexible in this respect, and generally, an algorithm based on them is more efficient.

2 Main results

We sloving the heat equation by nueral network of lattice architecture.

We define the weights of links to be used in the network for solving the heat equation having the form:

$$\frac{dU}{dt} = a^2 \triangle U + f(x, y, z)$$

or in finite differences:

$$\frac{u_{i,k+1} - u_{i,k}}{t} = a^2 \frac{u_{i+1,k} - 2u^{i,k} + u^{i-1,k}}{h} + f_i$$

We choose a feedback weight:

$$w_{ii} = b - \frac{1 - b}{k_i k_i}$$

Connection weight between neurons of one layer:

$$w_{ij} = \frac{1 - b}{k_j} * \frac{2k_i}{k_i + k_j}, \text{ where } b \in [0; 1), k_i \text{ depends of environmental properties}$$

Proof. In the stable state of the neural network, for each neuron i the inequality will be satisfied

$|U_i^{t+1} - U_i^t| < \epsilon$, where U_i^t is the output value of the neuron i at time t, ϵ is arbitrarily small value.

So $\int_\Omega (\triangle U(x) + f(x))^2 dx \approx 0$ ∎

References

[1] N. Korsunov, V. Mikhelev, A. Lomakin, "Application of Lattice Neural Networks for Modeling of Stationary Physical Fields," Proceedings of the 2013 IEEE 7th International Conference on Intelligent Data Acquisition and Advanced Computing Systems, 2013, Vol. 1, pp. 369-372.

[2] Nguyen, D., and B. Widrow, "Improving the learning speed of 2-layer neural networks by choosing initial values of the adaptive weights," Proceedings of the International Joint Conference on Neural Networks, vol 3, pp. 21-26, 1990.

The solution of the problem of classification of human brain pathologies on MRI images

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Abstract. The article discusses the methods of classification of MRI images of the human brain. The methods are trained binary classifiers to determine the presence or absence of a high stage of brain pathologies. The study aims to identify the best classification method on the processed BRATS data set. The paper analyzes three classification methods: image classification by basic contour primitives, a method based on a convolutional neural network, with a binary classifier, and a classification method based on a convolutional pretrained neural network Xception. The paper shows that the use of pre-trained neural convolutional networks allows to reduce the time for calculating the time and resources of a computer system for training a neural network. The results of a computational experiment are presented and it is shown that the best accuracy in solving the problem of classifying pathologies of the human brain in MRI images is achieved using the Xception neural convolutional pre-trained neural network.

Keywords. machine learning; convolutional neural networks; medical images; neural networks.

MSC2010. 68T05, 68Q32

1   Introduction

Currently, according to the World Health Organization[1], 9.6 million people died of cancer in 2018. According to statistics, every sixth death in the world is a cancer disease. Cancer is the second leading cause of death in the world. So in comparison with 2008, the number of deaths from cancer in the world increased by about 28%. One of the main methods for detecting various brain tumors is magnetic resonance imaging, which allows you to create a detailed image of various structures (tissues, bones, etc.) of the studied area. Magnetic resonance imaging allows you to form a three-dimensional shape of the surveyed area, which consists of a set of ordered images (slices) along the axes X, Y, Z. This structure allows the specialist to isolate the vascular network, as well as individual nerve trunks, which allows you to establish the diagnosis in the early stages and promptly prescribe a course of treatment.

The data set used was a processed BRATS data set. Processing consists in the formation of images for the input layer of the neural network, since one MRI scan is a cube whose planes are slices (images). The formed training set is single-channel 8-bit images. It is important to note that for further training the pre-trained network Xception[4], images from 3 planes were used immediately, i.e. the network has input parameters for 299x299x3 images, where 299x299 is the resolution of the input image, and 3 are RGB levels, which are used for simultaneous loading from 3 planes.

2   Experiment

After processing the BRATS data, the formed training and test datasets totaled 2,200 images. Training and test data are 85 and 15 percent, respectively. It should be noted that the learning time of the neural network depends on the hardware capacity. The table 1 shows the results of accuracy of teaching methods and time spent on training.

Table 1: Comparison of methods

Method	Time for train	Accuracy %
contour based method[2]	10	64
convolutional neural network 2018[3]	25	74
current neural network	15	91

Acknowledgments

In this paper, a comparison and analysis of the use of three methods for the classification of MRI images of the human brain for the presence or absence of a high degree of gliomas was made. These are the method of image classification by basic contour primitives, a method based on a convolutional neural network with a binary classifier, and a classification method based on a convolutional pre-trained neural network Xception. The results of the computational experiment showed that the best accuracy in solving the classification problem is the presence or absence of a high degree of gliomas using the Neural Convolutional Pre-trained Neural Network Xception. The maximum accuracy of the classifier was 91% on the formed test sample. These results prove the need for and the possibility of developing additional software for diagnosing and analyzing MRI scans.

Further work is aimed at improving the accuracy of classification and segmentation of bad objects in the MRI image.

References

[1] Statistics of oncological diseases, - https://www.who.int/ru/news-room/fact-sheets/detail/cancer

[2] Miroshnichenko A.S., Mikhelev V.M. - "Method of image classification" // XIX International Conference "Computer Science: Problems, Methodology, Technologies" (IPMT-2019) and X School-Conference "Computer Science in Education" (INED-2019), 14-15 February, Voronezh.

[3] Miroshnichenko A.S., Mikhelev V.M. - "The method of recognition of objects in MRI images based on a convolutional neural network" // XVIII International Conference "Computer Science: problems, methodology, technologies" (IPMT-2018) and IXED – 2018, P. 181-185, 8-9 February, Voronezh.

[4] Francois Chollet, Xception: Deep Learning with Depthwise Separable Convolutions.2017, arXiv:1610.02357v3 [cs.CV].

THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A MULTI-TERM NONLINEAR SEQUENTIAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We study a new class of boundary value problems of the problem

$$
\begin{cases}
\mathcal{L}x(t) = f\left(t, x(t), (\Phi x)(t), (\Psi x)(t), D_{0^+}^{\beta_1} x(t), ..., D_{0^+}^{\beta_n} x(t)\right) & t \in [0,1], \\
\\
D_{0^+}^{\frac{(\alpha-\beta)}{2}} x(0) = 0, & \\
\\
a D_{0^+}^{\frac{(\alpha-\beta)}{2}} x(1) + x(\eta) = 0 & 0 < \eta < 1.
\end{cases}
\qquad (1)
$$

Some existence and uniqueness results are obtained by using standard fixed point theorems. An example is given to illustrate our results.

Keywords. Fractional integro-differential equation; sequential fractional derivative; fractional boundary conditions; fixed point theorems.

MSC2010. 34A08, 34B10, 34B15.

1 Introduction

Boundary value problems for fractional differential equations have been extensively studied in the recent years. The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions.

In [1] Ahmed Alsaedi and Bachir Ahmad investigated the existence of solutions for a nonlinear fractional integro-differential equations, with fractional nonlocal integral boundary conditions given by

$$
\begin{cases}
D^q x(t) + f(t, x(t), (\Phi x)(t), (\Psi x)(t)) = 0, & 0 < t < 1, 1 < q \le 2, \\
\\
D^{\frac{q-1}{2}} x(0) = 0, & a D^{\frac{q-1}{2}} x(1) + x(\eta) = 0, \quad 0 < \eta < 1,
\end{cases}
$$

where D^q denotes the Riemann-Liouville fractional derivative of order q, f:$[0;1] \times X \times X \times X \to X$

is a continuous, for $\gamma, \delta : [0,1] \times [0,1] \to [0,\infty)$,

$$(\Phi x)(t) = \int_0^t \gamma(t,s)x(s)ds, \quad (\Psi x)(t) = \int_0^t \delta(t,s)x(s)ds,$$

and $a \in \mathbb{R}$ satisfies the condition $a\Gamma(q) + \eta^{q-1}\Gamma(\frac{q+1}{2}) \neq 0$. Here, $(X, \|.\|)$ is a Banach space and $C = C([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|.\|$.

Very recently the authors Ying Wang and Lishan Liu in [15] have investigated existence and uniqueness of positive solutions for the following fractional integro-differential equation

$$\begin{cases} {}^C D^\alpha u(t) + f(t, u(t), Tu(t), Su(t)) = 0, \quad t \in [0,1], \\ \\ u(0) = b_0, \quad u'(0) = b_1, ..., u^{(n-3)} = b_{n-3}, \\ \\ u^{(n-1)}(0) = b_{n-1}, u(1) = \mu \int_0^1 u(s)ds, \end{cases}$$

where $n-1 < \alpha \le n$, $\quad 0 \le \mu < n-1$, $\quad n \ge 3$, $\quad b_i \ge 0 (i = 1, 2, ..., n-3, n-1)$, ${}^C D^\alpha$ is the Caputo fractional derivative. $f \colon [0;T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and

$$(Tu)(t) = \int_0^t K(t,s)u(s)ds, \quad (Su)(t) = \int_0^t H(t,s)u(s)ds.$$

Motivated by the above works, we consider in this paper the existence and uniqueness of solutions for the problem(1).

2 Main results

In this section, we give some existence and uniqueness results for the problem (1) and then give an example illustrating the usefulness of our results.
Let $C(I)$ be the space of all continuous real-valued functions on $I = [0,1]$ and

$$X = \left\{ x : x \in C(I) \quad and \quad D^{\beta_i}x(t) \in C(I), \quad 0 < \beta_i < 1, \quad i = 1, 2, ..., n \right\}$$

endowed with the norm $\|x\|_X = \max_{t \in I}|x(t)| + \Sigma_{i=1}^n \max_{t \in I} |D^{\beta_i}x(t)|$. It is known that $(X, \|.\|)$ is a Banach space.

$$\left| f(t, u, v, w, x_1, x_2, ..., x_n) - f(t, u', v', w', y_1, y_2, ..., y_n) \right|$$
$$\le \mu(t) \left(|u - u'| + |v - v'| + |w - w'| + |x_1 - y_1| + |x_2 - y_2| + ... + |x_n - y_n| \right),$$

for all $t \in [0,1]$ and $u, v, w, u', v', w', x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in \mathbb{R}$. Then problem (1) has a unique solution whenever $\Delta < 1$

Proof. For any $x, y \in X$ and for each $t \in [0,1]$, by using the Holder inequality, for $i = 1, 2, ..., n$. Hence, we get

$$\begin{aligned} \|Fx - Fy\| \quad \le \quad & \left\{ \frac{|1 + \lambda_1||k|}{\Gamma(\alpha - \beta + 1)} + \frac{|\lambda_1 ak|}{\Gamma(\frac{\alpha-\beta}{2} + 1)} + (1 + \gamma_0 + \lambda_0)\Delta_1 \right. \\ & + \sum_{i=1}^n \left(\frac{|k|}{\Gamma(\alpha - \beta - \beta_i + 1)} + \frac{|\lambda_1 ak|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2} + 1)\Gamma(\alpha - \beta_i)} \right. \\ & \left. \left. + \frac{|\lambda_1 k|\Gamma(\alpha)}{\Gamma(\alpha - \beta_i)\Gamma(\alpha - \beta + 1)} \right) \right\} \|x - y\| = \Delta \|x - y\|. \end{aligned}$$

Since $\Delta < 1$, F is a contraction mapping, therefore, by using the Banach contraction principle, F has a unique fixed point, which is the unique solution of problem (1).

References

[1] Ahmed Alsaedi, Bashir Ahmad,*Existence of solutions for nonlinear fractional integro-differential equations with three-point nonlocal fractional boundary conditions*. Hindawi Publishing Corporation Advances in Difference Equations,(2010), $1 - 10$.

[2] N. Adjeroud; Existence of positive solutions for nonlinear fractional differential equations with multi-point boundary conditions, The Australian journal of mathematical analysis and applications, Volume 14, Issue 2, article 5, pp. (2017), $1 - 14$.

Application of mathematical regulation methods to assess the optical state of urban air

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Abstract. In this paper the mathematical regulation methods to assess the optical state of urban air is applied. For this, the entropy prices, which are the indicator of the informativeness the optical state of the city air, have been used.

Keywords. urban air; regulation methods; entropy .

MSC2010. 94A12

This is an assessment of the optical state of the city air with mathematical regulation methods. Here, the optical position of the city air is investigated in different optical intervals and the various aspects of observation. We will use mathematical methods that are widely utilized in practice in public decision-making technologies. The essence of these methods is to determine the order of preference among the elements of a set of properties [1,2].

For the purpose of evaluating the optical state of the city air, a series of measurements are first done with the use of mobile media (pyrometer, actinometer, photocenter, etc.) to obtain reprinted information about the air pollution spatial change. The results are generated primarily by statistical data. It is known that the use of entropy indicators is a common approach to evaluating informativeness in many research areas. The process of obtaining any information can be explained as a result of uncertainty changes in transmitting signals. If any manifestation passes from a situation (eg the state of the situation), transitional information is understood to be the difference between these situations and the uncertainties [3]:

$$I\left(A, B\right) = H\left(A\right) - H\left(B\right),$$

here $H\left(A\right)$ and $H\left(B\right)$ in accordance with the entropy prices of manifestations "A" and B situations. As an example, the informality of the optical state of different destinations of Sumgayit city air has been assessed, and the importance of informativeness among the directions has been determined. The calculations use the Condorcet, Schulze, Borda, Copeland and Simpson rules of mathematical regulation.

References

[1] O.D. Kichmarenko and A.P. Ogulenko, Theory of decision making, ONU named after I.I. Mechnikov, 2012, p. 52.

[2] I.V. Kozin, Decision-making rules based on pair domination relation, Bulletin of Zaporizhzhya national university, Ukraine, **1** (2008), 97–104.

[3] V.N. Chernyshov and A.V. Chernyshov, Theory of Systems and Systems Analysis, Publishing House Tambov Sstate Tech. University, 2008, p. 96.

Geometric Interpretation of Solution of Differential Equation in Thermodynamics

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Abstract. We consider the problem of a wording of thermodynamic provisions for the spaces of ideal gas on the basis of analysis of solution of quasilinear partial differential equation of the first order using method of characteristics. Differential geometry tools and means of computer mathematics are also applied. The result is presented in the form of connection between physical content of thermodynamic values and their mathematical analogs. By numerical methods using the means of computer mathematics it is shown the possibility of establishing patterns of implementation of thermodynamic processes as functions of time.

Keywords. ideal gas; geometric interpretation; thermodynamic provisions.

MSC2010. 34M45; 34A26; 35R50; 35R15.

1   Introduction

As thermodynamics is a basis for many physical sciences its theory must be clear and logical. But nowdays the axiomatic creation of thermodynamics is not completed. The problem of entropy and its existence has various aspects of interpretation. And the acceptable solution is not found yet. The second problem is related with the proceeding of thermodynamic processes in time. Classical science does not give the answer to a quastion what is the place of time in the theory. So we try to establish the communication between the physical content of thermodynamic values and their mathematical analogs on the basis of application of methods of differential geometry and means of computer mathematics.

2   Main results

We consider a partial differential equation [2]:

$$\frac{v}{2c_p}\frac{\partial Q}{\partial v} + \frac{p}{2c_v}\frac{\partial Q}{\partial p} = T, \tag{1}$$

where Q - amount of heat is the physical quantity characterizing process of heat exchange between thermodynamic system and the environment. Temperature T is called the measure of a deviation of a condition of a studied thermodynamic system from a condition of a reference body in the standartized conditions. We assume that each state of ideal gas is unambiguously defined by values of specific volume v and pressure p which are presented by the parametrical equations concerning time τ: $v = v(\tau)$ and $p = p(\tau)$.

The solution $Q = Q(v, p)$ of equation (1) geometrically represents a surface in three-dimentional space (v, p, Q) which is called an integrated surface. We use method of characteristics which are defined by the system of ordinary differential equations [1]:

$$2c_p \frac{dv}{v} = 2c_v \frac{dp}{p} = \frac{dQ}{T} = ds, \tag{2}$$

where s - any real parameter, changing along the characteristic curve.

From the first two equations (2) the dependence for the value ds turns out. It has a form of thermodynamic equation which is used in definition of entropy. Thus, in geometric representation entropy is an arch length for the characteristic curves corresponding to the field of directions defined by system (2).

The integrated solution of the equation (1) can be found in an analitical way. The general solution of equations (2) concerning entropy has a look:

$$v = v_l \exp\left(\frac{s}{2c_p}\right); p = p_l \exp\left(\frac{s}{2c_v}\right); Q = Q_l + c_p \beta_1 \frac{p_l v_l}{R_i} \left(\exp\left(\frac{s}{c_p \beta_1}\right) - 1\right); \beta_1 = \frac{2c_v}{c_p + c_v}.$$

We set a curve of any process l in a parametrical form concerning time τ: $v_l = v_l(\tau)$, $p_l = p_l(\tau)$, $Q_l = Q_l(\tau)$. Similar results can be received by numerical methods, using means of computer mathematics.

The integrated surface of the equation (1) can be covered by the collection of characteristics. The functions $f_1 = \frac{v}{2c_p}, f_2 = \frac{p}{2c_v}$, $f_3 = T$ define a field of directions in space (v, p, Q). In each point of this space there is a direction which directional cosines are propotional to f_1, f_2, f_3. So in each point of integrated surface vector determined by the field of directions stated above have to be in the tangent plane to this surface.

We find a projection l_0 of a curve of process l on vOp plane. For the equation (1) Cauchy problem in vOp plane is formulated in a form: to find the integrated surface passing through the curve l in the neighborhood of l_0. Geometric interpretation of Cauchy problem in a space (v, p, Q) assumes that through each point of process l it is necessary to carry out characteristic of the equation (1) and "to stick together" the integrated surface from them.

Let's assume on vOp plane the projection l_0 passes from the point A_0 to the point B_0. Through these points characteristics, isoterms and adiabatic curves pass. The collection of characteristics is described by the equation $p = C_1 v^k$ (k is adiabatic degree), the collection of isoterms – $pv = C_2$, the collection of adiabatic curves – $pv^k = C_3$. The entropy in such representation will be the characteristic and s – its arc length. Adiabatic curves will be the lines of level for the characteristics at $s = const$. Adiabatic curves, isoterms and characteristics form network of curvilinear not ortogonal coordinates on the vOp plane.

The geometric solution of Cauchy problem in (v, p, Q) space can be constructed as follows. Through any point on vOp plane characteristic is carried out until it crossing with projection l_0. Then it is necessary to put $Q = Q_l(\tau)$ taking into account the parametrical equations of process l. The integrated surface will characterize amount of heat for all set of conditions of ideal gas in the neighborhood of process l or its projection l_0.

References

[1] Koshlyakov I.S., *Partial Differential Equations of Mathematical Physics*, Moscow, High School, 1970.

[2] Gennadiy V. Averin, Anna V. Zviagintseva, Maria V. Shevtsova and Liliana N. Kurtova, Justification of Provisions of Thermodynamics by Methods of Differential Geometry of Multidimensional Spaces. Journal of Engineering and Applied Sciences. **12** (2017), 7830–7835.

Asymptotics of one-dimensional random walks with finite velocity

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Abstract. We consider.

Keywords. random walks; asymptotic analysis; integral equations, integral transforms, probability theory

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

We consider the problem of random walks with finite velocity, described by the integral equation (in one-dimensional form)

$$\psi(x,t) = \frac{v}{2}\int\limits_0^t [\psi(x-v\tau, t-\tau) + \psi(x+v\tau, t-\tau)]p(v\tau)d\tau + \frac{\delta(x-vt)+\delta(x+vt)}{2}P(vt) \quad (1)$$

where $p(\xi)$ is probability density on $[0,+\infty)$ and $P(\xi) = \int\limits_\xi^{+\infty} p(\eta)d\eta$ - survival probability function. Asymptotic solution of this equation at $t \to +\infty$ were investagated in several works, for example [1, 2, 3], but some cases of $p(\xi)$ wasn't studied as well. We also investigate cases $t \to 0$ and behaviour of $\psi(x,t)$ at $x \to vt$. The problem of rate of convergence are discussed briefly.

2   Main results

We obtain new asymptotics solution $\psi(x,t)$ in case of $t \to +\infty$ for functions $p(\xi)$ of the form

$$p(\xi) = C\xi^{-2}(1 + O(\xi^{-1})) \quad (2)$$

$$p(\xi) = C\xi^{-1}\ln^{-n}(\xi)(1 + O(\ln^{-1}\xi)) \quad (3)$$

For $p(\xi) = C\xi^{-\alpha}(1 + O(\xi^{-1}))$ with $2 > \alpha > 1$ the rates of convergence were studied.

Asymptotic behaviour of $\psi(x,t)$ near the "ballistic cone" $x = vt$ and at $t \to 0$ were studied using collision decomposition technique.

For certain $p(\xi)$ new exact solutions were found, for example for gamma-distribution $p(\xi) = \mu^k\Gamma^{-1}(k)\xi^{n-1}e^{-\mu\xi}$ and $p(\xi) = \mu^k\xi^{\alpha-1}E_{\alpha,\alpha}(-\mu\xi^\alpha)$ where $E_{\alpha,\beta}(x)$ - Mittag-Leffler function.

References

[1] V. V. Uchaikin and R. T. Sibatov, *Fractional Kinetics in Space: Anomalous Transport Models*, World Scientific, Singapore, 2018.

[2] V. V. Uchaikin and R. T. Sibatov, One-dimensional fractal walk at a finite free motion velocity, Technical Physics Letters **30** (2004), 316–318.

[3] R. T. Sibatov and V. V. Uchaikin, Statistics of photocounts of blinking fluorescence of quantum dots, Nonlinear and Quantum Optics **108** (2010), 761–767.

EXISTENCE AND UNIQUENESS FOR MULTI-TERM SEQUENTIAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract. In this paper we investigated a new kind of non-local multi-point boundary value problems of Caputo type sequential fractional integro-differential equations. Some new existence and uniqueness results are obtained by using some standard tools of fixed point theory. Examples are given to illustrate the results.

Keywords. Sequential fractional derivative, multi-point, Caputo fractional derivative, fixed point theorem, fractional differential equation.

MSC2010. 34A12; 34B10; 34B15.

1 Introduction

we study the existence and uniqueness of solutions for the nonlinear fractional integro-differential equation with m-point multi-term fractional integral boundary conditions.

$$
\begin{cases}
(^cD^q + k\,^cD^{q-1})u(t) = f(t, u(t), (\phi u)(t), (\psi u)(t), ^cD^{\beta_1}u(t), \ldots, ^cD^{\beta_n}u(t)), t \in [0,1] \\
\\
u(0) = 0, \quad \sum_{i=1}^{m-1} a_i u(\xi_i) = \beta \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} u(s)ds,
\end{cases}
\tag{1}
$$

where $^cD^q$ is the standard Caputo fractional derivative of order q, with $1 < q \leq 2, 0 < \beta_i < 1, k > 0, 0 < \eta < \xi_1 < \xi_2 < \ldots < \xi_{m-1} < 1, \beta, a_i, i = 1, \ldots, m$ are real constants, $f : [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ is continuous and for the mappings $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ with the property $\sup_{t \in [0,1]} |\int_0^t \lambda(t,s)ds| < \infty$ and $\sup_{t \in [0,1]} |\int_0^t \gamma(t,s)ds| < \infty$, the maps ϕ and ψ are defined by $(\phi u)(t) = \int_0^t \gamma(t,s)u(s)ds$ and $(\psi u)(t) = \int_0^t \lambda(t,s)u(s)ds$.

2 Main results

The main results concerning the existence and uniqueness of solution for the problem (**??**). Observe that problem (**??**) has solutions if the operator has fixed point.

Theorem 1 *Assume that $f : [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ be a continuous function satisfying the assumption*

(H_1) $|f(t,x,y,w,u_1,u_2,\ldots,u_n) - f(t,x',y',w',v_1,v_2,\ldots,v_n)|$

$\leq L_1|x-x'| + L_2|y-y'| + L_3|w-w'| + d_1|u_1-v_1| + d_2|u_2-v_2| + \ldots + d_n|u_n-v_n|,$

for all $t \in [0,1]$ and $x,y,w,x',y',w',u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n \in \mathbb{R}$.

where $L_i, d_j > 0, \forall i = 1,2,3, \forall j = 1,2,\ldots,n$ are Lipschitz constants.

*Then problem (**??**) has a unique solution if $\left(\Lambda_1 + \Lambda_2 \sum\limits_{i=1}^{n} \dfrac{1}{\Gamma(2-\beta_i)} \right) \omega < 1, , \zeta_1 = \sup\{L_1, d_1, d_2, \ldots, d_n\}, \zeta_2 = \sup\{L_2, L_3\}, \zeta = \sup\{\zeta_1, \zeta_2\}.$*

Proof.

$$\|F(u) - F(v)\| \;\leq\; \left(\Lambda_1 + \Lambda_2 \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)} \right) \omega \|u - v\|.$$

As : $\left(\Lambda_1 + \Lambda_2 \sum\limits_{i=1}^{n} \dfrac{1}{\Gamma(2-\beta_i)} \right) \omega < 1$, F is a contraction. Thus the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. . ∎

References

[1] Agarwal, R.P., Meehan, M. and OâRegan, D. *Fixed Point Theory and Applications* Cambridge University Press Beijing World Publishing Corporation.

[2] Ahmad, B, Sivasundaram, S: *On four-point nonlocal boundary value problems of nonlinear integro-diï¬erential equations of fractional order Appl. Math. Comput.* 217, 480 − 487(2010)

Application of the Fourier Method for the Numerical Solution of Stochastic Differential Equations

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Abstract. It is well known, that Ito stochastic differential equations (SDEs) are adequate mathematical models of dynamic systems under the influence of random disturbances. One of the effective approaches to numerical integration of Ito SDEs is an approach based on Taylor-Ito and Taylor-Stratonovich expansions. The most important feature of such expansions is a presence in them of so called iterated Ito or Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Ito SDEs. We successfully use the tool of generalized multiple Fourier series, built in the space L_2, for the mean-square approximation of iterated stochastic integrals.

Keywords. iterated Ito stochastic integral; iterated Stratonovich stochastic inegral; Taylor-Ito expansion; strong approximation; numerical modeling.

MSC2010. 60H10; 60H05; 60H35; 65C30.

1   Introduction

Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a fixed probability space and $\mathbf{W}_t$ — is F_t-measurable $\forall t \in [0, T]$ Wiener process with independent components $\mathbf{W}_t^{(i)}$; $i = 1, \ldots, m$. Consider an Ito SDE:

$$d\mathbf{X}_t = \mathbf{a}(\mathbf{X}_t, t)dt + B(\mathbf{X}_t, t)d\mathbf{W}_t, \ \mathbf{X}_0 = \mathbf{X}(0, \omega), \ \omega \in \Omega, \tag{1}$$

where $\mathbf{a} : \Re^n \times [0, T] \to \Re^n$, $B : \Re^n \times [0, T] \to \Re^{n \times m}$ satisfy to standard conditions of existence and uniqueness of strong solution $\mathbf{X}_t \in \Re^n$ of the SDE (1); $\mathbf{X}_0$ and $\mathbf{W}_t - \mathbf{W}_0$ $(t > 0)$ — are independent. In theorems 1 – 3 we solve the problem of combined mean-square approximation of stochastic integrals from Taylor-Ito and Taylor-Stratonovich expansions for the prosess $\mathbf{X}_t$.

2   Main results

Theorem 1 [1, p. 252]. *Assume, that $\psi_i(\tau) \in C_{[t,T]}$ $(i = 1, \ 2, \ldots, k)$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a complete orthonormal system of continuous functions in $L_2([t, T])$. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\mathrm{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \left(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k} \prod_{s=1}^{k} \phi_{j_s}(\tau_{l_s})\Delta\mathbf{W}_{\tau_{l_s}}^{(i_s)} \right),$$

where $J[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \ldots \int_t^{t_2} \psi_1(t_1)d\mathbf{W}_{t_1}^{(i_1)} \ldots d\mathbf{W}_{t_k}^{(i_k)}$ (iterated Ito stochastic integral), $\Delta\mathbf{W}_{\tau_j}^{(i)} = \mathbf{W}_{\tau_{j+1}}^{(i)} - \mathbf{W}_{\tau_j}^{(i)}$ $(i = 0, \ 1, \ \ldots, m)$, $\mathbf{W}_\tau^{(0)} = \tau$, $\zeta_j^{(i)} = \int_t^T \phi_j(\tau)d\mathbf{W}_\tau^{(i)}$ — are independent

standard Gaussian random variables for various i or j (if $i \neq 0$), $\{\tau_j\}_{j=0}^{N-1}$ — is a partition of $[t,T]$, satisfying to conditions: $t = \tau_0 < \ldots < \tau_N = T$, $\max_{0 \leq j \leq N-1}(\tau_{j+1} - \tau_j) \to 0$ if $N \to \infty$, $C_{j_k \ldots j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k$, $K(t_1, \ldots, t_k) = \mathbf{1}_{\{t_1 < \ldots < t_k\}} \psi_1(t_1) \ldots \psi_k(t_k)$ $(t_1, \ldots, t_k \in [t,T])$, $\mathbf{1}_A$ — is an indicator of the set A, $\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k$, $\mathrm{L}_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N-1; l_g \neq l_r (g \neq r); g, r = 1, \ldots, k\}$, $\mathrm{H}_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N-1\}$, l.i.m. — is a limit in the mean-square sense.

Consider particular cases of the theorem 1 for $k = 2, 3, 4$ [1, pp. 261-262]:

$$J[\psi^{(2)}]_{T,t} = \sum_{j_1,j_2=0}^{\infty} C_{j_2 j_1}(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2 \neq 0, j_1=j_2\}}),$$

$$J[\psi^{(3)}]_{T,t} = \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3 j_2 j_1}(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2 \neq 0, j_1=j_2\}} \zeta_{j_3}^{(i_3)} -$$
$$-\mathbf{1}_{\{i_2=i_3 \neq 0, j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3 \neq 0, j_1=j_3\}} \zeta_{j_2}^{(i_2)}),$$

$$J[\psi^{(4)}]_{T,t} = \sum_{j_1,\ldots,j_4=0}^{\infty} C_{j_4 \ldots j_1}(\prod_{l=1}^{4} \zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_1=i_2 \neq 0, j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3 \neq 0, j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_1=i_4 \neq 0, j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3 \neq 0, j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_2=i_4 \neq 0, j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} -$$
$$-\mathbf{1}_{\{i_3=i_4 \neq 0, j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_2 \neq 0, j_1=j_2\}} \mathbf{1}_{\{i_3=i_4 \neq 0, j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3 \neq 0, j_1=j_3\}} \mathbf{1}_{\{i_2=i_4 \neq 0, j_2=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_4 \neq 0, j_1=j_4\}} \mathbf{1}_{\{i_2=i_3 \neq 0, j_2=j_3\}}).$$

Consider some estimates for the mean-square error of approximation, based on the theorem 1.

Theorem 2 [1, pp. 500-501]. *Under conditions of the theorem 1:*

$$\mathsf{M}\{(J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} - J[\psi^{(k)}]_{T,t})^2\} \leq k!(\int_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1,\ldots,j_k=0}^{p_1,\ldots,p_k} C_{j_k \ldots j_1}^2)$$
$$(i_1, \ldots, i_k = 0, 1, \ldots, m \text{ and } 0 < T - t < 1 \text{ or } i_1, \ldots, i_k = 1, \ldots, m \text{ and } 0 < T - t < \infty)$$
$$and$$
$$\mathsf{M}\{(J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k} - J[\psi^{(k)}]_{T,t})^2\} = \int_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k - \sum_{j_1,\ldots,j_k=0}^{p_1,\ldots,p_k} C_{j_k \ldots j_1}^2$$
$$(i_1, \ldots, i_k = 1, \ldots, m \text{ and pairwise different}; 0 < T - t < \infty),$$

where $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ is a truncated series from the theorem 1 with upper limits $p_1, \ldots, p_k$, and M — is a mathematical expectation.

The following theorem adapts the theorem 1 for iterated Stratonovich stochastic integrals.

Theorem 3 [1, pp. 284-428]. *Let function $\psi_2(s)$ — is continuously differentiated at $[t,T]$ and functions $\psi_1(s)$, $\psi_3(s)$ — are two times continuously differentiated at $[t,T]$; $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a complete orthonormal system of Legendre polynomials or trigonometric functions in $L_2([t,T])$. Then*

$$J^*[\psi^{(2)}]_{T,t} = \operatorname*{l.i.m.}_{p_1,p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \quad J^*[\psi^{(k)}]_{T,t} = \operatorname*{l.i.m.}_{p \to \infty} \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_k}^{(i_k)},$$

where $J^*[\psi^{(k)}]_{T,t} = \int_t^{*T} \psi_k(t_k) \ldots \int_t^{*t_2} \psi_1(t_1) d\mathbf{W}_{t_1}^{(i_1)} \ldots d\mathbf{W}_{t_k}^{(i_k)}$ (iterated Stratonovich stochastic integral); $k = 3, 4, 5$ (for $k = 3$: $i_1, i_2, i_3 = 1, \ldots, m$; for $k = 4, 5$: $i_1, \ldots, i_5 = 0, 1, \ldots, m$ and $\psi_1(\tau), \ldots, \psi_5(\tau) \equiv 1$); the meaning of notations from the theorem 1 is remained.

References

[1] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practic of Numerical Solution. With MATLAB Programs. 6th Ed. (In Russian). *Electronic Journal "Differential Equations and Control Processes"*. 2018, no. 4. pp. A.1–A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

Mathematical Model of Learning System

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Abstract. The paper considers a mathematical model of a learning system. The mathematical model contains components that describe the learning objectives, content component, teacher's activity within the system, organization of the learning process, student's activity within the system, statistics gathering and analysis component, control component and optimization component. The model considers the new educational process components, such as the variety of competencies formed in the student as a result of taking the course. The didactic units of the course and the evaluative knowledge background are discussed. Operations with the courses are introduced that make it possible to create integrated courses and distinguish the general part. Statements about the integrated courses are considered. It turns out that creating an integrated course based on some course is equivalent to creating an integrated course based on the very first course, which means that it doesn't matter what course to take as a basis and add elements to.

Keywords. learning system; integrated course; operations of union and intersection.

MSC2010. 26A33; 34A60; 34G25; 93B05.

1   Introduction

Presently, computer technology-based learning systems have gained wide acceptance. Distance learning has become ingrained in the educational practice; learning management systems have been created in order to support it.

The mathematical model of the learning system can be represented as follows:

$$URS = \langle K, C, PP, PO \rangle, \tag{1}$$

where K – set of training courses within the learning system, C – set of competences formed in the students as a result of taking the system's courses, PP – set of the teacher's specializations, PO – set of the specializations of the students within the system.

Let the set of the training courses

$$K = \{K_i\}, \quad i = 1, \dots, n, \tag{2}$$

where K_i – the training course represented by the tuple

$$K_i = \langle G_i, CT_i, EM_i, Q_i \rangle, \quad i = 1, \dots, n, \tag{3}$$

where G_i – set of competences formed as a result of taking the course, CT_i – course content graph, EM_i – course training content, Q_i – evaluative means background.

The training content graph can be represented as follows:

$$CT_i = \langle CTN_i, CTR_i \rangle, \tag{4}$$

where CTN_i – set of the graph vertices, each corresponding to a didactic unit of a given course, CTR_i – set of the graph edges, reflecting the connections between the didactic units of the course. Element $(ctn_{ir}, ctn_{it}) \in CTR_i$ indicates the presence of a direct connection between the didactic units ctn_{ir} and ctn_{it}.

2   Main results

The thesis determines the elementary operations on the courses.

The operation of union (uniting) of the courses is determined as follows:

$$K_{un} = K_i \cup K_j = \langle G_i \cup G_j,\ CT_i \cup CT_j,\ EM_i \cup EM_j,\ Q_i \cup Q_j \rangle. \tag{5}$$

The operation is implemented as uniting of the respective sets for the model's components that represent the sets. The operation of union of the training courses is used when creating integrated training courses, as well as when expanding the content of the existing course. For the training content graphs, the operation is performed based on the definition of graph union operation as follows:

$$CTN_{un} = CTN_i \cup CTN_j,$$

$$CTR_{un} = CTR_i \cup CTR_j,$$

where CTN_i, CTN_j – sets of vertices of the respective graphs, CTR_i, CTR_j, – sets of edges of the respective graphs.

Based on the course union operation, the concept of an integrated course has been introduced.

Integrated training course $K_{int}(K_i, K_j)$, created based on the course K_i by addition of the course K_j, will be the course obtained with the help of the operation of union of the courses K_i, K_j.

Statement 1. The integrated course $K_{int}(K_i, K_j)$, created based on the course K_i by addition of the course K_j, coincides with the integrated course $K_{int}(K_j, K_i)$, created based on the course K_j, by addition of the course K_i, that is $K_{int}(K_i, K_j) = K_{int}(K_j, K_i)$.

The operation of intersection of the courses is determined as follows:

$$K_{int} = K_i \cap K_j = \langle G_i \cap G_j,\ CT_i \cap CT_j,\ EM_i \cap EM_j,\ Q_i \cap Q_j \rangle.$$

For the components of the model G_i, EM_i, Q_i the operation of intersection is the intersection of the respective sets. Intersection of the components CT_i and CT_j is determined based on the graph intersection operation.

Statement 2. The course intersection operation possesses the properties of commutativity and associativity.

References

[1] A. A. Bessonov, V. Ya. Mamaev and P. P Paramonov The intellectual training systems. GUAP (Sankt-Petersburg) (2016), 172.

Statistical mechanics of dilute vector model

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Abstract. It is analyzed the dilute vector model of statistical mechanics. The model is based on the Gibbs' probability measure of random field on the lattice $\mathbb{Z}^3$. It is proposed the approach for investigation of the correspondent probability distribution by the method that is analogous to the group expansion in the gas theory.

Keywords. Gibbs' distribution, lattice systems, dilute vector model, random vector field.

MSC2010. 82B20.

1 Introduction

The dilute vector model of statistical mechanics is under study. It is one of the classical lattice models which are investigated in frames of statistical mathematical physics (1). The model is defined by the ordered triple $\langle \Xi, \mathsf{H}, \mathsf{P} \rangle$ where Ξ is the phase space, H is the functional on Ξ which is named the model hamiltonian. The space Ξ consists of all pairs $\langle \tilde{c}(\boldsymbol{x}), \boldsymbol{n}(\boldsymbol{x}) \rangle$, $\boldsymbol{x} \in \mathbb{Z}^2$ where $\tilde{c}(\boldsymbol{x})$ is the random dichotomic functions which take values $\{0, 1\}$ at all points $\boldsymbol{x} \in V$, V is a finite part of the crystal lattice $\mathbb{Z}^3$. The $\boldsymbol{n}(\boldsymbol{x})$ is a unimodal vector field on the set $\{\boldsymbol{x} : \tilde{c}(\boldsymbol{x}) = 1\}$ that is generated by the function $\tilde{c}(\boldsymbol{x})$ such that $\boldsymbol{n}^2(\boldsymbol{x}) = 1$. Each value of the functional $\mathsf{H}[\tilde{c}, \boldsymbol{n}]$ for the pair $\langle \tilde{c}(\boldsymbol{x}), \boldsymbol{n}(\boldsymbol{x}) \rangle$ is defined by the formula

$$\mathsf{H}[\tilde{c}, \boldsymbol{n}] = -\sum_{\boldsymbol{x} \in V} \tilde{c}(\boldsymbol{x})(\boldsymbol{\mu}, \boldsymbol{n}(\boldsymbol{x})) + \frac{1}{2} \sum_{\langle \boldsymbol{x}, \boldsymbol{y} \rangle \in V^2} U(\boldsymbol{x} - \boldsymbol{y})\tilde{c}(\boldsymbol{x})\tilde{c}(\boldsymbol{y})\big(\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{y})\big) \tag{1}$$

where $U(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{Z}^3$ is real-valued function that is named the interaction potential. At last, P is the Gibbs probability measure defined by the formula for its differential

$$d\mathsf{P}[\tilde{c}, \boldsymbol{n}, d\boldsymbol{n}(\boldsymbol{x})] = \frac{1}{Z} \exp\Big(-\mathsf{H}[\tilde{c}, \boldsymbol{n}]/T\Big) \prod_{\boldsymbol{x} : \tilde{c}(\boldsymbol{x})=1} d\Omega(\boldsymbol{n}(\boldsymbol{x})) \tag{2}$$

where T is statistical temperature, $d\Omega(\boldsymbol{n}(\boldsymbol{x}))$ are measures on unit spheres $\mathfrak{S}$ in $\mathbb{R}^3$ each of which corresponds to the point $\boldsymbol{x} \in \mathbb{Z}^3$ and is connected with the direction $\boldsymbol{n}(\boldsymbol{x})$,

$$Z = \sum_{\{\tilde{c}(\boldsymbol{x})\}} \prod_{\boldsymbol{x} : \tilde{c}(\boldsymbol{x})=1} \int_{\mathfrak{S}^n} \exp\Big(-\mathsf{H}[\tilde{c}, \boldsymbol{n}]/T\Big) \prod_{\boldsymbol{x} \in : \tilde{c}(\boldsymbol{x})=1} d\Omega(\boldsymbol{x}).$$

The described system of statistical mechanics models some different statistical physical systems, for example, it models the nematic liquid crystal in lattice approximation (2).

2 Main result

We calculate the equation $P(\rho, T, \boldsymbol{\mu})$ of system state which is determined as the function on T and $\boldsymbol{\mu}$

$$P(\rho, T, \boldsymbol{\mu}) = \lim_{|V| \to \infty} |V|^{-1} \ln Z,$$

in the form of series that is analogous to the so-called group expansion. This expansion is given by the following formula

$$P(\rho, T, \boldsymbol{\mu}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{X_{n-1} \subset (\mathbb{Z}^3)^{n-1}} \int_{\mathfrak{S}^n} \sum_{\mathfrak{G}^{(n)}} \times$$

$$\times \left[\prod_{\{j,k\} \in \mathfrak{G}^{(n)}} \left(\exp(-U(\boldsymbol{x}_j - \boldsymbol{x}_k)(\boldsymbol{n}(\boldsymbol{x}_j) \cdot \boldsymbol{n}(\boldsymbol{x}_k))) - 1 \right) \right] \prod_{j=1}^{n} d\Omega(\boldsymbol{x}_j).$$

Here, $X_{n-1} = \langle \boldsymbol{x}_1, ..., \boldsymbol{x}_{n-1} \rangle$ and the summing $\sum_{\mathfrak{G}^{(n)}}$ is done over all connected graphs with n enumerated vertices and the internal production $\prod_{\{j,k\} \in \mathfrak{G}^{(n)}}$ is fulfilled over the edges of fixed graph $\mathfrak{G}^{(n)}$.

References

[1] Ruelle D. Statistical Mechanics. Rigorous Results.- New York-Amsterdam: W.A.Benjamin, 1969.

[2] Khachaturian A.G., Badalian D.A. Theory of phase transition from the isotropic liquid phase to the nematic one with account of correlations between molecules orientations. Cryatalography.- 1977.- 22;4.- P.677-685.

Choosing the duration of the audio communication session

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Abstract. The article presents the research results on the development of the decision-making support algorithm for choosing the duration of audio communication sessions, which allows transmission of audio streams with a given quality level in a mobile ad-hoc network. The proposed algorithm implementation will allow the ad-hoc node to display a message about the recommended audio communication session length.

Keywords. decision-making supporting algorithm, duration of the audio communication session, mobile ad-hoc network.

MSC2010. 68T01; 68T50.

1   Introduction

In the process of conducting search and rescue operations and emergency situations countermeasures, the great importance acquires the forces and means management organization through the provision of effective information exchange. At the same time, tasks are often carried out under the conditions of an inoperable (damaged) or missing telecommunications infrastructure and possible destructive external influences. To transfer information in such situations, it is preferable to deploy a mobile ad-hoc network (MANET), which is capable of operating without base stations and has an arbitrary decentralized topology [1-4]. MANET belongs to the class of packet data networks. With the packet transmission of audio streams, slight losses of individual packets are allowed. Lost voice information can be recovered using an approximation based on previous and subsequent data. However, significant packet loss cannot be recovered using this method. This leads to an unacceptable reduction in audio communication quality. The main causes of packet loss in MANET are network nodes random movement, their on and off switching, as well as the bit errors occur due to the low power of transmitting devices. Under conditions of significant packet loss, it is recommended to limit the length of transmitted messages or reduce the length of audio streaming sessions.

The purpose of the research presented in the article is to provide the choice of the audio communication sessions duration, with which it is possible to transfer an audio stream with a given quality level in a MANET. To achieve this goal, it is required to develop a decision-making supporting algorithm on the choice of audio streams transmission duration in a mobile ad-hoc network.

2 Main results

Assume that the audio stream consisting of a packets is transmitted with acceptable quality if no packets were lost in the delivery process or the data of all lost packets were successfully restored. In this case, to recover the loss of one packet at the receiving node, you need to get at least b previous packets and at least b subsequent packets. Also, are given the following values: P is the required probability of audio stream transmitting with acceptable quality; L is packet bit length; C is information bit rate; t is required to develop an algorithm for choosing the value T is the duration of the audio stream transmission. The audio stream transmission duration depends on the value a is the number of packets in a stream. The recommended value of a can be obtained based on the mathematical models, which describes the average volume of a multimedia message which can be transmitted with the required quality with the specified operation characteristics of the MANET [5].

Thus, in the article was developed an algorithm for estimating the recommended duration of audio stream transmission in a MANET. The calculation of this value is based on mathematical models, which allow obtaining the value of packets number with which the audio communication session will be carried out with acceptable quality.

As a result of the algorithm execution, the message with the recommended length of the audio communication session will be displayed on the ad-hoc node screen.

The direction of further research will be the development of software for implementing the proposed algorithm and obtaining experimental results.

References

[1] Cheong S.H., Lee K.I., Si Y.W., U L.H., 2011. Lifeline: Emergency Ad Hoc Network. Proc. of 7th International Conference Computational Intelligence and Security (CIS). Hainan: 283-289.

[2] Polshchykov K.O., Lazarev S.A., Zdorovtsov A.D., 2017. Limitary request queue choice mathematical model for the real time streams transfer by means of the mobile ad hoc network radio channel. Journal of Fundamental and Applied Sciences, **9**(7S): 1317-1327.

[3] Basagni S., Conti M., Giordano S., Stojmenovic I., 2004. Mobile Ad Hoc Networking. IEEE Press, 461.

[4] Polshchykov K.A., Lazarev S.A., Zdorovtsov A.D., 2018. Evaluation of channel accessibility waiting period prediction channel to transmit real-time streams in mobile ad hoc network. Journal of Advanced Research in Dynamical and Control Systems, **10** (4): 1460-1467.

[5] Polshchykov K.O., Lazarev S.A., Kiseleva E.D., 2018. Mathematical Model of Multimedia Information Exchange in Real Time Within a Mobile Ad Hoc Network. International Journal of Computer Science and Network Security, **18** (6): 20-24.

On attractors of GRN networks
with triangle regulatory matrix

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Abstract. We consider simplified model of genetic regulatory network with a triangle regulatory matrix W. We study asymptotic properties of the respective system of ordinary differential equations and focus on the structure of attractors. We conclude that attractors are a set of stable critical points and no critical points of the type focus are possible.

Keywords. dynamical systems, gene regulatory networks, qualitative analysis.

MSC2010. 34C10; 34D45; 92C42.

1   Introduction

We are concerned with the system

$$
\begin{cases}
\dfrac{dx_1}{dt} = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 + ... + w_{1n}x_n - \theta_1)}} - x_1, \\[2mm]
\dfrac{dx_2}{dt} = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{22}x_2 + ... + w_{2n}x_n - \theta_2)}} - x_2, \\[1mm]
... \\[1mm]
\dfrac{dx_n}{dt} = \dfrac{1}{1 + e^{-\mu_n(w_{n1}x_1 + w_{n2}x_2 + ... + w_{nn}x_n - \theta_n)}} - x_n,
\end{cases}
\tag{1}
$$

that is supposed to model interrelation between elements of a gene regulatory network ([2]) and hopefully will help to construct a system of pathways in telecommunication networks([1]). The regulatory matrix $W = (w_{ij})$ describes relations between elements of a network. An element w_{ij} describes the influence of the element x_j on the element x_i. There are three types of relations, namely, activation (w_{ij} is positive), no relation (w_{ij} is zero) and inhibition (w_{ij} is negative). Often only three values are used, -1, 0 and 1. A number of authors use the continuous scale that allows to measure intensity of interreltions.

In what follows we consider the general system of n differential equations, but with the specific triangle matrix

$$
W = \begin{vmatrix}
w_{11} & w_{12} & ... & w_{1n} \\
0 & w_{22} & ... & w_{2n} \\
... & & & \\
0 & 0 & ... & w_{nn}
\end{vmatrix}
\tag{2}
$$

Constants w_{ij} take values in $[-1,1] \setminus 0$. The system (1) takes the form

$$
\begin{cases}
x_1' = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 + \ldots + w_{1n}x_n - \theta_1)}} - x_1, \\
x_2' = \dfrac{1}{1 + e^{-\mu_2(\phantom{w_{11}x_1 + }w_{22}x_2 + \ldots + w_{2n}x_n - \theta_2)}} - x_2, \\
\ldots \\
x_n' = \dfrac{1}{1 + e^{-\mu_n(\phantom{w_{11}x_1 + w_{12}x_2 + \ldots + }w_{nn}x_n - \theta_n)}} - x_n.
\end{cases}
\tag{3}
$$

2 Main results

Critical points of system (3) are to be determined from

$$
\begin{cases}
x_1 = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 + \ldots + w_{1n}x_n - \theta_1)}}, \\
x_2 = \dfrac{1}{1 + e^{-\mu_2(\phantom{w_{11}x_1 + }w_{22}x_2 + \ldots + w_{2n}x_n - \theta_2)}}, \\
\ldots \\
x_n = \dfrac{1}{1 + e^{-\mu_n(\phantom{w_{11}x_1 + w_{12}x_2 + \ldots + }w_{nn}x_n - \theta_n)}}.
\end{cases}
\tag{4}
$$

Since the right sides in (4) are positive but less than unity, all critical points locate in the cube $(0;1) \times (0;1) \times \ldots \times (0;1)$.

Lemma 1 *There are at most three values x_n for the critical points of the system (3).*

Theorem 2 *The system (3) has at most n^3 critical points. Any critical point has the following characteristic numbers:*

$$
\begin{cases}
\lambda_1 = -1 + \mu_1 w_{11} g_1, \\
\lambda_2 = -1 + \mu_2 w_{22} g_2, \\
\ldots \\
\lambda_n = -1 + \mu_n w_{nn} g_n,
\end{cases}
\tag{5}
$$

where g_i are computable values depending on the coordinates of a critical point.

Corollary 3 *The system (3) cannot have critical points of the type focus.*

References

[1] Y. Koizumi et al., Adaptive Virtual Network Topology Control Based on Attractor Selection, Journal of Lightwave Technology, **28 (06/2010)**, Issue 11, (2010), 1720 − 1731. DOI:10.1109/JLT.2010.2048412

[2] H. de Jong , Modeling and simulation of genetic regulatory systems: a literature review, J Comput Biol. **9(1)** (2002), 67–103.

On Building a Model of a Protected Virtual Environment for Information Cooperation of Scientific and Educational Organizations

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Abstract. The paper deals with the construction of the set-theoretic model of a protected virtual environment of information interaction of scientific and educational organizations, based on description of the relationship of entities and the rules of information exchange. The composition, structure of classes and their hierarchy are described, which make it possible to present, using the set theory and predicate calculus, the object model of the organizational-technical association of subjects of information interaction within the framework of a single management model and security policy.

Keywords. secure virtual environment; object system model; description of object models.

MSC2010. 03E75; 03B80; 68U35; 94A62.

1 Introduction

Currently, the dominant methodology used in developing applications and information systems is an object-oriented approach. In this paper, the authors consider the entity relationship model when describing the virtual environment of information interaction between scientific and educational organizations. The main link of interaction is the portal, which is considered as a set of interrelated sections (access objects) that have a hierarchical tree structure of subordination. For each section of the portal, only one access rule can be assigned to the entire branch of the object tree due to the inheritance mechanism of the child access rights of the parent object. The inheritance mechanism can be disabled for the child object and new access rules are assigned, which will be inherited by its descendants. The set theory and mathematical logic are used as a mathematical tool for describing the set-theoretic model of a protected virtual environment of information interaction of scientific and educational organizations [1, 2].

2 Main results

Let P denote a vector consisting of sets of portal objects, and O a vector of sets of subjects and entities of information interaction within associations, consortia or network structures of scientific and educational organizations. The protected virtual environment of their interaction is a set $S = \{O, P\}$, and the set of objects of the vector O is an injection into P: $O \rightarrow P$.

The authors identified the main elements of the vectors O and P: $O = \langle O_i \rangle$, $i = \overline{1,7}$, $P = \langle P_j \rangle$, $j = \overline{1,10}$, where O_1– regulating governmental organs, O_2– external subjects, O_3– associations,

O_4- members of the association, O_5- units of the association, O_6- employees of the division of the association, O_7- security policy of the association, P_1- institution, P_2- management server, P_3- access control server, P_4- users domain, P_5- portal resource, P_6- access sections, P_7- access level group, P_8- access permissions, P_9- account, $P_{10}-$ request to the site.

The elements of the vectors O and P are formed, which are sets with a description of their structural elements and methods of their information interaction. Structural elements are basically also vectors consisting of vectors or typed data sets. For example, for $O_4 = \langle O_{4i} \rangle$, $i = \overline{1,5}$, where $O_{41}-$ is the vector with the names of the members of the association, $O_{42}-$ is the vector with the identifier of the member of the association, $O_{43}-$is the vector with the legal address, $O_{44}-$is the vector with bank details, $O_{45}-$is the vector with the role of the member of the association.

The execution of session requests by authentication in the portal can be formally defined as $P = \langle A, C, D \rangle$, where $A = \{a_i\}-$ is a set of access control nodes ($A \in P_3$), $i = \overline{1,n}$; $C = \{c_j\}-$ set of network control nodes ($C \in P_2$), $j = \overline{1,m}$; $D = \{d_i\}-$ is a set of user domains in the network ($D \in P_4$). A user domain is a uniquely named user group, which is represented by a tuple: $d_i = \langle U_i, D_i' \rangle$, where U_i- the set of users of the i-th domain, $i = \overline{1,n}$, $D_i'-$ is a subset of domains in a network with domain trust relationships d_i, $D_i' \subseteq D$ and $d_i \in D_i'$. Each user domain corresponds to a specific network access control node and conversely, $A \leftrightarrow D$. Each user domain d_i has a subset of currently logged in users $U_i' \subset U_i$. The network management node is a tuple $c_i = \langle S^0, R^0, D \rangle$, where S^0- the set of all active user sessions of the portal network; R^0- the set of all identified requests on the network. An access control node can be represented as a tuple: $a_i = \langle S_i, R_i, D_i' \rangle$, where S_i- is a set lot of active users of access control session, R_i- a set of identified requests to an access control node. Session access model:

$$\forall a_i : \exists s_{ik} \in S_i, u_{qz}' \in U_q', d_q \in D_i' \Rightarrow T : s_{ik} \to u_{qz}, ' \tag{1}$$

where a_i- access control node, $u_{qz}'-$ the z-th authorized user of the domain d_q session s_{ik}, $q = \overline{1,n}$, $T-$ the matching function of the user and his session.

A user request of a secure virtual information interaction environment is considered identified when it is possible to determine its initiator based on condition (1) by the active session:

$$\forall r_{ikx} \in R_i : \exists s_{ik} \in S_i \Rightarrow E : r_{ikx} \to s_{ik}, \tag{2}$$

where $r_{ikx}-$ x-th request to the i-th node containing the label k of the user session, $E-$ the function of identifying the user session on request.

Thus, to identify a user session (2), it is necessary and enough to have session data only on the access control node processing the request.

The authors obtained the relationship for all elements of the vectors O and P and their components.

References

[1] S.A. Lazarev and A.V. Demidov, The Concept of Construction of a Control System of an Information Exchange in The Network of Corporate Portals, Information Systems and Technologies, **4(60)** (2010), 123–129.

[2] S.A. Lazarev, I.S. Konstantinov and O.V. Mihalev, Realization of a single model session access in the distributed network portals, Vestnik komp'iuternykh i informatsionnykh tekhnologii (Herald of computer and information technologies), **6** (2014), 44–49.

Two dimensional autonomous dynamical system modeling the thermal breakdown of thin semiconductor film

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Abstract. It is analyzed the two-dimensional dynamical system that describes the generation of thermal breakdown in thin semiconductor film included in electrical circuit. On the basis of the system analysis it is calculated the time during thermal breakdown occurs and the radius of melted channel in the film which a breakdown result.

Keywords. autonomous dynamical systems, phase plane, saddle point, instability.

MSC2010. 70K05.

1   Introduction

We follow the theory of thermal breakdown proposed in [1] from the physical viewpoint, but we use more simple mathematical model that was analyzed in [2]. The model is based on supposition that thermal breakdown is generated due fluctuations over thermal equilibrium state connected with the temporally changing temperature distribution $T(\boldsymbol{x}, t)$ in the film if they exceed a threshold value. When such a fluctuation realized in the film, it increases very fast due rapidly increasing dependence of the electrical conductivity $\sigma(T)$ on temperature of the film material. At this, the temperature dependence of thermal conductivity $\varkappa(T)$ of the material is more slow. Therefore, in the nonlinear evolution equation arises the aggravation evolution due to a positive feedback. It is modeled the thermal breakdown since through a short time interval the temperature attains locally the melting temperature of the material in so-called thermal channel in the film which has small radius.

In the proposed approach we describe the thermal breakdown generation from a unique fluctuation and we construct the dynamical system in terms of two variables: $\Theta(t)$ is the amplitude of the temperature fluctuation, $r(t)$ is the radius of cylindrical thermal channel which arises in the thin semiconductor film due to the fluctuation. The resulting two-dimensional dynamical system based on some physical arguments has the form:

$$c\rho \frac{d\Theta(t)}{dt} = E^2 \Theta(t)\left(\sigma_1 + \frac{1}{2}\sigma_2\Theta(t)\right) - \frac{2}{r^2(t)}\Theta(t)\left(\varkappa_0 + \frac{1}{2}\varkappa_1\Theta(t)\right), \tag{1}$$

$$c\rho \dot{r}^2(t) = \left(\varkappa_1 - r^2(t)\frac{\sigma_2}{2}E^2\right)\Theta(t) \tag{2}$$

where c, ρ, E are some physical values which characterize the material properties and the experimental conditions. Here, we use the simplest nonlinearities in differential equations choosing above pointed out dependencies in the form

$$\varkappa(T_0 + \Theta) = \varkappa_0 + \varkappa_1 \Theta, \quad \sigma(T_0 + \Theta) = \sigma_0 + \sigma_1 \Theta + \frac{1}{2}\sigma_2 \Theta^2.$$

The problem consists of the setting when the aggravation mode in this dynamical system arises and to calculate the important breakdown characteristics.

2 Main result

The system (1), (2) is integrable by quadratures but its solution are not expressed by elementary functions. However, for the breakdown characterization it is sufficient the qualitative analysis of the system on its phase plane (Θ, r^2). Fixed points of the system fill the axe $\Theta = 0$. It is fulfilled the analysis of the following linearized system near the axe $\Theta = 0$

$$c\rho\delta\dot{\Theta}(t) = \left(E^2 \sigma_1 - \frac{2\varkappa_0}{\zeta r^2}\right)\delta\Theta(t) = \sigma_1 E^2 \left(1 \mp \frac{r'^2_*}{r^2}\right)\delta\Theta(t),$$

$$c\rho\delta\dot{r}^2(t) = \left(\frac{2}{\zeta}\varkappa_1 - r^2 E^2 \sigma_2\right)\delta\Theta(t) = \sigma_2 E^2 (r^2_* - r^2)\delta\Theta(t),$$

$$\frac{2}{\zeta E^2}\frac{\varkappa_1}{\sigma_2} = r^2_*, \quad \frac{2}{\zeta E^2}\frac{\varkappa_0}{|\sigma_1|} = r'^2_*.$$

where the signs (-) and (+) correspond to $\sigma_1 > 0$ and $\sigma_1 < 0$. Fixed points with $r < r_*$ are the saddle ones at $\sigma_1 < 0$, but $\delta\Theta(t) \to 0$, $\delta r^2(t)$ increases and they are not describe the breakdown. At $\sigma_1 > 0$ there are two different situations: at $r'^2_* > r^2_*$, all point $(0, r^2)$ with $r < r_*$ are saddle and $\delta\Theta(t) \to 0$ for them, but at $r'^2_* < r^2_*$, the points $(0, r^2)$ with $r < r'_*$ have the same property and the points with $r'^2_* > r^2_*$ are unstable, for which $\delta\Theta(t)$, $\delta r^2(t)$ are the increasing functions. The latter correspond to the thermal breakdown. At this, the threshold temperature is defined by the expression

$$\theta_* = \left(\frac{\sigma_1}{\sigma_2} - \frac{\varkappa_0}{\varkappa_1}\right) > 0$$

if it is positive.

References

[1] Virchenko Yu.P., Vodyanitskii A.A. Heat localization and formation of heat breakdown structure in semiconductor materials. I. Nonlinear model // Functional Materials. – 2001. – 8;3. – P.428-434.

[2] Andreyeva N.V., Virchenko Yu.P. Nonequilibrium Thermodynamics of Thermal Breakdown Generation, in Nonequilibrium Phenomena / Plasma Combustion Atmosphere. ed. G.D.Roy, S.M.Frolov, A.M. Starik. – 2009. – P.176-182.

About numerical approximation of diffusion problems in a porous medium

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Abstract. This paper is devoted to the numerical solution of new mathematical models describing the convective-diffusion propagation of a liquid in a porous medium. To describe the process we use homogenized equations, which are based on the basic laws of continuum mechanics. We consider the different regimes of motion, depending of the behaviour of small parameters arising in problem. The main goal is to show the difference in the difficulty of finding the unknown values for the homogenized and non-homogenized models.

Keywords. Numerical methods, diffusion, convection.

MSC2010. 76R10

1 Introduction

Currently, there are many mathematical models to describe the convective-diffusion propagation of a liquid in a porous medium on a macroscopic level. These models are generally based on Darcy's Law and various modifications of the diffusion equation [1]-[3], where the basic parameters have unclear physical meaning and differential equations derived on the basis of speculative conclusions. Correction coefficients are introduced to increase the stability of difference schemes for numerical simulations.

We propose a numerical approximation of the new mathematical models described in [4], where, first of all, the process is described at the microscopic level using the fundamental laws of continuum mechanics. Then the mathematical model is simplified and the exact asymptotic approximation is derived, which adequately describes the physical process under consideration at a macroscopic level.

2 Main results

Efficient and accurate numerical simulation on micro and macroscopic scales is fundamental to understanding the basic physical features of the diffusion process in porous media. There are many difference methods to solve such problems. Finite-difference methods are widely used because of their simplicity and efficiency on structured grids. However, in the presence of small geometric characteristics, their direct application leads to unrealistic demands on computer resources. This is especially noticeable in calculations in the area of several tens (hundreds)

meters. And in order to take into account the scale of pores of several tens of microns in three-dimensional modeling, you need about a hundred petabytes of RAM. Therefore, for an accurate (microscopic) model, numerical calculations make sense only in the two-dimensional case.

All calculations are based on special numerical methods. As a rule, these methods rely on decomposing the computational domain into elementary subregions, each of which can be loaded into a number of processor modules that share the same area of RAM. Then, when this domain decomposition is done, you can use various technologies to solve the problem numerically: explicit and implicit difference schemes, finite elements, an alternating Schwartz method. The key point of the approach is the use of different scales in different sub-areas.

It is well known that a combination of different grid scales produces numerical artifacts and the quality of the numerical method used is determined by the amplitude of the artifacts. In order to guarantee a reasonable level of numerical artifacts, we used the "intellectual" version of high-quality interpolation based on the fast Fourier transform with a relative error of approximately 0.0001 - 0.001. The obtained results allowed us to estimate the selected set of small parameters, scale them and make the appropriate changes in the original models in order to preserve the basic physical properties of the reservoir flows.

It should be noted that the numerical solution of the homogenized (macroscopic) problem requires significantly less computational power in the absence of the mentioned geometric characteristics. It is revealed that the homogenization characteristics differ in different sub-areas of the reservoir, therefore, it required the use of different spatial grids in the area [5].

Acknowledgments

References

[1] P. M. Bommer and R. S. Schechter, Mathematical modeling of in-situ uranium leaching. SPE Journal. **19**, (1979), 376–390.

[2] R. D. Schmidt and S. E. Follin and K. A. Peterson and E. V. Level, Geochemical kinetics model for in-situ leach mining. SPE Journal. **198**, (1981), 1140–1146.

[3] L. N. Krichevets, *Mathematical models and programs for hydrogeological and geotechnological calculations on computers*, MGRI, Moscow, 1987.

[4] A. M. Meirmanov and O. V. Galtsev and R. N. Zimin, *Free boundaries in Rock Mechanics*, De Gruyter, Berlin, 2017.

[5] C. Chardaire-Riviere and G. Chavent and J. Jafre and J. Liu, Multiscale representation for simultaneous estimation of relative permeabilities and capillary pressure. Society of Petroleum Engineers. **20**, (1990), 303–312.

The Use of Description Logic in The System-Objective Modeling of Organizational Knowledge

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Abstract. This article provides a brief description of the "Union-Function-Object" system-object approach, proposes a new way to formalize the "Union-Function-Object" three-element construction (UFO-element) using descriptive logic (DL). Considered the possibility of using DL for representing systems in an unambiguous, formalized form. Such logics combine rich expressive capabilities and relatively low computational complexity. DL uses the concepts of individual, concept and role. The basic theoretical concepts of description logic (DL) are described on the basis of the $\mathcal{ALC}$ DL and it's extension $\mathcal{ALCOQ}$. A definition of each part of the UFO element in form of $\mathcal{ALCOQ}$ DL was given. The described method of system-object knowledge modeling allows to obtain logical chains of concepts and connect them with roles.

Keywords. system-object approach; formalization; element "Union-Function-Object"; $\mathcal{ALC}$ and $\mathcal{ALCOQ}$ descriptive logic.

MSC2010. 62B10; 93C62; 93A30; 94A15.

1   Introduction

Nowadays, systems theory (general or abstract) is at the stage of forming its foundation. The existing theoretical constructions, no matter how they were called, do not constitute a complete scientific theory. This situation encourages researchers to propose various options for a systematic approach and use them to create a theory of systems. One of the variants of such a systematic approach, based on the fact that there is no set-theoretic system theory, is the "Union-Function-Object" system-object approach. The system-object UFO-approach is described by the following main points [1].

The system is represented as *a functional object*. The function of this object is due to the function of an object of a higher tier (super-system). The phenomenon of conditionality of a function of a system as a function of a supersystem is considered as a functional request of a supersystem for a system with a certain function - *an external determinant of the system*. It is the purpose of existence and the cause of the system, i.e. *universal system-forming factor*, as it determines the structural, functional and substantial properties of the system. The functioning of the system under the influence of the external determinant is its *internal determinant* and establishes between the system and the super-system *the relation of maintaining the functional ability of a more whole*. The process of moving of the internal determinant of the system to its external determinant is considered as an adaptation of the system to the supersystem request.

In addition, it is assumed that each system is necessarily connected to other systems. These links are flows of elements of the deep tier of related systems. The links between the subsystems of the system - supporting, the links between this system and external systems - are functional.

$$s_i = [(L_i?, L_i!); fs(L_i?)L_i!; (O_i?, O_i!, O_if)] \tag{1}$$

Expression (1) is one of the main ways of formalizing the UFO-approach using the Abadi-Kardeli calculus objects. In this calculus, an abstract object is a collection of methods and fields. $L_i?$ - field of a special object for describing the set of incoming interface flows corresponding to the incoming connections of the system s_i; $L_i!$ - field special object to describe the set of outgoing interface flows; fs - a special object method describing the function of the system s_i, i.e. process of converting incoming interface flows (system inbound) $L_i?$ in the outgoing $L_i!$. In accordance with the accepted notation in the theory of objects. $O_i?$ - a set of fields that contains the interface input characteristics of a special object (system s_i), $O_i!$ - a set of fields, which contains the interface output characteristics of a special object (system s_i), O_if - a set of fields, which contains the transfer characteristics of a special object (system si). At the same time, there are many fields for describing the object characteristics of the system $O_i = O_i? \cup O_i! \cup O_if$.

In the same time, description logic and an ontological approach can be used to describe information system architectures [1]. Descriptive logic is a family of languages for the formal description of knowledge. In this sense, the use of DL allows you to describe the elements of the system with some logical expressions. From the point of view of the authors of this article, the use of DL to create a new way of formalizing the system-object approach is a promising direction.

2   Main results

Descriptive logic combine rich expressive capabilities and relatively low computational complexity. DL uses the objects of *individual*, *concept* and *role*. One of the most well-known and basic DL is the $\mathcal{ALC}$ logic and it's extension $\mathcal{ALCOQ}$. $\mathcal{ALCOQ}$ logic concept syntax is following [2]:

$$\top|\bot|A|\neg C|C \sqcap D|C \sqcup D|\exists R.C|\forall R.C|\forall R| \le NR| \ge NR|\exists R|\{a\} \tag{2}$$

Next, we describe correspondence between UFO-elements and expressions of $\mathcal{ALCOQ}$ descriptive logic (2). So, the system s_i formulates as following expression:

$$s_i = U_i \sqcap F_i \sqcap \exists R_p.O_i, \tag{3}$$

where $U_i = (L_i? \sqcup L_i!) \sqcap \exists R_{in}.L_i? \sqcap R_{out}.L_i!$ and $F_i = L_i! \sqcap \exists R_f.L_i?$ and $O_i = \{a_1\} \sqcap ... \sqcap \{a_n\}$. R_{in}, R_{out}, R_f - roles that shows relationship between concepts.

The formalization (3) of the UFO-element using the $\mathcal{ALCOQ}$ logic allows us to simulate the nodal, functional and substantive (object) characteristics similar to (1). Consideration of the UFO-element in the form of the intersection of composite concepts (concept-object O_i, concept-function F_i, concept-node U_i). Such formalization makes it possible to describe the system as a whole in the form of expressions of the form (1).

References

[1] V. V. Mikhelev, S. I. Matorin, Formalization of System-Objective Representation of Knowledge With The Use of Descriptional Logic. - Scientific Gazette of BSU. Ser. Economy. Computer science. - 2019. - 2. Volume 46 (VAK). (accepted for printing).

[2] F. Baader, D. Calvanese, L. McGuinness, D. Nardi, P. F. Patel-Schneider, The Description Logic Handbook: Theory, Implementation, and Applications. - Cambridge University Press: 576. - 2003.

Solving leukocytes classification task on blood cell images with computational intelligence

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Abstract. This article describes a research that examined the leukocytes classification task on images of blood cells. At the initial stage of this research, the problem of solving the problem of leukocyte classification task in the hematological analysis of blood was identified and was performed a review of existing methods for the leukocytes classification and were marked their deficiencies. According to these deficiencies, was chosen the leukocytes classification method based on the convolutional neural networks using, also was implemented a computer system which can be used as a support tool for solving leukocytes classification task on blood cell images. The computational experiment and testing showed high efficiency of the implemented computer system relative to the majority of existing the leukocytes classification methods. In this project we used Python language with TensorFlow and Keras libraries.

Keywords. white blood cell; leukocytes classification; computational intelligence; deep learning; convolutional neural network;

MSC2010. 68T05; 97R40; 92C55; 92B20.

1   Introduction

The calculation and evaluation of leukocyte formula, which shows the total concentration of leukocytes contained in the blood and the percentage ratio of their various types, are among the most important aspects of a clinical blood test. The leukocytes classification task remains relevant and important to this day, since even relatively small deviations in the leukocyte formula can show important information about the human health. Determination of quantitative and morphological features of leukocytes in the blood is used to diagnose not only blood diseases, but a wide range of other diseases, and also to assess of the state of human health.

As an support tool of medical diagnostics, automated white blood cells classification methods on blood cell images have been actively used for a long time. The accuracy of these methods is relatively high, but due to the wide variability of the cells, it is impossible to fully rely on the results obtained using these methods. It is easy to notice that those existing leukocytes classification methods, which show the greatest efficiency in practice, are often based on the principles of using artificial neural networks and machine learning methods.

Using of mathematical model based on functionality of the real biological structures of nerve cells has a large number of advantages for solving a wide class of tasks. Using of deep learning eliminates the need to manual determination the features and attributes of an object for its analysis and classification with replacing complex calculations with simple automatically trained models.

2   Program implementation

In this research was solved the leukocytes classification task on blood cell image relative to its two classes: mononuclear and polynuclear. For program implementation was used the convolutional neural network architecture, designed for image processing. Figure 1 shows the used convolutional neural network model, which contains three pairs of convolutional layers and subsampling layers, followed by fully-connected model with two layers. This fully-connected model are used as a classifier for the binary classification task. The created model was trained by an free data set from the Kaggle platform, containing more the 7 thousand classified leukocyres images. The training process took place over twenty epochs.
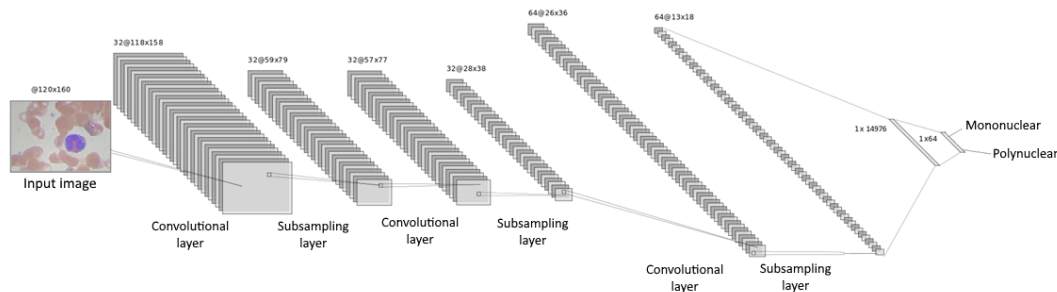

Figure 1: Used CNN model

3   Research results

The final network accuracy based on test data is 97%. Figure 2 shows obtained values of model error and accuracy while it was training.
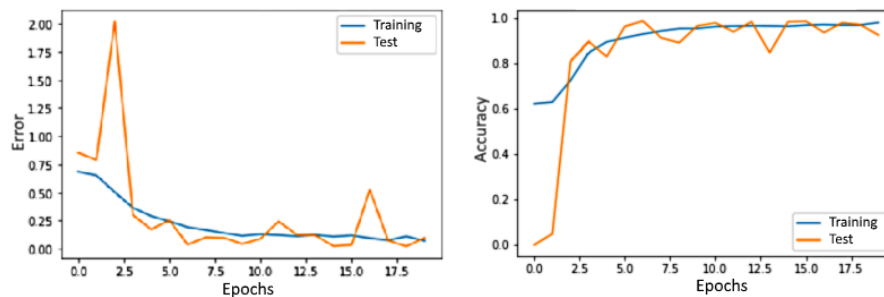

Figure 2: Model error and accuracy values

This computer system can be used as an support tool for the white blood cells classification in the framework of performing hematological blood analysis.

Acknowledgments

Fractional differential dynamics of planar microsupercapacitors based on arrays of carbon nanotubes

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Abstract. Planar micro-supercapacitors (PMSC) have small sizes and can be placed directly on a chip and integrated with other microelectronic devices. The present work is devoted to modeling of PMSC with electrodes consisting of multi-walled carbon nanotubes (MWCNT) filled by solid organic electrolyte. Using the anomalous diffusion model, we derive an equivalent circuit of PMSC. It consists of constant phase elements corresponding to ion subdiffusion in the CNT array and in the inter-electrode space, a capacitor and a resistor related to electric double layer capacity and electrolyte resistance. Based on measured impedance spectra, parameters of equivalent circuit were determined for different samples and conditions. Memory effect peculiar to fractional-order systems and observed in PMSC at discharging process is discussed.

Keywords. anomalous diffusion, lithium-ion cell, fractional derivative, equivalent circuit

MSC2010. 26A33; 76R50

1   Introduction

Recently, several studies have been reported on supercapacitors based on planar configuration (Fig. 1,a) utilizing graphene, carbon nanotubes (CNT) and conducting polymers with different forms and architecture (see review [1]). Planar micro-supercapacitors (PMSC) have typical sizes about 1 cm or even several millimeters and can be placed directly on a chip and integrated with other microelectronic devices. PMSC studied in this work were fabricated in "Technological Center" (Moscow) using standard microelectronics technologies [2]. Electrodes consist of multi-walled carbon nanotubes (MWCNT) (Fig. 1,b) synthesized by catalytic PECVD process, the system is filled by solid organic electrolyte. The measured voltammograms for produced PMSC demonstrate forms typical for devices with electric double layer without Faradaic reaction on electrode-electrolyte interface.

2   Main results

Using the anomalous diffusion model, we derive an equivalent circuit of PMSC (Fig. 1,c, inset). It consists of constant phase elements corresponding to ion subdiffusion in the CNT array and in the interelectrode space, a capacitor and a resistor related to electric double layer capacity and
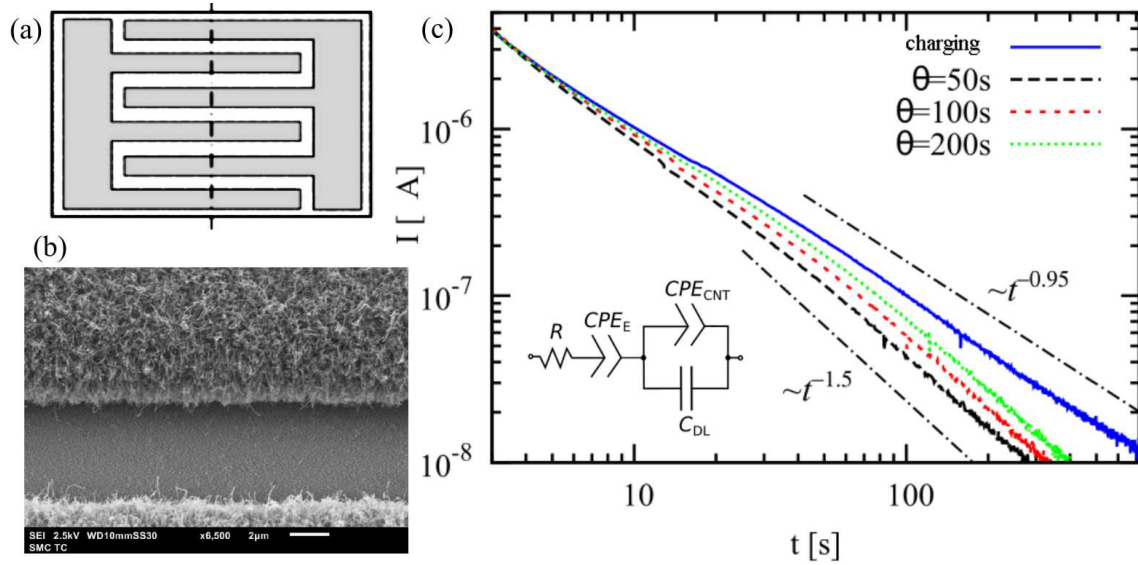
Figure 1: Planar configuration of studied PMSC (a); SEM image of MWCNT in PMSC (b), charging and discharging curves for different charging time θ (c).

electrolyte resistance. Based on measured impedance spectra, parameters of equivalent circuit were determined for different samples and conditions. Parameters of constant phase elements are determined by fitting impedance spectra. Their typical values are $\nu_E \approx 1$ and $\nu_{CNT} \approx 0.7$, that correspond to normal ion diffusion in inter-electrode space and subdiffusion in CNT arrays with characteristic exponent $\alpha = 2(1 - \nu) \approx 0.6$.

Under assumption of linear response, the equation for charging-discharging process is obtained by the inverse Fourier transformation of the impedance model. Due to the presence of constant phase elements, the equation contains fractional order derivatives that are operators non-local in time [3]. As a consequence, memory effect is peculiar to discharging process. The presence of memory is confirmed by potentiostatic measurements (see Fig. 1,c).

Acknowledgments

References

[1] Hu, H., Pei Z., Ye C., Recent advances in designing and fabrication of planar microsupercapacitors for on-chip energy storage. Energy Storage Materials 1 (2015) 82.

[2] Lebedev, E., Alekseyev, A., Gavrilin, I., Sysa, A., Kitsyuk, E., Ryazanov, R., Gromov, D. Integral planar supercapacitor with CNT-based composite electrodes for heat-sensitive MEMS and NEMS. In International Conference on Micro-and Nano-Electronics 2016 (Vol. 10224).

[3] Uchaikin, V. V., Sibatov, R.T. (2013). Fractional kinetics in solids: Anomalous charge transport in semiconductors, dielectrics, and nanosystems. World Scientific.

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